2

Basic Concepts and Examples in Finance

In this chapter, we present briefly the main concepts in mathematical finance as well as some straightforward applications of stochastic calculus for continuous-path processes. We study in particular the general principle for valuation of contingent claims, the Feynman-Kac approach, the Ornstein-Uhlenbeck and Vasicek processes, and, finally, the pricing of European options.

Derivatives are products whose payoffs depend on the prices of the traded underlying assets. In order for the model to be arbitrage free, the link between derivatives and underlying prices has to be made precise. We shall present the mathematical setting of this problem, and give some examples. In this area we recommend Portait and Poncet [723], Lipton [596], Overhaus et al. [689], and Brockhaus et al. [131].

Important assets are the zero-coupon bonds which deliver one monetary unit at a terminal date. The price of this asset depends on the interest rate. We shall present some basic models of the dynamics of the interest rate (Vasicek and CIR) and the dynamics of associated zero-coupon bonds. We refer the reader to Martellini et al. [624] and Musiela and Rutkowski [661] for a study of modelling of zero-coupon prices and pricing derivatives.

2.1 A Semi-martingale Framework

In a first part, we present in a general setting the modelling of the stock market and the hypotheses in force in mathematical finance. The dynamics of prices are semi-martingales, which is justified from the hypothesis of no-arbitrage (see the precise definition in $\rightarrowtail$ Subsection 2.1.2). Roughly speaking, this hypothesis excludes the possibility of starting with a null amount of money and investing in the market in such a way that the value of the portfolio at some fixed date T is positive (and not null) with probability 1. We shall comment upon this hypothesis later.

We present the definition of self-financing strategies and the concept of hedging portfolios in a case where the tradeable asset prices are given as

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semi-martingales. We give the definition of an arbitrage opportunity and we state the fundamental theorem which links the non-arbitrage hypothesis with the notion of equivalent martingale measure. We define a complete market and we show how this definition is related to the predictable representation property.

In this first section, we do not require path-continuity of asset prices.

An important precision: Concerning all financial quantities presented in that chapter, these will be defined up to a finite horizon T, called the maturity. On the other hand, when dealing with semi-martingales, these will be implicitly defined on $\mathbb{R}^+$.

2.1.1 The Financial Market

We study a financial market where assets (stocks) are traded in continuous time. We assume that there are d assets, and that the prices $S^i, i = 1, \dots, d$ of these assets are modelled as semi-martingales with respect to a reference filtration $\mathbf{F}$. We shall refer to these assets as **risky assets** or as **securities**. We shall also assume that there is a **riskless asset** (also called the savings account) with dynamics

$$dS_t^0 = S_t^0 r_t dt, \; S_0^0 = 1$$

where r is the (positive) **interest rate**, assumed to be $\mathbf{F}$-adapted. One monetary unit invested at time 0 in the riskless asset will give a payoff of $\exp\left(\int_0^t r_s ds\right)$ at time t. If r is deterministic, the price at time 0 of one monetary unit delivered at time t is

$$\boxed{R_t := \exp\left(-\int_0^t r_s ds\right) .}$$

The quantity $R_t = (S_t^0)^{-1}$ is called the **discount factor**, whether or not it is deterministic. The discounted value of S_t^i is $S_t^i R_t$; in the case where r and S_t^i are deterministic, this is the monetary value at time 0 of S_t^i monetary units delivered at time t. More generally, if a process $(V_t, t \geq 0)$ describes the value of a financial product at any time t, its discounted value process is $(V_t R_t, t \geq 0)$. The asset that delivers one monetary unit at time T is called a **zero-coupon bond** (ZC) of maturity T. If r is deterministic, its price at time t is given by

$$P(t,T) = \exp\left(-\int_t^T r(s) ds\right) ,$$

the dynamics of the ZC's price is then $d_t P(t,T) = r_t P(t,T) dt$ with the terminal condition $P(T,T) = 1$. If r is a stochastic process, the problem of giving the price of a zero-coupon bond is more complex; we shall study this case later. In that setting, the previous formula for $P(t,T)$ would be absurd, since $P(t,T)$ is known at time t (i.e., is $\mathcal{F}_t$-measurable), whereas the quantity $\exp\left(-\int_t^T r(s)ds\right)$ is not. Zero-coupon bonds are traded and are at the core of trading in financial markets.

Comment 2.1.1.1 In this book, we assume, as is usual in mathematical finance, that borrowing and lending interest rates are equal to $(r_s, s \geq 0)$: one monetary unit borrowed at time 0 has to be reimbursed by $S_t^0 = \exp\left(\int_0^t r_s ds\right)$ monetary units at time t. One monetary unit invested in the riskless asset at time 0 produces $S_t^0 = \exp\left(\int_0^t r_s ds\right)$ monetary units at time t. In reality, borrowing and lending interest rates are not the same, and this equality hypothesis, which is assumed in mathematical finance, oversimplifies the "real-world" situation. Pricing derivatives with different interest rates is very similar to pricing under constraints. If, for example, there are two interest rates with $r_1 < r_2$, one has to assume that it is impossible to borrow money at rate r_1 (see also $\rightarrowtail$ Example 2.1.2.1). We refer the reader to the papers of El Karoui et al. [307] for a study of pricing with constraints.

A **portfolio** (or a strategy) is a $(d+1)$-dimensional $\mathbf{F}$-predictable process $(\widehat{\pi}_t = (\pi_t^i, i = 0, \dots, d) = (\pi_t^0, \pi_t);\ t \geq 0)$ where π_t^i represents the number of shares of asset i held at time t. Its time-t value is

$$V_t(\widehat{\pi}) := \sum_{i=0}^{d} \pi_t^i S_t^i = \pi_t^0 S_t^0 + \sum_{i=1}^{d} \pi_t^i S_t^i \,.$$

We assume that the integrals $\int_0^t \pi_s^i dS_s^i$ are well defined; moreover, we shall often place more integrability conditions on the portfolio $\widehat{\pi}$ to avoid arbitrage opportunities (see $\rightarrowtail$ Subsection 2.1.2).

We shall assume that the market is liquid: there is no transaction cost (the buying price of an asset is equal to its selling price), the number of shares of the asset available in the market is not bounded, and **short-selling** of securities is allowed (i.e., $\pi^i, i \geq 1$ can take negative values) as well as borrowing money $(\pi^0 < 0)$.

We introduce a constraint on the portfolio, to make precise the idea that instantaneous changes to the value of the portfolio are due to changes in prices, not to instantaneous rebalancing. This **self-financing** condition is an extension of the discrete-time case and we impose it as a constraint in continuous time. *We emphasize that this constraint is not a consequence of Itô's lemma* and that, if a portfolio $(\widehat{\pi}_t = (\pi_t^i, i = 0, \dots, d) = (\pi_t^0, \pi_t);\ t \geq 0)$ is given, this condition has to be satisfied.

Definition 2.1.1.2 *A portfolio $\widehat{\pi}$ is said to be **self-financing** if*

$$dV_t(\widehat{\pi}) = \sum_{i=0}^{d} \pi_t^i dS_t^i \,,$$

or, in an integrated form, $V_t(\widehat{\pi}) = V_0(\widehat{\pi}) + \sum_{i=0}^d \int_0^t \pi_s^i dS_s^i$.

If $\widehat{\pi} = (\pi^0, \pi)$ is a self-financing portfolio, then some algebraic computation establishes that

$$\begin{aligned} dV_t(\widehat{\pi}) &= \pi_t^0 S_t^0 r_t dt + \sum_{i=1}^{d} \pi_t^i dS_t^i = r_t V_t(\widehat{\pi})dt + \sum_{i=1}^{d} \pi_t^i (dS_t^i - r_t S_t^i dt) \\ &= r_t V_t(\widehat{\pi})dt + \pi_t(dS_t - r_t S_t dt) \end{aligned}$$

where the vector $\pi = (\pi^i; i = 1, \dots, d)$ is written as a $(1, d)$ matrix. We prove now that the self-financing condition holds for discounted processes, i.e., if all the processes V and S^i are discounted (note that the discounted value of S_t^0 is 1):

Proposition 2.1.1.3 *If $\widehat{\pi}$ is a self-financing portfolio, then*

$$R_t V_t(\widehat{\pi}) = V_0(\widehat{\pi}) + \sum_{i=1}^{d} \int_0^t \pi_s^i d(R_s S_s^i) \,. \tag{2.1.1}$$

Conversely, if x is a given positive real number, if $\pi = (\pi^1, \dots, \pi^d)$ is a vector of predictable processes, and if V^π denotes the solution of

$$\boxed{dV_t^\pi = r_t V_t^\pi dt + \pi_t(dS_t - r_t S_t dt) \,, \; V_0^\pi = x \,,} \tag{2.1.2}$$

then the $\mathbb{R}^{d+1}$-valued process $(\widehat{\pi}_t = (V_t^\pi - \pi_t S_t, \pi_t);\, t \geq 0)$ is a self-financing strategy, and $V_t^\pi = V_t(\widehat{\pi})$.

Proof: Equality (2.1.1) follows from the integration by parts formula:

$$d(R_t V_t) = R_t dV_t - V_t r_t R_t dt = R_t \pi_t (dS_t - r_t S_t dt) = \pi_t d(R_t S_t) \,.$$

Conversely, if $(x, \pi = (\pi^1, \dots, \pi^d))$ are given, then one deduces from (2.1.2) that the value V_t^π of the portfolio at time t is given by

$$V_t^\pi R_t = x + \int_0^t \pi_s d(R_s S_s)$$

and the wealth invested in the riskless asset is

$$\pi_t^0 S_t^0 = V_t^\pi - \sum_{i=1}^{d} \pi_t^i S_t^i = V_t^\pi - \pi_t S_t \,.$$

The portfolio $\widehat{\pi} = (\pi^0, \pi)$ is obviously self-financing since

$$dV_t = r_t V_t dt + \pi_t (dS_t - r_t S_t dt) = \pi_t^0 S_t^0 r_t dt + \pi_t dS_t \, .$$

The process $(\sum_{i=1}^d \int_0^t \pi_s^i d(R_s S_s^i),\ t \geq 0)$ is the discounted gain process. □

This important result proves that a self-financing portfolio is characterized by its initial value $V_0(\widehat{\pi})$ and the strategy $\pi = (\pi^i, i = 1, \ldots, d)$ which represents the investment in the risky assets. The equality (2.1.1) can be written in terms of the savings account S^0 as

$$\boxed{\frac{V_t(\widehat{\pi})}{S_t^0} = V_0(\widehat{\pi}) + \sum_{i=1}^d \int_0^t \pi_s^i \, d\left(\frac{S_s^i}{S_s^0}\right)} \tag{2.1.3}$$

or as

$$dV_t^0 = \sum_{i=1}^d \pi_t^i dS_t^{i,0}$$

where

$$V_t^0 = V_t R_t = V_t / S_t^0, \quad S_t^{i,0} = S_t^i R_t = S_t^i / S_t^0$$

are the prices in terms of time-0 monetary units. We shall extend this property in ↣ Section 2.4 by proving that the self-financing condition does not depend on the choice of the numéraire.

By abuse of language, we shall also call $\pi = (\pi^1, \ldots, \pi^d)$ a self-financing portfolio.

The investor is said to have a **long position** at time t on the asset S if $\pi_t \geq 0$. In the case $\pi_t < 0$, the investor is **short**.

Exercise 2.1.1.4 Let $dS_t = (\mu dt + \sigma dB_t)$ and $r = 0$. Is the portfolio $\widehat{\pi}(t, 1)$ self-financing? If not, find π^0 such that $(\pi_t^0, 1)$ is self-financing. ◁

2.1.2 Arbitrage Opportunities

Roughly speaking, an **arbitrage opportunity** is a self-financing strategy π with zero initial value and with terminal value $V_T^\pi \geq 0$, such that $\mathbb{E}(V_T^\pi) > 0$.

From Dudley's result (see Subsection 1.6.3), it is obvious that we have to impose conditions on the strategies to exclude arbitrage opportunities. Indeed, if B is a BM, for any constant A, it is possible to find an adapted process φ such that $\int_0^T \varphi_s dB_s = A$. Hence, in the simple case $dS_s = \sigma S_s dB_s$ and null interest rate, it is possible to find π such that $\int_0^T \pi_s dS_s = A > 0$. The process $V_t = \int_0^t \pi_s dS_s$ would be the value of a self-financing strategy, with null initial wealth and strictly positive terminal value, therefore, π would be an arbitrage opportunity. These strategies are often called doubling strategies, by extension to an infinite horizon of a tossing game: a player with an initial wealth 0 playing such a game will have, with probability 1, at some time a wealth equal to 10^{63} monetary units: he only has to wait long enough (and to agree to lose a large amount of money before that). It "suffices" to play in continuous time to win with a BM.

Example 2.1.2.1 If there are two riskless assets in the market with interest rates r_1 and r_2, then in order to exclude arbitrage opportunities, we must have $r_1 = r_2$: otherwise, if $r_1 < r_2$, an investor might borrow an amount k of money at rate r_1, and invest the same amount at rate r_2. The initial wealth is 0 and the wealth at time T would be $ke^{r_2 T} - ke^{r_1 T} > 0$. So, in the case of different interest rates with $r_1 < r_2$, one has to restrict the strategies to those for which the investor can only borrow money at rate r_2 and invest at rate r_1. One has to add one dimension to the portfolio; the quantity of shares of the savings account, denoted by π^0 is now a pair of processes $\pi^{0,1}, \pi^{0,2}$ with $\pi^{0,1} \geq 0, \pi^{0,2} \leq 0$ where the wealth in the bank account is $\pi_t^{0,1} S_t^{0,1} + \pi_t^{0,2} S_t^{0,2}$ with $dS_t^{0,j} = r_j S_t^{0,j} dt$.

Exercise 2.1.2.2 There are many examples of relations between prices which are obtained from the absence of arbitrage opportunities in a financial market. As an exercise, we give some examples for which we use call and put options (see ↣ Subsection 2.3.2 for the definition). The reader can refer to Cox and Rubinstein [204] for proofs. We work in a market with constant interest rate r. We emphasize that these relations are model-independent, i.e., they are valid whatever the dynamics of the risky asset.

- Let C (resp. P) be the value of a European call (resp. a put) on a stock with current value S, and with strike K and maturity T. Prove the put-call parity relationship

$$\boxed{C = P + S - Ke^{-rT} \, .}$$

- Prove that $S \geq C \geq \max(0, S - K)$.
- Prove that the value of a call is decreasing w.r.t. the strike.
- Prove that the call price is concave w.r.t. the strike.
- Prove that, for $K_2 > K_1$,

$$K_2 - K_1 \geq C(K_2) - C(K_1) \, ,$$

where $C(K)$ is the value of the call with strike K.

◁

2.1.3 Equivalent Martingale Measure

We now introduce the key definition of equivalent martingale measure (or risk-neutral probability). It is a major tool in giving the prices of derivative products as an expectation of the (discounted) terminal payoff, and the existence of such a probability is related to the non-existence of arbitrage opportunities.

Definition 2.1.3.1 *An* **equivalent martingale measure** *(e.m.m.) is a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ on $\mathcal{F}_T$, such that the discounted prices $(R_t S_t^i, t \le T)$ are $\mathbb{Q}$-local martingales.*

It is proved in the seminal paper of Harrison and Kreps [421] in a discrete setting and in a series of papers by Delbaen and Schachermayer [233] in a general framework, that the existence of e.m.m. is more or less equivalent to the absence of arbitrage opportunities. One of the difficulties is to make precise the choice of "admissible" portfolios. We borrow from Protter [726] the name of Folk theorem for what follows:

Folk Theorem: *Let S be the stock price process. There is absence of arbitrage essentially if and only if there exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the discounted price process is a $\mathbb{Q}$-local martingale.*

From (2.1.3), we deduce that not only the discounted prices of securities are local-martingales, but that more generally, any price, and in particular prices of derivatives, are local martingales:

Proposition 2.1.3.2 *Under any e.m.m. the discounted value of a self-financing strategy is a local martingale.*

Comment 2.1.3.3 Of course, it can happen that discounted prices are strict local martingales. We refer to Pal and Protter [692] for an interesting discussion.

2.1.4 Admissible Strategies

As mentioned above, one has to add some regularity conditions on the portfolio to exclude arbitrage opportunities. The most common such condition is the following admissibility criterion.

Definition 2.1.4.1 *A self-financing strategy π is said to be admissible if there exists a constant A such that $V_t(\pi) \ge -A$, a.s. for every $t \le T$.*

Definition 2.1.4.2 *An arbitrage opportunity on the time interval $[0,T]$ is an admissible self-financing strategy π such that $V_0^\pi = 0$ and $V_T^\pi \ge 0, \mathbb{E}(V_T^\pi) > 0$.*

In order to give a precise meaning to the fundamental theorem of asset pricing, we need some definitions (we refer to Delbaen and Schachermayer [233]). In the following, we assume that the interest rate is equal to 0. Let us define the sets

$$\begin{aligned}
\mathcal{K} &= \left\{ \int_0^T \pi_s dS_s \;:\; \pi \text{ is admissible } \right\}, \\
\mathcal{A}_0 &= \mathcal{K} - L^0_+ = \left\{ X = \int_0^T \pi_s dS_s - f \;: \pi \text{ is admissible}, f \geq 0, f \text{ finite} \right\}, \\
\mathcal{A} &= \mathcal{A}_0 \cap L^\infty \,, \\
\bar{\mathcal{A}} &= \text{closure of } \mathcal{A} \ \text{ in } L^\infty \,.
\end{aligned}$$

Note that $\mathcal{K}$ is the set of terminal values of admissible self-financing strategies with zero initial value. Let L^∞_+ be the set of positive random variables in L^∞.

Definition 2.1.4.3 *A semi-martingale S satisfies the no-arbitrage condition if $\mathcal{K} \cap L^\infty_+ = \{0\}$. A semi-martingale S satisfies the* **No-Free Lunch with Vanishing Risk** *(NFLVR) condition if $\bar{\mathcal{A}} \cap L^\infty_+ = \{0\}$.*

Obviously, if S satisfies the no-arbitrage condition, then it satisfies the NFLVR condition.

Theorem 2.1.4.4 (Fundamental Theorem.) *Let S be a locally bounded semi-martingale. There exists an equivalent martingale measure $\mathbb{Q}$ for S if and only if S satisfies NFLVR.*

PROOF: The proof relies on the Hahn-Banach theorem, and goes back to Harrison and Kreps [421], Harrison and Pliska [423] and Kreps [545] and was extended by Ansel and Stricker [20], Delbaen and Schachermayer [233], Stricker [809]. We refer to the book of Delbaen and Schachermayer [236], Theorem 9.1.1. □

The following result (see Delbaen and Schachermayer [236], Theorem 9.7.2.) establishes that the dynamics of asset prices have to be semi-martingales:

Theorem 2.1.4.5 *Let S be an adapted càdlàg process. If S is locally bounded and satisfies the no free lunch with vanishing risk property for simple integrands, then S is a semi-martingale.*

Comments 2.1.4.6 (a) The study of the absence of arbitrage opportunities and its connection with the existence of e.m.m. has led to an extensive literature and is fully presented in the book of Delbaen and Schachermayer [236]. The survey paper of Kabanov [500] is an excellent presentation of arbitrage theory. See also the important paper of Ansel and Stricker [20] and Cherny [167] for a slightly different definition of arbitrage.

