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Bridging the Gap: Solutions

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Inequalities

Test Yourself

1. 5, 6, 7. We include 5 because the inequality sign in "$5 \leq x$" is "greater than or equal to", whereas in "$x < 8$" we are demanding *strictly* less than.

2.

$$\begin{aligned} x+2 &< 7 \\ x+2-2 &< 7-2 \\ x &< 5 \end{aligned}$$

3.

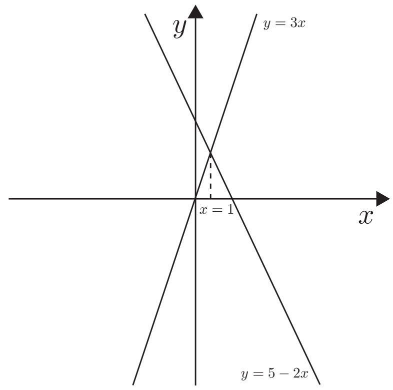
$$\begin{aligned} 3x+4 &\geq 5x+2 \\ 3x+2 &\geq 5x \\ 2 &\geq 2x \\ 1 &\geq x \\ \text{So } x &\leq 1 \end{aligned}$$

4.

$$\begin{aligned} -3x &< -12 \\ -x &< -4 \\ x &> 4 \end{aligned}$$

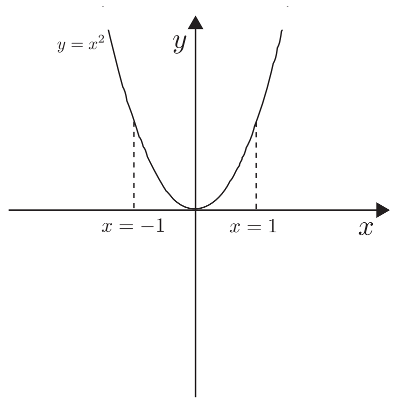
(We reverse the direction of the inequality sign because we multiply through by -1).

5. We need to draw the lines $y = 3x$ and $y = 5 - 2x$, and see when the first line lies "above" the second.



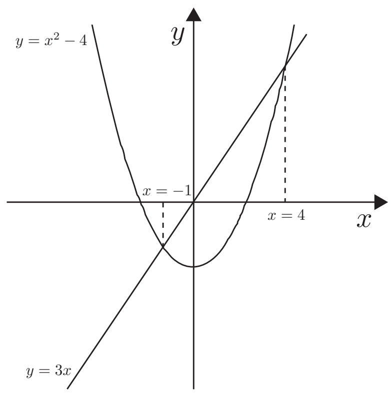
So $x \geq 1$.

6. We draw $y = x^2$ and $y = 1$.



Here we have 2 critical points and the graph shows us that there are 2 parts to the solution: $x < -1$ and $x > 1$.

7. We will require the following graph:



This time, only the "middle" part satisfies the inequality, so we have that $-1 < x < 4$.

8. $x^2 > 3x$. We need to solve $x^2 = 3x$:

$$x = 0 \text{ or } x = 3$$

We are interested in the regions where $y = x^2$ is above $y = 3x$, so we need $x < 0$ and $x > 3$.

9.
$$\frac{x-4}{x+10} < 0$$

$x = 4$ and $x = -10$ are the critical values (by inspection of the numerator and denominator respectively). So the different regions of interest are:

$$x < -10 \ , \ -10 < x < 4 \ , \ x > 4$$

By direct substitution, we see that the solution is $-10 < x < 4$, because if we choose any value of x in this region, our fraction is indeed negative.

10.

$$
\begin{aligned}
\frac{x+3}{x-2} &< x+6 \\
\frac{x+3}{x-2} - x - 6 &< 0 \\
\frac{x+3-(x+6)(x-2)}{x-2} &< 0 \\
\frac{x+3-x^2-4x+12}{x-2} &< 0 \\
\frac{-x^2-3x+15}{x-2} &< 0 \\
\frac{x^2+3x-15}{2-x} &< 0
\end{aligned}
$$

(We get this last step by multiplying numerator and denominator by -1. Note that this *doesn't* reverse the inequality: we didn't multiply the whole expression by -1, only the numerator and denominatorof the fraction).

