Starlight by Numbers

They say that mathematicians drink a toast which goes: “Here’s to pure mathematics – may it never be of any use to anyone.” Well by that score, I’m definitely not a mathematician; at least not a pure mathematician. Let’s face it, for many people (perhaps myself among them) mathematics reaches the parts of the brain that hurt, so when we do seek to solve a mathematical equation as we will from time to time, you can be sure that there’s a real reason for doing this.

One obvious reason is that the number that results from solving an equation may be of real importance to us; a less obvious reason, but one that is just as important and perhaps even more important to the learner is that a simple equation can be used to explore some part of astrophysics. The basic idea is to use a pocket calculator to try out or to plug different numbers into the equation; this enables you to get a “feel” for the kind of numbers that are involved in the solution to the equation (are they huge numbers or very small ones for example?). This process of “equation exploration” will also show you how the all-important solution to the equation actually depends on the different numbers that get plugged in. For example, will doubling an input number simply double the value of the answer or maybe multiply it by four. The result is that by doing this kind of thing you are guaranteed to gain a much deeper understanding of that particular bit of astrophysics.

You can if you wish ignore the equations we encounter without really losing anything, but if you have a calculator, then do have a go at using it to explore an equation; you’ll soon come to realize just how valuable and even enjoyable this is. As for the kinds of equations that we will come

K. Robinson, *Starlight*, Patrick Moore’s Practical Astronomy Series,
DOI 10.1007/978-1-4419-0708-0_2, © Springer Science+Business Media, LLC 2009

across, have a look further down at Equation (10); if an equation like this presents you with no problems, then feel free to go to the final section of this chapter on “Star Distances by Numbers.” The main purpose of this fairly short chapter is to show you how to solve equations such as this and thus hopefully give you a solid foundation and a smooth read through the rest of the book. For any other mathematical points that come up, we’ll deal with them only when the need arises in order to prevent you from getting “mathematical indigestion.” So here goes, starting with some very basic stuff about numbers.

Large Numbers and Small Numbers

Start with the number 100; a "1" followed by two zeros, which of course also equals 10×10, or two number 10s multiplied together. Similarly the number 1,000 – a "1" followed by three zeros is the same as three number 10 s multiplied together. The way that mathematicians and scientists write a number like, for example, 100,000 is 10^5. This is a shorthand way of writing the number "1" followed by five (5) zeros or five number 10 s multiplied together; so "100" becomes 10^2 and "1,000" becomes 10^3. This clearly avoids the need to write long strings of zeros, but it does more, as you might expect. One way of saying the number 10^3 (besides saying "one thousand") is of course "ten cubed," but a more precise way is to say "*ten to the power three*" or just "ten to the three," and then, for example, 10^5 can be spoken of as "ten to the power five" or "ten to the five," and so on. The process of taking a quantity "x" of the same number and multiplying them together is called *raising the number to the power* "x" and in particular, numbers such as 10^7, 10^8, etc., are often referred to as *powers of 10*. The actual number "x" – for example, the "5" in 10^5 – is called the *index of the power*, or just the index and the plural of index here is *indices*.

The Rule of Indices

If we multiply 100 by 1,000, we get 100,000, or using our new "*powers of 10*" notation,

$$10^2 \times 10^3 = 10^5 \tag{1}$$

So when we multiply two different powers of 10 together, we simply *add the indices together* to get the resulting power of 10. This is a very important and powerful rule in mathematics called the *rule of indices*, and it can be applied to numbers other than 10. For example, a very important number that is used a lot by both astronomers and physicists is the number 2.718 (to 3 places of decimals). Mathematicians give this number the symbol "e" just like they give the number 3.142 the symbol "π." So, for example,

$$e^7 \times e^5 = e^{12} = 2.718^7 \times 2.718^5 = 2.718^{12} \tag{2}$$

Provided the number whose powers are being taken (in this case "e" or 2.718) is the same throughout the equation, the rule works. The number 2.718^{12} is very large, by the way, and we'll see shortly how to write such a number, but first let's extend this powerful rule of indices.

