Chapter 2
The Fourier method

9 Derivation of the heat equation

We consider a straight homogeneous metal rod of length l, cross-section S, and density ρ. We choose the axis x along the rod, and let $x = 0$ be the left end of the rod, so that $x = l$ is its right end. Denote by $u(x,t)$ the temperature of the rod at a point x at the moment $t > 0$. We assume that the cross-section is small, so that u depends only on x. It turns out that $u(x,t)$ satisfies the differential equation called the heat equation,

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}(x,t) + bf(x,t), \tag{9.1}$$

where $f(x,t)$ is the density of the external heat source at the point x at the moment t. This means that the piece $[x, x+\Delta x]$ of the rod during the time interval from t until $t+\Delta t$ receives from the outside the amount of heat equal to

$$Q_{external} = f(x,t)\Delta x \Delta t. \tag{9.2}$$

Let us derive (9.1). To do this, we write the equation of the heat balance for the piece of the rod $[x, x+\Delta x]$ as the time changes from t to $t+\Delta t$:

$$cm\Delta T = Q. \tag{9.3}$$

Here c is the specific heat capacity of the material, $m = \rho S \Delta x$ is the mass of the piece, and ΔT is the temperature increase:

$$\Delta T \approx u(x, t+\Delta t) - u(x,t). \tag{9.4}$$

Q is the total amount of heat received by the piece:

$$Q = Q_{external} + Q_l + Q_r, \tag{9.5}$$

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where $Q_{external}$ is the heat received from the external sources, Q_l is the amount of heat received from the left (that is, through the section of the rod at the point x), while Q_r is the amount of heat received from the right (that is, through the section of the rod at the point $x+\Delta x$). See Fig. 9.1.

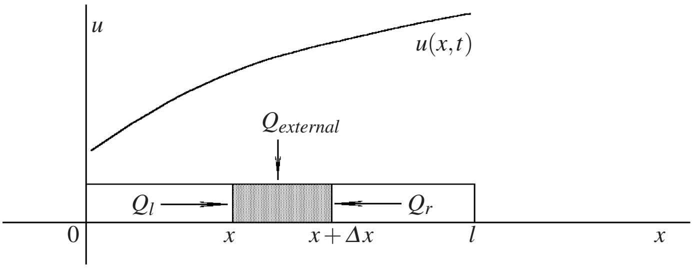

Fig. 9.1

According to the Fourier law of heating,

$$Q_l = -\lambda S \frac{\partial u}{\partial x}(x,t)\Delta t, \qquad Q_r = \lambda S \frac{\partial u}{\partial x}(x+\Delta x,t)\Delta t, \tag{9.6}$$

where λ is the heat transfer coefficient and S is the cross-section area of the rod. The relation (9.6) means that the rate of the heat transfer through the cross-section of the rod at the point x is proportional to the rate of change of the temperature, $\frac{\partial u}{\partial x}(x,t)$. Signs in (9.6) are chosen so that the heat is transferred from warmer bodies to cooler ones (the second law of thermodynamics). For example, for $u(x,t)$ on Fig. 9.1, $Q_l < 0$, $Q_r > 0$, while $\frac{\partial u}{\partial x} > 0$ everywhere, hence the signs in the left- and right-hand sides of (9.6) coincide.

Substituting (9.6) and (9.2) into (9.5), and then (9.5) and (9.4) into (9.3), we get

$$c\rho S\Delta x\Big(u(x,t+\Delta t)-u(x,t)\Big) \approx f(x,t)\Delta x\Delta t + \lambda S\Big(\frac{\partial u}{\partial x}(x+\Delta x,t)-\frac{\partial u}{\partial x}(x,t)\Big)\Delta t.$$

From here, dividing by $\Delta x\Delta t$ and considering the limit $\Delta x \to 0$ and $\Delta t \to 0$, we get

$$c\rho S\frac{\partial u}{\partial t} = \lambda S\frac{\partial^2 u}{\partial x^2} + f(x,t). \tag{9.7}$$

Then (9.1) follows, with the values of the constants being $a^2 = \frac{\lambda}{c\rho}$ and $b = \frac{1}{c\rho S}$.

10 Mixed problem for the heat equation

Here we will describe the basic idea of the Fourier method.

To determine the temperature of the rod, besides equation (9.1), one needs to specify the initial temperature

$$u(x,0) = \varphi(x), \qquad 0 < x < l \tag{10.1}$$

and the boundary conditions. For example, if the ends of the rod are submerged into the melting ice, then their temperature will be equal to zero (0° C):

$$u(0,t) = 0, \qquad u(l,t) = 0, \qquad t > 0. \tag{10.2}$$

The problem (9.1), (10.1), (10.2) is called the mixed problem for the heat equation.

For simplicity, we first assume that $f(x,t) \equiv 0$. The general case with the nonhomogeneity $f(x,t) \neq 0$ is considered in Section 15 below. Let us write the problem (9.1), (10.1), (10.2) (with $f \equiv 0$) in the operator form:

$$\begin{cases} \frac{d}{dt}\hat{u} = a^2 A\hat{u}(t), \qquad t > 0; \\ \hat{u}(0) = \hat{\varphi}. \end{cases} \tag{10.3}$$

Here $A = \frac{d^2}{dx^2}$, $\hat{u}(t) \equiv u(x,t)$, and $\hat{\varphi} \equiv \varphi(x)$. As it follows from the boundary conditions (10.2), $\hat{u}(t) \in C_0^2[0,l]$ for all $t > 0$, where

$$C_0^2[0,l] \equiv \{u(x) \in C^2[0,l] \,:\, u(0) = u(l) = 0\}.$$

Thus, we consider the operator $A = \frac{d^2}{dx^2}$ on the domain $D(A) = C_0^2[0,l]$.

The idea of the Fourier method is to try to find a solution to the problem (10.3) in the form of the sum of particular solutions of the form $T(t)X(x)$. Let us illustrate this idea on an example of the system of n ordinary differential equations with n unknown functions, also written in the vector form (10.3):

$$\begin{cases} \frac{d}{dt}\hat{u}(t) = A\hat{u}(t), \qquad \hat{u}(t) = \big(\hat{u}_1(t), \ldots, \hat{u}_n(t)\big) \in \mathbb{R}^n, \qquad t > 0; \\ \hat{u}(0) = \hat{\varphi} = (\hat{\varphi}_1, \ldots, \hat{\varphi}_n) \in \mathbb{R}^n, \end{cases} \tag{10.4}$$

where A is a matrix of size $n \times n$. Assume that there is a basis of the eigenvectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ of the matrix A, with the eigenvalues λ_k:

$$A\boldsymbol{e}_k = \lambda_k \boldsymbol{e}_k, \qquad k = 1, \ldots, n. \tag{10.5}$$

Then the solution $\hat{u}(t)$ we are looking for, as well as the initial vector $\hat{\varphi}$, can be represented as

$$\hat{u}(t) = \sum_{k=1}^{n} T_k(t)\boldsymbol{e}_k, \qquad \hat{\varphi} = \sum \varphi_k \boldsymbol{e}_k.$$

Substituting into (10.4) we get

$$\sum_{k=1}^{n} \frac{dT_k(t)}{dt}\boldsymbol{e}_k = \sum_{k=1}^{n} \lambda_k T_k(t)\boldsymbol{e}_k, \qquad \sum_{k=1}^{n} T_k(0)\boldsymbol{e}_k = \sum_{k=1}^{n} \varphi_k \boldsymbol{e}_k,$$

hence

$$\frac{dT_k(t)}{dt} = \lambda_k T_k(t), \quad t > 0; \quad T_k(0) = \varphi_k.$$

We see that $T_k(t) = \varphi_k e^{\lambda_k t}$, and, therefore,

$$\hat{u}(t) = \sum_{k=1}^{n} \varphi_k e^{\lambda_k t} \boldsymbol{e}_k. \tag{10.6}$$

In what follows we will obtain the analogs of formulas (10.5)–(10.6) for the operator $A = \frac{d^2}{dx^2}$.

11 The Sturm – Liouville problem

Let us find in $D(A) = C_0^2[0,l]$ the eigenfunctions $X_1(x),\ X_2(x), \ldots$ of the operator A:

$$\begin{cases} AX_k = \lambda_k X_k, & k \in \mathbb{N}; \\ X_k \in D(A), & X_k \neq 0. \end{cases} \tag{11.1}$$

The relation (11.1) means that

$$\begin{cases} X_k''(x) = \lambda_k X_k(x), & 0 < x < l; \\ X_k(0) = X_k(l) = 0, & X_k(x) \not\equiv 0. \end{cases} \tag{11.2}$$

Remark 11.1. We will show below in Section 13 that the solution to the problem (10.3) in the basis $X_1, \ldots, X_k, \ldots$ of the eigenfunctions of the operator A has the form analogous to (10.6):

$$u(x,t) = \sum_{k=1}^{\infty} e^{a^2 \lambda_k t} \varphi_k X_k(x), \tag{11.3}$$

where φ_k are the components of $\hat{\varphi}$ in the basis $\{X_k \;:\; k \in \mathbb{N}\}$. Let us point out that in view of (11.1) each term in the series (11.3) satisfies the operator equation (10.3). Therefore any finite (partial) sum of this series also satisfies (10.3). The entire series (11.3) satisfies equation (10.3) if it allows termwise differentiation: once in t and twice in x. This is the case when the series converges sufficiently fast.

We introduce the notation

$$\langle u, v \rangle = \int_0^l u(x)\, v(x)\, dx \quad \text{for} \quad \forall u, v \in L^2[0,l].$$

Lemma 11.1. The operator $A = \frac{d^2}{dx^2}$ with the domain $D(A) = C_0^2[0,l]$ is symmetric and negative:

$$\left\langle \frac{d^2u}{dx^2}, v \right\rangle = \left\langle u, \frac{d^2v}{dx^2} \right\rangle, \qquad \forall u, v \in D(A), \tag{11.4}$$

$$\left\langle \frac{d^2u}{dx^2}, u \right\rangle < 0, \quad \forall u \in D(A), \qquad u(x) \not\equiv 0. \tag{11.5}$$

Proof. (i) The equality (11.4) means that

$$\int_0^l u''(x)\,v(x)\,dx = \int_0^l u(x)\,v''(x)\,dx. \tag{11.6}$$

To prove it, we integrate both sides of (11.6) by parts:

$$\int_0^l u''(x)\,v(x)\,dx = u'\,v\Big|_0^l \quad - \int_0^l u'(x)\,v'(x)\,dx, \tag{11.7}$$

$$\int_0^l u(x)\,v''(x)\,dx = u\,v'\Big|_0^l \quad - \int_0^l u'(x)\,v'(x)\,dx. \tag{11.8}$$

The boundary terms in the right-hand sides of (11.7) and (11.8) vanish since $v(0) = v(l) = 0$ and $u(0) = u(l) = 0$. Thus, the relation (11.6) is proved.
(ii) When $u = v$, it follows from (11.7) that

$$\left\langle \frac{d^2u}{dx^2}, u \right\rangle = \int_0^l u''(x)\,u(x)\,dx = -\int_0^l \big(u'(x)\big)^2\,dx \le 0.$$

This proves (11.5). Indeed, if $\int_0^l \big(u'(x)\big)^2\,dx = 0$, then $u'(x) \equiv 0$, $u(x) \equiv \text{const}$. But because of the boundary conditions $u(0) = u(l) = 0$ one concludes that $u(x) \equiv 0$, contradicting the condition $u(x) \not\equiv 0$ in (11.5).

Corollary 11.2. All the eigenvalues of the operator $A = d^2/dx^2$ are negative. Indeed, as it follows from (11.5),

$$0 > \left\langle \frac{d^2X_k}{dx^2}, X_k \right\rangle = \lambda_k \langle X_k, X_k \rangle.$$

The eigenfunctions X_k, X_n with different eigenvalues $\lambda_k \ne \lambda_n$ are orthogonal:

$$\int_0^l X_k(x)\,X_n(x)\,dx = 0.$$

Indeed, it follows from (11.4) that

$$\lambda_k \langle X_k, X_n \rangle = \langle AX_k, X_n \rangle = \langle X_k, AX_n \rangle = \lambda_n \langle X_k, X_n \rangle,$$

implying that $\langle X_k, X_n \rangle = 0$.