(b) Some authors (e.g., Karatzas [510], Levental and Skorokhod [583]) give the name of tame strategies to admissible strategies.

(c) It should be noted that the condition for a strategy to be admissible is restrictive from a financial point of view. Indeed, in the case $d = 1$, it excludes short position on the stock. Moreover, the condition depends on the choice of numéraire. These remarks have led Sin [799] and Xia and Yan [851, 852] to introduce allowable portfolios, i.e., by definition there exists $a \geq 0$ such that $V_t^\pi \geq -a \sum_i S_t^i$. The authors develop the fundamental theory of asset pricing in that setting.

(d) Frittelli [364] links the existence of e.m.m. and NFLVR with results on optimization theory, and with the choice of a class of utility functions.

(e) The condition $\mathcal{K} \cap L_+^\infty = \{0\}$ is too restrictive to imply the existence of an e.m.m.

2.1.5 Complete Market

Roughly speaking, a market is complete if any derivative product can be perfectly hedged, i.e., is the terminal value of a self-financing portfolio.

Assume that there are d risky assets S^i which are $\mathbf{F}$-semi-martingales and a riskless asset S^0. A **contingent claim** H is defined as a square integrable $\mathcal{F}_T$-random variable, where T is a fixed horizon.

Definition 2.1.5.1 *A contingent claim H is said to be* **hedgeable** *if there exists a predictable process $\pi = (\pi^1, \ldots, \pi^d)$ such that $V_T^\pi = H$. The self-financing strategy $\hat{\pi} = (V^\pi - \pi S, \pi)$ is called the* ***replicating strategy*** *(or the* ***hedging strategy****) of H, and $V_0^\pi = h$ is the initial price. The process V^π is the price process of H.*

In some sense, this initial value is an equilibrium price: the seller of the claim agrees to sell the claim at an initial price p if he can construct a portfolio with initial value p and terminal value greater than the claim he has to deliver. The buyer of the claim agrees to buy the claim if he is unable to produce the same (or a greater) amount of money while investing the price of the claim in the financial market.

It is also easy to prove that, if the price of the claim is not the initial value of the replicating portfolio, there would be an arbitrage in the market: assume that the claim H is traded at v with $v > V_0$, where V_0 is the initial value of the replicating portfolio. At time 0, one could

- ▶ invest V_0 in the financial market using the replicating strategy
- ▶ sell the claim at price v
- ▶ invest the amount $v - V_0$ in the riskless asset.

The terminal wealth would be (if the interest rate is a constant r)

- ▶ the value of the replicating portfolio, i.e., H
- ▶ minus the value of the claim to deliver, i.e., H
- ▶ plus the amount of money in the savings account, that is $(v - V_0)e^{rT}$

and that quantity is strictly positive. If the claim H is traded at price v with $v < V_0$, we invert the positions, buying the claim at price v and selling the replicating portfolio.

Using the characterization of a self-financing strategy obtained in Proposition 2.1.1.3, we see that the contingent claim H is hedgeable if there exists a pair (h, π) where h is a real number and π a d-dimensional predictable process such that

$$H/S_T^0 = h + \sum_{i=1}^d \int_0^T \pi_s^i d(S_s^i/S_s^0) \,.$$

From (2.1.3) the discounted value at time t of this strategy is given by

$$V_t^\pi/S_t^0 = h + \sum_{i=1}^d \int_0^t \pi_s^i dS_s^{i,0} \,.$$

We shall say that V_0^π is the initial value of H, and that π is the **hedging portfolio**. Note that the discounted price process $V^{\pi,0}$ is a $\mathbb{Q}$-local martingale under any e.m.m. $\mathbb{Q}$.

To give precise meaning the notion of market completeness, one needs to take care with the measurability conditions. The filtration to take into account is, in the case of a deterministic interest rate, the filtration generated by the traded assets.

Definition 2.1.5.2 *Assume that r is deterministic and let $\mathbf{F}^S$ be the natural filtration of the prices. The market is said to be* ***complete*** *if any contingent claim $H \in L^2(\mathcal{F}_T^S)$ is the value at time T of some self-financing strategy π.*

If r is stochastic, the standard attitude is to work with the filtration generated by the discounted prices.

Comments 2.1.5.3 (a) We emphasize that the definition of market completeness depends strongly on the choice of measurability of the contingent claims (see $\rightarrowtail$ Subsection 2.3.6) and on the regularity conditions on strategies (see below).

(b) It may be that the market is complete, but there exists no e.m.m. As an example, let us assume that a riskless asset S^0 and two risky assets with dynamics

$$dS_t^i = S_t^i(b_i dt + \sigma dB_t), \; i = 1, 2$$

are traded. Here, B is a one-dimensional Brownian motion, and $b_1 \neq b_2$. Obviously, there does not exist an e.m.m., so arbitrage opportunities exist, however, the market is complete. Indeed, any contingent claim H can be written as a stochastic integral with respect to S^1/S^0 (the market with the two assets S^0, S^1 is complete).

(c) In a model where $dS_t = S_t(b_t dt + \sigma dB_t)$, where b is $\mathbf{F}^B$-adapted, the value of the trend b has no influence on the valuation of hedgeable contingent claims. However, if b is a process adapted to a filtration bigger than the filtration $\mathbf{F}^B$, there may exist many e.m.m.. In that case, one has to write the dynamics of S in its natural filtration, using filtering results (see $\rightarrowtail$ Section 5.10). See, for example, Pham and Quenez [711].

Theorem 2.1.5.4 *Let $\tilde{S}$ be a process which represents the discounted prices. If there exists a unique e.m.m. $\mathbb{Q}$ such that $\tilde{S}$ is a $\mathbb{Q}$-local martingale, then the market is complete and arbitrage free.*

PROOF: This result is obtained from the fact that if there is a unique probability measure such that $\tilde{S}$ is a local martingale, then the process $\tilde{S}$ has the representation property. See Jacod and Yor [472] for a proof or $\rightarrowtail$ Subsection 9.5.3. □

Theorem 2.1.5.5 *In an arbitrage free and complete market, the time-t price of a (bounded) contingent claim H is*

$$\boxed{V_t^H = R_t^{-1}\mathbb{E}_{\mathbb{Q}}(R_T H|\mathcal{F}_t)} \tag{2.1.4}$$

where $\mathbb{Q}$ is the unique e.m.m. and R the discount factor.

PROOF: In a complete market, using the predictable representation theorem, there exists π such that $HR_T = h + \sum_{i=1}^d \int_0^T \pi_s dS_s^{i,0}$, and $S^{i,0}$ is a $\mathbb{Q}$-martingale. Hence, the result follows. □

Working with the historical probability yields that the process Z defined by $Z_t = L_t R_t V_t^H$, where L is the Radon-Nikodým density, is a $\mathbb{P}$-martingale; therefore we also obtain the price V_t^H of the contingent claim H as

$$V_t^H R_t L_t = \mathbb{E}_{\mathbb{P}}(L_T R_T H|\mathcal{F}_t)\,. \tag{2.1.5}$$

Remark 2.1.5.6 Note that, in an incomplete market, if H is hedgeable, then the time-t value of the replicating portfolio is $V_t^H = R_t^{-1}\mathbb{E}_{\mathbb{Q}}(R_T H|\mathcal{F}_t)$, for any e.m.m. $\mathbb{Q}$.

2.2 A Diffusion Model

In this section, we make precise the dynamics of the assets as Itô processes, we study the market completeness and, in a Markovian setting, we present the PDE approach.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We assume that an n-dimensional Brownian motion B is constructed on this space and we denote by $\mathbf{F}$ its natural filtration. We assume that the dynamics of the assets of the financial market are as follows: the dynamics of the savings account are

$$dS_t^0 = r_t S_t^0 dt,\ S_0^0 = 1\,, \tag{2.2.1}$$

and the vector valued process $(S^i, 1 \leq i \leq d)$ consisting of the prices of d risky assets is a d-dimensional diffusion which follows the dynamics

$$dS_t^i = S_t^i(b_t^i dt + \sum_{j=1}^n \sigma_t^{i,j} dB_t^j)\,, \tag{2.2.2}$$

where r, b^i, and the volatility coefficients $\sigma^{i,j}$ are supposed to be given $\mathbf{F}$-predictable processes, and satisfy for any t, almost surely,

$$r_t > 0,\ \int_0^t r_s ds < \infty, \quad \int_0^t |b_s^i| ds < \infty\,, \quad \int_0^t (\sigma_s^{i,j})^2 ds < \infty\,.$$

The solution of (2.2.2) is

$$S_t^i = S_0^i \exp\left(\int_0^t b_s^i ds + \sum_{j=1}^n \int_0^t \sigma_s^{i,j} dB_s^j - \frac{1}{2}\sum_{j=1}^n \int_0^t (\sigma_s^{i,j})^2 ds\right)\,.$$

In particular, the prices of the assets are strictly positive. As usual, we denote by

$$R_t = \exp\left(-\int_0^t r_s ds\right) = 1/S_t^0$$

the discount factor. We also denote by $S^{i,0} = S^i/S^0$ the discounted prices and $V^0 = V/S^0$ the discounted value of V.

2.2.1 Absence of Arbitrage

Proposition 2.2.1.1 *In the model* (2.2.1–2.2.2)*, the existence of an e.m.m. implies absence of arbitrage.*

PROOF: Let π be an admissible self-financing strategy, and assume that $\mathbb{Q}$ is an e.m.m. Then,

$$dS_t^{i,0} = R_t\,(dS_t^i - r_t S_t^i dt) = S_t^{i,0} \sum_{j=1}^n \sigma_t^{i,j} dW_t^j$$

where W is a $\mathbb{Q}$-Brownian motion. Then, the process $V^{\pi,0}$ is a $\mathbb{Q}$-local martingale which is bounded below (admissibility assumption), and therefore, it is a supermartingale, and $V_0^{\pi,0} \geq \mathbb{E}_{\mathbb{Q}}(V_T^{\pi,0})$. Therefore, $V_T^{\pi,0} \geq 0$ implies that the terminal value is null: there are no arbitrage opportunities. □

2.2.2 Completeness of the Market

In the model (2.2.1, 2.2.2) when $d = n$ (i.e., the number of risky assets equals the number of driving BM), and when σ is invertible, the e.m.m. exists and is unique as long as some regularity is imposed on the coefficients. More precisely, we require that we can apply Girsanov's transformation in such a way that the d-dimensional process W where

$$dW_t = dB_t + \sigma_t^{-1}(b_t - r_t\mathbf{1})dt = dB_t + \theta_t dt\,,$$

is a $\mathbb{Q}$-Brownian motion. In other words, we assume that the solution L of

$$dL_t = -L_t\sigma_t^{-1}(b_t - r_t\mathbf{1})dB_t = -L_t\theta_t dB_t, \quad L_0 = 1$$

is a martingale (this is the case if θ is bounded). The process

$$\boxed{\theta_t = \sigma_t^{-1}(b_t - r_t\mathbf{1})}$$

is called the **risk premium**[1]. Then, we obtain

$$dS_t^{i,0} = S_t^{i,0}\sum_{j=1}^d \sigma_t^{i,j} dW_t^j\,.$$

We can apply the predictable representation property under the probability $\mathbb{Q}$ and find for any $H \in L^2(\mathcal{F}_T)$ a d-dimensional predictable process $(h_t, t \leq T)$ with $\mathbb{E}_{\mathbb{Q}}(\int_0^T |h_s|^2 ds) < \infty$ and

$$HR_T = \mathbb{E}_{\mathbb{Q}}(HR_T) + \int_0^T h_s\, dW_s\,.$$

Therefore,

$$HR_T = \mathbb{E}_{\mathbb{Q}}(HR_T) + \sum_{i=1}^d \int_0^T \pi_s^i\, dS_s^{i,0}$$

where π satisfies $\sum_{i=1}^d \pi_s^i S_s^{i,0}\sigma_s^{i,j} = h_s^j$. Hence, the market is complete, the price of H is $\mathbb{E}_{\mathbb{Q}}(HR_T)$, and the hedging portfolio is $(V_t - \pi_t S_t, \pi_t)$ where the time-t discounted value of the portfolio is given by

$$V_t^0 = R_t^{-1}\mathbb{E}_{\mathbb{Q}}(HR_T|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(HR_T) + \int_0^t R_s\pi_s(dS_s - r_sS_sds)\,.$$

Remark 2.2.2.1 In the case $d < n$, the market is generally incomplete and does not present arbitrage opportunities. In some specific cases, it can be reduced to a complete market as in the $\rightarrowtail$ Example 2.3.6.1.

In the case $n < d$, the market generally presents arbitrage opportunities, as shown in Comments 2.1.5.3, but is complete.

[1] In the one-dimensional case, σ is, in finance, a positive process. Roughly speaking, the investor is willing to invest in the risky asset only if $b > r$, i.e., if he will get a positive "premium."

2.2.3 PDE Evaluation of Contingent Claims in a Complete Market

In the particular case where $H = h(S_T)$, r is deterministic, h is bounded, and S is an inhomogeneous diffusion

$$dS_t = DS_t(b(t, S_t)dt + \Sigma(t, S_t)dB_t)\,,$$

where DS is the diagonal matrix with S^i on the diagonal, we deduce from the Markov property of S under $\mathbb{Q}$ that there exists a function $V(t, x)$ such that

$$\mathbb{E}_{\mathbb{Q}}(R(T)h(S_T)|\mathcal{F}_t) = R(t)V(t, S_t) = V^0(t, S_t)\,.$$

The process $(V^0(t, S_t),\, t \geq 0)$ is a martingale, hence its bounded variation part is equal to 0. Therefore, as soon as V is smooth enough (see Karatzas and Shreve [513] for conditions which ensure this regularity), Itô's formula leads to

$$\begin{aligned} V^0(t, S_t) &= V^0(0, S_0) + \sum_{i=1}^{d} \int_0^t \partial_{x_i} V^0(s, S_s)(dS^i_s - r(s)S^i_s ds) \\ &= V(0, S_0) + \sum_{i=1}^{d} \int_0^t \partial_{x_i} V(s, S_s) dS^{i,0}_s\,, \end{aligned}$$

where we have used the fact that

$$\partial_{x_i} V^0(t, x) = R(t)\, \partial_{x_i} V(t, x)\,.$$

We now compare with (2.1.1)

$$V^0(t, S_t) = \mathbb{E}_{\mathbb{Q}}(HR(T)) + \sum_{i=1}^{d} \int_0^t \pi^i_s dS^{i,0}_s$$

and we obtain that $\pi^i_s = \partial_{x_i} V(s, S_s)$.

Proposition 2.2.3.1 *Let*

$$dS^i_t = S^i_t\,(r(t)dt + \sum_{j=1}^{d} \sigma_{i,j}(t, S_t)dB^j_t)\,,$$

be the risk-neutral dynamics of the d risky assets where the interest rate is deterministic. Assume that V solves the PDE, for $t < T$ and $x_i > 0, \forall i$,

$$\boxed{\partial_t V + r(t) \textstyle\sum_{i=1}^{d} x_i \partial_{x_i} V + \displaystyle\frac{1}{2} \sum_{i,j} x_i x_j \partial_{x_i x_j} V \sum_{k=1}^{d} \sigma_{i,k}\sigma_{j,k} = r(t)V} \tag{2.2.3}$$

with terminal condition $V(T, x) = h(x)$. Then, the value at time t of the contingent claim $H = h(S_T)$ is equal to $V(t, S_t)$.

The hedging portfolio is $\pi^i_t = \partial_{x_i} V(t, S_t), i = 1, \ldots, d$.

In the one-dimensional case, when $dS_t = S_t(b(t,S_t)dt + \sigma(t,S_t)dB_t)$, the PDE reads, for $x > 0, t \in [0,T[$,

$$\boxed{\partial_t V(t,x) + r(t)x\partial_x V(t,x) + \frac{1}{2}\sigma^2(t,x)x^2\partial_{xx}V(t,x) = r(t)V(t,x)} \tag{2.2.4}$$

with the terminal condition $V(T,x) = h(x)$.

Definition 2.2.3.2 *Solving the equation (2.2.4) with the terminal condition is called the Partial Derivative Equation (PDE) evaluation procedure.*

In the case when the contingent claim H is path-dependent (i.e., when the payoff $H = h(S_t, t \le T)$ depends on the past of the price process, and not only on the terminal value), it is not always possible to associate a PDE to the pricing problem (see, e.g., Parisian options (see ↣ Section 4.4) and Asian options (see ↣ Section 6.6)).

Thus, we have two ways of computing the price of a contingent claim of the form $h(S_T)$, either we solve the PDE, or we compute the conditional expectation (2.1.5). The quantity RL is often called **the state-price density** or the pricing kernel. Therefore, in a complete market, we can characterize the processes which represent the value of a self-financing strategy.

Proposition 2.2.3.3 *If a given process V is such that VR is a $\mathbb{Q}$-martingale (or VRL is a $\mathbb{P}$-martingale), it defines the value of a self-financing strategy.*

In particular, the process $(N_t = 1/(R_tL_t),\ t \ge 0)$ is the value of a portfolio (NRL is a $\mathbb{P}$-martingale), called the **numéraire portfolio** or the growth optimal portfolio. It satisfies

$$dN_t = N_t((r(t) + \theta_t^2)dt + \theta_t dB_t)\,.$$

(See Becherer [63], Long [603], Karatzas and Kardaras [511] and the book of Heath and Platen [429] for a study of the numéraire portfolio.) It is a main tool for consumption-investment optimization theory, for which we refer the reader to the books of Karatzas [510], Karatzas and Shreve [514], and Korn [538].

2.3 The Black and Scholes Model

We now focus on the well-known Black and Scholes model, which is a very particular and important case of the diffusion model.

2.3.1 The Model

The **Black and Scholes model** [105] (see also Merton [641]) assumes that there is a riskless asset with interest rate r and that the dynamics of the price of the underlying asset are

$$dS_t = S_t(bdt + \sigma dB_t)$$

under the historical probability $\mathbb{P}$. Here, the risk-free rate r, the trend b and the volatility σ are supposed to be constant (note that, for valuation purposes, b may be an $\mathbf{F}$-adapted process). In other words, the value at time t of the risky asset is

$$S_t = S_0 \exp\left(bt + \sigma B_t - \frac{\sigma^2}{2}t\right).$$

From now on, we fix a finite horizon T and our processes are only indexed by $[0, T]$.

Notation 2.3.1.1 In the sequel, for two semi-martingales X and Y, we shall use the notation $X \stackrel{\text{mart}}{=} Y$ (or $dX_t \stackrel{\text{mart}}{=} dY_t$) to mean that $X - Y$ is a local martingale.

Proposition 2.3.1.2 *In the Black and Scholes model, there exists a unique e.m.m. $\mathbb{Q}$, precisely $\mathbb{Q}|_{\mathcal{F}_t} = \exp(-\theta B_t - \frac{1}{2}\theta^2 t)\mathbb{P}|_{\mathcal{F}_t}$ where $\theta = \frac{b-r}{\sigma}$ is the risk-premium. The risk-neutral dynamics of the asset are*

$$dS_t = S_t(rdt + \sigma dW_t)$$

where W is a $\mathbb{Q}$-Brownian motion.