What about the number 10 itself? It's simply the number 1 followed by one zero, so we should be able to write it as 10^1. We can check that this is okay by making sure it satisfies the rule of indices; so, for example, 10×100, which equals 1,000, can also be written $10^1 \times 10^2 = 10^3$; and yes, the indices do add together correctly. Also by virtue of our example using the number 2.718, we can say that any *number raised to the power* "1" *is just the number itself*; so "e^1" just equals "e." What about the number "1," or 1 followed by no zeros? In the powers of 10 notation this would be written 10^0, and this too satisfies the rule of indices because, for example, $10^0 \times 10^2 = 10^2$, which is the same as saying 1×100 equals 100. Once again the rule extends to all numbers so that *any number raised to the power zero is equal to* 1; so again, for example, $e^0 = 1$.

With what we've learned so far we can make very large numbers by raising a smaller number such as 10 or "e" to a very high power. But what about very small numbers? Start with the number 100,000 or 10^5; If we divide this by 100 or 10^2, we get 1,000 or 10^3. In other words,

$$\frac{10^5}{10^2} = 10^3 \tag{3}$$

So when we *divide* one power of 10 by another we have to subtract the index at the bottom; i.e., in the denominator from that at the top in the numerator. Another way to write Equation (3) is like this:

$$\frac{10^5}{10^2} = 10^5 \times \frac{1}{10^2} = 10^3 \tag{4}$$

So here we've turned the division of two powers of 10 into the multiplication of one power of 10 with another number that involves the reciprocal (the reciprocal of any number simply equals the number 1 divided by that number) of a power of 10. This has to satisfy the rule of indices and the only way that it can do this is to make $1/10^2$ equal to 10^{-2} because then we get

$$10^5 \times 10^{-2} = 10^3 \tag{5}$$

The indices check out because 5 + (–2) is the same as 5 – 2, which equals 3. This has also told us that a small number such as 1/100,000, or $1/10^5$, is written as 10^{-5}. Extending the idea again to our friend the number 2.718 or "e," the reciprocal of 2.718 or 1/e would be written as e^{-1}; it equals 0.368.

The Rule of Indices for All Indices

Any number can in fact be raised to a power that does not have to be either a positive or a negative whole number; an important example of this kind of power would be the number $x^{1/2}$. We can easily see the meaning of this number by multiplying it by itself and applying the rule of indices because then we get; $x^{1/2} \times x^{1/2} = x^1$, which just equals x. So $x^{1/2}$ is just the square root of x, and using the same procedure, $x^{1/3}$ is the cube root of x and so on.

A trickier problem is the meaning of something like $x^{3/8}$. We can in fact "kill two birds with one stone" here by thinking about a number such as $(10^5)^3$. Notice that this is not the same as $10^5 \times 10^3$, which would of course equal 10^8. Instead, this is the number 10^5 multiplied by itself 3 times – in other words, it's the number 10^5 *raised to the power* 3 (a number raised to a power, which is then itself raised to some other power). The number 10^5 multiplied by itself 3 times is the same as $10^5 \times 10^5 \times 10^5$, which of course equals 10^{15}. See how the number 15 is just equal to 5×3? So if we have a number that is raised to some power and *we raise it again* to some other power, we multiply the two powers together to get the final answer. So in general terms; $(x^y)^z$ is equal to $x^{y \times z}$ or just x^{yz}. This idea in fact extends to any number of indices, so for example $((e^2)^3)^4$ is equal to e^{24}. We see now that $x^{3/8}$ is the same as $(x^{1/8})^3$; i.e., the eighth root of x multiplied by itself 3 times. We shall use this important application of the rule of indices in chapter *A Star Story – 10 Billion Years in the Making*, where we need to be okay with the fact that $(x^3)^{1/2}$ is the same as $x^{3/2}$ or $x^{1.5}$.

Finally we can have numbers like $e^{-0.43}$; i.e., $2.718^{-0.43}$. This, however, is not the kind of thing to try and visualize in any way, nor to try and work out with pencil and paper. We shall find a need to be able to work out this sort of thing in chapter *Space – The Great Radiation Field*, and the best way is to find the key on your calculator labeled "x^y" or maybe "y^x." (If your calculator is not a scientific one then do give serious consideration to purchasing one – it will become your great friend.)

Try, for example, tapping in the number 2.512, then press the "x^y" key; now tap in the number 2.4 and finally press the "=" key to get the answer 9.121. You've just calculated the ratio of brightness for two stars whose magnitudes differ by 2.4.

Fortunately, working just with powers of 10 is much simpler, but the need to do so crops up all the time, so it pays to be comfortable when using them. Following are a few examples to illustrate how things work.