Solution of the Sturm – Liouville problem

From equation (11.2) we get

$$X_k(x) = A_k e^{\sqrt{\lambda_k}x} + B_k e^{-\sqrt{\lambda_k}x}. \tag{11.9}$$

Substituting this into the boundary conditions (11.2), we get

$$\begin{cases} A_k + B_k = 0, \\ A_k e^{\sqrt{\lambda_k}l} + B_k e^{-\sqrt{\lambda_k}l} = 0. \end{cases} \tag{11.10}$$

The matrix of this system should be degenerate, or else $A_k = B_k = 0$ and $X_k(x) \equiv 0$, contradicting (11.2). Thus, λ_k satisfy the characteristic equation

$$\det \begin{bmatrix} 1 & 1 \\ e^{\sqrt{\lambda_k}l} & e^{-\sqrt{\lambda_k}l} \end{bmatrix} = e^{-\sqrt{\lambda_k}l} - e^{\sqrt{\lambda_k}l} = 0.$$

It then follows that $e^{-\sqrt{\lambda_k}l} = e^{\sqrt{\lambda_k}l}$, hence $e^{2\sqrt{\lambda_k}l} = 1$. Therefore, $2\sqrt{\lambda_k}l = 2k\pi i$, $k \in \mathbb{Z}$, leading to

$$\sqrt{\lambda_k} = \frac{k\pi i}{l} \quad \Rightarrow \quad \lambda_k = -\left(\frac{k\pi}{l}\right)^2. \tag{11.11}$$

Here we may assume that $k \geq 0$. As one might have expected, $\lambda_k \leq 0$. Thus, the eigenvalues λ_k are found. Now let us find the eigenfunctions $X_k(x)$. For this, we take into account that the system (11.10) is degenerate. Therefore, these two equations are linearly dependent, and it suffices to consider only the first one: $B_k = -A_k$. In view of (11.11), we get:

$$X_k(x) = A_k\left(e^{\frac{k\pi i}{l}x} - e^{-\frac{k\pi i}{l}x}\right) = A_k 2i \sin\frac{k\pi x}{l}.$$

Here we applied the Euler formula

$$e^{i\varphi} - e^{-i\varphi} = (\cos\varphi + i\sin\varphi) - (\cos\varphi - i\sin\varphi) = 2i\sin\varphi.$$

Since the eigenfunctions X_k are defined up to a factor, we can finally set

$$X_k(x) = \sin\frac{k\pi x}{l}, \qquad k = 1, 2, \ldots.$$

Here we can assume that $k > 0$, since for $k = 0$ we have $X_0(x) \equiv 0$.
Answer.

$$\lambda_k = -\left(\frac{k\pi}{l}\right)^2, \qquad X_k(x) = \sin\frac{k\pi x}{l}, \qquad k = 1, 2, \ldots.$$

Properties of solutions to the Sturm – Liouville problem

Property 11.3. Completeness: $X_k(x)$ form a complete orthogonal set in $L^2(0,l)$ (this property is known from the theory of the Fourier series).

Property 11.4. Orthogonality:

$$\langle X_k, X_n \rangle = \int_0^l X_k(x) X_n(x)\, dx = 0 \quad \text{for} \quad k \neq n. \tag{11.12}$$

Property 11.5. Asymptotics: $\lambda_k \sim -k^2$ for $k \to \infty$. That is, there exists a limit

$$\lim_{k\to\infty} \frac{\lambda_k}{-k^2} > 0.$$

Problem 11.6. Check directly the orthogonality property (11.12) for X_k.

Solution. Since $k \neq n$,

$$\int_0^l \sin\frac{k\pi x}{l} \sin\frac{n\pi x}{l}\, dx = \frac{1}{2}\int_0^l \left(\cos\frac{(k-n)\pi x}{l} - \cos\frac{(k+n)\pi x}{l}\right) dx = 0.$$

Problem 11.7. Find the norm of X_k in $L^2(0,l)$.

Solution.

$$||X_k||^2 \equiv \int_0^l X_k^2(x)\, dx = \int_0^l \sin^2\frac{k\pi x}{l}\, dx = \int_0^l \frac{1-\cos\frac{2k\pi x}{l}}{2}\, dx = \frac{l}{2}. \tag{11.13}$$

Problem 11.8. Plot the graph of $X_k(x)$.

Solution. See Fig. 11.1.

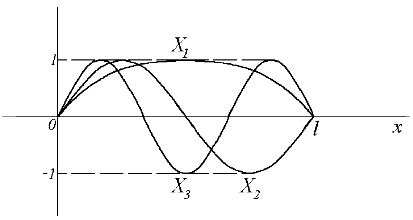

Fig. 11.1

Problem 11.9. Solve the Sturm – Liouville problem, that is, find the eigenfunctions of the operator $A \equiv \frac{d^2}{dx^2}$ on the interval $[0,l]$ for each of the boundary conditions:

$$X_k(0) = X_k'(l) = 0, \tag{11.14}$$
$$X_k'(0) = X_k(l) = 0, \tag{11.15}$$
$$X_k'(0) = X_k'(l) = 0. \tag{11.16}$$

Answer.
For (11.14), $\lambda_k = -\left(\frac{(k+\frac{1}{2})\pi}{l}\right)^2$, $X_k(x) = \sin\frac{(k+\frac{1}{2})\pi x}{l}$, $k = 0, 1, 2, \dots$. See Fig. 11.2.

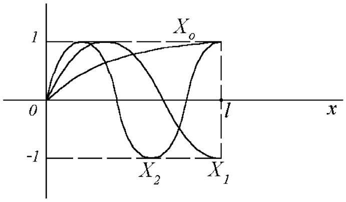

Fig. 11.2

For (11.15), $\lambda_k = -\left(\frac{(k+\frac{1}{2})\pi}{l}\right)^2$, $X_k(x) = \cos\frac{(k+\frac{1}{2})\pi x}{l}$, $k = 0, 1, 2, \dots$. See Fig. 11.3.

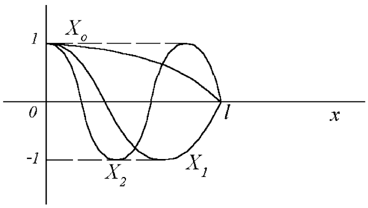

Fig. 11.3

For (11.16), $\lambda_k = -(\frac{k\pi}{l})^2$, $X_k(x) = \cos\frac{k\pi x}{l}$, $k = 0, 1, 2, \dots$. See Fig. 11.4.

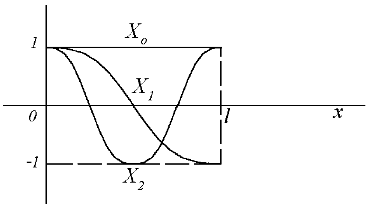

Fig. 11.4

One can also consider arbitrary boundary conditions of the form

$$\alpha_0 X_k'(0) + \beta_0 X_k(0) = 0, \qquad \alpha_1 X_k'(l) + \beta_1 X_k(l) = 0, \tag{11.17}$$

where $\alpha_{0,1}$ and $\beta_{0,1}$ are real numbers such that

$$\alpha_0^2 + \beta_0^2 \neq 0, \qquad \alpha_1^2 + \beta_1^2 \neq 0.$$

Problem 11.10. Prove that the operator $\frac{d^2}{dx^2}$ with the boundary conditions (11.17) is symmetric.

Remark 11.11. The eigenfunctions and the eigenvalues corresponding to each of the boundary conditions (11.14), (11.15), and (11.16) possess all the properties 11.3, 11.4, and 11.5 (completeness, orthogonality, the asymptotics of the eigenvalues) of solutions to the Sturm – Liouville problem (11.1). See [Pet91, SD64, TS90, Vla84].

Multidimensional eigenvalue problem

Let us consider an arbitrary bounded region $\Omega \subset \mathbb{R}$ with a smooth boundary $\partial\Omega$ and the problem of finding the eigenfunctions of the Laplace operator in Ω with the Dirichlet boundary conditions:

$$\triangle X_k(x) = \lambda_k X_k(x), \qquad x \in \Omega,$$

$$X_k\Big|_{\partial\Omega} = 0. \tag{11.18}$$

It turns out that its eigenfunctions corresponding to different λ_k are also orthogonal in $L^2(\Omega)$, while its eigenvalues λ_k are negative.

Problem 11.12. Prove that the Laplace operator with the boundary conditions (11.18) is symmetric and negative.

Problem 11.13. Prove that if instead of (11.18) one takes the Neumann boundary conditions,

$$\frac{\partial X_k}{\partial \boldsymbol{n}}\Big|_{\partial\Omega} = 0,$$

where $\dfrac{\partial}{\partial \boldsymbol{n}}$ stands for the derivative normal to $\partial\Omega$, then the Laplace operator is symmetric and non-positive, with $\lambda = 0$ the eigenvalue corresponding to the eigenfunction $X_0(x) \equiv 1$.

12 Eigenfunction expansions

As we already pointed out, the eigenfunctions $\sin\frac{k\pi x}{l}$, $k = 1, 2, \ldots$ form a complete orthogonal set in $L^2(0,l)$. Therefore they make up an orthogonal basis in $L^2(0,l)$ and, consequently, any function $\varphi(x) \in L^2(0,l)$ could be decomposed over this basis:

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k X_k(x). \tag{12.1}$$

Let us find the formula for the coefficients φ_k. This is accomplished with the aid of the orthogonality conditions (11.12): we multiply (12.1) by $X_k(x)$ and integrate from 0 to l. Then we get

$$\int_0^l \varphi(x)X_n(x)\,dx = \sum_{k=1}^{\infty} \varphi_k \int_0^l X_k(x)X_n(x)\,dx = \varphi_n \int_0^l X_n^2(x)\,dx, \tag{12.2}$$

since all the terms in the summation in (12.2) with numbers $k \neq n$ are equal to zero! Termwise integration of the series in (12.2) is justified since the series in (12.1) converges in $L^2(0,l)$, while the scalar product in $L^2(0,l)$ is continuous in each of the two arguments.

Then from (12.2) we get the desired expression for the coefficients:

$$\varphi_n = \frac{\int_0^l \varphi(x)X_n(x)\,dx}{\int_0^l X_n^2(x)\,dx} = \frac{2}{l}\int_0^l \varphi(x)X_n(x)\,dx, \tag{12.3}$$

where we took into account that $\|X_n\|^2 = l/2$ by (11.13).

Problem 12.1. Find the conditions on the function $\varphi(x)$ so that the following is true:

(i) The series (12.1) converges uniformly on the interval $[0,l]$;
(ii) The series (12.1) is termwise differentiable two times.