Proof: If $\mathbb{Q}$ is equivalent to $\mathbb{P}$, there exists a strictly positive martingale L such that $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$. From the predictable representation property under $\mathbb{P}$, there exists a predictable ψ such that

$$dL_t = \psi_t dB_t = L_t \phi_t dB_t$$

where $\phi_t L_t = \psi_t$. It follows that

$$d(LRS)_t \stackrel{\text{mart}}{=} (LRS)_t(b - r + \phi_t \sigma)dt.$$

Hence, in order for $\mathbb{Q}$ to be an e.m.m., or equivalently for LRS to be a $\mathbb{P}$-local martingale, there is one and only one process ϕ such that the bounded variation part of LRS is null, that is

$$\phi_t = \frac{r - b}{\sigma} = -\theta,$$

where θ is the risk premium. Therefore, the unique e.m.m. has a Radon-Nikodým density L which satisfies $dL_t = -L_t\theta dB_t, L_0 = 1$ and is given by $L_t = \exp(-\theta B_t - \frac{1}{2}\theta^2 t)$.

Hence, from Girsanov's theorem, $W_t = B_t + \theta t$ is a $\mathbb{Q}$-Brownian motion, and

$$dS_t = S_t(bdt + \sigma dB_t) = S_t(rdt + \sigma(dB_t + \theta dt)) = S_t(rdt + \sigma dW_t)\,.$$

□

In a closed form, we have

$$S_t = S_0 \exp\left(bt + \sigma B_t - \frac{\sigma^2}{2}t\right) = S_0 e^{rt} \exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right) = S_0 e^{\sigma X_t}$$

with $X_t = \nu t + W_t$, and $\nu = \frac{r}{\sigma} - \frac{\sigma}{2}$.

In order to price a contingent claim $h(S_T)$, we compute the expectation of its discounted value under the e.m.m.. This can be done easily, since $\mathbb{E}_{\mathbb{Q}}(h(S_T)e^{-rT}) = e^{-rT}\,\mathbb{E}_{\mathbb{Q}}(h(S_T))$ and

$$\mathbb{E}_{\mathbb{Q}}(h(S_T)) = \mathbb{E}\left[h(S_0 e^{rT - \frac{\sigma^2}{2}T} \exp(\sigma\sqrt{T}\,G))\right]$$

where G is a standard Gaussian variable.

We can also think about the expression $\mathbb{E}_{\mathbb{Q}}(h(S_T)) = \mathbb{E}_{\mathbb{Q}}(h(xe^{\sigma X_T}))$ as a computation for the drifted Brownian motion $X_t = \nu t + W_t$. As an exercise on Girsanov's transformation, let us show how we can reduce the computation to the case of a standard Brownian motion. The process X is a Brownian motion under $\mathbb{Q}^*$, defined on $\mathcal{F}_T$ as

$$\mathbb{Q}^* = \exp\left(-\nu W_T - \frac{1}{2}\nu^2 T\right)\mathbb{Q} = \zeta_T \mathbb{Q}\,.$$

Therefore,

$$\mathbb{E}_{\mathbb{Q}}(h(xe^{\sigma X_T})) = \mathbb{E}_{\mathbb{Q}^*}(\zeta_T^{(-1)} h(xe^{\sigma X_T}))\,.$$

From

$$\zeta_T^{(-1)} = \exp\left(\nu W_T + \frac{1}{2}\nu^2 T\right) = \exp\left(\nu X_T - \frac{1}{2}\nu^2 T\right)\,,$$

we obtain

$$\mathbb{E}_{\mathbb{Q}}\left(h(xe^{\sigma X_T})\right) = \exp\left(-\frac{1}{2}\nu^2 T\right)\mathbb{E}_{\mathbb{Q}^*}(\exp(\nu X_T) h(xe^{\sigma X_T}))\,, \tag{2.3.1}$$

where on the left-hand side, X is a $\mathbb{Q}$-Brownian motion with drift ν and on the right-hand side, X is a $\mathbb{Q}^*$-Brownian motion. We can and do write the quantity on the right-hand side as $\exp(-\frac{1}{2}\nu^2 T)\,\mathbb{E}(\exp(\nu W_T)h(xe^{\sigma W_T}))$, where W is a generic Brownian motion.

We can proceed in a more powerful way using Cameron-Martin's theorem, i.e., the absolute continuity relationship between a Brownian motion with drift and a Brownian motion. Indeed, as in Exercise 1.7.5.5

$$\mathbb{E}_{\mathbb{Q}}(h(xe^{\sigma X_T})) = \mathbf{W}^{(\nu)}(h(xe^{\sigma X_T})) = \mathbb{E}\left(e^{\nu W_T - \frac{\nu^2}{2}T} h(xe^{\sigma W_T})\right) \tag{2.3.2}$$

which is exactly (2.3.1).

Proposition 2.3.1.3 *Let us consider the Black and Scholes framework*

$$dS_t = S_t(rdt + \sigma dW_t),\ S_0 = x$$

where W is a $\mathbb{Q}$-Brownian motion and $\mathbb{Q}$ is the e.m.m. or risk-neutral probability. In that setting, the value of the contingent claim $h(S_T)$ is

$$\mathbb{E}_{\mathbb{Q}}(e^{-rT}h(S_T)) = e^{-(r+\frac{\nu^2}{2})T}\mathbf{W}\left(e^{\nu X_T}h(xe^{\sigma X_T})\right)$$

where $\nu = \frac{r}{\sigma} - \frac{\sigma}{2}$ and X is a Brownian motion under $\mathbf{W}$.

The time-t value of the contingent claim $h(S_T)$ is

$$\mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}h(S_T)|\mathcal{F}_t) = e^{-(r+\frac{\nu^2}{2})(T-t)}\mathbf{W}\left(e^{\nu X_{T-t}}h(ze^{\sigma X_{T-t}})\right)|_{z=S_t}\,.$$

The value of a path-dependent contingent claim $\Phi(S_t, t \le T)$ is

$$\mathbb{E}_{\mathbb{Q}}(e^{-rT}\Phi(S_t, t \le T)) = e^{-(r+\frac{\nu^2}{2})T}\mathbf{W}\left(e^{\nu X_T}\Phi(xe^{\sigma X_t}, t \le T)\right)\,.$$

PROOF: It remains to establish the formula for the time-t value. From

$$\mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}h(S_T)|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}h(S_t S_T^t)|\mathcal{F}_t)$$

where $S_T^t = S_T/S_t$, using the independence between S_T^t and $\mathcal{F}_t$ and the equality $S_T^t \overset{\text{law}}{=} S_{T-t}^1$, where S^1 has the same dynamics as S, with initial value 1, we get

$$\mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}h(S_T)|\mathcal{F}_t) = \Psi(S_t)$$

where

$$\Psi(x) = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}h(xS_T^t)) = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}h(xS_{T-t}))\,.$$

This last quantity can be computed from the properties of BM. Indeed,

$$\mathbb{E}_{\mathbb{Q}}\left(h(S_T)|\mathcal{F}_t\right) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dy\, h\left(S_t e^{r(T-t)+\sigma\sqrt{T-t}y-\sigma^2(T-t)/2}\right)e^{-y^2/2}\,.$$

(See Example 1.5.4.7 if needed.) □

Notation 2.3.1.4 In the sequel, when working in the Black and Scholes framework, we shall use systematically the notation $\nu = \frac{r}{\sigma} - \frac{\sigma}{2}$ and the fact that for $t \geq s$ the r.v. $S_t^s = S_t/S_s$ is independent of S_s.

Exercise 2.3.1.5 The payoff of a power option is $h(S_T)$, where the function h is given by $h(x) = x^\beta(x-K)^+$. Prove that the payoff can be written as the difference of European payoffs on the underlying assets $S^{\beta+1}$ and S^β with strikes depending on K and β . ◁

Exercise 2.3.1.6 We consider a contingent claim with a terminal payoff $h(S_T)$ and a continuous payoff $(x_s, s \leq T)$, where x_s is paid at time s. Prove that the price of this claim is

$$V_t = \mathbb{E}_\mathbb{Q}(e^{-r(T-t)}h(S_T) + \int_t^T e^{-r(s-t)}x_s ds|\mathcal{F}_t)\,.$$

◁

Exercise 2.3.1.7 In a Black and Scholes framework, prove that the price at time t of the contingent claim $h(S_T)$ is

$$C_h(x, T-t) = e^{-r(T-t)}\mathbb{E}_\mathbb{Q}(h(S_T)|S_t = x) = e^{-r(T-t)}\mathbb{E}_\mathbb{Q}(h(S_T^{t,x}))$$

where $S_s^{t,x}$ is the solution of the SDE

$$dS_s^{t,x} = S_s^{t,x}(rds + \sigma dW_s),\ S_t^{t,x} = x$$

and the hedging strategy consists of holding $\partial_x C_h(S_t, T-t)$ shares of the underlying asset.

Assuming some regularity on h, and using the fact that $S_T^{t,x} \stackrel{\text{law}}{=} xe^{\sigma X_{T-t}}$, where X_{T-t} is a Gaussian r.v., prove that

$$\partial_x C_h(x, T-t) = \frac{1}{x}\mathbb{E}_\mathbb{Q}\left(h'(S_T^{t,x})S_T^{t,x}\right)e^{-r(T-t)}\,.$$

◁

2.3.2 European Call and Put Options

Among the various derivative products, the most popular are the European Call and Put Options, also called vanilla[2] options.

A **European call** is associated with some underlying asset, with price $(S_t, t \geq 0)$. At maturity (a given date T), the holder of a call receives $(S_T - K)^+$ where K is a fixed number, called the strike. The price of a call is the amount of money that the buyer of the call will pay at time 0 to the seller. The time-t price is the price of the call at time t, equal to $\mathbb{E}_\mathbb{Q}(e^{-r(T-t)}(S_T-K)^+|\mathcal{F}_t)$, or, due to the Markov property, $\mathbb{E}_\mathbb{Q}(e^{-r(T-t)}(S_T-K)^+|S_t)$. At maturity (a given date T), the holder of a European put receives $(K - S_T)^+$.

[2] To the best of our knowledge, the name "vanilla" (or "plain vanilla") was given to emphasize the standard form of these products, by reference to vanilla, a standard flavor for ice cream, or to plain vanilla, a standard font in printing.

Theorem 2.3.2.1 Black and Scholes formula.
Let $dS_t = S_t(bdt + \sigma dB_t)$ be the dynamics of the price of a risky asset and assume that the interest rate is a constant r. The value at time t of a European call with maturity T and strike K is $\mathcal{BS}(S_t, \sigma, t)$ where

$$\boxed{\begin{aligned}\mathcal{BS}(x,\sigma,t) := x\mathcal{N}&\left[d_1\left(\frac{x}{Ke^{-r(T-t)}}, T-t\right)\right] \\ &- Ke^{-r(T-t)}\mathcal{N}\left[d_2\left(\frac{x}{Ke^{-r(T-t)}}, T-t\right)\right]\end{aligned}} \tag{2.3.3}$$

where

$$d_1(y,u) = \frac{1}{\sqrt{\sigma^2 u}}\ln(y) + \frac{1}{2}\sqrt{\sigma^2 u}, \quad d_2(y,u) = d_1(y,u) - \sqrt{\sigma^2 u}\,,$$

where we have written $\sqrt{\sigma^2}$ so that the formula does not depend on the sign of σ.

PROOF: It suffices to solve the evaluation PDE (2.2.4) with terminal condition $C(x,T) = (x-K)^+$. Another method is to compute the conditional expectation, under the e.m.m., of the discounted terminal payoff, i.e., $\mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T-K)^+|\mathcal{F}_t)$. For $t=0$,

$$\mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T-K)^+) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}S_T\mathbb{1}_{\{S_T\geq K\}}) - Ke^{-rT}\mathbb{Q}(S_T\geq K)\,.$$

Under $\mathbb{Q}$, $dS_t = S_t(rdt + \sigma dW_t)$ hence, $S_T \stackrel{\text{law}}{=} S_0 e^{rT-\sigma^2 T/2}e^{\sigma\sqrt{T}G}$, where G is a standard Gaussian law, hence

$$\mathbb{Q}(S_T\geq K) = \mathcal{N}\left[d_2\left(\frac{x}{Ke^{-rT)}}, T\right)\right]\,.$$

The equality

$$\mathbb{E}_{\mathbb{Q}}(e^{-rT}S_T\mathbb{1}_{\{S_T\geq K\}}) = x\mathcal{N}\left(d_1\left(\frac{S_0}{Ke^{-rT}}, T\right)\right)$$

can be proved using the law of S_T, however, we shall give in $\rightarrowtail$ Subsection 2.4.1 a more pleasant method.

The computation of the price at time t is carried out using the Markov property. □

Let us emphasize that a pricing formula appears in Bachelier [39, 41] in the case where S is a drifted Brownian motion. The central idea in Black and Scholes' paper is the hedging strategy. Here, the hedging strategy for a call is to keep a long position of $\Delta(t,S_t) = \frac{\partial C}{\partial x}(S_t, T-t)$ in the underlying asset (and to have $C - \Delta S_t$ shares in the savings account). It is well known that this quantity is equal to $\mathcal{N}(d_1)$. This can be checked by a tedious differentiation of (2.3.3). One can also proceed as follows: as we shall see in $\rightarrowtail$ Comments 2.3.2.2

$$C(x, T-t) = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}(S_T - K)^+ | S_t = x) = \mathbb{E}_{\mathbb{Q}}(R_T^t (x S_T^t - K)^+)\,,$$

where $S_T^t = S_T/S_t$, so that $\Delta(t,x)$ can be obtained by a differentiation with respect to x under the expectation sign. Hence,

$$\Delta(t,x) = \mathbb{E}(R_T^t S_T^t \mathbb{1}_{\{x S_T^t \geq K\}}) = \mathcal{N}\left(d_1(S_t/(Ke^{-r(T-t)}), T-t)\right)\,.$$

This quantity, called the "Delta" (see $\rightarrowtail$ Subsection 2.3.3) is positive and bounded by 1. The second derivative with respect to x (the "Gamma") is $\frac{1}{\sigma x \sqrt{T-t}} \mathcal{N}'(d_1)$, hence $C(x, T-t)$ is convex w.r.t. x.

Comment 2.3.2.2 It is remarkable that the PDE evaluation was obtained in the seminal paper of Black and Scholes [105] without the use of any e.m.m.. Let us give here the main arguments. In this paper, the objective is to replicate the risk-free asset with simultaneous positions in the contingent claim and in the underlying asset. Let (α, β) be a replicating portfolio and

$$V_t = \alpha_t C_t + \beta_t S_t$$

the value of this portfolio assumed to satisfy the self-financing condition, i.e.,

$$dV_t = \alpha_t dC_t + \beta_t dS_t$$

Then, assuming that C_t is a smooth function of time and underlying value, i.e., $C_t = C(S_t, t)$, by relying on Itô's lemma the differential of V is obtained:

$$dV_t = \alpha_t(\partial_x C dS_t + \partial_t C dt + \frac{1}{2}\sigma^2 S_t^2 \partial_{xx} C dt) + \beta_t dS_t\,,$$

where $\partial_t C$ (resp. $\partial_x C$) is the derivative of C with respect to the second variable (resp. the first variable) and where all the functions $C, \partial_x C, \dots$ are evaluated at (S_t, t). From $\alpha_t = (V_t - \beta_t S_t)/C_t$, we obtain

$$\begin{aligned} dV_t = {} & ((V_t - \beta_t S_t)(C_t)^{-1} \partial_x C + \beta_t)\sigma S_t dB_t \\ & + \left(\frac{V_t - \beta_t S_t}{C_t}\left(\partial_t C + \frac{1}{2}\sigma^2 S_t^2 \partial_{xx} C + b S_t \partial_x C\right) + \beta_t S_t b\right) dt\,. \end{aligned} \tag{2.3.4}$$

If this replicating portfolio is risk-free, one has $dV_t = V_t r dt$: the martingale part on the right-hand side vanishes, which implies

$$\beta_t = (S_t \partial_x C - C_t)^{-1} V_t \partial_x C$$

and

$$\frac{V_t - \beta_t S_t}{C_t}\left(\partial_t C + \frac{1}{2}\sigma^2 S_t^2 \partial_{xx} C + S_t b \partial_x C\right) + \beta_t S_t b = rV_t \,. \tag{2.3.5}$$

Using the fact that

$$(V_t - \beta_t S_t)(C_t)^{-1}\partial_x C \, + \beta_t = 0$$

we obtain that the term which contains b, i.e.,

$$bS_t\left(\frac{V_t - \beta S_t}{C_t}\partial_x C + \beta_t\right)$$

vanishes. After simplifications, we obtain

$$\begin{aligned} rC &= \left(1 + \frac{S\partial_x C}{C - S\partial_x C}\right)\left(\partial_t C + \frac{1}{2}\sigma^2 x^2 \partial_{xx} C\right) \\ &= \frac{C}{C - S\partial_x C}\left(\partial_t C + \frac{1}{2}\sigma^2 x^2 \partial_{xx} C\right) \end{aligned}$$

and therefore the PDE evaluation

$$\begin{aligned} &\partial_t C(x,t) + rx\partial_x C(x,t) + \frac{1}{2}\sigma^2 x^2 \partial_{xx} C(x,t) \\ &\qquad = rC(x,t),\ x > 0, t \in [0, T[\end{aligned} \tag{2.3.6}$$

is obtained. Now,

$$\beta_t = V_t \partial_x C (S\partial_x C - C)^{-1} = V_0 \frac{\mathcal{N}(d_1)}{Ke^{-rT}\mathcal{N}(d_2)} \,.$$

Note that the hedging ratio is

$$\frac{\beta_t}{\alpha_t} = -\partial_x C(t, S_t) \,.$$

Reading carefully [105], it seems that the authors assume that there exists a self-financing strategy $(-1, \beta_t)$ such that $dV_t = rV_t dt$, which is not true; in particular, the portfolio $(-1, \mathcal{N}(d_1))$ is not self-financing and its value, equal to $-C_t + S_t\mathcal{N}(d_1) = Ke^{-r(T-t)}\mathcal{N}(d_2)$, is not the value of a risk-free portfolio.

Exercise 2.3.2.3 Robustness of the Black and Scholes formula. Let

$$dS_t = S_t(bdt + \sigma_t dB_t)$$

where $(\sigma_t, t \geq 0)$ is an adapted process such that for any t, $0 < a \leq \sigma_t \leq b$. Prove that

$$\forall t,\ \mathcal{BS}(S_t, a, t) \leq \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t) \leq \mathcal{BS}(S_t, b, t) \,.$$

`Hint:` This result is obtained by using the fact that the $\mathcal{BS}$ function is convex with respect to x. ◁

Comment 2.3.2.4 The result of the last exercise admits generalizations to other forms of payoffs as soon as the convexity property is preserved, and to the case where the volatility is a given process, not necessarily **F**-adapted. See El Karoui et al. [301], Avellaneda et al. [29] and Martini [625]. This convexity property holds for a d-dimensional price process only in the geometric Brownian motion case, see Ekström et al. [296]. See Mordecki [413] and Bergenthum and Rüschendorf [74], for bounds on option prices.