Working with Powers of Ten

So far we've learned how to multiply together two powers of 10; so for example

$$10^8 \times 10^5 = 10^{13} \tag{6}$$

This kind of operation can be extended to any number of terms on the left-hand side, so, for example

$$10^8 \times 10^5 \times 10^3 \times 10^7 = 10^{23} \tag{7}$$

A quantity such as $10^{11}/10^4$ can also be written $10^{11} \times 10^{-4}$, which of course equals 10^7, but note also that you may come across numbers such as $10^{-11} \times 10^4$, which in this case equals 10^{-7} $(-11 + 4 = -7)$. Finally, consider an expression such as $\frac{10^{11}}{10^{-4}}$ This is equivalent to $10^{11} \times \frac{1}{10^{-4}}$. Go back to old habits and think of the number 10^{-4} as one ten thousandth, and ask "How many times does one ten thousandth go into 1?" The answer is, of course, ten thousand times, or 10^4. So, a number such as $1/10^{-4}$ becomes 10^4. The general rule here is that an index in the bottom line or denominator of an equation can be simply moved up to the numerator or top line *provided you change the sign of the index*. So the above expression becomes

$$\frac{10^{11}}{10^{-4}} = 10^{11} \times 10^4 = 10^{15} \tag{8}$$

We can now put these ideas together; for example, $\frac{10^{-34}}{10^{-7}} \times 10^8$ becomes $10^{-34} \times 10^8 \times 10^7$, which equals 10^{-19}. The key to all of this is making sure that the signs (+ or –) of your indices are correct and that you add all the indices *algebraically* – i.e., you take the sign of each index into account.

Scientific Notation

Of course, most numbers do not consist simply of multiples or powers of 10; here, for example, is a fairly large number, 299792.0. This is, in fact, the speed of light in kilometers per second, and note that we've included the decimal point. If we divide this number by 10, the decimal point moves one place to the left to give us 29979.2, and dividing again by 10 we get 2997.92, giving us the speed of light divided by 100.

To get our speed of light back we could simply reverse the process and move the decimal point two places to the right, but another way to write the restored number would be 2997.92×100, or 2997.92×10^2. Pursuing this idea further, we can write the speed of light as 2.99792×10^5 km/s. This is the standard way to write down the speed of light (or indeed any large number) in what is called *scientific notation*. A very small number such as 0.0000005 can be written as 5.0×10^{-7} in this notation by multiplying by 10^7, thereby moving the decimal point 7 places to the right and then multiplying by 10^{-7} to give us back the original number. This number is approximately equal to the wavelength of green light in meters (more on this in chapter *From Light to Starlight*). The number 2.718^{12}, which we encountered earlier, can be evaluated using your calculator by entering 2.718 followed by pressing the "x^y" key followed by the number 12, and finally followed by the "=" key to give the number 162552.416. In scientific notation this number is written as 1.62552416×10^5.

Far from being just an esoteric way to write down large and small numbers, scientific notation, as we'll now see, makes it much easier to get a useful number out of an equation if the numbers we plug in are in this form. Here in normal number form is an equation that tells us how much energy is carried by the green light we just mentioned:

$$E(\text{energy}) = \frac{0.0000000000000000000000000000000006626 \times 299700000}{0.0000005} \tag{9}$$

This is not a pretty sight; but when the numbers are written in scientific notation it looks like this:

$$E = \frac{6.626 \times 10^{-34}}{5.0 \times 10^{-7}} \times 2.997 \times 10^8 \tag{10}$$

To get the actual value for the energy out of this equation is very straightforward; we divide the whole thing into two parts; one consisting of the ordinary numbers like the 6.626 and the other part consisting of the powers of 10 terms. Now "E" becomes

$$E = \frac{6.626 \times 2.997}{5.0} \times \frac{10^{-34}}{10^{-7}} \times 10^{8} \qquad (11)$$

We can now deal with this one a bit at a time. First use your pocket calculator to solve the "normal numbers" part and get the answer 3.972 to three decimal places, then put this number "to one side" while we do the powers of ten part. Using the rule of indices, we can put the 10^{-7} onto the top row by changing the –7 to +7 and then we just have to carefully add the indices together, taking into account their respective signs; this gives us

$$10^{-34} \times 10^{8} \times 10^{7} = 10^{-19} \qquad (12)$$

All we have to do now is to multiply this by the first bit of the calculation (the 3.972) to get the final answer – 3.972×10^{-19}.