Solution. (i) It is sufficient (but not necessary) that

$$\sum_{k=1}^{\infty} |\varphi_k| < \infty. \tag{12.4}$$

For this inequality to hold, it suffices to require that

$$\varphi(x) \in C^1[0,l], \qquad \varphi(0) = \varphi(l) = 0. \tag{12.5}$$

Let us derive (12.4) from (12.5). Integrating by parts, we get:

$$\varphi_k = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi x}{l}\,dx = \frac{2}{l}\int_0^l \varphi(x)\frac{(-\cos\frac{k\pi x}{l})'}{\frac{k\pi}{l}}\,dx$$
$$= \frac{2}{k\pi}\left(-\varphi(x)\cos\frac{k\pi x}{l}\Big|_0^l + \int_0^l \varphi'(x)\cos\frac{k\pi x}{l}\,dx\right). \tag{12.6}$$

Above, the boundary terms are equal to zero due to the boundary conditions in (12.5). Therefore $\varphi_k = \frac{2}{k\pi}\varphi_k'$, where $\varphi_k' = \int_0^l \varphi'(x)\cos\frac{k\pi x}{l}\,dx$. But $\{\cos\frac{k\pi x}{l} \,:\, k\in\mathbb{N}\}$ is the orthogonal system in $L^2(0,l)$, with $\int_0^l \cos^2\frac{k\pi x}{l}\,dx = \frac{l}{2}$, hence, due to the Bessel inequality,

$$\sum_{k=1}^{\infty}|\varphi_k'|^2 \le \frac{2}{l}\int_0^l |\varphi'(x)|^2\,dx < \infty. \tag{12.7}$$

Therefore, applying the Cauchy – Bunyakovsky inequality, we get:

$$\sum_{k=1}^{\infty}|\varphi_k| = \sum_{k=1}^{\infty}\left|\frac{2}{k\pi}\varphi_k'\right| \le \left(\sum_{k=1}^{\infty}\left|\frac{2}{k\pi}\right|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}|\varphi_k'|^2\right)^{\frac{1}{2}} < \infty. \tag{12.8}$$

(ii) For the series (12.1) to be twice differentiable, it suffices to have the series for $\varphi''(x)$ converge uniformly in x. The latter takes place if

$$\sum_{k=1}^{\infty}k^2|\varphi_k| < \infty. \tag{12.9}$$

For this, we require that in addition to (12.5) we also have

$$\varphi(x)\in C^3[0,l] \quad \text{and} \quad \varphi''(0) = \varphi''(l) = 0. \tag{12.10}$$

Let us derive (12.9) from (12.10) and (12.5). For this, we remark that, due to (12.5) and (12.6),

$$\varphi_k = \frac{2}{k\pi}\int_0^l \varphi'(x)\cos\frac{k\pi x}{l}\,dx = \frac{2l}{(k\pi)^2}\left(\varphi'(x)\sin\frac{k\pi x}{l}\Big|_0^l - \int_0^l \varphi''(x)\sin\frac{k\pi x}{l}\,dx\right)$$
$$= \frac{2l^2}{(k\pi)^3}\left(\varphi''(x)\cos\frac{k\pi x}{l}\Big|_0^l - \int_0^l \varphi'''(x)\cos\frac{k\pi x}{l}\,dx\right).$$

The boundary terms vanished due to the boundary conditions (12.10) and due to $\sin\frac{k\pi x}{l}$ equal zero at $x=0$ and $x=l$. Therefore, $\varphi_k = \frac{-2l^2}{(k\pi)^3}\varphi_k'''$, where $\varphi_k''' = \int_0^l \varphi'''(x)\cos\frac{k\pi x}{l}\,dx$. But $\varphi'''\in L^2(0,l)$; thus, by (12.7),

$$\sum_{k=1}^{\infty}|\varphi_k'''|^2 \le \frac{2}{l}\int_0^l |\varphi'''(x)|^2\,dx < \infty,$$

and, similarly to (12.8),

$$\sum_{k=1}^{\infty} k^2|\varphi_k| = \sum_{k=1}^{\infty} k^2 \left| \frac{-2l}{(k\pi)^3}\varphi_k''' \right| \leq \frac{2l^2}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{|k|}|\varphi_k'''| < \infty.$$

Problem 12.2. Show that for a function $\varphi(x) \in C^{(N)}[0,l]$ the estimates

$$|\varphi_k| \leq \frac{C}{|k|^N}, \qquad k = 1, 2, \ldots \tag{12.11}$$

are satisfied if and only if

$$\varphi(0) = \varphi(l) = 0, \quad \varphi''(0) = \varphi''(l) = 0, \quad \ldots, \quad \varphi^{2n}(0) = \varphi^{2n}(l) = 0 \tag{12.12}$$

for all $2n \leq N-2$, $n = 0, 1, 2, \ldots$.

Let us point out that the boundary conditions (12.12) are satisfied, in particular, for all the eigenfunctions $\sin\frac{k\pi x}{l}$. On the other hand, under the condition (12.11), the series (12.1) is convergent on the interval $[0,l]$ uniformly together with its derivatives up to the order $N-2$. Therefore, since the homogeneous boundary conditions (12.12) are satisfied for the eigenfunctions $\sin\frac{k\pi x}{l}$, it follows that the same boundary conditions are also satisfied for the sum of the series (12.1). This proves the necessity of conditions (12.12) for (12.11).

Remark 12.3. Similarly, let us consider the decomposition of the function $\varphi(x)$ over a system of eigenfunctions $X_k(x)$ which corresponds to each of the boundary conditions (11.14), (11.15), and (11.16). For estimate (12.11) for the Fourier coefficients φ_k of this decomposition to be true, it is necessary that $\varphi(x)$ satisfy the same homogeneous boundary conditions as the eigenfunctions $X_k(x)$ and their derivatives up to the order $N-2$. When $\varphi \in C^{(N)}[0,l]$, it is easy to check that these conditions are not only necessary but also sufficient for (12.11).

Problem 12.4. Solve Problem 12.2 for the decomposition over the eigenfunctions of the Sturm – Liouville problem with each of the boundary conditions (11.14), (11.15), and (11.16).

Problem 12.5. Decompose over the set $\{\sin\frac{k\pi x}{l} \ : \ k \in \mathbb{N}\}$, the following functions:

a. $\varphi(x) \equiv 1$, $0 < x < l$. See Fig. 12.1.

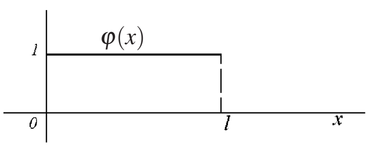

Fig. 12.1

Solution. $\varphi_k = \frac{2}{l}\int\limits_0^l \sin\frac{k\pi x}{l}\,dx = -\frac{2}{l}\frac{\cos\frac{k\pi x}{l}}{\frac{k\pi}{l}}\Big|_0^l = \frac{2}{k\pi}\left[1-(-1)^k\right].$

Let us point out that now the condition (12.4) is not satisfied. This is because $\varphi(x)\equiv 1$ is not equal to zero at the ends of the interval.

b. $\varphi(x)\equiv x$, $0<x<l$. See Fig. 12.2.

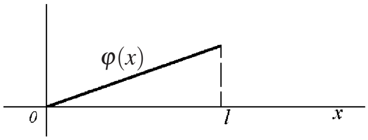

Fig. 12.2

Solution.

$$\varphi_k = \frac{2}{l}\int\limits_0^l x\sin\frac{k\pi x}{l}\,dx = \frac{2}{l}\int\limits_0^l x\frac{\left(-\cos\frac{k\pi x}{l}\right)'}{\frac{k\pi}{l}}\,dx = \ldots = -\frac{2}{k\pi}l(-1)^k.$$

Here $|\varphi_k|\sim\frac{1}{k}$ because $\varphi(l)\neq 0$ (see (12.11) and (12.12)).

c. $\varphi(x)=x(l-x)$. See Fig. 12.3.

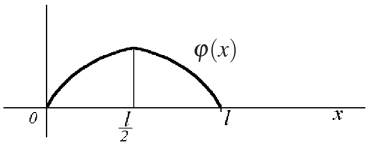

Fig. 12.3

Is it true that $\varphi_k = O(\frac{1}{k})$, or $O(\frac{1}{k^2})$, or $O(\frac{1}{k^3})$, ...?

Problem 12.6. Decompose the functions $\varphi(x)=1, x, x^2, x(l-x)$ over the eigenfunctions of the Sturm – Liouville problem with each of the boundary conditions (11.14), (11.15), and (11.16). In each of these cases, find the asymptotics:

$$\varphi_k = O\left(\frac{1}{k}\right),\quad O\left(\frac{1}{k^2}\right),\quad \ldots.$$

Hint. Use Remark 12.3.

13 The Fourier method for the heat equation

So, let us solve the problem (10.3):

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(l,t) = 0, \qquad t > 0, \\ u(x,0) = \varphi(x), \qquad 0 < x < l. \end{cases} \tag{13.1}$$

Let us look for a solution to the problem (13.1) in the form of the series

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) X_k(x), \qquad X_k(x) = \sin \frac{k\pi x}{l}. \tag{13.2}$$

Due to the completeness of the set of eigenfunctions

$$\left\{ \sin \frac{k\pi x}{l} \, : \, k \in \mathbb{N} \right\} \tag{13.3}$$

in $L^2(0,l)$, one can write in the form (13.2) any function $u(x,t)$ as long as $u(x,t) \in L^2(0,l)$ for each fixed t. The choice of the basis (13.3) is dictated by the boundary conditions which appear in (13.1). Namely, each term of the series (13.2) satisfies these boundary conditions since $\sin \frac{k\pi x}{l}$, $k \in \mathbb{N}$, satisfy the boundary conditions in (11.2).

To find the solution $u(x,t)$, it remains to determine *temporal functions* $T_k(t)$ (the functions $X_k(x)$ are called the *spatial functions*). $T_k(t)$ are found substituting the series (13.2) into the equation and the initial condition in (13.1).

Determining the temporal functions

A. We substitute the series (13.2) into equation (13.1): For $t > 0$,

$$\sum_{k=1}^{\infty} T_k'(t) \sin \frac{k\pi x}{l} = -a^2 \sum_{k=1}^{\infty} T_k(t) \left(\frac{k\pi}{l} \right)^2 \sin \frac{k\pi x}{l}, \qquad 0 < x < l. \tag{13.4}$$

Here we interchanged the operators of differentiation, $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x^2}$, with the summation of the series. Below we will discuss why this interchange is allowed. The justification of the Fourier method is based on proving the validity of this interchange. In (13.4) we also used the identity

$$\frac{\partial^2}{\partial x^2} \sin \frac{k\pi x}{l} = -\left(\frac{k\pi}{l} \right)^2 \sin \frac{k\pi x}{l}$$

satisfied by the eigenfunctions of the Sturm – Liouville problem (11.1)–(11.2). Let us point out that the boundary conditions for the Sturm – Liouville problem have already been used.

Further, if the series in (13.4) converge in $L^2(0,l)$, then, due to the orthogonality of the basis $\{\sin\frac{k\pi x}{l} \,:\, k\in\mathbb{N}\}$, we get the following equations on the temporal functions $T_k(t)$:

$$T_k'(t) = -a^2\Big(\frac{k\pi}{l}\Big)^2 T_k(t) = -\Big(\frac{ak\pi}{l}\Big)^2 T_k(t), \quad t>0, \quad k=1,2,\ldots. \tag{13.5}$$

For each $k\in\mathbb{N}$, (13.5) is a homogeneous linear differential equation with constant coefficients. Let us write its characteristic equation:

$$\lambda = -\Big(\frac{ak\pi}{l}\Big)^2.$$

Then the general solution of (13.5) is given by

$$T_k(t) = C_k e^{-(\frac{ak\pi}{l})^2 t}. \tag{13.6}$$

Substituting this expression into (13.2), we get

$$u(x,t) = \sum_{k=1}^{\infty} C_k e^{-(\frac{ak\pi}{l})^2 t} \sin\frac{k\pi x}{l}. \tag{13.7}$$

B. The unknown constants C_k in (13.7) are found from the initial conditions. Namely, substituting the series (13.2) into the initial conditions in (13.1), we find:

$$\sum_{k=1}^{\infty} T_k(0)\sin\frac{k\pi x}{l} = \varphi(x), \qquad 0<x<l. \tag{13.8}$$

Hence, $T_k(0)$ coincide with the Fourier coefficients of the decomposition of the function $\varphi(x)$ over the set $\{\sin\frac{k\pi x}{l} \,:\, k\in\mathbb{N}\}$ (see (12.3)):

$$T_k(0) = \varphi_k \equiv \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi x}{l}\,dx. \tag{13.9}$$

Comparing with (13.6), we find

$$C_k = \varphi_k.$$

Thus, (13.7) takes the form

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k e^{-(\frac{ak\pi}{l})^2 t} \sin\frac{k\pi x}{l}. \tag{13.10}$$

Justification of the Fourier method for the heat equation

Does the series (13.10) indeed represent the solution to the problem (13.1)?
A. For $t > 0$, the series (13.10) converge for each $x \in [0,l]$. For example, let

$$\varphi(x) \in L^2(0,l). \tag{13.11}$$

Then the series (13.8) converge in the same space $L^2(0,l)$. Indeed, from the Cauchy – Bunyakovsky inequality,

$$|\varphi_k| \leq \frac{2}{l}\int_0^l |\varphi(x)|\,dx \leq \frac{2}{l}\Big(\int_0^l dx\Big)^{\frac{1}{2}}\Big(\int_0^l \varphi^2(x)\,dx\Big)^{\frac{1}{2}} \leq \text{const.}$$

Therefore, the series (13.10) for each fixed $t > 0$ is dominated by the series

$$\text{const}\sum_{k=1}^{\infty} e^{-(\frac{ak\pi}{l})^2 t} = \text{const}\sum_{k=1}^{\infty} e^{-\varepsilon k^2},$$

where $\varepsilon = (\frac{a\pi}{l})^2 t > 0$, which in turn is dominated by the convergent geometric series. Hence, according to the Weierstrass Theorem, the functional series (13.10) converges uniformly on $[0,l]$ for $\forall t > 0$ to a function which is continuous in x.