Exercise 2.3.2.5 Suppose that the dynamics of the risky asset are given by $dS_t = S_t(bdt + \sigma(t)dB_t)$, where σ is a deterministic function. Characterize the law of S_T under the risk-neutral probability $\mathbb{Q}$ and prove that the price of a European option on the underlying S, with maturity T and strike K, is $\mathcal{BS}(x, \Sigma(t), t)$ where $(\Sigma(t))^2 = \frac{1}{T-t}\int_t^T \sigma^2(s)ds$. ◁

Exercise 2.3.2.6 Assume that, under $\mathbb{Q}$, S follows a Black and Scholes dynamics with $\sigma = 1, r = 0, S_0 = 1$. Prove that the function $t \to C(1, t; 1) := \mathbb{E}_{\mathbb{Q}}((S_t - 1)^+)$ is a cumulative distribution function of some r.v. X; identify the law of X.
`Hint:` $\mathbb{E}_{\mathbb{Q}}((S_t - 1)^+) = \mathbb{Q}(4B_1^2 \leq t)$ where B is a $\mathbb{Q}$-BM. See Bentata and Yor [72] for more comments. ◁

2.3.3 The Greeks

It is important for practitioners to have a good knowledge of the sensitivity of the price of an option with respect to the parameters of the model.

The **Delta** is the derivative of the price of a call with respect to the underlying asset price (the spot). In the BS model, the Delta of a call is $\mathcal{N}(d_1)$. The Delta of a portfolio is the derivative of the value of the portfolio with respect to the underlying price. A portfolio with zero Delta is said to be delta neutral. Delta hedging requires continuous monitoring and rebalancing of the hedge ratio.

The **Gamma** is the derivative of the Delta w.r.t. the underlying price. In the BS model, the Gamma of a call is $\mathcal{N}'(d_1)/S\sigma\sqrt{T-t}$. It follows that the BS price of a call option is a convex function of the spot. The Gamma is important because it makes precise how much hedging will cost in a small interval of time.

The **Vega** is the derivative of the option price w.r.t. the volatility. In the BS model, the Vega of a call is $\mathcal{N}'(d_1)S\sqrt{T-t}$.

2.3.4 General Case

Let us study the case where

$$dS_t = S_t(\alpha_t dt + \sigma_t dB_t)\,.$$

Here, B is a Brownian motion with natural filtration $\mathbf{F}$ and α and σ are bounded $\mathbf{F}$-predictable processes. Then,

$$S_t = S_0 \exp\left(\int_0^t \left(\alpha_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dB_s\right)$$

and $\mathcal{F}_t^S \subset \mathcal{F}_t$. We assume that r is the constant risk-free interest rate and that $\sigma_t \geq \epsilon > 0$, hence the risk premium $\theta_t = \dfrac{\alpha_t - r}{\sigma_t}$ is bounded. It follows that the process

$$L_t = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right),\ t \leq T$$

is a uniformly integrable martingale. We denote by $\mathbb{Q}$ the probability measure satisfying $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$ and by W the Brownian part of the decomposition of the $\mathbb{Q}$-semi-martingale B, i.e., $W_t = B_t + \int_0^t \theta_s ds$. Hence, from integration by parts formula, $d(RS)_t = R_t S_t \sigma_t dW_t$.

Then, from the predictable representation property (see Section 1.6), for any square integrable $\mathcal{F}_T$-measurable random variable H, there exists an $\mathbf{F}$-predictable process ϕ such that $HR_T = \mathbb{E}_{\mathbb{Q}}(HR_T) + \int_0^T \phi_s dW_s$ and $\mathbb{E}(\int_0^T \phi_s^2 ds) < \infty$; therefore

$$HR_T = \mathbb{E}_{\mathbb{Q}}(HR_T) + \int_0^T \psi_s d(RS)_s$$

where $\psi_t = \phi_t/(R_t S_t \sigma_t)$. It follows that H is hedgeable with the self-financing portfolio $(V_t - \psi_t S_t, \psi_t)$ where

$$V_t = R_t^{-1}\mathbb{E}_{\mathbb{Q}}(HR_T|\mathcal{F}_t) = H_t^{-1}\mathbb{E}_{\mathbb{P}}(HH_T|\mathcal{F}_t)$$

with $H_t = R_t L_t$. The process H is called the deflator or the pricing kernel.

2.3.5 Dividend Paying Assets

In this section, we suppose that the owner of one share of the stock receives a dividend. Let S be the stock process. Assume in a first step that the stock pays dividends Δ_i at fixed increasing dates $T_i, i \leq n$ with $T_n \leq T$. The price of the stock at time 0 is the expectation under the risk-neutral probability $\mathbb{Q}$ of the discounted future payoffs, that is

$$S_0 = \mathbb{E}_{\mathbb{Q}}(S_T R_T + \sum_{i=1}^{n} \Delta_i R_{T_i}) .$$

We now assume that the dividends are paid in continuous time, and let D be the cumulative dividend process (that is D_t is the amount of dividends received between 0 and t). The discounted price of the stock is the risk-neutral expectation (one often speaks of risk-adjusted probability in the case of dividends) of the future dividends, that is

$$S_t R_t = \mathbb{E}_{\mathbb{Q}} \left(S_T R_T + \int_t^T R_s dD_s | \mathcal{F}_t \right) .$$

Note that the discounted price $R_t S_t$ is no longer a $\mathbb{Q}$-martingale. On the other hand, the discounted cum-dividend price[3]

$$S_t^{cum} R_t := S_t R_t + \int_0^t R_s dD_s$$

is a $\mathbb{Q}$-martingale. Note that $S_t^{cum} = S_t + \frac{1}{R_t} \int_0^t R_s dD_s$. If we assume that the reference filtration is a Brownian filtration, there exists σ such that

$$d(S_t^{cum} R_t) = \sigma_t S_t R_t dW_t ,$$

and we obtain

$$d(S_t R_t) = -R_t dD_t + S_t R_t \sigma_t dW_t .$$

Suppose now that the asset S pays a proportional dividend, that is, the holder of one share of the asset receives $\delta S_t dt$ in the time interval $[t, t+dt]$. In that case, under the risk-adjusted probability $\mathbb{Q}$, the discounted value of an asset equals the expectation on the discounted future payoffs, i.e.,

$$R_t S_t = \mathbb{E}_{\mathbb{Q}}(R_T S_T + \delta \int_t^T R_s S_s ds | \mathcal{F}_t) .$$

Hence, the discounted cum-dividend process

$$R_t S_t + \int_0^t \delta R_s S_s ds$$

is a $\mathbb{Q}$-martingale so that the risk-neutral dynamics of the underlying asset are given by

$$dS_t = S_t \left((r - \delta) dt + \sigma dW_t \right) . \tag{2.3.7}$$

One can also notice that the process $(S_t R_t e^{\delta t}, t \geq 0)$ is a $\mathbb{Q}$-martingale.

[3] Nothing to do with scum!

If the underlying asset pays a proportional dividend, the self-financing condition takes the following form. Let

$$dS_t = S_t(b_t dt + \sigma_t dB_t)$$

be the historical dynamics of the asset which pays a dividend at rate δ. A trading strategy π is self-financing if the wealth process $V_t = \pi_t^0 S_t^0 + \pi_t^1 S_t$ satisfies

$$dV_t = \pi_t^0 dS_t^0 + \pi_t^1(dS_t + \delta S_t dt) = rV_t dt + \pi_t^1(dS_t + (\delta - r)S_t dt)\,.$$

The term $\delta \pi_t^1 S_t$ makes precise the fact that the gain from the dividends is reinvested in the market. The process VR satisfies

$$d(V_t R_t) = R_t \pi_t^1(dS_t + (\delta - r)S_t dt) = R_t \pi_t^1 S_t \sigma dW_t$$

hence, it is a (local) $\mathbb{Q}$-martingale.

2.3.6 Rôle of Information

When dealing with completeness the choice of the filtration is very important; this is now discussed in the following examples:

Example 2.3.6.1 Toy Example. Assume that the riskless interest rate is a constant r and that the historical dynamics of the risky asset are given by

$$dS_t = S_t(bdt + \sigma_1 dB_t^1 + \sigma_2 dB_t^2)$$

where $(B^i, i = 1, 2)$ are two independent BMs and b a constant[4]. It is not possible to hedge every $\mathcal{F}_T^{B^1,B^2}$-measurable contingent claim with strategies involving only the riskless and the risky assets, hence the market consisting of the $\mathcal{F}_T^{B^1,B^2}$-measurable contingent claims is incomplete.

The set $\mathcal{Q}$ of e.m.m's is obtained via the family of Radon-Nikodým densities $dL_t = L_t(\psi_t dB_t^1 + \gamma_t dB_t^2)$ where the predictable processes ψ, γ satisfy $b + \psi_t \sigma_1 + \gamma_t \sigma_2 = r$. Thus, the set $\mathcal{Q}$ is infinite.

However, writing the dynamics of S as a semi-martingale in its own filtration leads to $dS_t = S_t(bdt + \sigma dB_t^3)$ where B^3 is a Brownian motion and $\sigma^2 = \sigma_1^2 + \sigma_2^2$. Note that $\mathcal{F}_t^{B^3} = \mathcal{F}_t^S$. It is now clear that any $\mathcal{F}_T^S$-measurable contingent claim can be hedged, and the market is $\mathbf{F}^S$-complete.

Example 2.3.6.2 More generally, a market where the riskless asset has a price given by (2.2.1) and where the d risky assets' prices follow

$$dS_t^i = S_t^i(b^i(t, S_t)dt + \sum_{j=1}^n \sigma^{i,j}(t, S_t)dB_t^j),\ S_0^i = x_i\,, \tag{2.3.8}$$

[4] Of course, the superscript 2 is not a power!

where B is a n-dimensional BM, with $n > d$, can often be reduced to the case of an $\mathcal{F}_T^S$-complete market. Indeed, it may be possible, under some regularity assumptions on the matrix σ, to write the equation (2.3.8) as

$$dS_t^i = S_t^i(b^i(t,S_t)dt + \sum_{j=1}^{d} \widetilde{\sigma}^{i,j}(t,S_t)d\widetilde{B}_t^j),\ S_0^i = x_i\,,$$

where $\widetilde{B}$ is a d-dimensional Brownian motion. The concept of a strong solution for an SDE is useful here. See the book of Kallianpur and Karandikar [506] and the paper of Kallianpur and Xiong [507].

When

$$dS_t^i = S_t^i\left(b_t^i dt + \sum_{j=1}^{n} \sigma_t^{i,j} dB_t^j\right),\ S_0^i = x_i,\ i = 1,\dots,d, \tag{2.3.9}$$

and $n > d$, if the coefficients are adapted with respect to the Brownian filtration $\mathbf{F}^B$, then the market is generally incomplete, as was shown in Exercice 2.3.6.1 (for a general study, see Karatzas [510]). Roughly speaking, a market with a riskless asset and risky assets is complete if the number of sources of noise is equal to the number of risky assets.

An important case of an incomplete market (the stochastic volatility model) is when the coefficient σ is adapted to a filtration different from $\mathbf{F}^B$. (See $\rightarrowtail$ Section 6.7 for a presentation of some stochastic volatility models.)

Let us briefly discuss the case $dS_t = S_t\sigma_t dW_t$. The square of the volatility can be written in terms of S and its bracket as $\sigma_t^2 = \frac{d\langle S\rangle_t}{S_t^2 dt}$ and is obviously $\mathbf{F}^S$-adapted. However, except in the particular case of regular local volatility, where $\sigma_t = \sigma(t, S_t)$, the filtration generated by S is not the filtration generated by a one-dimensional BM. For example, when $dS_t = S_t e^{B_t} dW_t$, where B is a BM independent of W, it is easy to prove that $\mathcal{F}_t^S = \mathcal{F}_t^W \vee \mathcal{F}_t^B$, and in the warning (1.4.1.6) we have established that the filtration generated by S is not generated by a one-dimensional Brownian motion and that S does not possess the predictable representation property.

2.4 Change of Numéraire

The value of a portfolio is expressed in terms of a monetary unit. In order to compare two numerical values of two different portfolios, one has to express these values in terms of the same **numéraire**. In the previous models, the numéraire was the savings account. We study some cases where a different choice of numéraire is helpful.

2.4.1 Change of Numéraire and Black-Scholes Formula

Definition 2.4.1.1 *A numéraire is any strictly positive price process. In particular, it is a semi-martingale.*

As we have seen, in a Black and Scholes model, the price of a European option is given by:

$$\begin{aligned} C(S_0,T) &= \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T-K)\mathbb{1}_{\{S_T\geq K\}}) \\ &= \mathbb{E}_{\mathbb{Q}}(e^{-rT}S_T\mathbb{1}_{\{S_T\geq K\}}) - e^{-rT}K\mathbb{Q}(S_T\geq K)\,. \end{aligned}$$

Hence, if

$$k = \frac{1}{\sigma}\left(\ln(K/x) - (r-\frac{1}{2}\sigma^2)T\right)\,,$$

using the symmetry of the Gaussian law, one obtains

$$\mathbb{Q}(S_T\geq K) = \mathbb{Q}(W_T\geq k) = \mathbb{Q}(W_T\leq -k) = \mathcal{N}\left(d_2\left(\frac{x}{Ke^{-rT}},T\right)\right)$$

where the function d_2 is given in Theorem 2.3.2.1.

From the dynamics of S, one can write:

$$e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T\mathbb{1}_{\{S_T\geq K\}}) = S_0\mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{W_T\geq k\}}\exp\left(-\frac{\sigma^2}{2}T+\sigma W_T\right)\right)\,.$$

The process $(\exp(-\frac{\sigma^2}{2}t+\sigma W_t), t\geq 0)$ is a positive $\mathbb{Q}$-martingale with expectation equal to 1. Let us define the probability $\mathbb{Q}^*$ by its Radon-Nikodým derivative with respect to $\mathbb{Q}$:

$$\mathbb{Q}^*|_{\mathcal{F}_t} = \exp\left(-\frac{\sigma^2}{2}t+\sigma W_t\right)\mathbb{Q}|_{\mathcal{F}_t}\,.$$

Hence,

$$e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T\mathbb{1}_{\{S_T\geq K\}}) = S_0\mathbb{Q}^*(W_T\geq k)\,.$$

Girsanov's theorem implies that the process $(\widehat{W}_t = W_t - \sigma t, t\geq 0)$ is a $\mathbb{Q}^*$-Brownian motion. Therefore,

$$\begin{aligned} e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T\mathbb{1}_{\{S_T\geq K\}}) &= S_0\mathbb{Q}^*\left(W_T-\sigma T\geq k-\sigma T\right) \\ &= S_0\mathbb{Q}^*\left(\widehat{W}_T\leq -k+\sigma T\right)\,, \end{aligned}$$

i.e.,

$$e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T\mathbb{1}_{\{S_T\geq K\}}) = S_0\mathcal{N}\left(d_1\left(\frac{x}{Ke^{-rT}},T\right)\right)\,.$$

Note that this change of probability measure corresponds to the choice of $(S_t, t\geq 0)$ as numéraire (see $\rightarrowtail$ Subsection 2.4.3).

2.4.2 Self-financing Strategy and Change of Numéraire

If N is a numéraire (e.g., the price of a zero-coupon bond), we can evaluate any portfolio in terms of this numéraire. If V_t is the value of a portfolio, its value in the numéraire N is V_t/N_t. The choice of the numéraire does not change the fundamental properties of the market. We prove below that the set of self-financing portfolios does not depend on the choice of numéraire.

Proposition 2.4.2.1 *Let us assume that there are d assets in the market, with prices $(S^i_t;\, i = 1, \ldots, d,\, t \geq 0)$ which are continuous semi-martingales with S^1 there to be strictly positive.(We do not require that there is a riskless asset.) We denote by $V^\pi_t = \sum_{i=1}^d \pi^i_t S^i_t$ the value at time t of the portfolio $\pi_t = (\pi^i_t, i = 1, \ldots, d)$. If the portfolio $(\pi_t,\, t \geq 0)$ is self-financing, i.e., if $dV^\pi_t = \sum_{i=1}^d \pi^i_t dS^i_t$, then,choosing S^1_t as a numéraire, and*

$$dV^{\pi,1}_t = \sum_{i=2}^d \pi^i_t dS^{i,1}_t$$

where $V^{\pi,1}_t = V^\pi_t/S^1_t,\; S^{i,1}_t = S^i_t/S^1_t$.

Proof: We give the proof in the case $d = 2$ (for two assets). We note simply V (instead of V^π) the value of a self-financing portfolio $\pi = (\pi^1, \pi^2)$ in a market where the two assets $S^i, i = 1, 2$ (there is no savings account here) are traded. Then

$$\begin{aligned} dV_t &= \pi^1_t dS^1_t + \pi^2_t dS^2_t = (V_t - \pi^2_t S^2_t) dS^1_t / S^1_t + \pi^2_t dS^2_t \\ &= (V^1_t - \pi^2_t S^{2,1}_t) dS^1_t + \pi^2_t dS^2_t \,. \end{aligned} \tag{2.4.1}$$

On the other hand, from $V^1_t S^1_t = V_t$ one obtains

$$dV_t = V^1_t dS^1_t + S^1_t dV^1_t + d\langle S^1, V^1 \rangle_t \,, \tag{2.4.2}$$

hence,

$$\begin{aligned} dV^1_t &= \frac{1}{S^1_t} \left(dV_t - V^1_t dS^1_t - d\langle S^1, V^1 \rangle_t \right) \\ &= \frac{1}{S^1_t} \left(\pi^2_t dS^2_t - \pi^2_t S^{2,1}_t dS^1_t - d\langle S^1, V^1 \rangle_t \right) \end{aligned}$$

where we have used (2.4.1) for the last equality. The equality $S^{2,1}_t S^1_t = S^2_t$ implies

$$dS^2_t - S^{2,1}_t dS^1_t = S^1_t dS^{2,1}_t + d\langle S^1, S^{2,1} \rangle_t$$

hence,

$$dV^1_t = \pi^2_t dS^{2,1}_t + \frac{\pi^2_t}{S^1_t} d\langle S^1, S^{2,1} \rangle_t - \frac{1}{S^1_t} d\langle S^1, V^1 \rangle_t \,.$$

This last equality implies that

$$\left(1+\frac{1}{S_t^1}\right)d\langle V^1, S^1\rangle_t = \pi_t^2\left(1+\frac{1}{S_t^1}\right)d\langle S^1, S^{2,1}\rangle_t$$

hence, $d\langle S^1, V^1\rangle_t = \pi_t^2 d\langle S^1, S^{2,1}\rangle_t$, hence it follows that $dV_t^1 = \pi_t^2 dS_t^{2,1}$. □

Comment 2.4.2.2 We refer to Benninga et al. [71], Duffie [270], El Karoui et al. [299], Jamshidian [478], and Schroder [773] for details and applications of the change of numéraire method. Change of numéraire has strong links with optimization theory, see Becherer [63] and Gourieroux et al. [401]. See also an application to hedgeable claims in a default risk setting in Bielecki et al. [89]. We shall present applications of change of numéraire in ↣ Subsection 2.7.1 and in the proof of symmetry relations (e.g., ↣ formula (3.6.1.1)).