It can often happen when doing a calculation this way that you end up with an answer that (just as an example) may look something like this: 122.434×10^{7}. You can if you wish leave it like this, but if you're a fussy sort of person then you should rearrange it to look like this: 1.22434×10^{9}.

If you're unfamiliar with doing calculations in scientific notation then using the above method exactly as described is good for practice; it gets you familiar with handling powers of 10 and with the way in which scientific notation works. However you can, if you wish, enter a number in scientific notation directly into your calculator. To do this with, for example, the number 6.672×10^{-11}, first enter 6.672 in the normal way. Then press the key marked "Exp" on your calculator; you'll now have two small zeros at the top right corner of your calculator display. Now enter the number 11; this will appear instead of the two zeros. Finally, press the key marked "+/–;" this will turn the "11" into a "–11." If you now want to multiply or divide this number by another scientific notation number simply press the "×" or the "÷" and enter the new number, remembering, of course, that if this number involves a positive power of 10 then there's no need to use the "+/–" key. Happy calculating!

So now we've calculated the amount of energy carried by light having a wavelength of 5.0×10^{-7} m, which is equal to 3.972×10^{-19}. But 3.972×10^{-19} what? The quantity of energy that we have calculated here is measured in units called *joules*, and judging by the fact that the answer we got is a very tiny number, maybe we should ask ourselves if the joule is perhaps too large a unit of energy for the kind of situation we are dealing with. On the other hand it may be that green light simply doesn't carry much energy and leave it at that. We'll say more about this kind of thing as the need arises; the important thing here is that you can handle a calculation like Equation (10) and be confident of getting the correct answer.

Star Distances by Numbers

One of the most important things that can be known about any star is its distance. This precious number enables astronomers to determine the true brightness of a star and thus its total power output, or *luminosity*. It can even enable an estimate to be made of the star's temperature. We'll see in due course how these things are done, but for the moment and really for the sake of completeness, we'll say a word or two about the meaning of the numbers, which are used when discussing the distances to the stars.

As amateur astronomers we fairly quickly get to know that, when talking about distances out beyond the Solar System we use the *light year* – the distance that a beam of light travels in 1 year. Light travels at a speed of 2.998×10^8 m/s, and the length of a "year" is 365.242 days (to three decimal places), or 3.15569088×10^7 s, and so it is a fairly straightforward calculation to work out that a light year is the mighty distance of about 9.461×10^{15} m. Even so, the distances to the visible stars are of the order of around 10 to around 1,000 light years, with many stars in our galaxy being at much greater distances than this, of course. Professional astronomers, though, tend to use an alternative star distance measure – the *parsec*, or "pc" for short, which stands for "parallax second."

The parsec comes straight out of the only direct way to measure the distance to a star, which involves determining with the utmost care the apparent change in position of the star against a background of more distant stars (a background of distant galaxies is even better) over a period of six months – i.e., as Earth swings between opposite sides of its orbit around the Sun, as shown in Fig. 1. Again as amateur astronomers we learn very soon that the only way to measure the "distance," or separation between any two points in the sky, is in terms of the angle between two lines drawn from these points, which intersect at our eyes as shown in Fig. 2.

Using this method, the angular separation between the two pointer stars of the Big Dipper (the Plough in the U.K.) is about 5°; the angular sizes of the Sun and Moon are both about half a degree, or 30 *arc minutes*. The apparent diameters of the planets are of the order of a few tens of *arc seconds* (or "arcsec," for short), where 1 arcsec is 1/3,600th of a degree. By the time we get to the separations of close double stars, we're talking of the order of maybe a few arc seconds, and it's well known to amateur observers that one of the most demanding tests for a telescope is its ability to separate or resolve close double stars.

The half yearly shift of even relatively nearby stars amounts to less than 1 arcsec, and so determining these shifts puts even large Earth-based telescopes at the very limits of their performance. The result is that until the *Hipparcos* satellite considerably improved things, it was very much the norm for star distances to be accurate only to about 10–20% and the limiting distance was about 300 light years, or about 100 pc.