Corollary 13.1. The series (13.10) satisfies the boundary conditions (10.2).

B. The series (13.10) is a differentiable function in $x \in [0,l]$ for any $t > 0$. Indeed, according to the theorem about the termwise differentiation of a series,

$$\frac{\partial u}{\partial x}(x,t) = \sum_{k=1}^{\infty} \varphi_k e^{-(\frac{ak\pi}{l})^2 t}\Big(-\cos\frac{k\pi x}{l}\Big)\frac{k\pi}{l}, \tag{13.12}$$

as long as the series in the right-hand side converges uniformly in x on $[0,l]$. The last condition is satisfied for any $t > 0$ since the series (13.12) is dominated by the convergent series

$$\text{const}\frac{\pi}{l}\sum_{k=1}^{\infty} k e^{-\varepsilon k^2} < \infty.$$

C. The series (13.10) has derivatives in x and in t of all orders for $t > 0$. This is proved similarly to **B.**

Corollary 13.2. All termwise differentiations of series in (13.4) are justified, hence the series (13.10) satisfies the heat equation (9.1).

Finally, for $t = 0$ the series (13.10) satisfies the initial condition (10.1) in view of (13.8) and (13.9) in the following sense (prove this!):

$$||u(x,t) - \varphi(x)||_{L^2(0,l)} \to 0 \quad \text{for} \quad t \to 0+.$$

Remark 13.3. The condition (13.11) allows the function $\varphi(x)$ to have discontinuities: For example, let $\varphi(x) \equiv 0$ for $x < \frac{l}{2}$, $u(x) \equiv 1$ for $x < \frac{l}{2}$. Then the function $u(x,0) = \varphi(x)$ will be discontinuous. At the same time, the solution $u(x,t)$ for any $t > 0$ will be a smooth function on $[0,l]$! One says that the heat equation (9.1) "smoothens" the initial data.

Problem 13.4. Find the solution to the mixed problem

$$\begin{cases} \frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x < 5, \qquad t > 0; \\ u(0,t) = u(5,t) = 0; \\ u(x,0) = 1. \end{cases}$$

Solution. According to (13.10),

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k e^{-(\frac{3k\pi}{5})^2 t} \sin\frac{k\pi x}{5}, \tag{13.13}$$

where φ_k are found using (13.9):

$$\varphi_k = \frac{2}{5}\int_0^5 \sin\frac{k\pi x}{5}\,dx = \frac{2}{k\pi}\left[1-(-1)^k\right].$$

Problem 13.5. Find the limit of the solution (13.13) for $t \to \infty$.

Solution.

$$\begin{aligned} \lim_{t\to+\infty} u(x,t) &= \lim_{t\to+\infty} \sum_{k=1}^{\infty} \varphi_k e^{-(\frac{3k\pi}{5})^2 t} \sin\frac{k\pi x}{5} \\ &= \sum_{k=1}^{\infty} \varphi_k \lim_{t\to+\infty} e^{-(\frac{3k\pi}{5})^2 t} \sin\frac{k\pi x}{5} = \sum_{k=1}^{\infty} 0 = 0. \end{aligned} \tag{13.14}$$

Problem 13.6. Justify the interchange of taking the limit and the summation in (13.14).

Problem 13.7. Find the solution to the mixed problem

$$\begin{cases} u_t(x,t) = 4u_{xx}(x,t), \qquad 0 < x < 3, \qquad t > 0; \\ u(0,t) = 0, \qquad u_x(3,t) = 0; \\ u(x,0) = x. \end{cases} \tag{13.15}$$

Solution. Here the solution should be decomposed over the eigenfunctions of the Sturm – Liouville problem (11.14) (see Fig. 11.2):

$$u(x,t) = \sum_{k=0}^{\infty} T_k(t) \sin\frac{(k+\frac{1}{2})\pi x}{3}. \tag{13.16}$$

Substituting this series into (13.15), we obtain

$$\sum_{k=0}^{\infty} T_k'(t)\sin\frac{(k+\frac{1}{2})\pi x}{3} = 4\sum_{k=0}^{\infty} -\Big(\frac{(k+\frac{1}{2})\pi}{3}\Big)^2 T_k(t)\sin\frac{(k+\frac{1}{2})\pi x}{3}.$$

From this relation, for any $k = 0, 1, 2, \ldots$,

$$T_k'(t) = -\Big(\frac{2(k+\frac{1}{2})\pi}{3}\Big)^2 T_k(t) \quad \Rightarrow \quad T_k(t) = C_k e^{-\left(\frac{2(k+\frac{1}{2})\pi}{3}\right)^2 t}. \tag{13.17}$$

Substituting (13.16) into the initial condition of the problem (13.15), we get

$$\sum_{k=0}^{\infty} T_k(0)\sin\frac{(k+\frac{1}{2})\pi x}{3} = x \quad \Longrightarrow$$

$$T_k(0) = \frac{2}{3}\int_0^3 x\sin\frac{(k+\frac{1}{2})\pi x}{3}\,dx$$

$$= \frac{2}{3}x\frac{-\cos\frac{(k+\frac{1}{2})\pi x}{3}}{\frac{(k+\frac{1}{2})\pi}{3}}\Bigg|_0^3 + \frac{2}{3}\int_0^3 \frac{\cos\frac{(k+\frac{1}{2})\pi x}{3}}{\frac{(k+\frac{1}{2})\pi}{3}}\,dx = 0 + \frac{2}{3}\frac{\sin\frac{(k+\frac{1}{2})\pi x}{3}}{\left(\frac{(k+\frac{1}{2})\pi}{3}\right)^2}\Bigg|_0^3$$

$$= \frac{\frac{2}{3}\sin(k+\frac{1}{2})\pi}{\left(\frac{(k+\frac{1}{2})\pi}{3}\right)^2} = \frac{2}{3}\frac{(-1)^k 9}{(k+\frac{1}{2})^2\pi^2} = \frac{6(-1)^k}{(k+\frac{1}{2})^2\pi^2}.$$

Since $C_k = T_k(0)$, we may now substitute $T_k(t)$ given by (13.17) into (13.16), getting

$$u(x,t) = \sum_{k=0}^{\infty} \frac{6(-1)^k}{(k+\frac{1}{2})^2\pi^2} e^{-\frac{4\pi^2(k+\frac{1}{2})^2 t}{9}} \sin\frac{(k+\frac{1}{2})\pi x}{3}.$$

Problem 13.8. Find the solution to the mixed problem

$$\begin{cases} u_t(x,t) = 16u_{xx}(x,t), & 0 < x < 3, \qquad t > 0; \\ u_x(0,t) = u_x(3,t) = 0; \\ u(x,0) = x. \end{cases}$$

Problem 13.9. Find the limit $t \to \infty$ of the solution of the previous problem.

Answer.

$$\lim_{t\to\infty} u = \varphi_0 \equiv \frac{1}{3}\int_0^3 x\,dx = \frac{1}{3}\cdot\frac{9}{2} = \frac{3}{2}.$$

14 Mixed problem for the d'Alembert equation

Let us solve the mixed problem

$$\begin{cases} u_{tt}(x,t) = a^2 u_{xx}(x,t), \qquad 0 < x < l, \qquad t > 0; \\ u(0,t) = 0, \qquad u(l,t) = 0; \\ u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x). \end{cases} \tag{14.1}$$

Similarly to (10.3), it is written in the operator form as

$$\begin{cases} \frac{\partial^2 \hat{u}}{\partial t^2}(t) = a^2 A\hat{u}(t), \quad t > 0; \\ \hat{u}(0) = \varphi, \quad \frac{\partial \hat{u}}{\partial t}(0) = \psi. \end{cases}$$

Solution of the problem (14.1)

We will look for the solution in the form of the series (13.2):

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) \sin \frac{k\pi x}{l}. \tag{14.2}$$

A. Substituting (14.2) into (14.1), we formally get

$$\sum_{k=1}^{\infty} T_k''(t) \sin \frac{k\pi x}{l} = a^2 \sum_{k=1}^{\infty} -\left(\frac{k\pi}{l}\right)^2 T_k(t) \sin \frac{k\pi x}{l}.$$

From here, as long as these series converge in $L^2(0,l)$, we find the equations on the temporal functions (compare with (13.5)):

$$T_k''(t) = -\left(\frac{ak\pi}{l}\right)^2 T_k(t), \quad \forall k = 1, 2, \dots. \tag{14.3}$$

The general solution is (compare with (13.6)):

$$T_k(t) = A_k \cos \frac{ak\pi}{l} t + B_k \sin \frac{ak\pi}{l} t. \tag{14.4}$$

B. The unknown constants A_k and B_k are found from the initial conditions in (14.1):

$$\begin{cases} u(x,0) = \sum_{k=1}^{\infty} T_k(0) \sin \frac{k\pi x}{l} = \varphi(x) \quad \Rightarrow T_k(0) = \varphi_k \quad (\text{see } (13.9)), \\ u_t(x,0) = \sum\limits_{k=1}^{\infty} T_k'(0) \sin \frac{k\pi x}{l} = \psi(x) \ \Rightarrow \ T_k'(0) = \psi_k \equiv \frac{2}{l} \int\limits_0^l \psi_k(x) \sin \frac{k\pi x}{l}\, dx. \end{cases}$$

Substituting (14.4) in the above relation, we find:

$$\begin{aligned}
&T_k(0) = A_k = \varphi_k, \\
&T_k'(0) = B_k \frac{ak\pi}{l} = \psi_k \quad \Rightarrow \quad B_k = \frac{\psi_k}{\left(\frac{ak\pi}{l}\right)}.
\end{aligned} \tag{14.5}$$

Therefore, according to (14.4),

$$T_k(t) = \varphi_k \cos \frac{ak\pi}{l} t + \frac{\psi_k}{\left(\frac{ak\pi}{l}\right)} \sin \frac{ak\pi}{l} t.$$

Finally, substituting (14.5) into (14.2), we obtain:

$$u(x,t) = \sum_{k=1}^{\infty} \left(\varphi_k \cos \frac{ak\pi}{l} t + \frac{\psi_k}{\left(\frac{ak\pi}{l}\right)} \sin \frac{ak\pi}{l} t \right) \sin \frac{k\pi x}{l} t. \tag{14.6}$$

Question 14.1. While deriving (14.3), we again interchanged differentiation in x and t with the summation. Is this justified?

Justification of the Fourier method for the d'Alembert equation

A. Does the series (14.6) converge? It is dominated by the series

$$\text{const} \sum_{k=1}^{\infty} \left(|\varphi_k| + \frac{|\psi_k|}{k} \right).$$

For the convergence of this series, it suffices that

$$\begin{cases} \varphi(x) \in C^1[0,l], & \varphi(0) = \varphi(l) = 0; \\ \psi(x) \in C[0,l]. \end{cases}$$

This is proved similarly to the derivation of (12.4) from (12.5).
B. We need to be able to differentiate the series (14.6) twice in x and in t. For this, the convergence of the following series suffices:

$$\sum_{k=1}^{\infty} \left(k^2 |\varphi_k| + k |\psi_k| \right) < \infty. \tag{14.7}$$

For the convergence of this series, it is sufficient to have

$$\begin{cases} \varphi(x) \in C^3[0,l], & \varphi(0) = \varphi(l) = 0, & \varphi''(0) = \varphi''(l) = 0; \\ \psi(x) \in C^2[0,l], & \psi(0) = \psi(l) = 0. \end{cases} \tag{14.8}$$

This is proved analogously to the derivation of (12.9) from (12.10).

Conclusion. The series (14.6) is a solution to the problem (14.1) if the functions φ and ψ satisfy the conditions (14.8).

Remark 14.2. More precise (less restrictive) conditions on φ, ψ are given in terms of the Sobolev spaces (see Section 26 below).