2.4.3 Change of Numéraire and Change of Probability

We define a change of probability associated with any numéraire Z. The numéraire is a price process, hence the process $(Z_t R_t, t \geq 0)$ is a strictly positive $\mathbb{Q}$-martingale. Define $\mathbb{Q}^Z$ as $\mathbb{Q}^Z|_{\mathcal{F}_t} := (Z_t R_t)\mathbb{Q}|_{\mathcal{F}_t}$.

Proposition 2.4.3.1 *Let $(X_t, t \geq 0)$ be the dynamics of a price and Z a new numéraire. The price of X, in the numéraire Z: $(X_t/Z_t, 0 \leq t \leq T)$, is a $\mathbb{Q}^Z$-martingale.*

Proof: If X is a price process, the discounted process $\widetilde{X}_t := X_t R_t$ is a $\mathbb{Q}$-martingale. Furthermore, from Proposition 1.7.1.1, it follows that X_t/Z_t is a $\mathbb{Q}^Z$-martingale if and only if $(X_t/Z_t)Z_t R_t = R_t X_t$ is a $\mathbb{Q}$-martingale. □

In particular, if the market is arbitrage-free, and if a riskless asset S^0 is traded, choosing this asset as a numéraire leads to the risk-neutral probability, under which X_t/S_t^0 is a martingale.

Comments 2.4.3.2 (a) If the numéraire is the numéraire portfolio, defined at the end of Subsection 2.2.3, i.e., $N_t = 1/R_t S_t$, then the risky assets are $\mathbb{Q}^N$-martingales.

(b) See ↣ Subsection 2.7.2 for another application of change of numéraire.

2.4.4 Forward Measure

A particular choice of numéraire is the zero-coupon bond of maturity T. Let $P(t,T)$ be the price at time t of a zero-coupon bond with maturity T. If the interest rate is deterministic, $P(t,T) = R_T/R_t$ and the computation of the value of a contingent claim X reduces to the computation of $P(t,T)\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t)$ where $\mathbb{Q}$ is the risk-neutral probability measure.

When the spot rate r is a stochastic process, $P(t,T) = (R_t)^{-1}\mathbb{E}_{\mathbb{Q}}(R_T|\mathcal{F}_t)$ where $\mathbb{Q}$ is the risk-neutral probability measure and the price of a contingent claim H is $(R_t)^{-1}\mathbb{E}_{\mathbb{Q}}(HR_T|\mathcal{F}_t)$. The computation of $\mathbb{E}_{\mathbb{Q}}(HR_T|\mathcal{F}_t)$ may be difficult and a change of numéraire may give some useful information. Obviously, the process

$$\zeta_t := \frac{1}{P(0,T)}\mathbb{E}_{\mathbb{Q}}(R_T|\mathcal{F}_t) = \frac{P(t,T)}{P(0,T)}R_t$$

is a strictly positive $\mathbb{Q}$-martingale with expectation equal to 1. Let us define the **forward measure** $\mathbb{Q}^T$ as the probability associated with the choice of the zero-coupon bond as a numéraire:

Definition 2.4.4.1 *Let $P(t,T)$ be the price at time t of a zero-coupon with maturity T. The T-forward measure is the probability $\mathbb{Q}^T$ defined on $\mathcal{F}_t$, for $t \leq T$, as*

$$\mathbb{Q}^T|_{\mathcal{F}_t} = \zeta_t\, \mathbb{Q}|_{\mathcal{F}_t}$$

where $\zeta_t = \dfrac{P(t,T)}{P(0,T)}R_t$.

Proposition 2.4.4.2 *Let $(X_t, t \geq 0)$ be the dynamics of a price. Then the forward price $(X_t/P(t,T), 0 \leq t \leq T)$ is a $\mathbb{Q}^T$-martingale.*
The price of a contingent claim H is

$$V_t^H = \mathbb{E}_{\mathbb{Q}}\left(H\exp\left(-\int_t^T r_s ds\right)|\mathcal{F}_t\right) = P(t,T)\mathbb{E}_{\mathbb{Q}^T}(H|\mathcal{F}_t)\,.$$

Remark 2.4.4.3 Obviously, if the spot rate r is deterministic, $\mathbb{Q}^T = \mathbb{Q}$ and the forward price is equal to the spot price.

Comment 2.4.4.4 A **forward contract** on H, made at time 0, is a contract that stipulates that its holder pays the deterministic amount K at the delivery date T and receives the stochastic amount H. Nothing is paid at time 0. The forward price of H is K, determined at time 0 as $K = \mathbb{E}_{\mathbb{Q}_T}(H)$. See Björk [102], Martellini et al. [624] and Musiela and Rutkowski [661] for various applications.

2.4.5 Self-financing Strategies: Constrained Strategies

We present a very particular case of hedging with strategies subject to a constraint. The change of numéraire technique is of great importance in characterizing such strategies. This result is useful when dealing with default risk (see Bielecki et al. [93]).

We assume that the $k \geq 3$ assets S^i traded in the market are continuous semi-martingales, and we assume that S^1 and S^k are strictly positive processes. We do not assume that there is a riskless asset (we can consider this case if we specify that $dS^1_t = r_t S^1_t dt$).

Let $\pi = (\pi^1, \pi^2, \dots, \pi^k)$ be a self-financing trading strategy satisfying the following constraint:

$$\sum_{i=\ell+1}^{k} \pi^i_t S^i_t = Z_t, \quad \forall\, t \in [0,T], \tag{2.4.3}$$

for some $1 \leq \ell \leq k-1$ and a predetermined, $\mathbf{F}$-predictable process Z. Let $\Phi_\ell(Z)$ be the class of all self-financing trading strategies satisfying the condition (2.4.3). We denote by $S^{i,1} = S^i/S^1$ and $Z^1 = Z/S^1$ the prices and the value of the constraint in the numéraire S^1.

Proposition 2.4.5.1 *The relative time-t wealth $V^{\pi,1}_t = V^\pi_t (S^1_t)^{-1}$ of a strategy $\pi \in \Phi_\ell(Z)$ satisfies*

$$V^{\pi,1}_t = V^{\pi,1}_0 + \sum_{i=2}^{\ell} \int_0^t \pi^i_u \, dS^{i,1}_u + \sum_{i=\ell+1}^{k-1} \int_0^t \pi^i_u \left(dS^{i,1}_u - \frac{S^{i,1}_u}{S^{k,1}_u} \, dS^{k,1}_u \right) + \int_0^t \frac{Z^1_u}{S^{k,1}_u} \, dS^{k,1}_u .$$

Proof: Let us consider discounted values of price processes $S^1, S^2, \dots, S^k$, with S^1 taken as a numéraire asset. In the proof, for simplicity, we do not indicate the portfolio π as a superscript for the wealth. We have the numéraire invariance

$$V^1_t = V^1_0 + \sum_{i=2}^{k} \int_0^t \pi^i_u \, dS^{i,1}_u . \tag{2.4.4}$$

The condition (2.4.3) implies that

$$\sum_{i=\ell+1}^{k} \pi^i_t S^{i,1}_t = Z^1_t ,$$

and thus

$$\pi^k_t = (S^{k,1}_t)^{-1} \Big(Z^1_t - \sum_{i=\ell+1}^{k-1} \pi^i_t S^{i,1}_t \Big). \tag{2.4.5}$$

By inserting (2.4.5) into (2.4.4), we arrive at the desired formula. $\square$

Let us take $Z = 0$, so that $\pi \in \Phi_\ell(0)$. Then the constraint condition becomes $\sum_{i=\ell+1}^{k} \pi^i_t S^i_t = 0$, and (2.4.4) reduces to

$$V^{\pi,1}_t = \sum_{i=2}^{\ell} \int_0^t \pi^i_s \, dS^{i,1}_s + \sum_{i=\ell+1}^{k-1} \int_0^t \pi^i_s \left(dS^{i,1}_s - \frac{S^{i,1}_s}{S^{k,1}_s} \, dS^{k,1}_s \right). \tag{2.4.6}$$

The following result provides a different representation for the (relative) wealth process in terms of correlations (see Bielecki et al. [92] for the case where Z is not null).

Lemma 2.4.5.2 *Let $\pi = (\pi^1, \pi^2, \dots, \pi^k)$ be a self-financing strategy in $\Phi_\ell(0)$. Assume that the processes S^1, S^k are strictly positive. Then the relative wealth process $V_t^{\pi,1} = V_t^\pi (S_t^1)^{-1}$ satisfies*

$$V_t^{\pi,1} = V_0^{\pi,1} + \sum_{i=2}^{\ell} \int_0^t \pi_u^i \, dS_u^{i,1} + \sum_{i=\ell+1}^{k-1} \int_0^t \widehat{\pi}_u^{i,k,1} \, d\widehat{S}_u^{i,k,1}, \quad \forall \, t \in [0,T],$$

where we denote

$$\widehat{\pi}_t^{i,k,1} = \pi_t^i (S_t^{1,k})^{-1} e^{\alpha_t^{i,k,1}}, \quad \widehat{S}_t^{i,k,1} = S_t^{i,k} e^{-\alpha_t^{i,k,1}}, \tag{2.4.7}$$

with $S_t^{i,k} = S_t^i (S_t^k)^{-1}$ and

$$\alpha_t^{i,k,1} = \langle \ln S^{i,k}, \ln S^{1,k} \rangle_t = \int_0^t (S_u^{i,k})^{-1} (S_u^{1,k})^{-1} \, d\langle S^{i,k}, S^{1,k} \rangle_u. \tag{2.4.8}$$

PROOF: Let us consider the relative values of all processes, with the price S^k chosen as a numéraire, and $V_t^k := V_t (S_t^k)^{-1} = \sum_{i=1}^k \pi_t^i S_t^{i,k}$ (we do not indicate the superscript π in the wealth). In view of the constraint we have that $V_t^k = \sum_{i=1}^\ell \pi_t^i S_t^{i,k}$. In addition, as in Proposition 2.4.2.1 we get

$$dV_t^k = \sum_{i=1}^{k-1} \pi_t^i \, dS_t^{i,k}.$$

Since $S_t^{i,k} (S_t^{1,k})^{-1} = S_t^{i,1}$ and $V_t^1 = V_t^k (S_t^{1,k})^{-1}$, using an argument analogous to that of the proof of Proposition 2.4.2.1, we obtain

$$V_t^1 = V_0^1 + \sum_{i=2}^{\ell} \int_0^t \pi_u^i \, dS_u^{i,1} + \sum_{i=\ell+1}^{k-1} \int_0^t \widehat{\pi}_u^{i,k,1} \, d\widehat{S}_u^{i,k,1}, \quad \forall \, t \in [0,T],$$

where the processes $\widehat{\pi}_t^{i,k,1}$, $\widehat{S}_t^{i,k,1}$ and $\alpha_t^{i,k,1}$ are given by (2.4.7)–(2.4.8). □

The result of Proposition 2.4.5.1 admits a converse.

Proposition 2.4.5.3 *Let an $\mathcal{F}_T$-measurable random variable H represent a contingent claim that settles at time T. Assume that there exist $\mathbf{F}$-predictable processes π^i, $i = 2, 3, \dots, k-1$ such that*

$$\begin{aligned} \frac{H}{S_T^1} &= x + \sum_{i=2}^{l} \int_0^T \pi_t^i \, dS_t^{i,1} \\ &\quad + \sum_{i=l+1}^{k-1} \int_0^T \pi_t^i \left(dS_t^{i,1} - \frac{S_t^{i,1}}{S_t^{k,1}} \, dS_t^{k,1} \right) + \int_0^T \frac{Z_t^1}{S_t^{k,1}} \, dS_t^{k,1}. \end{aligned}$$

Then there exist two $\mathbf{F}$*-predictable processes* π^1 *and* π^k *such that the strategy* $\pi = (\pi^1, \pi^2, \dots, \pi^k)$ *belongs to* $\Phi_\ell(Z)$ *and replicates* H*. The wealth process of* π *equals, for every* $t \in [0,T]$,

$$\begin{aligned}\frac{V_t^\pi)}{S_t^1} &= x + \sum_{i=2}^{l} \int_0^t \pi_u^i \, dS_u^{i,1} \\ &+ \sum_{i=l+1}^{k-1} \int_0^t \pi_u^i \left(dS_u^{i,1} - \frac{S_u^{i,1}}{S_u^{k,1}} \, dS_u^{k,1} \right) + \int_0^t \frac{Z_u^1}{S_u^{k,1}} \, dS_u^{k,1}.\end{aligned}$$

PROOF: The proof is left as an exercise. □

2.5 Feynman-Kac

In what follows, $\mathbb{E}_x$ is the expectation corresponding to the probability distribution of a Brownian motion W starting from x.

2.5.1 Feynman-Kac Formula

Theorem 2.5.1.1 *Let* $\alpha \in \mathbb{R}^+$ *and let* $k : \mathbb{R} \to \mathbb{R}^+$ *and* $g : \mathbb{R} \to \mathbb{R}$ *be continuous functions with* g *bounded. Then the function*

$$f(x) = \mathbb{E}_x \left[\int_0^\infty dt \, g(W_t) \exp \left(-\alpha t - \int_0^t k(W_s) ds \right) \right] \tag{2.5.1}$$

is piecewise C^2 *and satisfies*

$$(\alpha + k) f = \frac{1}{2} f'' + g \,. \tag{2.5.2}$$

PROOF: We refer to Karatzas and Shreve [513] p.271. □

Let us assume that f is a bounded solution of (2.5.2). Then, one can check that equality (2.5.1) is satisfied.

We give a few hints for this verification. Let us consider the increasing process Z defined by:

$$Z_t = \alpha t + \int_0^t k(W_s) ds \,.$$

By applying Itô's lemma to the process

$$U_t^\varphi := \varphi(W_t) e^{-Z_t} + \int_0^t g(W_s) e^{-Z_s} ds \,,$$

where φ is C^2, we obtain

$$dU_t^{\varphi} = \varphi'(W_t)e^{-Z_t}dW_t + \left(\frac{1}{2}\varphi''(W_t) - (\alpha + k(W_t))\,\varphi(W_t) + g(W_t)\right) e^{-Z_t}dt$$

Now let $\varphi = f$ where f is a bounded solution of (2.5.2). The process U^f is a local martingale:

$$dU_t^f = f'(W_t)e^{-Z_t}dW_t\,.$$

Since U^f is bounded, U^f is a uniformly integrable martingale, and

$$\mathbb{E}_x(U_\infty^f) = \mathbb{E}_x\left(\int_0^\infty g(W_s)e^{-Z_s}ds\right) = U_0^f = f(x)\,.$$

□

2.5.2 Occupation Time for a Brownian Motion

We now give Kac's proof of Lévy's arcsine law as an application of the Feynman-Kac formula:

Proposition 2.5.2.1 *The random variable $A_t^+ := \int_0^t \mathbb{1}_{[0,\infty[}(W_s)ds$ follows the arcsine law with parameter t:*

$$\mathbb{P}(A_t^+ \in ds) = \frac{ds}{\pi\sqrt{s(t-s)}}\mathbb{1}\{0 \le s < t\}\,.$$

PROOF: By applying Theorem 2.5.1.1 to $k(x) = \beta\mathbb{1}_{\{x\ge 0\}}$ and $g(x) = 1$, we obtain that for any $\alpha > 0$ and $\beta > 0$, the function f defined by:

$$f(x) := \mathbb{E}_x\left[\int_0^\infty dt \exp\left(-\alpha t - \beta\int_0^t \mathbb{1}_{[0,\infty[}(W_s)ds\right)\right] \tag{2.5.3}$$

solves the following differential equation:

$$\begin{cases} \alpha f(x) = \frac{1}{2}f''(x) - \beta f(x) + 1, & x \ge 0 \\ \alpha f(x) = \frac{1}{2}f''(x) + 1, & x \le 0 \end{cases}. \tag{2.5.4}$$

Bounded and continuous solutions of this differential equation are given by:

$$f(x) = \begin{cases} Ae^{-x\sqrt{2(\alpha+\beta)}} + \frac{1}{\alpha+\beta}, & x \ge 0 \\ Be^{x\sqrt{2\alpha}} + \frac{1}{\alpha}, & x \le 0 \end{cases}.$$

Relying on the continuity of f and f' at zero, we obtain the unique bounded C^2 solution of (2.5.4):

$$A = \frac{\sqrt{\alpha+\beta} - \sqrt{\alpha}}{(\alpha+\beta)\sqrt{\alpha}}\;,\;\; B = \frac{\sqrt{\alpha} - \sqrt{\alpha+\beta}}{\alpha\sqrt{\alpha+\beta}}\,.$$

The following equality holds:

$$f(0) = \int_0^\infty dt e^{-\alpha t} \mathbb{E}_0 \left[e^{-\beta A_t^+} \right] = \frac{1}{\sqrt{\alpha(\alpha+\beta)}} .$$

We can invert the Laplace transform using the identity

$$\int_0^\infty dt e^{-\alpha t} \left(\int_0^t du \frac{e^{-\beta u}}{\pi \sqrt{u(t-u)}} \right) = \frac{1}{\sqrt{\alpha(\alpha+\beta)}} ,$$

and the density of A_t^+ is obtained:

$$\mathbb{P}(A_t^+ \in ds) = \frac{ds}{\pi\sqrt{s(t-s)}} \mathbb{1}_{\{s<t\}} .$$

Therefore, the law of A_t^+ is the arcsine law on $[0,t]$, and its distribution function is, for $s \in [0,t]$:

$$\mathbb{P}(A_t^+ \leq s) = \frac{2}{\pi} \arcsin \sqrt{\frac{s}{t}} .$$

Note that, by scaling, $A_t^+ \stackrel{\text{law}}{=} t A_1^+$. □

Comment 2.5.2.2 This result is due to Lévy [584] and a different proof was given by Kac [502]. Intensive studies for the more general case $\int_0^t f(W_s) ds$ have been made in the literature. Biane and Yor [86] and Jeanblanc et al. [483] present a study of the laws of these random variables for particular functions f, using excursion theory and the Ray-Knight theorem for Brownian local times at an exponential time.