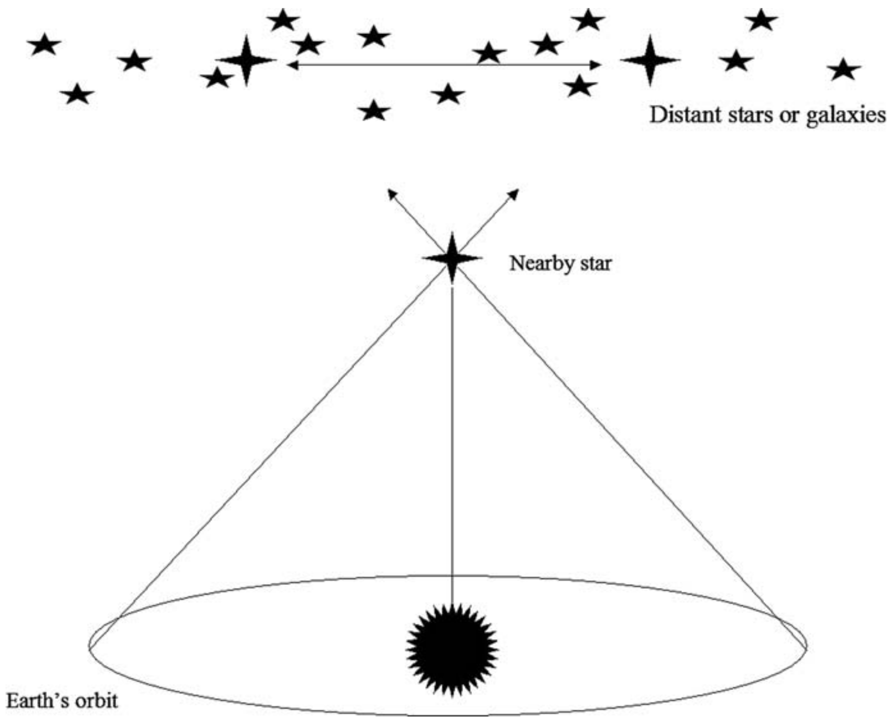

Figure 1. The apparent position of a nearby star shifts against a background of more distant stars or galaxies over a period six months as Earth swings between opposite points in its orbit. By measuring this (tiny) shift and knowing the Earth–Sun distance, simple trigonometry enables the distance to the star to be determined. In reality, the process is rather more involved than this simple diagram would suggest.

The actual *parallax* of a star is defined to be the angle between two lines running from the center of the Sun and a point in Earth's orbit whose distance from the Sun is 1 astronomical unit (A.U.) (1 A.U. is the average Earth–Sun distance, which is equal to 1.496×10^{11} m) and which intersect at the star itself (as shown in Fig. 3). Thus the parallax is equal to half of and not the whole apparent angular shift of a star over a period of six months. So the parallax is the angle at the apex of an extremely thin

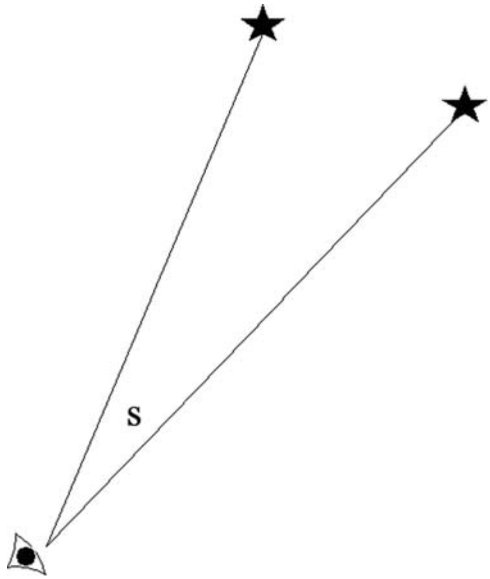

Figure 2. The apparent "separation" of two objects on the sky (this could also be the apparent shift of one star over a period of six months) is measured by the angle they subtend at the eye (or telescope). This angle is measured in degrees, minutes (or *arcminutes*), and seconds (or *arcseconds*).

right-angled triangle. The length of the base of this triangle is just 1 A.U., and if the parallax of a star were in fact equal to 1 arcsec (1/3,600th or 2.778×10^{-4} degrees), then its distance, using high school trigonometry, would be equal to 1 A.U. divided by the tangent of 1 arcsec. If you try this on your calculator you should get the answer 3.086×10^{16} m. *This distance is defined to be equal to* 1 *parsec*, and if we divide it by the number of meters in 1 light year, then we can see that 1 parsec is equal to 3.262 light years.