Problem 14.3. Find the solution of the mixed problem

$$\begin{cases} u_t = 9u_{xx}(x,t), & 0 < x < 4, \quad t > 0; \\ u_x(0,t) = 0, & u(4,t) = 0; \\ u(x,0) = 0, & u_t(x,0) = 16 - x^2. \end{cases} \tag{14.9}$$

Solution. One needs to decompose the solution over the eigenfunctions of the Sturm – Liouville problem (11.15) (see Fig. 11.3):

$$u(x,t) = \sum_{k=0}^{\infty} T_k(t) \cos \frac{(k+\frac{1}{2})\pi x}{4}.$$

Substitution into (14.9) gives, similarly to (14.3),

$$T_k''(t) = -9\Big(\frac{(k+\frac{1}{2})\pi}{4}\Big)^2 T_k(t). \tag{14.10}$$

The initial conditions in (14.9) give

$$\begin{cases} T_k(0) = \varphi_k = 0, \\ T_k'(0) = \psi_k \equiv \dfrac{2}{4}\displaystyle\int_0^4 (16 - x^2) \cos \frac{(k+\frac{1}{2})\pi x}{4}\, dx = \dfrac{4^3(-1)^k}{(k+\frac{1}{2})^3 \pi^3}. \end{cases} \tag{14.11}$$

Let us point out that here $\varphi_k \equiv 0$, while $\psi(x)$ satisfies conditions similar to (14.8): $\psi(x) \equiv 16 - x^2 \in C^2[0,4]$; $\psi'(0) = \psi(4) = 0$, that is, $\psi(x)$ satisfies the same homogeneous boundary conditions as the eigenfunctions $X_k(x) = \cos \frac{(k+\frac{1}{2})\pi x}{4}$ do, and $|\psi_k| \le C/k^3$ due to Remark 12.3. Therefore, the estimate (14.7) takes place.

From (14.10) and (14.11) we find, similarly to (14.4) and (14.5):

$$T_k(t) = \frac{\psi_k \sin \frac{3(k+\frac{1}{2})\pi t}{4}}{\frac{3(k+\frac{1}{2})\pi}{4}}.$$

Answer.

$$u(x,t) = \sum_{k=1}^{\infty} \frac{256(-1)^k}{3\big((k+\frac{1}{2})\pi\big)^4} \sin \frac{3(k+\frac{1}{2})\pi t}{4} \cos \frac{(k+\frac{1}{2})\pi x}{4}.$$

15 The Fourier method for nonhomogeneous equations

The heat equation

A. Let us consider the mixed problem for the nonhomogeneous heat equation with the homogeneous boundary conditions (nonhomogeneous boundary conditions in Section 16 below will be the next step in developing the Fourier method):

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad 0 < x < l; \\ u(0,t) = 0, \qquad u(l,t) = 0; \\ u(x,0) = \varphi(x). \end{cases} \tag{15.1}$$

Again, we look a solution of this problem in the form (13.2):

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) \sin \frac{k\pi x}{l}. \tag{15.2}$$

The new step will be the decomposition of $f(x,t)$ over the eigenfunctions of the Sturm – Liouville problem:

$$f(x,t) = \sum_{k=1}^{\infty} f_k(t) \sin \frac{k\pi x}{l}; \qquad f_k(t) = \frac{2}{l} \int_0^l f(x,t) \sin \frac{k\pi x}{l}\, dx. \tag{15.3}$$

This decomposition is possible due to the completeness of the family of eigenfunctions $\sin \frac{k\pi x}{l}$, $k \in \mathbb{N}$, in the space $L^2(0,l)$ as long as $f(x,t) \in L^2(0,l)$ for each fixed $t > 0$.
B. For finding the temporal functions $T_k(t)$, we substitute decompositions (15.2), (15.3) into (15.1):

$$\sum_{k=1}^{\infty} T_k'(t) \sin \frac{k\pi x}{l} = -a^2 \sum_{k=1}^{\infty} \left(\frac{k\pi}{l}\right)^2 T_k(t) \sin \frac{k\pi x}{l} + \sum_{k=1}^{\infty} f_k(t) \sin \frac{k\pi x}{l}. \tag{15.4}$$

From here, due to the orthogonality of the family of eigenfunctions, we get

$$T_k'(t) = -\left(\frac{ak\pi}{l}\right)^2 T_k(t) + f_k(t), \qquad t > 0, \quad k = 1, 2, \ldots. \tag{15.5}$$

Thus, the differential equation on the temporal functions is obtained. For the unique determination of these functions, one needs to take into account the initial condition from (15.1):

$$\sum_{k=1}^{\infty} T_k(0) \sin \frac{k\pi x}{l} = \varphi(x) \quad \Rightarrow \quad T_k(0) = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{k\pi x}{l}\, dx.$$

Let us point out that the boundary conditions in (15.1) are automatically satisfied due to decomposition (15.2) (since they are satisfied for the eigenfunctions $\sin\frac{k\pi x}{l}$) as long as $T_k(t) = O\left(\frac{1}{k^2}\right)$.

C. Let us apply this scheme to particular problems.

Problem 15.1. Solve the mixed problem

$$\begin{cases} u_t = 16u_{xx} + 2, \qquad 0 < x < 7, \quad t > 0; \\ u_x(0,t) = u(7,t) = 0; \\ u(x,0) = 0. \end{cases} \tag{15.6}$$

Solution. As it follows from the boundary conditions, the solution should be decomposed over the eigenfunctions of the Sturm – Liouville problem (11.15) (see Fig. 11.3):

$$u(x,t) = \sum_{k=0}^{\infty} T_k(t)\cos\frac{(k+\frac{1}{2})\pi x}{7}. \tag{15.7}$$

Substituting this series into (15.6), we get the equation similar to (15.5):

$$T_k'(t) = -\left(\frac{4(k+\frac{1}{2})\pi}{7}\right)^2 T_k + f_k, \qquad t > 0, \quad k = 1, 2, \dots, \tag{15.8}$$

where

$$f_k \equiv \frac{2}{7}\int_0^7 2\cos\frac{(k+\frac{1}{2})\pi x}{7}\,dx = \frac{4}{7}\,\frac{\sin\frac{(k+\frac{1}{2})\pi x}{7}}{\frac{(k+\frac{1}{2})\pi}{7}}\Bigg|_0^7 = 4\frac{(-1)^k}{(k+\frac{1}{2})\pi}. \tag{15.9}$$

As it follows from the initial condition of the problem,

$$T_k(0) = 0. \tag{15.10}$$

Let us solve the problem (15.8), (15.10). The general solution to (15.8) has the form

$$T_k(t) = T_k^0(t) + T_k^p(t), \tag{15.11}$$

where $T_k^0(t)$ is the general solution to the homogeneous equation,

$$T_k^0(t) = C_k e^{-\left(\frac{4(k+\frac{1}{2})\pi}{7}\right)^2 t}, \tag{15.12}$$

and a particular solution $T_k^p(t)$ to the nonhomogeneous equation (15.8) is a constant. Substituting $T_k^p(t) = A_k$. into (15.8), we get

$$0 = -\left(\frac{4(k+\frac{1}{2})\pi}{7}\right)^2 A_k + f_k,$$

$$A_k = \frac{49 f_k}{16\big((k+\frac{1}{2})\pi\big)^2} = \frac{49(-1)^k}{4\big((k+\frac{1}{2})\pi\big)^3}. \tag{15.13}$$

Substituting (15.12) and (15.13) into (15.11), we get

$$T_k(t) = C_k e^{-\left(\frac{4(k+\frac{1}{2})\pi}{7}\right)^2 t} + \frac{49(-1)^k}{4\big((k+\frac{1}{2})\pi\big)^3}. \tag{15.14}$$

Now we need to take into account the initial condition (15.10):

$$0 = C_k + \frac{49(-1)^k}{4\big((k+\frac{1}{2})\pi\big)^3} \quad \Rightarrow \quad C_k = -\frac{49(-1)^k}{4\big((k+\frac{1}{2})\pi\big)^3}.$$

Finally, substituting (15.14) into (15.7), we get

$$u(x,t) = \sum_{k=0}^{\infty} (-1)^k \frac{49}{4\big((k+\frac{1}{2})\pi\big)^3} \Big(-e^{-(\frac{4(k+\frac{1}{2}\pi}{7})^2 t} + 1\Big) \cos\frac{(k+\frac{1}{2})\pi x}{7}. \tag{15.15}$$

Problem 15.2. Find the limit of the solution to the problem (15.6) as $t \to +\infty$.

Solution. Taking the limit $t \to \infty$ in each term in the series (15.15), we get (justify!)

$$u_\infty(x) \equiv \lim_{t\to+\infty} u(x,t) = \sum_{k=0}^{\infty} \frac{49(-1)^k}{4\big((k+\frac{1}{2})\pi\big)^3} \cos\frac{(k+\frac{1}{2})\pi x}{7}. \tag{15.16}$$

Let us compute the sum of this series. For this, we notice that

$$u'_\infty(x) = -\sum_{k=0}^{\infty} \frac{7}{4} \frac{(-1)^k}{\big((k+\frac{1}{2})\pi\big)^2} \sin\frac{(k+\frac{1}{2})\pi x}{7}, \tag{15.17}$$

$$u''_\infty(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k}{4(k+\frac{1}{2})\pi} \cos\frac{(k+\frac{1}{2})\pi x}{7} = -\frac{1}{8} \tag{15.18}$$

where the last equality follows from decomposition (see (15.9))

$$2 = \sum_{k=0}^{\infty} \frac{4(-1)^k}{(k+\frac{1}{2})\pi} \cos\frac{(k+\frac{1}{2})\pi x}{7}.$$

Integrating twice the identity (15.18), we get

$$u_\infty(x) = \frac{1}{16}(-x^2 + C_1 x + C_2). \tag{15.19}$$

To find C_1 and C_2, we notice that due to (15.16) and (15.17)

$$u_\infty(7) = 0, \qquad u'_\infty(0) = 0.$$

Substituting the expression (15.19) into the above relations, we find $C_1 = 0,\ C_2 = 49$; hence, $u_\infty(x) = \frac{1}{16}(49 - x^2)$.

Remark 15.3. We could obtain u_∞ directly from (15.6), without using the nonstationary solution (15.15). To do so, we substitute u_t by 0 and solve the problem

$$\begin{cases} 0 = 16u''_\infty(x) + 2, & 0 < x < 7; \\ u'_\infty(0) = 0, & u_\infty(7) = 0. \end{cases} \tag{15.20}$$

Remark 15.4. The important property of the heat equation is that under stationary external conditions (that is, when the nonhomogeneous terms of the equation and the boundary conditions do not depend explicitly on t) the solution $u(x,t)$ stabilizes as $t \to +\infty$:

$$u(x,t) \to u_\infty(x), \qquad t \to +\infty. \tag{15.21}$$

The limit function $u_\infty(x)$ is the solution to the corresponding stationary problem. The only exception is the case of perfect insulation at both ends ($\frac{\partial u}{\partial x} = 0$ at $x = 0$ and $x = l$) and the nonhomogeneity is nonzero. In this case, there is no limit stationary state $u_\infty(x)$. If, for example, the nonhomogeneity in the equation is a positive constant (permanent heat influx), then the temperature growth is unbounded.

Problem 15.5. Find the limit as $t \to +\infty$ of the solution to the mixed problem

$$\begin{cases} u_t = 25u_{xx}(x,t) + 3x^2, & 0 < x < 6; \\ u(0,t) = 0, \quad u'(6,t) = 1; \\ u(x,0) = \sin x. \end{cases} \tag{15.22}$$

Solution. As we said above, we get from (15.22) and (15.21) the following boundary value problem for $u_\infty(x) = \lim_{t\to\infty} u(x,t)$:

$$\begin{cases} 0 = 25u''_\infty(x) + 3x^2, & 0 < x < 6; \\ u_\infty(0) = 0, & u'_\infty(6) = 1. \end{cases}$$

Integrating this equation, we get $u_\infty(x) = -\frac{x^4}{100} + C_1x + C_2$. From the boundary conditions we get $C_2 = 0$, $-\frac{6^3}{25} + C_1 = 1$.

Answer. $u_\infty(x) = -\frac{x^4}{100} + \frac{241}{25}x$.

The wave equation

Let us consider the nonhomogeneous wave equation.