2.5.3 Occupation Time for a Drifted Brownian Motion

The same method can be applied in order to compute the density of the occupation times above and below a level $L > 0$ up to time t for a Brownian motion with drift ν, i.e.,

$$A_t^{+,L,\nu} = \int_0^t ds \mathbb{1}_{\{X_s > L\}}, \quad A_t^{-,L,\nu} = \int_0^t ds \mathbb{1}_{\{X_s < L\}}$$

where $X_t = \nu t + W_t$. We start with the computation of Ψ where

$$\Psi(\alpha, \beta) := \mathbf{W}_0^{(\nu)} \left(\int_0^\infty dt \exp\left(-\alpha t - \beta \int_0^t ds \mathbb{1}_{\{X_s < 0\}} \right) \right) .$$

From the Feynman-Kac result, Ψ is the unique bounded solution of the equation (see Akahori [1] for details)

$$-\frac{1}{2}f'' - \nu f' + \alpha f + \beta \mathbb{1}_{\{x<0\}} f = 1\,.$$

Hence,

$$\begin{aligned}\Psi(\alpha,\beta) = &\frac{\nu}{2\alpha}\frac{\sqrt{\nu^2+2(\alpha+\beta)}}{\alpha+\beta} - \frac{\nu}{2(\alpha+\beta)}\frac{\sqrt{\nu^2+2\alpha}}{\alpha} \\ &+ \frac{1}{2}\frac{\sqrt{\nu^2+2(\alpha+\beta)}}{\alpha+\beta}\frac{\sqrt{\nu^2+2\alpha}}{\alpha} - \frac{\nu^2}{2}\frac{1}{\alpha(\alpha+\beta)}\,.\end{aligned}$$

Inverting the Laplace transform, we get

$$\begin{aligned}\mathbb{P}(A_t^{-,0,\nu} \in du)/du = &\left[\sqrt{\frac{2}{\pi u}}\exp\left(-\frac{\nu^2}{2}u\right) - 2\nu\,\Theta(\nu\sqrt{u})\right] \\ \times &\left[\nu + \frac{1}{\sqrt{2\pi(t-u)}}\exp\left(-\frac{\nu^2}{2}(t-u)\right) - \nu\,\Theta(\nu\sqrt{t-u})\right] \quad (2.5.5)\end{aligned}$$

where $\Theta(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp(-\frac{y^2}{2})dy$. More generally, the law of $A_t^{-,L,\nu}$ for $L > 0$ is obtained from

$$\mathbb{P}(A_t^{-,L,\nu} \le u) = \int_0^u ds\,\varphi(s,L;\nu)\mathbb{P}(A_{t-s}^{-,0,\nu} < u-s)$$

where $\varphi(s,L;\nu)$ is the density $\mathbb{P}(T_L(X) \in ds)/ds$ (see $\rightarrowtail$ (3.2.3) for its closed form). The law of $A_t^{+,L,\nu}$ follows from $A_t^{+,L,\nu} + A_t^{-,L,\nu} = t$.

The law of $A_t^{+,L,0}$ can also be obtained in a more direct way. It is easy to compute the double Laplace transform

$$\Psi(\alpha,\beta;L) := \int_0^\infty dt\,e^{-\alpha t}\mathbb{E}_0\left(e^{-\beta A_t^{+,L,0}}\right)$$

as follows: let, for $L > 0$, $T_L = \inf\{t : X_t = L\}$. Then,

$$\begin{aligned}\Psi(\alpha,\beta;L) &= \mathbb{E}_0\left(\int_0^{T_L} dt\,e^{-\alpha t} + \int_{T_L}^\infty dt\,e^{-\alpha t}\exp\left(-\beta\int_{T_L}^\infty ds\,\mathbb{1}_{\{W_s>L\}}\right)\right) \\ &= \frac{1}{\alpha}\,\mathbb{E}_0(1-e^{-\alpha T_L}) + \frac{1}{\sqrt{\alpha(\alpha+\beta)}}\,\mathbb{E}_0(e^{-\alpha T_L}) \\ &= \frac{1}{\alpha}\,(1-e^{-L\sqrt{2\alpha}}) + \frac{1}{\sqrt{\alpha(\alpha+\beta)}}\,e^{-L\sqrt{2\alpha}}\,.\end{aligned}$$

This quantity is the double Laplace transform of

$$f(t,u)du := \mathbb{P}(T_L > t)\,\delta_0(du) + \frac{1}{\sqrt{u(t-u)}}\,e^{-L^2/(2(t-u))}\mathbb{1}_{\{u<t\}}du\,,$$

i.e.,

$$\Psi(\alpha,\beta;L) = \int_0^\infty\int_0^\infty e^{-\alpha t}e^{-\beta u}f(t,u)dtdu\,.$$

Comment 2.5.3.1 For a general presentation of Feynman-Kac formula, we refer to Durrett [286] and Karatzas and Shreve [513]. For extensions, see Chung [185], Evans [337], Fusai and Tagliani [371] and Pitman and Yor [718, 719]. Occupation time densities for CEV processes (see $\rightarrowtail$ Section 6.4) are presented in Leung and Kwok [582].

2.5.4 Cumulative Options

Let S be a given process. The occupation time of S above (resp. below) a level L up to time t is the random variable $A_t^{+,L} := \int_0^t ds \mathbb{1}_{\{S_s \geq L\}}$ (resp. $A_t^{-,L} = \int_0^t ds \mathbb{1}_{\{S_s \leq L\}}$). An occupation time derivative is a contingent claim whose payoff depends on the terminal value of the underlying asset and on an occupation time. We are mainly interested in terminal payoff of the form $f(S_T, A_T^{-,L})$, (or $f(S_T, A_T^{+,L})$). In a **Black and Scholes model**, as given in Proposition 2.3.1.3, the price of such a claim is

$$\mathbb{E}_{\mathbb{Q}}(e^{-rT} f(S_T, A_T^{-,L})) = e^{-rT-\nu^2 T/2}\mathbb{E}\left(e^{\nu W_T} f(xe^{\sigma W_T}, A_T^{-,\ell}(W))\right)$$

where $\ell = \sigma^{-1}\ln(L/x)$.

We study the particular case $f(x,a) = (x-K)^+ e^{-\rho a}$, called a step option by Linetsky [590]. Let

$$C_{step}(x) = e^{-(r+\nu^2/2)T}\mathbb{E}\left(e^{\nu W_T}(xe^{\sigma W_T} - K)^+ e^{-\rho A_T^{-,\ell}}\right)$$

where W starts from 0. Setting $\gamma = r + \nu^2/2$, we obtain

$$\begin{aligned} C_{step}(x) &= e^{-\gamma T}\mathbb{E}_{-\ell}\left(e^{\nu(W_T+\ell)}(xe^{\sigma(W_T+\ell)} - K)^+ e^{-\rho A_T^{-,0}}\right) \\ &= e^{-\gamma T + \nu\ell}(xe^{\sigma\ell}\Psi(-\ell, \nu+\sigma) - K\Psi(-\ell,\nu)) \end{aligned}$$

where $\Psi(x,a) = \mathbb{E}_x\left(e^{aW_T}\mathbb{1}_{\{W_T \geq \frac{1}{\sigma}\ln(K/L)\}} e^{-\rho A_T^{-,0}}\right)$. The function Ψ can be computed from the joint law of $(A_T^{-,0}, W_T)$.

Proposition 2.5.4.1 *The density of the pair* $(A_t^{-,0}, W_t)$ *is*

$$\mathbb{P}(A_t^{-,0} \in du, W_t \in dx) = du\, dx \frac{|x|}{\sqrt{2\pi}} \int_u^t \frac{1}{\sqrt{s^3(t-s)^3}} e^{-x^2/(2(t-s))} ds\, \mathbb{1}_{\{u<t\}}.$$

PROOF: Let, for $a > 0, \rho > 0$,

$$f(t,x) = \mathbb{E}\left(\mathbb{1}_{[a,\infty[}(x+W_t)\exp\left(-\rho\int_0^t \mathbb{1}_{]-\infty,0]}(x+W_s)ds\right)\right).$$

From the Feynman-Kac theorem, the function f satisfies the PDE

$$\partial_t f = \frac{1}{2}\partial_{xx} f - \rho \mathbb{1}_{]-\infty,0]}(s) f, \quad f(0,x) = \mathbb{1}_{[a,\infty[}(x)\,.$$

Letting $\widehat{f}$ be the Laplace transform in time of f, i.e.,

$$\widehat{f}(\lambda, x) = \int_0^\infty e^{-\lambda t} f(x,t)dt\,,$$

we obtain

$$-\mathbb{1}_{[a,\infty[}(x) + \lambda \widehat{f} = \frac{1}{2}\partial_{xx}\widehat{f} - \rho \mathbb{1}_{]-\infty,0]}(x)\widehat{f}\,.$$

Solving this ODE with the boundary conditions at 0 and a leads to

$$\widehat{f}(\lambda, 0) = \frac{\exp(-a\sqrt{2\lambda})}{\sqrt{\lambda}\left(\sqrt{\lambda} + \sqrt{\lambda+\rho}\right)} = \widehat{f_1}(\lambda)\widehat{f_2}(\lambda)\,, \tag{2.5.6}$$

with

$$\widehat{f_1}(\lambda) = \frac{1}{\sqrt{\lambda}\left(\sqrt{\lambda} + \sqrt{\lambda+\rho}\right)}, \ \widehat{f_2}(\lambda) = \exp(-a\sqrt{2\lambda})\,.$$

Then, one gets

$$-\partial_a \widehat{f}(\lambda, 0) = \sqrt{2}\frac{\exp(-a\sqrt{2\lambda})}{\sqrt{\lambda} + \sqrt{\lambda+\rho}}\,.$$

The right-hand side of (2.5.6) may be recognized as the product of the Laplace transforms of the functions

$$f_1(t) = \frac{1-e^{-\rho t}}{\rho\sqrt{2\pi t^3}}, \ \text{ and } \ f_2(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-a^2/2t}\,,$$

hence, it is the Laplace transform of the convolution of these two functions. The result follows. □

Comment 2.5.4.2 Cumulative options are studied in Chesney et al. [175, 196], Dassios [211], Detemple [251], Fusai [370], Hugonnier [451] and Moraux [657]. In [370], Fusai determines the Fourier transform of the density of the occupation time $\tau = \int_0^T \mathbb{1}_{\{a<\nu s + W_s < b\}} ds$, in order to compute the price of a corridor option, i.e., an option with payoff $(\tau - K)^+$. The joint law of W_T and A_T^+ can be found in Fujita and Miura [366] where the authors present, among other results, options which are knocked-out at the time

$$\tau = \inf\left\{t \,:\, \int_{T_a}^t \mathbb{1}_{\{S_u \le a\}}\, du \ge \alpha(T - T_a)\right\}.$$

Exercise 2.5.4.3 (1) Deduce from Proposition 2.5.4.1 $\mathbb{P}(A_t^{-,0} \in du | W_t = x)$.
(2) Recover the formula (2.5.5) for $\mathbb{P}(A_t^{-,0} \in du)$. ◁

2.5.5 Quantiles

Proposition 2.5.5.1 *Let $X_t = \mu t + \sigma W_t$ and $M_t^X = \sup_{s \leq t} X_s$. We assume $\sigma > 0$. Define, for a fixed t, $\theta_t^X = \sup\{s \leq t \, : \, X_s = M_s^X\}$. Then*

$$\theta_t^X \stackrel{\text{law}}{=} \int_0^t \mathbb{1}_{\{X_s > 0\}} ds \,.$$

PROOF: We shall prove the result in the case $\sigma = 1, \mu = 0$ in $\rightarrowtail$ Exercise 4.1.7.5. The drifted Brownian motion case follows from an application of Girsanov's theorem. □

Proposition 2.5.5.2 *Let $X_t = \mu t + \sigma W_t$ with $\sigma > 0$, and*

$$q^X(\alpha, t) = \inf \left\{ x \, : \, \int_0^t \mathbb{1}_{\{X_s \leq x\}} ds \, > \alpha t \right\} .$$

Let $X^i, i = 1, 2$ be two independent copies of X. Then

$$q^X(\alpha, t) \stackrel{\text{law}}{=} \sup_{0 \leq s \leq \alpha t} X_s^1 + \inf_{0 \leq s \leq (1-\alpha t)} X_s^2 \,.$$

PROOF: We give the proof for $t = 1$. We note that

$$A^X(x) = \int_0^1 \mathbb{1}_{\{X_s > x\}} ds = \int_{T_x}^1 \mathbb{1}_{\{X_s > x\}} ds = 1 - \int_0^{1-T_x} \mathbb{1}_{\{X_{s+T_x} \leq x\}} ds$$

where $T_x = \inf\{t \, : \, X_t = x\}$. Then, denoting $q(\alpha) = q^X(\alpha, 1)$, one has

$$\mathbb{P}(q(\alpha) > x) = \mathbb{P}(A^X(x) > 1 - \alpha) = \mathbb{P}\left(\int_0^{1-T_x} \mathbb{1}_{\{X_{s+T_x} - x > 0\}} ds > 1 - \alpha \right) .$$

The process $(X_s^1 = X_{s+T_x} - x, \, s \geq 0)$ is independent of $(X_s, s \leq T_x; T_x)$ and has the same law as X. Hence,

$$\mathbb{P}(q(\alpha) > x) = \int_0^\alpha \mathbb{P}(T_x \in du) \mathbb{P}\left(\int_0^{1-u} \mathbb{1}_{\{X_s^1 > 0\}} ds > 1 - \alpha \right) .$$

Then, from Proposition 2.5.5.1,

$$\mathbb{P}\left(\int_0^{1-u} \mathbb{1}_{\{X_s^1 > 0\}} ds > 1 - \alpha \right) = \mathbb{P}(\theta_{1-u}^{X^1} > 1 - \alpha) \,.$$

From the definition of θ_s^1, for $s > a$,

$$\mathbb{P}(\theta_s^{X^1} > a) = \mathbb{P}\left(\sup_{u \leq a} (X_u^1 - X_a^1) < \sup_{a \leq v \leq s} (X_v^1 - X_a^1) \right) .$$

It is easy to check that

$$\left(\sup_{u\leq a}(X_u^1 - X_a^1),\ \sup_{a\leq v\leq s}(X_v^1 - X_a^1)\right) \stackrel{\text{law}}{=} \left(-\inf_{u\leq a} X_u^2,\ \sup_{0<v\leq s-a} X_v^3\right)$$

where X^2 and X^3 are two independent copies of X. The result follows. □

Exercise 2.5.5.3 Prove that, in the case $\nu = 0$, setting $\beta = ((1-\alpha)/\alpha)^{1/2}$, and $\Phi^*(x) = \sqrt{2/\pi}\int_x^\infty e^{-y^2/2}dy$

$$\mathbb{P}(q(\alpha) \in dx) = \begin{cases} \sqrt{2/\pi}\, e^{-x^2/2}\Phi^*(\beta x)dx \text{ for } x \geq 0 \\ \sqrt{2/\pi}\, e^{-x^2/2}\Phi^*(-x\beta^{-1})dx \text{ for } x \leq 0 \end{cases}.$$

◁

Comment 2.5.5.4 See Akahori [1], Dassios [211, 212], Detemple [252], Embrechts et al. [324], Fujita and Yor [368], Fusai [370], Miura [653] and Yor [866] for results on quantiles and pricing of quantile options.

2.6 Ornstein-Uhlenbeck Processes and Related Processes

In this section, we present a particular SDE, the solution of which was used to model interest rates. Even if this kind of model is nowadays not so often used by practitioners for interest rates, it can be useful for modelling underlying values in a real options framework.

2.6.1 Definition and Properties

Proposition 2.6.1.1 *Let k, θ and σ be bounded Borel functions, and W a Brownian motion. The solution of*

$$dr_t = k(t)(\theta(t) - r_t)dt + \sigma(t)dW_t \tag{2.6.1}$$

is

$$r_t = e^{-K(t)}\left(r_0 + \int_0^t e^{K(s)}k(s)\theta(s)ds + \int_0^t e^{K(s)}\sigma(s)dW_s\right)$$

where $K(t) = \int_0^t k(s)ds$. The process $(r_t, t \geq 0)$ is a Gaussian process with mean

$$\mathbb{E}(r_t) = e^{-K(t)}\left(r_0 + \int_0^t e^{K(s)}k(s)\theta(s)ds\right)$$

and covariance

$$e^{-K(t)-K(s)}\int_0^{t\wedge s} e^{2K(u)}\sigma^2(u)du\,.$$

PROOF: The solution of (2.6.1) is a particular case of Example 1.5.4.8. The values of the mean and of the covariance follow from Exercise 1.5.1.4. □

The **Hull and White model** corresponds to the dynamics (2.6.1) where k is a positive function. In the particular case where k, θ and σ are constant, we obtain

Corollary 2.6.1.2 *The solution of*

$$\boxed{dr_t = k(\theta - r_t)dt + \sigma dW_t} \tag{2.6.2}$$

is

$$r_t = (r_0 - \theta)e^{-kt} + \theta + \sigma \int_0^t e^{-k(t-u)} dW_u .$$

The process $(r_t, t \geq 0)$ *is a Gaussian process with mean* $(r_0 - \theta)e^{-kt} + \theta$ *and covariance*

$$\mathrm{Cov}(r_s, r_t) = \frac{\sigma^2}{2k} e^{-k(s+t)} (e^{2ks} - 1) = \frac{\sigma^2}{k} e^{-kt} \sinh(ks)$$

for $s \leq t$*.*

In finance, the solution of (2.6.2) is called a **Vasicek** process. In general, k is chosen to be positive, so that $\mathbb{E}(r_t) \to \theta$ as $t \to \infty$ (this is why this process is said to enjoy the **mean reverting** property). The process (2.6.1) is called a **Generalized Vasicek process** (GV). Because r is a Gaussian process, it takes negative values. This is one of the reasons why this process is no longer used to model interest rates. When $\theta = 0$, the process r is called an **Ornstein-Uhlenbeck** (OU) process. Note that, if r is a Vasicek process, the process $r - \theta$ is an OU process with parameter k. More formally,

Definition 2.6.1.3 *An Ornstein-Uhlenbeck (OU) process driven by a BM follows the dynamics* $dr_t = -kr_t dt + \sigma dW_t$*.*

An OU process can be constructed in terms of time-changed BM (see also ↣ Section 5.1):

Proposition 2.6.1.4 (i) *If W is a BM starting from x and* $a(t) = \sigma^2 \frac{e^{2kt}-1}{2k}$*, the process* $Z_t = e^{-kt} W_{a(t)}$ *is an OU process starting from x.*

(ii) *Conversely, if U is an OU process starting from x, then there exists a BM W starting from x such that* $U_t = e^{-kt} W_{a(t)}$*.*

PROOF: Indeed, the process Z is a Gaussian process, with mean xe^{-kt} and covariance $e^{-k(t+s)}(a(t) \wedge a(s))$. □

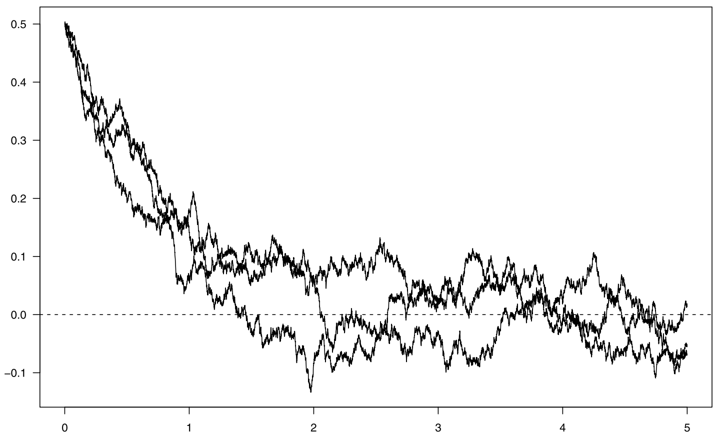

Fig. 2.1 Simulation of Ornstein-Uhlenbeck paths $\theta = 0, k = 3/2, \sigma = 0.1$

More generally, one can define an Ornstein-Uhlenbeck process driven by a Lévy process (see $\rightarrowtail$ Chapter 11). Here, we note that the Vasicek process defined in (2.6.2) is an OU process, driven by the Brownian motion with drift $\sigma W_t + k\theta t$.