The advantage of using parsecs is that if you know the parallax of a star, then its distance in parsecs is just equal to the reciprocal of the parallax. So, as we've seen, a parallax of 1 arcsec corresponds to a distance of 1 parsec; a parallax of 0.5 arcsec results in a distance of 2 parsecs and so on. So, for example, the parallax of Proxima Centauri the nearest star is about 0.75 arcsec; the reciprocal of this is 1.333, which is the distance to Proxima in parsecs that, when multiplied by 3.262, gives us 4.35, or its distance in light years. The reason that this simple relationship works is because the baseline of these "parallax right-angled triangles" all have the

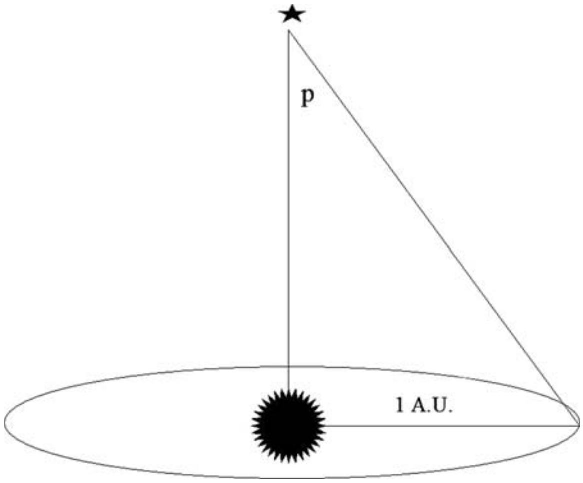

Figure 3. The angle "p" defines the "parallax" of a star; it is equal to half (not the whole) of the annual apparent shift of the star's position. This tiny angle thus forms the apex of a very narrow right-angled triangle whose base has a length of 1 Astronomical Unit.

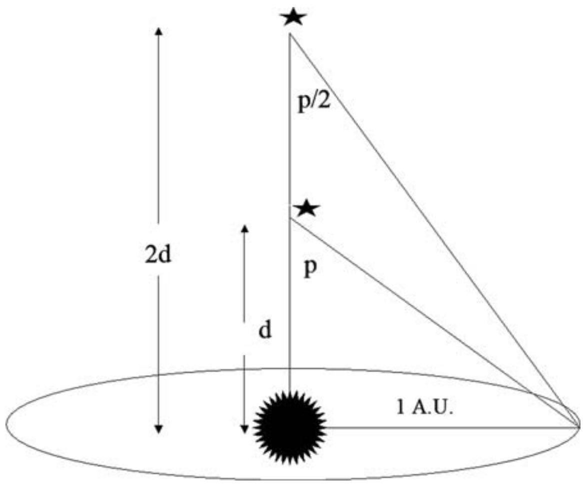

Figure 4. The use of right-angled triangles makes parallax trigonometry very simple. If the distance to a star is doubled, its parallax is halved; triple the distance, and its parallax becomes one third, and so on.

same base length, i.e., 1 A.U. (they are, in fact, what mathematicians call similar triangles), and so if the distance doubles, the parallax halves, etc., as shown in Fig. 4.

So there we have it; this is pretty well all the mathematics you'll need for now (oh! and do remember that number "e" or 2.718, which we'll meet again later) to hopefully get that extra bit out of this book. Read on!

Key Points

- When the same number is multiplied by itself "x" times, the number is said to be raised to the power "x."
- The number "x" is called the index of the power, or simply the index (plural "indices").
- If a number that is raised to the power "x" is multiplied by the same number raised to the power "y," then the result is the same number raised to the power "$x + y$;" the indices simply add together, and this is called the rule of indices.
- For any number "z, "z^1" equals "z" and "z^0" equals 1.
- For any number "z" and index "x," the number "z^{-x}" is equal to the reciprocal of z^x; i.e., $1/z^x$.
- Any number such as $(x^y)^z$ is equal to x^{yz}, i.e., when a number is successively raised to several different powers, the final result is equal to the number raised to the *product* of all the powers.
- Very large and very small numbers are usually written in scientific notation. For example, 5,000 is written as 5.0×10^3 and 0.005 is written as 5.0×10^{-3}.
- The number "e," which equals 2.718, is very important and should be remembered.
- A star at a distance of 1 parsec, or 1 pc, would have a parallax equal to 1 arcsec; no star is as close as this.
- The distance in parsecs to any star whose parallax is known is simply the reciprocal of the parallax.

http://www.springer.com/978-1-4419-0707-3

Starlight
An Introduction to Stellar Physics for Amateurs
Robinson, K.
2009, IX, 277 p. 74 illus., Softcover
ISBN: 978-1-4419-0707-3