Problem 15.6. Solve the following mixed problem (where $\omega > 0$):

$$\begin{cases} u_{tt}(x,t) = 25u_{xx} + x(3-x)\sin\omega t, & 0 < x < 3, \quad t > 0; \\ u(0,t) = u(3,t) = 0; \\ u(x,0) = 0, & u_t(x,0) = 0. \end{cases} \tag{15.23}$$

Solution. **A.** In view of the boundary conditions in (15.23), we are looking for the solution u in form of the decomposition over the eigenfunctions of the Sturm – Liouville problem (11.1):

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) \sin \frac{k\pi x}{3}. \tag{15.24}$$

For this, the function $x(3-x)\sin \omega t$ in equation (15.23) is also decomposed into the series over the system $\sin \frac{k\pi x}{3}$:

$$x(3-x)\sin \omega t = \sum_{k=1}^{\infty} g_k \sin \frac{k\pi x}{3} \sin \omega t, \tag{15.25}$$

where $g_k = \frac{2}{3}\int\limits_0^3 x(3-x)\sin \frac{k\pi x}{3}\, dx = \frac{36}{(k\pi)^3}\left(1-(-1)^k\right)$.

B. Finding the temporal functions $T_k(t)$. Substituting decomposition (15.24) and (15.25) into equation (15.23) and using the orthogonality of the family $\sin \frac{k\pi x}{3}$, we get, similarly to (14.3),

$$T_k''(t) = -\left(\frac{5k\pi}{3}\right)^2 T_k(t) + g_k \sin \omega t. \tag{15.26}$$

Substitution of the series (15.24) into the initial conditions (15.23) gives

$$T_k(0) = 0, \qquad T_k'(0) = 0. \tag{15.27}$$

The Cauchy problem (15.26)–(15.27) uniquely determines the temporal functions $T_k(t)$.

It is known that the general solution to equation (15.26) has the form

$$T_k(t) = T_k^0(t) + T_k^p(t), \tag{15.28}$$

where $T_k^0(t)$ is the general solution of the corresponding homogeneous equation

$$T_k^0(t) = A_k \cos \frac{5k\pi}{3}t + B_k \sin \frac{5k\pi}{3}t,$$

while $T_k^p(t)$ is a particular solution to the nonhomogeneous equation (15.26).

When finding a particular solution, one needs to distinguish two cases: the *resonant case* and the *non-resonant case*.

1. Non-resonant case: For all $k \in N$,

$$\omega \neq \frac{5k\pi}{3}. \tag{15.29}$$

Then $T_k^p(t)$ are to be looked for in the form

$$T_k^p(t) = C_k \sin \omega t.$$

Substitution into (15.26) gives

$$-\omega^2 C_k \sin \omega t = -\Big(\frac{5k\pi}{3}\Big)^2 C_k \sin \omega t + g_k \sin \omega t,$$

from where, in view of (15.29),

$$C_k = \frac{g_k}{\big(\frac{5k\pi}{3}\big)^2 - \omega^2}.$$

Then (15.28) takes the form

$$T_k(t) = A_k \cos \frac{5k\pi}{3} t + B_k \sin \frac{5k\pi}{3} t + \frac{g_k \sin \omega t}{\big(\frac{5k\pi}{3}\big)^2 - \omega^2}.$$

Finally, the initial conditions (15.27) yield

$$A_k = 0, \quad B_k \frac{5k\pi}{3} + \frac{g_k \omega}{(\frac{5k\pi}{3})^2 - \omega^2} = 0 \quad \Rightarrow \quad B_k = -\frac{g_k \omega}{\frac{5k\pi}{3}\big((\frac{5k\pi}{3})^2 - \omega^2\big)}.$$

Thus, in the case when (15.29) is satisfied for all $k = 1, 2, \ldots$, we have

$$u(x,t) = \sum_{k=1}^{\infty} \frac{g_k}{\big(\frac{5k\pi}{3}^2\big) - \omega^2} \Big(-\frac{\omega}{\big(\frac{5k\pi}{3}\big)} \sin\Big(\frac{5k\pi}{3} t\Big) + \sin \omega t\Big) \sin \frac{k\pi x}{3}. \tag{15.30}$$

2. Resonant case: For some $m \in \mathbb{N}$,

$$\omega = \frac{5m\pi}{3}. \tag{15.31}$$

In this case,

$$T_m^p(t) = t\big(C_m \cos \omega t + D_m \sin \omega t\big).$$

Taking $k = m$ and substituting into (15.26), we get

$$\begin{aligned} &2\big(-C_m \omega \sin \omega t + D_m \omega \cos \omega t\big) + t\big(-C_m \omega^2 \cos \omega t - D_m \omega^2 \sin \omega t\big) \\ &= -\big(\frac{5m\pi}{3}\big)^2 t\big(C_m \cos \omega t + D_m \sin \omega t\big) + g_m \sin \omega t. \end{aligned} \tag{15.32}$$

Here in the left-hand side we used the Leibniz formula for computing

$$\frac{d^2}{dt^2}\Big[t\big(C_m \cos \omega t + D_m \sin \omega t\big)\Big].$$

Taking into account (15.31) and collecting the terms in (15.32), we get

$$2\big(-C_m \omega \sin \omega t + D_m \omega \cos \omega t\big) = g_m \sin \omega t.$$

We compare the coefficients at $\cos \omega t$ and $\sin \omega t$ on the left and on the right:

$$2D_m\omega = 0, \qquad -2C_m\omega = g_m.$$

Since $\omega > 0$,

$$D_m = 0, \qquad C_m = -\frac{g_m}{2\omega}.$$

Thus,

$$T_m^p(t) = -t\frac{g_m}{2\omega}\cos\omega t.$$

Therefore

$$T_m(t) = A_m\cos\frac{5k\pi}{3}t + B_m\sin\frac{5k\pi}{3}t - t\frac{g_m}{2\omega}\cos\omega t.$$

Substituting into the initial conditions (15.27), we get

$$A_m = 0; \quad B_m\frac{5m\pi}{3} - \frac{g_m}{2\omega} = 0 \Longrightarrow B_m = \frac{3g_m}{10m\pi\omega}.$$

Therefore,

$$T_m^p(t) = \frac{3g_m}{10m\pi\omega}\sin\left(\frac{5k\pi}{3}t\right) - t\frac{g_m}{3}\cos\omega t.$$

Thus, if for some $m \in \mathbb{N}$ the condition (15.31) is satisfied, we get (compare with (15.30)):

$$u(x,t) = \sum_{k\in\mathbb{N}, k\neq m} \frac{g_k}{\left(\frac{5k\pi}{3}\right)^2 - \omega^2}\left(-\frac{\omega}{\left(\frac{5k\pi}{3}\right)}\sin\frac{5k\pi}{3}t + \sin\omega t\right)\sin\frac{k\pi x}{3}$$
$$+\left(\frac{3g_m}{10m\pi\omega}\sin\frac{5m\pi}{3}t - t\frac{g_m}{2\omega}\cos\omega t\right)\sin\frac{m\pi x}{3}. \tag{15.33}$$

Remark 15.7. In the non-resonant case, all the terms in the series (15.30) are bounded functions of x, t, while in the resonant case (15.31) one of the terms in (15.33) is unbounded when $t \to +\infty$. Therefore, for large t, the solution will be represented mainly by the last term in (15.33). As t grows, the solution becomes unboundedly large. If it were the amplitude of a string, the string would break. As the matter of fact, when the solution becomes large, it is no longer described by the linear wave equation, and the formula (15.33) is no longer valid.

Problem 15.8. Find the solution to the mixed problem

$$\begin{cases} u_{tt}(x,t) = 16u_{xx} + \sin\frac{7\pi x}{10}, & 0 < x < 5, \quad t > 0; \\ u(0,t) = 0, \qquad u_x(5,t) = 0; \\ u(0,x) = 0, \qquad u_t(0,x) = 0. \end{cases}$$

16 The Fourier method for nonhomogeneous boundary conditions

Up to now, we were using the Fourier method only for problems with homogeneous boundary conditions. It turns out that the problem with nonhomogeneous boundary conditions is easily reduced to a problem with homogeneous boundary conditions.

The heat equation

Problem 16.1. Find the solution to the mixed problem

$$\begin{cases} u_t = 9u_{xx}, & 0 < x < 4, \quad t > 0; \\ u(0,t) = f(t), & u(4,t) = g(t); \\ u(x,0) = 0. \end{cases} \tag{16.1}$$

Solution. Let us find an auxiliary function $v(x,t)$ which satisfies the given boundary conditions:

$$v(0,t) = f(t), \qquad v(4,t) = g(t), \qquad t > 0.$$

Such a function can easily be found, for example, using a linear interpolation

$$v(x,t) = \frac{x}{4} g(t) + \frac{4-x}{4} f(t).$$

Denote $w = u - v$. Then w satisfies the homogeneous boundary conditions

$$w(0,t) = 0, \qquad w(4,t) = 0, \qquad t > 0. \tag{16.2}$$

Question 16.2. What equation and boundary conditions does the function w satisfy?

Answer. We substitute $u = w + v$ into (16.1); then

$$\begin{cases} w_t + v_t = 9(w_{xx} + v_{xx}), \\ w(x,0) + v(x,0) = 0, \end{cases}$$

leading to

$$\begin{cases} w_t = 9w_{xx} + 9(v_{xx} - v_t), \\ w(x,0) = -v(x,0). \end{cases}$$

Thus, unlike u, the function w satisfies the nonhomogeneous heat equation! But the boundary conditions (16.2) are now homogeneous, hence w could be found using the method of Section 15; then $u = w + v$ is the solution to the problem (16.1). Thus, we sent the nonhomogeneity from the boundary conditions into the differential equation (16.1) and into the initial condition.

The wave equation

Problem 16.3. Solve the mixed problem

$$\begin{cases} u_{tt} = 16u_{xx}, & 0 < x < 5, \quad t > 0; \\ u(0,t) = 0, & u_x(5,t) = \sin \omega t; \\ u(x,0) = 0, & u_t(x,0) = 0. \end{cases} \tag{16.3}$$

Solution. **A.** The auxiliary function

$$v(x,t) = x \sin \omega t$$

satisfies the specified boundary conditions. For $w \equiv u - v$, we have:

$$\begin{cases} w_{tt} = 16w_{xx} + \omega^2 x \sin \omega t, & 0 < x < 5, \quad t > 0; \\ w(0,t) = 0, \qquad w_x(5,t) = 0; \\ w(x,0) = -v(x,0) = 0, & w_t(x,0) = -v_t(x,0) = -x\omega. \end{cases} \tag{16.4}$$

B. Following the method of Section 15, we are looking for w in the form

$$w(x,t) = \sum_{k=0}^{\infty} T_k(t) \sin \frac{(k+\frac{1}{2})\pi x}{5}. \tag{16.5}$$

For this, we expand the right-hand side of equation (16.4):

$$\omega^2 x \sin \omega t = \omega^2 \sin \omega t \sum_{k=0}^{\infty} x_k \sin \frac{(k+\frac{1}{2})\pi x}{5},$$

where

$$\begin{aligned} x_k &= \frac{2}{5} \int_0^5 x \sin \frac{(k+\frac{1}{2})\pi x}{5}\, dx = -\frac{2}{5} \frac{5}{(k+\frac{1}{2})\pi} \int_0^5 x\, d \cos \frac{(k+\frac{1}{2})\pi x}{5} \\ &= -\frac{5}{(k+\frac{1}{2})\pi} \left[x \cos \frac{(k+\frac{1}{2})\pi x}{5} \Big|_0^5 - \int_0^5 \cos \frac{(k+\frac{1}{2})\pi x}{5}\, dx \right] \\ &= \frac{2 \cdot 5}{(k+\frac{1}{2})^2 \pi^2} \sin \frac{(k+\frac{1}{2})\pi x}{5} \Big|_0^5 = \frac{10}{(k+\frac{1}{2})^2 \pi^2} \cdot (-1)^k. \end{aligned} \tag{16.6}$$

C. Substituting (16.5)–(16.6) into equation (16.4), we find the equations on the temporal functions $T_k(t)$:

$$T_k''(t) = -16 \left(\frac{(k+\frac{1}{2})\pi}{5} \right)^2 T_k(t) + \omega^2 x_k \sin \omega t, \qquad k = 0, 1, 2, \ldots. \tag{16.7}$$

From the initial conditions (16.4) we find $T_k(0) = 0$ and

$$T_k'(0) = \frac{2}{5}\int_0^5 (-\omega x)\sin\frac{(k+\frac{1}{2})\pi x}{5}\,dx = -\omega\frac{10\cdot(-1)^k}{(k+\frac{1}{2})^2\pi^2}. \tag{16.8}$$

In the last equality, we took into account (16.6). The problem (16.7)–(16.8) could be solved in the same way as in Section 15. Again, two cases are possible: resonant and non-resonant.