From the Markov and Gaussian properties of a Vasicek process r we deduce:

Proposition 2.6.1.5 *Let r be a Vasicek process, the solution of* (2.6.2) *and let* $\mathbf{F}$ *be its natural filtration. For $s < t$, the conditional expectation and the conditional variance of r_t, with respect to $\mathcal{F}_s$ (denoted as* $\mathrm{Var}_s(r_t)$*) are given by*

$$\mathbb{E}(r_t|\mathcal{F}_s) = \mathbb{E}(r_t|r_s) = (r_s - \theta)e^{-k(t-s)} + \theta$$
$$\mathrm{Var}_s\ (r_t) = \frac{\sigma^2}{2k}(1 - e^{-2k(t-s)})\,.$$

Note that the filtration generated by the process r is equal to that of the driving Brownian motion. Owing to the Gaussian property of the process r, the law of the integrated process $\int_0^t r_s ds$ can be characterized as follows:

Proposition 2.6.1.6 *Let r be a solution of* (2.6.2)*.*
The process $\left(\int_0^t r_s ds,\ t \geq 0\right)$ is Gaussian with mean and variance given by

$$\mathbb{E}\left(\int_0^t r_s ds\right) = \theta t + (r_0 - \theta)\frac{1-e^{-kt}}{k},$$
$$\mathrm{Var}\left(\int_0^t r_s ds\right) = -\frac{\sigma^2}{2k^3}(1-e^{-kt})^2 + \frac{\sigma^2}{k^2}\left(t - \frac{1-e^{-kt}}{k}\right)$$

and covariance (for $s < t$)

$$\frac{\sigma^2}{k^2}\left(s - e^{-kt}\frac{e^{ks}-1}{k} - \frac{1-e^{-ks}}{k} + e^{-k(t+s)}\frac{e^{2ks}-1}{2k}\right).$$

Proof: From the definition, $r_t = r_0 + k\theta t - k\int_0^t r_s ds + \sigma W_t$, hence

$$\begin{aligned}\int_0^t r_s ds &= \frac{1}{k}[-r_t + r_0 + k\theta t + \sigma W_t]\\ &= \frac{1}{k}[k\theta t + (r_0-\theta)(1-e^{-kt}) - \sigma e^{-kt}\int_0^t e^{ku}dW_u + \sigma W_t].\end{aligned}$$

Obviously, from the properties of the Wiener integral, the right-hand side defines a Gaussian process. It remains to compute the expectation and the variance of the Gaussian variable on the right-hand side, which is easy, since the variance of a Wiener integral is well known. □

Note that one can also justify directly the Gaussian property of an integral process $(\int_0^t y_s ds, t \geq 0)$ where y is a Gaussian process.

More generally, for $t \geq s$,

$$\mathbb{E}\left(\int_s^t r_u\, du|\mathcal{F}_s\right) = \theta(t-s) + (r_s - \theta)\frac{1-e^{-k(t-s)}}{k} := M(s,t)\,, \quad (2.6.3)$$
$$\begin{aligned}\mathrm{Var}_s\left(\int_s^t r_u\, du\right) &= -\frac{\sigma^2}{2k^3}(1-e^{-k(t-s)})^2\\ &\quad + \frac{\sigma^2}{k^2}\left(t - s - \frac{1-e^{-k(t-s)}}{k}\right) := V(s,t)\,. \quad (2.6.4)\end{aligned}$$

Exercise 2.6.1.7 Compute the transition probability for an OU process. ◁

Exercise 2.6.1.8 (1) Let B be a Brownian motion, and define the probability P^b via

$$\mathbb{P}^b|_{\mathcal{F}_T} := \exp\left\{-b\int_0^T B_s\, dB_s - \frac{b^2}{2}\int_0^T B_s^2\, ds\right\}\mathbb{P}|_{\mathcal{F}_T}\,.$$

Prove that the process $(B_t, t \geq 0)$ is a $\mathbb{P}^b$-Ornstein-Uhlenbeck process and that

$$\mathbb{E}\left(\exp\left(-\alpha B_t^2 - \frac{b^2}{2}\int_0^t B_s^2\, ds\right)\right) = \mathbb{E}^b\left(\exp\left(-\alpha B_t^2 + \frac{b}{2}(B_t^2 - t)\right)\right),$$

where $\mathbb{E}^b$ is the expectation w.r.t. the probability measure $\mathbb{P}^b$. One can also prove that if B is an n-dimensional BM starting from a

$$\mathbb{E}_a \left(\exp(-\alpha |B_t|^2 - \frac{b^2}{2}\int_0^t |B_s|^2 ds) \right)$$

$$= \left(\cosh bt + \frac{2\alpha}{b} \sinh bt \right)^{-n/2} \exp\left(-\frac{|a|^2 b}{2} \frac{1 + \frac{2\alpha}{b}\coth bt}{\coth bt + 2\alpha/b} \right),$$

where $\mathbb{E}_a$ is the expectation for a BM starting from a. (See Yor [864].)

(2) Use the Gaussian property of the variable B_t to obtain that

$$\mathbb{E}_{\mathbb{P}} \left(\exp\left(-\alpha B_t^2 - \frac{b^2}{2}\int_0^t B_s^2\, ds \right) \right) = \left(\cosh\, bt + 2\frac{\alpha}{b} \sinh\, bt \right)^{-\frac{1}{2}}.$$

If B and C are two independent Brownian motions starting form 0, prove that

$$\mathbb{E}_{\mathbb{P}} \left(\exp(-\alpha(B_t^2 + C_t^2) - \frac{b^2}{2}\int_0^t (B_s^2 + C_s^2)\, ds) \right) = \left(\cosh\, bt + 2\frac{\alpha}{b} \sinh\, bt \right)^{-1}.$$

(3) Deduce Lévy's area formula:

$$\begin{aligned} \mathbb{E}(\exp i\lambda \mathcal{A}_t \big| \, |Z_t|^2 = r^2) &= \mathbb{E}\left(\exp -\frac{\lambda^2}{8}\int_0^t |Z_s|^2 ds \big| \, |Z_t|^2 = r^2 \right) \\ &= \frac{t\lambda/2}{\sinh(t\lambda/2)} \exp -\frac{r^2}{2}(\lambda t \coth \lambda t - 1), \end{aligned}$$

where

$$\mathcal{A}_t := \frac{1}{2}\int_0^t (B_s dC_s - C_s dB_s) = \frac{1}{2}\gamma\left(\int_0^t (B_s^2 + C_s^2) ds \right)$$

where γ is a Brownian motion independent of $|Z|^2 := B^2 + C^2$ (see $\rightarrowtail$ Exercise 5.1.3.9)

`Hint:` Note that $\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t)$. $\triangleleft$

2.6.2 Zero-coupon Bond

Suppose that the dynamics of the interest rate under the risk-neutral probability are given by (2.6.2). The value $P(t,T)$ of a zero-coupon bond maturing at date T is given as the conditional expectation under the e.m.m. of the discounted payoff. Using the Laplace transform of a Gaussian law (see Proposition 1.1.12.1), and Proposition 2.6.1.6, we obtain

$$P(t,T) = \mathbb{E}\left(\exp\left(-\int_t^T r_u\, du \right) \big| \mathcal{F}_t \right) = \exp\left(-M(t,T) + \frac{1}{2}V(t,T) \right),$$

where M and V are defined in (2.6.3) and (2.6.4).

Proposition 2.6.2.1 *In a Vasicek model, the price of a zero-coupon with maturity T is*

$$\begin{aligned}P(t,T) &= \exp\left[-\theta(T-t) - (r_t-\theta)\frac{1-e^{-k(T-t)}}{k} - \frac{\sigma^2}{4k^3}(1-e^{-k(T-t)})^2 \right.\\ &\qquad \left. + \frac{\sigma^2}{2k^2}\left(T-t-\frac{1-e^{-k(T-t)}}{k}\right)\right]\\ &= \exp(a(t,T) - b(t,T)r_t)\,,\end{aligned}$$

with $b(t,T) = \frac{1-e^{-k(T-t)}}{k}$.

Without any computation, we know that

$$d_tP(t,T) = P(t,T)(r_tdt - \sigma_t dW_t)\,,$$

since the discounted value of the zero-coupon bond is a martingale. It suffices to identify the volatility term. It is not difficult, using Itô's formula, to check that the risk-neutral dynamics of the zero-coupon bond are

$$d_tP(t,T) = P(t,T)(r_tdt - b(t,T)dW_t)\,.$$

2.6.3 Absolute Continuity Relationship for Generalized Vasicek Processes

Let W be a $\mathbb{P}$-Brownian motion starting from x, θ a bounded Borel function and L the solution of $dL_t = kL_t(\theta(t)-W_t)dW_t,\ L_0=1$, that is,

$$L_t = \exp\left(\int_0^t k(\theta(s)-W_s)dW_s - \frac{1}{2}\int_0^t k^2(\theta(s)-W_s)^2ds\right)\,. \tag{2.6.5}$$

This process is a martingale, from the non-explosion criteria. We define

$$\mathbb{P}^{k,\theta}|_{\mathcal{F}_t} = L_t\,\mathbb{P}|_{\mathcal{F}_t}\,.$$

Then,

$$W_t = x + \beta_t + \int_0^t k(\theta(s)-W_s)ds$$

where, thanks to Girsanov's theorem, β is a $\mathbb{P}^{k,\theta}$-Brownian motion starting from 0. Hence, we have proved that the $\mathbb{P}$-Brownian motion W is a GV process under $\mathbb{P}^{k,\theta}$ (and thus we generalize Exercise 2.6.1.8).

Proposition 2.6.3.1 *Let θ be a differentiable function and let $\mathbb{P}^{k,\theta}_x$ be the law of the GV process*

$$dr_t = dW_t + k(\theta(t)-r_t)dt,\ r_0 = x\,.$$

We denote by $\mathbf{W}_x$ *the law of a Brownian motion starting from* x. *Then the following absolute continuity relationship holds*

$$\mathbb{P}_x^{k,\theta}|_{\mathcal{F}_t} = \exp\left[\frac{k}{2}\left(t + x^2 - k\int_0^t \theta^2(s)ds - 2x\theta(0)\right)\right]$$
$$\times \exp\left[k\theta(t)X_t - \frac{k}{2}X_t^2 + \int_0^t (k^2\theta(s) - k\theta'(s))X_s ds - \frac{k^2}{2}\int_0^t X_s^2 ds\right] \mathbf{W}_x|_{\mathcal{F}_t}\,.$$

Proof: We have seen that $\mathbb{P}_x^{k,\theta}|_{\mathcal{F}_t} = L_t\,\mathbf{W}_x|_{\mathcal{F}_t}$ where L is given in (2.6.5). Since θ is differentiable, an integration by parts under $\mathbf{W}_x$ leads to

$$\int_0^t (\theta(s) - X_s)dX_s = \theta(t)X_t - x\theta(0) - \int_0^t \theta'(s)X_s ds - \frac{1}{2}(X_t^2 - x^2 - t)\,.$$

□

Corollary 2.6.3.2 *Let* r *be a Vasicek process*

$$dr_t = k(\theta - r_t)dt + \sigma dW_t,\ r_0 = x\,.$$

Then

$$\mathbb{P}_x^{k,\theta}|_{\mathcal{F}_t} = \exp\left(\frac{k}{2}\left(t + x^2 - k\theta^2 t - 2x\theta\right)\right) \tag{2.6.6}$$
$$\times \exp\left(-\frac{k}{2}X_t^2 + k\theta X_t + k^2\theta\int_0^t X_s ds - \frac{k^2}{2}\int_0^t X_s^2 ds\right)\mathbf{W}_x|_{\mathcal{F}_t}\,.$$

Proof: The absolute continuity relation (2.6.6) follows from Proposition 2.6.3.1. □

Example 2.6.3.3 As an exercise, we present the computation of

$$A = \mathbb{E}_x^{k,\theta}\left(\exp\left(-\alpha X_t - \lambda X_t^2 - \beta\int_0^t X_s ds - \frac{\gamma^2}{2}\int_0^t X_s^2 ds\right)\right),$$

where $(\alpha, \beta, \theta, \lambda, \gamma)$ are real numbers with $\lambda > 0$. From (2.6.6)

$$A = \exp\left(\frac{k}{2}\left(t + x^2 - k\theta^2 t - 2x\theta\right)\right)$$
$$\times\,\mathbf{W}_x\left(\exp\left(-\lambda_1 X_t^2 + \alpha_1 X_t + (k^2\theta - \beta)\int_0^t X_s ds - \frac{\gamma_1^2}{2}\int_0^t X_s^2 ds\right)\right)$$

where $\lambda_1 = \lambda + \frac{k}{2}$, $\alpha_1 = k\theta - \alpha$, $\gamma_1^2 = \gamma^2 + k^2$. From

$$(k^2\theta - \beta)\int_0^t X_s ds - \frac{\gamma_1^2}{2}\int_0^t X_s^2 ds = -\frac{\gamma_1^2}{2}\int_0^t (X_s + \beta_1)^2 ds + \frac{\beta_1^2\gamma_1^2}{2}t$$

with $\beta_1 = \frac{\beta - k^2\theta}{\gamma_1^2}$ and setting $Z_s = X_s + \beta_1$, one gets

$$A = \exp\left(\frac{k}{2}\left(t + x^2 - k\theta^2 t - 2x\theta\right) + \frac{\beta_1^2\gamma_1^2}{2}t\right) \times \mathbf{W}_{x+\beta_1}\left(\exp\left(-\lambda_1(Z_t - \beta_1)^2 + \alpha_1(Z_t - \beta_1) - \frac{\gamma_1^2}{2}\int_0^t Z_s^2 ds\right)\right).$$

Now,

$$-\lambda_1(Z_t - \beta_1)^2 + \alpha_1(Z_t - \beta_1) = -\lambda_1 Z_t^2 + (\alpha_1 + 2\lambda_1\beta_1)Z_t - \beta_1(\lambda_1\beta_1 + \alpha_1).$$

Hence,

$$\begin{aligned}&\mathbf{W}_{x+\beta_1}\left(\exp\left(-\lambda_1(Z_t - \beta_1)^2 + \alpha_1(Z_t - \beta_1) - \frac{\gamma_1^2}{2}\int_0^t Z_s^2 ds\right)\right)\\ &= e^{-\beta_1(\lambda_1\beta_1+\alpha_1)}\mathbf{W}_{x+\beta_1}\left(\exp\left(-\lambda_1 Z_t^2 + (\alpha_1 + 2\lambda_1\beta_1)Z_t - \frac{\gamma_1^2}{2}\int_0^t Z_s^2 ds\right)\right).\end{aligned}$$

From (2.6.6) again

$$\begin{aligned}&\mathbf{W}_{x+\beta_1}\left(\exp\left(-\lambda_1 Z_t^2 + (\alpha_1 + 2\lambda_1\beta_1)Z_t - \frac{\gamma_1^2}{2}\int_0^t Z_s^2 ds\right)\right)\\ &= \exp\left(-\frac{\gamma_1}{2}\left(t + (x+\beta_1)^2\right)\right)\\ &\qquad \times \mathbb{E}_{x+\beta_1}^{\gamma_1,0}\left(\exp\left((-\lambda_1 + \frac{\gamma_1}{2})X_t^2 + (\alpha_1 + 2\lambda_1\beta_1)X_t\right)\right).\end{aligned}$$

Finally

$$A = e^C\,\mathbb{E}_{x+\beta_1}^{\gamma_1,0}\left(\exp\left((-\lambda_1 + \frac{\gamma_1}{2})X_t^2 + (\alpha_1 + 2\lambda_1\beta_1)X_t\right)\right)$$

where

$$C = \frac{k}{2}\left(t + x^2 - k\theta^2 t - 2x\theta\right) + \frac{\beta_1^2\gamma_1^2}{2}t - \beta_1(\lambda_1\beta_1 + \alpha_1) - \frac{\gamma_1}{2}\left(t + (x+\beta_1)^2\right).$$

One can then finish the computation since, under $\mathbb{P}_{x+\beta_1}^{\gamma_1,0}$ the r.v. X_t is a Gaussian variable with mean $m = (x+\beta_1)e^{-\gamma_1 t}$ and variance $\frac{\sigma^2}{2\gamma_1}(1 - e^{-2\gamma_1 t})$. Furthermore, from Exercise 1.1.12.3, if $U \overset{\text{law}}{=} \mathcal{N}(m, \sigma^2)$

$$\mathbb{E}(\exp\{\lambda U^2 + \mu U\}) = \frac{\Sigma}{\sigma}\exp\left(\frac{\Sigma^2}{2}(\mu + \frac{m}{\sigma^2})^2 - \frac{m^2}{2\sigma^2}\right).$$

with $\Sigma^2 = \dfrac{\sigma^2}{1 - 2\lambda\sigma^2}$, for $2\lambda\sigma^2 < 1$.

2.6.4 Square of a Generalized Vasicek Process

Let r be a GV process with dynamics

$$dr_t = k(\theta(t) - r_t)dt + dW_t$$

and $\rho_t = r_t^2$. Hence

$$d\rho_t = (1 - 2k\rho_t + 2k\theta(t)\sqrt{\rho_t})dt + 2\sqrt{\rho_t}dW_t\,.$$

By construction, the process ρ takes positive values, and can represent a spot interest rate. Then, the value of the corresponding zero-coupon bond can be computed as an application of the absolute continuity relationship between a GV and a BM, as we present now.

Proposition 2.6.4.1 *Let*

$$d\rho_t = (1 - 2k\rho_t + 2k\theta(t)\sqrt{\rho_t})dt + 2\sqrt{\rho_t}dW_t,\ \rho_0 = x^2.$$

Then

$$\mathbb{E}\left[\exp\left(-\int_0^T \rho_s ds\right)\right] = A(T)\,\exp\left(\frac{k}{2}(T + x^2 - k\int_0^T \theta^2(s)ds - 2\theta(0)x)\right)$$

where

$$A(T) = \exp\left(\frac{1}{2}\left(\int_0^T f(s)ds + \int_0^T g^2(s)ds\right)\right)\,.$$

Here,

$$f(s) = K\frac{\alpha e^{Ks} + e^{-Ks}}{\alpha e^{Ks} - e^{-Ks}},$$
$$g(s) = k\frac{\theta(T)v(T) - \int_s^T (\theta'(u) - k\theta(u))v(u)du}{v(s)}$$

with $v(s) = \alpha e^{Ks} - e^{-Ks}$, $K = \sqrt{k^2 + 2}$ *and* $\alpha = \dfrac{k - K}{k + K}e^{-2TK}$.