Complete the solution of the problem (16.1).

Remark 16.4. For problems like (16.4) a condition analogous to (14.8) is not satisfied. Still, the new function $w(x,t)$ satisfies the initial and boundary conditions in the usual sense. It is only the first equation in (16.4) that is satisfied in the sense of distributions (see Section 26 below).

Problem 16.5. Find the resonance condition in the problem (16.3).

Answer. $\omega = \frac{4(m+\frac{1}{2})\pi x}{5}$ for some $m = 0, 1, 2, \ldots$.

17 The Fourier method for the Laplace equation

Boundary value problems in a rectangle

A. Let us consider the boundary value problem in the rectangle $\Omega = [0,a]\times[0,b]$:

$$\begin{cases} \triangle u(x,y) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0<x<a, \quad 0<y<b; \\ u(0,y) = 0, \qquad u(a,y) = 0; \\ u(x,0) = f(x), \quad u(x,b) = g(x). \end{cases} \tag{17.1}$$

This is the boundary value problem, or *the Dirichlet problem*: the function u is given at the boundary of the considered region. See Fig. 17.1.

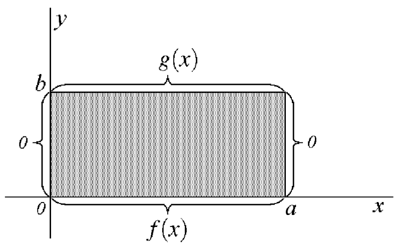

Fig. 17.1

Solution. The problem (17.1) can be solved by the method of Section 15, where the role of the variable t is now played by the variable y, as could be seen from comparing problems (17.1) and (15.1). We are looking for the solution in the following form:

$$u(x,y) = \sum_{k=1}^{\infty} Y_k(y) \sin \frac{k\pi x}{a}. \tag{17.2}$$

Then the boundary conditions at $x = 0$ and $x = a$ in (17.1) are automatically satisfied. We substitute (17.2) into equation (17.1). This gives equations on $Y_k(y)$:

$$-\left(\frac{k\pi}{a}\right)^2 Y_k(y) + Y_k''(y) = 0, \qquad 0 < y < b. \tag{17.3}$$

Substitution into the boundary conditions (17.1) at $y = 0$ and $y = b$ yields

$$\begin{cases} Y_k(0) = f_k \equiv \dfrac{2}{a}\displaystyle\int_0^a f(x) \sin \frac{k\pi x}{a}\, dx, \\ Y_k(b) = g_k \equiv \dfrac{2}{a}\displaystyle\int_0^a g(x) \sin \frac{k\pi x}{a}\, dx. \end{cases} \tag{17.4}$$

The general solution to equation (17.3) has the form

$$Y_k(y) = A_k e^{\frac{k\pi}{a}y} + B_k e^{-\frac{k\pi}{a}y}. \tag{17.5}$$

The constants A_k and B_k are found from the boundary conditions (17.4):

$$A_k + B_k = f_k, \qquad A_k e^{\frac{k\pi}{a}b} + B_k e^{-\frac{k\pi}{a}b} = g_k.$$

Solving this system, we find:

$$\begin{cases} A_k = \frac{1}{e^{\frac{k\pi}{a}b} - e^{-\frac{k\pi}{a}b}}\left(g_k - f_k e^{-\frac{k\pi}{a}b}\right), \\ B_k = \frac{1}{e^{\frac{k\pi}{a}b} - e^{-\frac{k\pi}{a}b}}\left(f_k e^{\frac{k\pi}{a}b} - g_k\right). \end{cases} \tag{17.6}$$

Thus, the solution of the problem (17.1) is given by (17.2), (17.5), and (17.6).

Let us check the validity of the solution (17.2). We need to justify the possibility of the termwise differentiation of the series (17.2). If $f(x)$ and $g(x)$ are integrable functions, then $f(x)$ and $g(x)$ are bounded:

$$|f_k| \le \frac{2}{a}\int_0^a |f(x)|\, dx, \qquad |g_k| \le \frac{2}{a}\int_0^a |g(x)|\, dx.$$

But then from (17.6) we see that $|A_k| \le \frac{c}{e^{\frac{k\pi}{a}b}}$, $|B_k| \le \text{const}$. Therefore, it follows from (17.5) that $|Y_k(y)| \le c e^{-\frac{k\pi}{a}(b-y)} + c e^{-\frac{k\pi}{a}y}$. As a consequence, for $\varepsilon < y < b - \varepsilon$, with $\varepsilon > 0$ small, one has $|Y_k(y)| \le c e^{-\frac{k\pi}{a}\varepsilon}$, and the series (17.2) for these values of y is dominated by the convergent geometric series $\sum_{k=1}^{\infty} c e^{-\frac{k\pi}{a}\varepsilon}$. It is easy to see that the derivatives of the second order in x and in y of the series (17.2) are dominated by the series $\sum_{k=1}^{\infty} ck^2 e^{-\frac{k\pi}{a}\varepsilon}$, which is also convergent. In the same way one proceeds with the derivatives of any order in x and y.

Conclusion. Solution of the Dirichlet problem (17.1) is a smooth function inside the rectangle Ω. Let us assume that, as in (12.5), $f(x)$, $g(x) \in C_0^2[0,a]$. Then, analogously to (12.4), $f_k, g_k = O(\frac{1}{k^2})$ and, consequently, $|Y_k(y)| \le \frac{c}{k^2}$, $y \in [0,b]$. Therefore, the series (17.2) converges uniformly in the rectangle $\Omega = [0,a] \times [0,b]$, and its sum is a function which is continuous in this rectangle and satisfies boundary conditions in (17.1).

B. More general boundary value problem of the Dirichlet type in the rectangle,

$$\begin{cases} \triangle u(x,y) = 0, & 0 < x < a, \quad 0 < y < b; \\ u(0,y) = \varphi(y), & u(a,y) = \psi(y); \\ u(x,0) = f(x), & u(x,b) = g(x), \end{cases} \tag{17.7}$$

could be solved by decomposing the solution u into two terms:

$$u = u_1 + u_2. \tag{17.8}$$

Here u_1 solves the problem (17.1), while u_2 solves the problem

$$\begin{cases} \triangle u_2 = 0, & 0 < x < a, \quad 0 < y < b; \\ u_2(0,y) = \varphi(y), & u_2(a,y) = \psi(y); \\ u_2(x,0) = 0, & u_2(x,b) = 0. \end{cases}$$

This problem takes the same form as (17.1) if one interchanges x and y. Therefore u_2 should be tried in the form (compare with (17.2)):

$$u_2(x,y) = \sum_{k=1}^{\infty} X_k(x) \sin \frac{k\pi y}{b}. \tag{17.9}$$

If $f,\ g \in C_0^2[0,a]$, while $\varphi,\ \psi \in C_0^2[0,b]$, then, according to what we said above, u_1 and u_2, and, consequently, $u = u_1 + u_2$ are continuous functions in Ω which satisfy the required boundary conditions.

In the general case, for the continuity of $u(x,y)$ in Ω, the following compatibility conditions are obviously required:

$$f(0) = \varphi(0), \quad \varphi(b) = g(0), \quad g(a) = \psi(b), \quad \psi(0) = f(a). \tag{17.10}$$

Problem 17.1. Prove that the problem (17.7) has a solution continuous in Ω if $f, g \in C^2[0,a]$, φ, $\psi \in C^2[0,b]$, and the compatibility condition (17.10) is satisfied.

Hint. Try to find the solution to equation $\triangle v = 0$ in Ω which coincides with the boundary values given by functions f, g, φ, and ψ at the boundary of the region Ω. Then the difference $u - v$ could be found using decomposition (17.8) described above.

C. Now we consider the nonhomogeneous Laplace equation (*the Poisson equation*).

Problem 17.2. Solve the boundary value problem

$$\begin{cases} \triangle u(x,y) = x^2 y, & 0<x<a, \quad 0<y<b; \\ u(0,y)=0, & u(a,y)=0; \\ u(x,0)=0, & \frac{\partial u}{\partial y}(x,b)=0. \end{cases} \tag{17.11}$$

Let us point out that here at $x=0$, $x=a$, and $y=0$ one has the boundary value of the Dirichlet type, while at $y=b$ one has the boundary value of the Neumann type (that is, the derivative of the solution in the normal direction is specified).

Solution. Homogeneous boundary conditions at $x=0$, and $x=a$ allow to write the solution in the form of the series over the eigenfunctions of the corresponding Sturm – Liouville problem:

$$u(x,y) = \sum_{k=1}^{\infty} Y_k(x) \sin \frac{k\pi y}{a}. \tag{17.12}$$

We also decompose over these functions the right-hand side:

$$x^2 y = y \sum_{k=1}^{\infty} g_k \sin \frac{k\pi y}{a}, \qquad g_k = \frac{2}{a}\int_0^a x^2 \sin \frac{k\pi y}{a}\, dx.$$

Substituting these decompositions into (17.11), we get for $\forall k = 1, 2, \ldots$

$$-\left(\frac{k\pi}{a}\right)^2 Y_k(y) + Y_k''(y) = y g_k, \quad 0<y<b; \qquad Y_k(0)=0, \quad Y_k'(b)=0. \tag{17.13}$$

Then

$$Y_k(y) = A_k e^{\frac{k\pi x}{a}} + B_k e^{-\frac{k\pi x}{a}} + \frac{y g_k}{-(\frac{k\pi}{a})^2}. \tag{17.14}$$

The constants A_k and B_k can be found after substituting this solution into the boundary conditions in (17.13):

$$A_k + B_k = 0, \qquad \frac{k\pi}{a} A_k e^{\frac{k\pi}{a} b} + \left(-\frac{k\pi}{a}\right) B_k e^{-\frac{k\pi}{a} b} + \frac{g_k}{-(\frac{k\pi}{a})^2} = 0.$$

Answer. The solution is given by the formulas (17.12), (17.14).

Boundary value problems in annulus and in disc

A. Let us solve the boundary value problem of the Dirichlet type in the annulus between the circles of radii r_1 and r_2:

$$\begin{cases} \triangle u(x,y) = 0, & r_1^2 < x^2+y^2 < r_2^2; \\ u|_{x^2+y^2=r_1^2} = f_1(\varphi), \quad u|_{x^2+y^2=r_2^2} = f_2(\varphi); & 0 \le \varphi \le 2\pi. \end{cases} \tag{17.15}$$

Here f_1 and f_2 are given continuous functions of the angular variable φ.

Solution. Let us convert to polar coordinates $r = \sqrt{x^2+y^2}$, $\varphi = \arctan \frac{y}{x}$.

Problem 17.3. Prove that in these coordinates the problem (17.15) takes the form

$$\begin{cases} \triangle u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \varphi^2} = 0, & r_1 < r < r_2; \\ u|_{r=r_1} = f_1(\varphi), \quad u|_{r=r_2} = f_2(\varphi); & 0 \le \varphi \le 2\pi. \end{cases} \tag{17.16}$$

This is a problem in a rectangle $[0,2\pi] \times [r_1,r_2]$ (Fig. 17.2). The boundary conditions are given at the lower and at the upper sides of the rectangle.

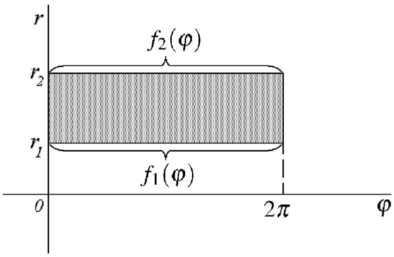

Fig. 17.2

Question 17.4. Are there boundary conditions at the left and right sides of the rectangle?

Answer. Yes, it is the periodicity condition in the variable φ:

$$u(0,r) = u(2\pi,r), \qquad \frac{\partial u}{\partial \varphi}(0,r) = \frac{\partial u}{\partial \varphi}(2\pi,r). \tag{17.17}$$

This follows from the fact that the points with the polar coordinates $(0,r)$ and $(2\pi,r)$ are identical. Analogous periodicity conditions in φ also hold for all partial derivatives of u in r and φ.