PROOF: From Proposition 2.6.3.1,

$$\mathbb{E}\left[\exp\left(-\int_0^T \rho_s ds\right)\right] = A(T)\,\exp\left(\frac{k}{2}(T + x^2 - k\int_0^T \theta^2(s)ds - 2\theta(0)x)\right)$$

where $A(T)$ is equal to the expectation, under $\mathbf{W}$, of

$$\exp\left(k\theta(T)X_T - \frac{k}{2}X_T^2 + \int_0^T (k^2\theta(s) - k\theta'(s))X_s ds - \frac{k^2 + 2}{2}\int_0^T X_s^2 ds\right)\,.$$

The computation of $A(T)$ follows from Example 1.5.7.1 which requires the solution of

$$f^2(s) + f'(s) = k^2 + 2, \; s \le T,$$
$$f(s)g(s) + g'(s) = k\theta'(s) - k^2\theta(s),$$

with the terminal condition at time T

$$f(T) = -k, \quad g(T) = k\theta(T) \, .$$

Let us set $K^2 = k^2 + 2$. The solution follows by solving the classical Ricatti equation $f^2(s) + f'(s) = K^2$ whose solution is

$$f(s) = K \frac{\alpha e^{Ks} + e^{-Ks}}{\alpha e^{Ks} - e^{-Ks}} \, .$$

The terminal condition yields $\alpha = \dfrac{k-K}{k+K} e^{-2TK}$. A straightforward computation leads to the expression of g given in the proposition. □

2.6.5 Powers of δ-Dimensional Radial OU Processes, Alias CIR Processes

In the case $\theta = 0$, the process

$$d\rho_t = (1 - 2k\rho_t)dt + 2\sqrt{\rho_t} dW_t$$

is called a one-dimensional square OU process which is justified by the computation at the beginning of this subsection. Let U be a δ-dimensional OU process, i.e., the solution of

$$U_t = u + B_t - k \int_0^t U_s ds$$

where B is a δ-dimensional Brownian motion and k a real number, and set $V_t = \|U_t\|^2$. From Itô's formula,

$$\boxed{dV_t = (\delta - 2kV_t)dt + 2\sqrt{V_t} dW_t}$$

where W is a one-dimensional Brownian motion. The process V is called either a squared δ-dimensional radial Ornstein-Uhlenbeck process or more commonly in mathematical finance a Cox-Ingersoll-Ross (CIR) process with dimension δ and linear coefficient k, and, for $\delta \ge 2$, does not reach 0 (see $\rightarrowtail$ Subsection 6.3.1).

Let $\gamma \neq 0$ be a real number, and $Z_t = V_t^{\gamma}$. Then,

$$Z_t = z + 2\gamma \int_0^t Z_s^{1-1/(2\gamma)} dW_s - 2k\gamma \int_0^t Z_s ds + \gamma(2(\gamma-1)+\delta) \int_0^t Z_s^{1-1/\gamma} ds \,.$$

In the particular case $\gamma = 1 - \delta/2$,

$$Z_t = z + 2\gamma \int_0^t Z_s^{1-1/(2\gamma)} dW_s - 2k\gamma \int_0^t Z_s ds \,,$$

or in differential notation

$$dZ_t = Z_t(\mu dt + \sigma Z_t^{\beta} dW_t) \,,$$

with

$$\mu = -2k\gamma, \beta = -1/(2\gamma) = 1/(\delta - 2), \sigma = 2\gamma \,.$$

The process Z is called a CEV process.

Comment 2.6.5.1 We shall study CIR processes in more details in $\rightarrowtail$ Section 6.3. See also Pitman and Yor [716, 717]. See $\rightarrowtail$ Section 6.4, where squares of OU processes are of major interest in constructing CEV processes.

2.7 Valuation of European Options

In this section, we give a few applications of Itô's lemma, changes of probabilities and Girsanov's theorem to the valuation of options.

2.7.1 The Garman and Kohlhagen Model for Currency Options

In this section, European currency options will be considered. It will be shown that the Black and Scholes formula corresponds to a specific case of the Garman and Kohlhagen [373] model in which the foreign interest rate is equal to zero. As in the Black and Scholes model, let us assume that trading is continuous and that the historical dynamics of the underlying (the currency) S are given by

$$dS_t = S_t(\alpha dt + \sigma dB_t) \,.$$

whereas the risk-neutral dynamics satisfy the Garman-Kohlhagen dynamics

$$dS_t = S_t((r-\delta)dt + \sigma dW_t) \,. \tag{2.7.1}$$

Here, $(W_t, t \geq 0)$ is a $\mathbb{Q}$-Brownian motion and $\mathbb{Q}$ is the risk-neutral probability defined by its Radon-Nikodým derivative with respect to $\mathbb{P}$ as $\mathbb{Q}|_{\mathcal{F}_t} = \exp(-\theta B_t - \frac{1}{2}\theta^2 t)\, \mathbb{P}|_{\mathcal{F}_t}$ with $\theta = \dfrac{\alpha - (r-\delta)}{\sigma}$. It follows that

$$S_t = S_0 e^{(r-\delta)t} e^{\sigma W_t - \frac{\sigma^2}{2}t}.$$

The domestic (resp. foreign) interest rate r (resp. δ) and the volatility σ are constant. The term δ corresponds to a dividend yield for options (see Subsection 2.3.5).

The method used in the Black and Scholes model will give us the PDE evaluation for a European call. We give the details for the reader's convenience.

In that setting, the PDE evaluation for a contingent claim $H = h(S_T)$ takes the form

$$-\partial_u V(x,T-t)+(r-\delta)x\partial_x V(x,T-t)+\frac{1}{2}\sigma^2x^2\partial_{xx}V(x,T-t) = rV(x,T-t) \tag{2.7.2}$$

with the initial condition $V(x,0) = h(x)$. Indeed, the process $e^{-rt}V(S_t,t)$ is a $\mathbb{Q}$-martingale, and an application of Itô's formula leads to the previous equality. Let us now consider the case of a European call option:

Proposition 2.7.1.1 *The time-t value of the European call on an underlying with risk-neutral dynamics (2.7.1) is $C_E(S_t, T-t)$. The function C_E satisfies the following PDE:*

$$\begin{aligned} -\frac{\partial C_E}{\partial u}(x,T-t) &+ \frac{1}{2}\sigma^2x^2\frac{\partial^2 C_E}{\partial x^2}(x,T-t) \\ &+ (r-\delta)x\frac{\partial C_E}{\partial x}(x,T-t) = rC_E(x,T-t) \end{aligned} \tag{2.7.3}$$

with initial condition $C_E(x,0) = (x-K)^+$, and is given by

$$C_E(x,u) = xe^{-\delta u}\mathcal{N}\left[d_1\left(\frac{xe^{-\delta u}}{Ke^{-ru}},u\right)\right] - Ke^{-ru}\mathcal{N}\left[d_2\left(\frac{xe^{-\delta u}}{Ke^{-ru}},u\right)\right], \tag{2.7.4}$$

where the d_i's are given in Theorem 2.3.2.1.

PROOF: The evaluation PDE (2.7.3) is obtained from (2.7.2). Formula (2.7.4) is obtained by a direct computation of $\mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T-K)^+)$, or by solving (2.7.3). □

2.7.2 Evaluation of an Exchange Option

An exchange option is an option to exchange one asset for another. In this domain, the original reference is Margrabe [623]. The model corresponds to an extension of the Black and Scholes model with a stochastic strike price, (see Fischer [345]) in a risk-adjusted setting. Let us assume that under the risk-adjusted neutral probability $\mathbb{Q}$ the stock prices' (respectively, S^1 and S^2) dynamics[5] are given by:

[5] Of course, 1 and 2 are only superscripts, not powers.

$$dS_t^1 = S_t^1\ ((r-\nu)dt + \sigma_1 dW_t)\ ,\ dS_t^2 = S_t^2\ ((r-\delta)dt + \sigma_2 dB_t)$$

where r is the risk-free interest rate and ν and δ are, respectively, the stock 1 and 2 dividend yields and σ_1 and σ_2 are the stock prices' volatilities. The correlation coefficient between the two Brownian motions W and B is denoted by ρ. It is assumed that all of these parameters are constant. The payoff at maturity of the **exchange call option** is $(S_T^1 - S_T^2)^+$. The option price is therefore given by:

$$\begin{aligned} C_{EX}(S_0^1, S_0^2, T) &= \mathbb{E}_{\mathbb{Q}}(e^{-rT}(S_T^1 - S_T^2)^+) = \mathbb{E}_{\mathbb{Q}}(e^{-rT}S_T^2(X_T - 1)^+) \\ &= S_0^2 \mathbb{E}_{\mathbb{Q}^*}(e^{-\delta T}(X_T - 1)^+)\ . \end{aligned} \tag{2.7.5}$$

Here, $X_t = S_t^1/S_t^2$, and the probability measure $\mathbb{Q}^*$ is defined by its Radon-Nikodým derivative with respect to $\mathbb{Q}$

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = e^{-(r-\delta)t}\frac{S_t^2}{S_0^2} = \exp\left(\sigma_2 B_t - \frac{\sigma_2^2}{2}t\right)\ . \tag{2.7.6}$$

Note that this change of probability is associated with a change of numéraire, the new numéraire being the asset S^2. Using Itô's lemma, the dynamics of X are

$$dX_t = X_t[(\delta - \nu + \sigma_2^2 - \rho\sigma_1\sigma_2)dt + \sigma_1 dW_t - \sigma_2 dB_t]\,.$$

Girsanov's theorem for correlated Brownian motions (see Subsection 1.7.4) implies that the processes $\widetilde{W}$ and $\widetilde{B}$ defined as

$$\widetilde{W}_t = W_t - \rho\sigma_2 t,\ \widetilde{B}_t = B_t - \sigma_2 t\,,$$

are $\mathbb{Q}^*$-Brownian motions with correlation ρ. Hence, the dynamics of X are

$$dX_t = X_t[(\delta-\nu)dt + \sigma_1 d\widetilde{W}_t - \sigma_2 d\widetilde{B}_t] = X_t[(\delta-\nu)dt + \sigma dZ_t]$$

where Z is a $\mathbb{Q}^*$-Brownian motion defined as

$$dZ_t = \sigma^{-1}(\sigma_1 d\widetilde{W}_t - \sigma_2 d\widetilde{B}_t)$$

and where

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\,.$$

As shown in equation (2.7.5), δ plays the rôle of a discount rate. Therefore, by relying on the Garman and Kohlhagen formula (2.7.4), the exchange option price is given by:

$$C_{EX}(S_0^1, S_0^2, T) = S_0^1 e^{-\nu T}\mathcal{N}(b_1) - S_0^2 e^{-\delta T}\mathcal{N}(b_2)$$

with

$$b_1 = \frac{\ln(S_0^1/S_0^2) + (\delta-\nu)T}{\Sigma\sqrt{T}} + \frac{1}{2}\Sigma\sqrt{T}\,,\ b_2 = b_1 - \Sigma\sqrt{T}\,.$$

This value is independent of the domestic risk-free rate r. Indeed, since the second asset is the numéraire, its dividend yield, δ, plays the rôle of the domestic risk-free rate. The first asset dividend yield ν, plays the rôle of the foreign interest rate in the foreign currency option model developed by Garman and Kohlhagen [373]. When the second asset plays the rôle of the numéraire, in the risk-neutral economy the risk-adjusted trend of the process $(S_t^1/S_t^2, t \geq 0)$ is the dividend yield differential $\delta - \nu$.

2.7.3 Quanto Options

In the context of the international diversification of portfolios, **quanto options** can be useful. Indeed with these options, the problems of currency risk and stock market movements can be managed simultaneously. Using the model established in El Karoui and Cherif [298], the valuation of these products can be obtained.

Let us assume that under the domestic risk-neutral probability $\mathbb{Q}$, the dynamics of the stock price S, in foreign currency units and of the currency price X, in domestic units, are respectively given by:

$$\begin{aligned} dS_t &= S_t\left((\delta - \nu - \rho\sigma_1\sigma_2)dt + \sigma_1 dW_t\right) \\ dX_t &= X_t\left((r-\delta)dt + \sigma_2 dB_t\right) \end{aligned} \tag{2.7.7}$$

where r, δ and ν are respectively the domestic, foreign risk-free interest rate and the dividend yield and σ_1 and σ_2 are, respectively, the stock price and currency volatilities. Again, the correlation coefficient between the two Brownian motions is denoted by ρ. It is assumed that the parameters are constant.

The trend in equation (2.7.7) is equal to $\mu_1 = \delta - \nu - \rho\sigma_1\sigma_2$ because, in the domestic risk-neutral economy, we want the trend of the stock price (in domestic units: XS) dynamics to be equal to $r - \nu$.

We now present four types of quanto options:

Foreign Stock Option with a Strike in a Foreign Currency

In this case, the payoff at maturity is $X_T(S_T - K)^+$, i.e., the value in the domestic currency of the standard Black and Scholes payoff in the foreign currency $(S_T - K)^+$. The call price is therefore given by:

$$C_{qt1}(S_0, X_0, T) := \mathbb{E}_{\mathbb{Q}}(e^{-rT}(X_T S_T - K X_T)^+) .$$

This quanto option is an exchange option, an option to exchange at maturity T, an asset of value $X_T S_T$ for another of value $K X_T$. By relying on the previous Subsection 2.7.2

$$C_{qt1}(S_0, X_0, T) = X_0 \mathbb{E}_{\mathbb{Q}^*}(e^{-\delta T}(S_T - K)^+)$$

where the probability measure $\mathbb{Q}^*$ is defined by its Radon-Nikodým derivative with respect to $\mathbb{Q}$, in equation (2.7.6).

Cameron-Martin's theorem implies that the two processes $(B_t - \sigma_2 t, t \geq 0)$ and $(W_t - \rho\sigma_2 t, t \geq 0)$ are $\mathbb{Q}^*$ -Brownian motions. Now, by relying on equation (2.7.7)

$$dS_t = S_t \left((\delta - \nu)dt + \sigma_1 d(W_t - \rho\sigma_2 t)\right) .$$

Therefore, under the $\mathbb{Q}^*$ measure, the trend of the process $(S_t, t \geq 0)$ is equal to $\delta - \nu$ and the volatility of this process is σ_1. Therefore, using the Garman and Kohlhagen formula (2.7.4), the exchange option price is given by

$$C_{qt1}(S_0, X_0, T) = X_0(S_0 e^{-\nu T}\mathcal{N}(b_1) - Ke^{-\delta T}\mathcal{N}(b_2))$$

with

$$b_1 = \frac{\ln(S_0/K) + (\delta - \nu)T}{\sigma_1\sqrt{T}} + \frac{1}{2}\sigma_1\sqrt{T}\,,\; b_2 = b_1 - \sigma_1\sqrt{T}\,.$$

This price could also be obtained by a straightforward arbitrage argument. If a stock call option (with payoff $(S_T - K)^+$) is bought in the domestic country, its payoff at maturity is the quanto payoff $X_T(S_T - K)^+$and its price at time zero is known. It is the Garman and Kohlhagen price (in the foreign risk-neutral economy where the trend is $\delta - \nu$ and the positive dividend yield is ν), times the exchange rate at time zero.

Foreign Stock Option with a Strike in the Domestic Currency

In this case, the payoff at maturity is $(X_T S_T - K)^+$. The call price is therefore given by

$$C_{qt2}(S_0, X_0, T) := \mathbb{E}_{\mathbb{Q}}(e^{-rT}(X_T S_T - K)^+)\,.$$

This quanto option is a standard European option, with a new underlying process XS, with volatility given by

$$\sigma_{XS} = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}$$

and trend equal to $r - \nu$ in the risk-neutral domestic economy. The risk-free discount rate and the dividend rate are respectively r and ν. Its price is therefore given by

$$C_{qt2}(S_0, X_0, T) = X_0 S_0 e^{-\nu T}\mathcal{N}(b_1) - Ke^{-rT}\mathcal{N}(b_2)$$

with

$$b_1 = \frac{\ln(X_0 S_0/K) + (r - \nu)T}{\sigma_{XS}\sqrt{T}} + \frac{1}{2}\sigma_{XS}\sqrt{T}\,,\; b_2 = b_1 - \sigma_1\sqrt{T}\,.$$

Quanto Option with a Given Exchange Rate

In this case, the payoff at maturity is $\bar{X}(S_T - K)^+$, where $\bar{X}$ is a given exchange rate ($\bar{X}$ is fixed at time zero). The call price is therefore given by:

$$C_{qt3}(S_0, X_0, T) := \mathbb{E}_{\mathbb{Q}}(e^{-rT}\bar{X}(S_T - K)^+)$$

i.e.,

$$C_{qt3}(S_0, X_0, T) = \bar{X}e^{-(r-\delta)T}\mathbb{E}_{\mathbb{Q}}(e^{-\delta T}(S_T - K)^+)\,.$$

We obtain the expectation, in the risk-neutral domestic economy, of the standard foreign stock option payoff discounted with the foreign risk-free interest rate. Now, under the domestic risk-neutral probability $\mathbb{Q}$, the foreign asset trend is given by $\delta - \nu - \rho\sigma_1\sigma_2$ (see equation (2.7.7)).

Therefore, the price of this quanto option is given by

$$C_{qt3}(S_0, X_0, T) = \bar{X}e^{-(r-\delta)T}\left[S_0 e^{-(\nu+\rho\sigma_1\sigma_2)T}\mathcal{N}(b_1) - Ke^{-\delta T}\mathcal{N}(b_2)\right]$$

with

$$b_1 = \frac{\ln(S_0/K) + (\delta - \nu - \rho\sigma_1\sigma_2)T}{\sigma_1\sqrt{T}} + \frac{1}{2}\sigma_1\sqrt{T},\ b_2 = b_1 - \sigma_1\sqrt{T}\,.$$

Foreign Currency Quanto Option

In this case, the payoff at maturity is $S_T(X_T - K)^+$. The call price is therefore given by

$$C_{qt4}(S_0, X_0, T) := \mathbb{E}_{\mathbb{Q}}(e^{-rT}S_T(X_T - K)^+)\,.$$

Now, the price can be obtained by relying on the first quanto option. Indeed, the stock price now plays the rôle of the currency price and vice-versa. Therefore, μ_1 and σ_1 can be used respectively instead of $r - \delta$ and σ_2, and vice versa. Thus

$$C_{qt4}(S_0, X_0, T) = S_0(X_0 e^{(r-\delta+\rho\sigma_1\sigma_2-(r-\mu_1))T}\mathcal{N}(b_1) - Ke^{-(r-\mu_1)T}\mathcal{N}(b_2))$$

or, in a closed form

$$C_{qt4}(S_0, X_0, T) = S_0(X_0 e^{-\nu T}\mathcal{N}(b_1) - Ke^{-(r-\delta+\nu+\rho\sigma_1\sigma_2)\,T}\mathcal{N}(b_2))$$

with

$$b_1 = \frac{\ln(X_0/K) + (r - \delta + \rho\sigma_1\sigma_2)T}{\sigma_2\sqrt{T}} + \frac{1}{2}\sigma_2\sqrt{T},\ b_2 = b_1 - \sigma_2\sqrt{T}\,.$$

Indeed, $r - \delta + \rho\sigma_1\sigma_2$ is the trend of the currency price under the probability measure $\mathbb{Q}^*$, defined by its Radon-Nikodým derivative with respect to $\mathbb{Q}$ as

$$\mathbb{Q}^*|_{\mathcal{F}_t} = \exp\left(\sigma_1 W_t - \frac{1}{2}\sigma_1^2 t\right)\mathbb{Q}|_{\mathcal{F}_t}\,.$$

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