Problem 17.5. Show that the conditions (17.17) together with equation (17.16) guarantee the periodicity in φ of all the derivatives of u in r and φ if $u(\varphi,r)$ is a smooth function in the rectangle $[0,2\pi] \times [r_1,r_2]$.

The Sturm – Liouville problem which corresponds to the homogeneous boundary conditions (17.17) has the form

$$\begin{cases} \dfrac{\partial^2}{\partial \varphi^2}\Phi(\varphi) = \lambda \Phi(\varphi), \qquad 0 < \varphi < 2\pi; \\ \Phi(0) = \Phi(2\pi), \qquad \Phi'(0) = \Phi'(2\pi). \end{cases} \tag{17.18}$$

Solving this problem, we find:

$$\lambda_k = -k^2, \qquad \Phi_k(\varphi) = A_k \cos k\varphi + B_k \sin k\varphi, \qquad k = 0, 1, 2, \ldots .$$

Therefore, for each $k \neq 0$ there are two linearly independent eigenfunctions: $\cos k\varphi$ and $\sin k\varphi$, while for $k = 0$ there is only one eigenfunction: $\Phi_0(\varphi) \equiv 1$. As it is known from the Fourier series theory, these eigenfunctions form a complete orthogonal set in $L^2(0, 2\pi)$ and are mutually orthogonal. The squares of their L^2-norms are given by

$$\begin{cases} \displaystyle\int_0^{2\pi} \Phi_0^2(\varphi)\, d\varphi = \int_0^{2\pi} d\varphi = 2\pi; \\ \displaystyle\int_0^{2\pi} \cos^2(k\varphi)\, d\varphi = \int_0^{2\pi} \sin^2(k\varphi)\, d\varphi = \pi, \quad k = 1, 2, 3, \ldots . \end{cases} \tag{17.19}$$

The Fourier method for the problem (17.16) in the annulus consists of finding the solution in the form of a series over the eigenfunctions of the problem (17.18):

$$u(\varphi, r) = \sum_{k=0}^{\infty} R_k(r) \cos k\varphi + \sum_{k=1}^{\infty} S_k(r) \sin k\varphi. \tag{17.20}$$

Substituting this series into equation (17.16), we get the following equations on the "radial" functions $R_k(r)$:

$$R_k'' + \frac{1}{r} R_k' + \frac{1}{r^2} R_k(-k^2) = 0, \qquad r_1 < r < r_2, \quad k = 0, 1, 2, \ldots \tag{17.21}$$

and the same equations on S_k:

$$S_k'' + \frac{1}{r} S_k' + \frac{1}{r^2} S_k(-k^2) = 0, \qquad r_1 < r < r_2, \quad k = 0, 1, 2, \ldots . \tag{17.22}$$

Let us solve the *radial* equations (17.21), (17.22). These are *the Euler equations*. Substituting $R_k = r^\lambda$ into (17.21), we get

$$\lambda(\lambda - 1) r^{\lambda-2} + \lambda r^{\lambda-2} - k^2 r^{\lambda-2} = 0,$$

and we get the characteristic equation $\lambda^2 - k^2 = 0$, hence $\lambda = \pm k$. If $k \neq 0$, then the roots are simple, and the general solutions to (17.21) and (17.22) have the following form:

$$R_k(r) = A_k r^k + B_k r^{-k}, \qquad k = 1, 2, 3, \ldots ; \tag{17.23}$$

$$S_k(r) = C_k r^k + D_k r^{-k}, \qquad k = 1, 2, 3, \ldots . \tag{17.24}$$

For $k = 0$, the root of the equation $\lambda = 0$ has multiplicity 2, hence

$$R_0(r) = A_0 + B_0 \ln r. \tag{17.25}$$

Substituting (17.23)–(17.25) into (17.20), we get the general solution of a homogeneous Laplace equation in the annulus:

$$u(\varphi, r) = A_0 + B_0 \ln r + \sum_{k=1}^{\infty} (A_k r^k + \frac{B_k}{r^k}) \cos k\varphi + \sum_{k=1}^{\infty} (C_k r^k + \frac{D_k}{r^k}) \sin k\varphi. \tag{17.26}$$

Remark 17.6. This is a general form of a harmonic function in the annulus.

The values of the constants in (17.26) are obtained from the boundary conditions (17.16):

$$\begin{cases} A_0 + B_0 \ln r_1 + \sum\limits_{k=1}^{\infty} (A_k r_1^k + B_k r_1^{-k}) \cos k\varphi + \sum\limits_{k=1}^{\infty} (C_k r_1^k + D_k r_1^{-k}) \sin k\varphi = f_1(\varphi), \\ A_0 + B_0 \ln r_2 + \sum\limits_{k=1}^{\infty} (A_k r_2^k + B_k r_2^{-k}) \cos k\varphi + \sum\limits_{k=1}^{\infty} (C_k r_2^k + D_k r_2^{-k}) \sin k\varphi = f_2(\varphi), \end{cases} \tag{17.27}$$

where $0 \le \varphi \le 2\pi$. Taking into account the orthogonality of the eigenfunctions of the Sturm – Liouville problem (17.18) and the relations (17.19), we get

$$\begin{cases} A_0 + B_0 \ln r_1 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f_1(\varphi)\, d\varphi, \\ A_0 + B_0 \ln r_2 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f_2(\varphi)\, d\varphi, \end{cases} \tag{17.28}$$

and, similarly, for $k = 1, 2, 3, \ldots$,

$$\begin{cases} A_k r_1^k + B_k r_1^{-k} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_1(\varphi) \cos k\varphi\, d\varphi, \\ A_k r_2^k + B_k r_2^{-k} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_2(\varphi) \cos k\varphi\, d\varphi; \end{cases} \tag{17.29}$$

$$\begin{cases} C_k r_1^k + D_k r_1^{-k} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_1(\varphi) \sin k\varphi\, d\varphi, \\ C_k r_2^k + D_k r_2^{-k} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_2(\varphi) \sin k\varphi\, d\varphi. \end{cases} \tag{17.30}$$

We find A_0 and B_0 from the system (17.28) and A_k, B_k from (17.29). C_k and D_k are found from (17.30). The problem (17.15) is solved.

Problem 17.7. Prove that the solution (17.26) of the problem (17.15) is infinitely differentiable in the interior of the annulus.

Problem 17.8. Solve the Dirichlet problem in the annulus:

$$\begin{cases} \triangle u(x,y) = 0, & 4 < x^2 + y^2 < 9; \\ u|_{x^2+y^2=4} = x, & u|_{x^2+y^2=9} = y. \end{cases}$$

Solution. Here $r_1 = 2$, $r_2 = 3$, so that

$$f_1(\varphi) = 2\cos\varphi, \qquad f_2(\varphi) = 3\sin\varphi. \tag{17.31}$$

Therefore, the right-hand sides in (17.28) are equal to zero and $A_0 = B_0 = 0$. Analogously, the right-hand sides of the systems (17.29) and (17.30) are equal to zero for all $k \neq 1$, thus

$$A_k = B_k = 0, \quad C_k = D_k = 0 \quad \text{for} \quad k \neq 1.$$

Hence, the series (17.26) contains only two terms:

$$u(\varphi, r) = (A_1 r + B_1 r^{-1})\cos\varphi + (C_1 r + D_1 r^{-1})\sin\varphi. \tag{17.32}$$

The remaining coefficients are obtained from the systems of equations

$$\begin{cases} A_1 2 + B_1 \frac{1}{2} = 2, \\ A_1 3 + B_1 \frac{1}{3} = 0, \end{cases} \qquad \begin{cases} C_1 2 + D_1 \frac{1}{2} = 0, \\ C_1 3 + D_1 \frac{1}{3} = 3, \end{cases} \tag{17.33}$$

which are derived directly from (17.31). Namely, (17.33) is obtained by substituting (17.31) into (17.27) and comparing the Fourier coefficients in both sides of the relations, instead of evaluating integrals in (17.29)–(17.30). From (17.33) we find

$$A_1 = -\frac{4}{5}, \qquad B_1 = \frac{36}{5}, \qquad C_1 = \frac{9}{5}, \qquad D_1 = -\frac{36}{5}. \tag{17.34}$$

Answer. $u(\varphi, r) = \left(-\frac{4}{5}r + \frac{36}{5}r^{-1}\right)\cos\varphi + \left(\frac{9}{5}r - \frac{36}{5}r^{-1}\right)\sin\varphi.$

B. Now let us consider the Dirichlet problem in the disc of radius R:

$$\begin{cases} \triangle u(x,y) = 0, & x^2 + y^2 < R^2; \\ u|_{x^2+y^2=R^2} = f(\varphi), & 0 < \varphi < 2\pi. \end{cases} \tag{17.35}$$

A solution of this problem also has the form (17.26), since the disc $x^2 + y^2 < R^2$ contains the (degenerate) annulus $0 < x^2 + y^2 < R^2$. But the disc also contains the point $(0,0)$, where the solution has to be finite:

$$|u(0,0)| < \infty. \tag{17.36}$$

It can be shown [TS90] that (17.36) holds if and only if all the terms which have the singularity at $(0,0)$ of the form $\ln r$ and r^{-k} are absent from (17.26). This means that $B_0 = B_k = D_k = 0,\ k = 1, 2, 3, \ldots$. Thus, (17.26) takes the form

$$u(x,y) = A_0 + \sum_{k=1}^{\infty} r^k (A_k \cos k\varphi + C_k \sin k\varphi). \tag{17.37}$$

This is the analog of the Taylor series for a harmonic function in a disc. The coefficients of the series (17.37) are found from the boundary condition of the problem (17.35).

Problem 17.9. Solve the Dirichlet problem in the disc:

$$\begin{cases} \triangle u(x,y) = 0, & x^2+y^2<4; \\ u|_{x^2+y^2=4} = x^2. \end{cases}$$

Solution. We are looking for the solution u in the form (17.37). The substitution of this series into the boundary condition gives:

$$A_0 + \sum_{k=1}^{\infty} 2^k (A_k \cos k\varphi + C_k \sin k\varphi) = 2 + 2\cos 2\varphi, \tag{17.38}$$

since $x^2|_{r=2} = (2\cos\varphi)^2 = 4\cos^2\varphi = 2+2\cos 2\varphi$. Comparing the Fourier coefficients in the left- and right-hand sides of (17.38), we see that all A_k and C_k with $k \neq 0$ and $k \neq 2$ are equal to zero, and and the formula (17.37) yields the answer: $A_0 = 2$, $A_2 = 1/2$, $C_2 = 0$. The formula (17.37) takes the form

$$u = 2 + r^2 \frac{1}{2}\cos 2\varphi = 2 + \frac{r^2}{2}(\cos^2\varphi - \sin^2\varphi) = 2 + \frac{x^2-y^2}{2}.$$

Problem 17.10. Solve the Dirichlet problem in the annulus:

$$\begin{cases} \triangle u(x,y) = x^2, & 9 < x^2+y^2 < 16; \\ u|_{x^2+y^2=9} = 0, & u|_{x^2+y^2=16} = 0. \end{cases}$$

Hint. Both the solution that we are looking for and the right-hand side of the equation are to be decomposed into the series of the form (17.20). Equations on the radial functions R_k and S_k will be the nonhomogeneous Euler equations.

Problem 17.11. Solve the Neumann problem in the disc:

$$\begin{cases} \triangle u(x,y) = 0, & x^2+y^2<9; \\ \frac{\partial u}{\partial \boldsymbol{n}}|_{x^2+y^2=9} = y, \end{cases}$$

where $\frac{\partial}{\partial \boldsymbol{n}}$ is the derivative normal to the boundary of the disc.

Hint. Solution is to be looked for in the form of the series (17.37); moreover, in the polar coordinates one has $\frac{\partial u}{\partial \boldsymbol{n}} = \frac{\partial u}{\partial r}$.

Conclusion. The heat equation, the wave equation, and the Laplace equation possess different properties. As it follows from the results of Chapter 2, solutions of the homogeneous Laplace equation and the heat equation are smooth inside the regions where they are considered, even if the boundary values are discontinuous. At the same time, solutions of the homogeneous wave equation could be discontinuous if, for example, the initial data are discontinuous functions.

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