Now, using the quadratic formula on the numerator:

$$
\begin{aligned}
x &= \frac{-3 \pm \sqrt{9+60}}{2} \\
&= \frac{-3+\sqrt{69}}{2}, \frac{-3-\sqrt{69}}{2}
\end{aligned}
$$

and the final critical value is $x = 2$, which we obtain from the denominator.

By substituting in values from appropriate regions, we find that the solution is

$$
\frac{-3-\sqrt{69}}{2} < x < 2 \text{ and } x > \frac{-3+\sqrt{69}}{2}
$$

1.1 What are Inequalities?

1. 8, 9, 10.

2.

$$
\begin{aligned}
2x &> 22 \\
\frac{2x}{2} &> \frac{22}{2} \\
x &> 11
\end{aligned}
$$

3.

$$\begin{aligned} x + 4 &> 11 \\ x + 4 - 4 &> 11 - 4 \\ x &> 7 \end{aligned}$$

4.

$$\begin{aligned} 2x + 5 &\leq 15 \\ 2x &\leq 10 \\ x &\leq 5 \end{aligned}$$

5.

$$\begin{aligned} 3x - 4 &< x + 6 \\ 3x &< x + 10 \\ 2x &< 10 \\ x &< 5 \end{aligned}$$

6.

$$\begin{aligned} x - 7 &\geq 2x + 3 \\ x &\geq 2x + 10 \\ -10 &\geq x \\ x &\leq -10 \end{aligned}$$

7.

$$\begin{aligned} 3x - 5 - 4x &> 7 \\ -5 - x &> 7 \\ -x &> 12 \\ x &< -12 \end{aligned}$$

8.

$$\begin{aligned} \frac{-x}{2} + 3 &\geq 3x - 4 \\ \frac{-x}{2} + 7 &\geq 3x \\ -x + 14 &\geq 6x \\ 14 &\geq 7x \\ x &\leq 2 \end{aligned}$$

9.

$$3xy + 4y > 2y$$

(We know $y > 0$, so we can divide through by it, and preserve the inequality.)

$$\begin{aligned} 3x + 4 &> 2 \\ 3x &> -2 \\ x &> -\frac{2}{3} \end{aligned}$$

10.

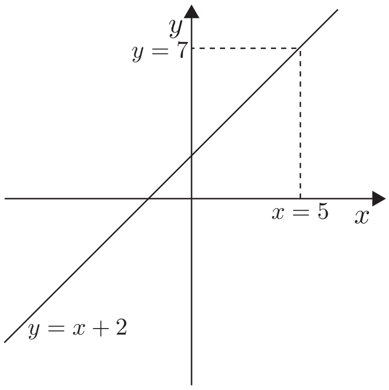
$$\begin{aligned} 2xyz + yz &> 3xyz \\ 2xz + z &> 3xz \text{ , because } y > 0 \\ 2x + 1 &< 3x \end{aligned}$$

We have the previous step because $z < 0$, so dividing through by it reverses the inequality.

$$\begin{aligned} 1 &< x \\ x &> 1 \end{aligned}$$

1.2 Using Graphs

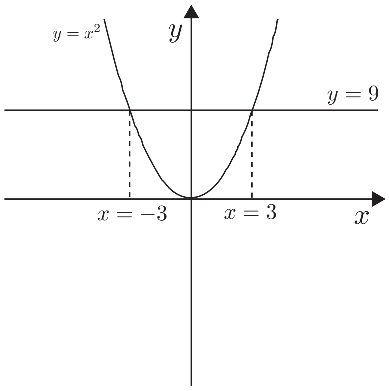
1. Our required graph is:



So, whenever $x < 5$, the line $y = x + 2$ lies below the line $y = 7$.

Hence $x < 5$

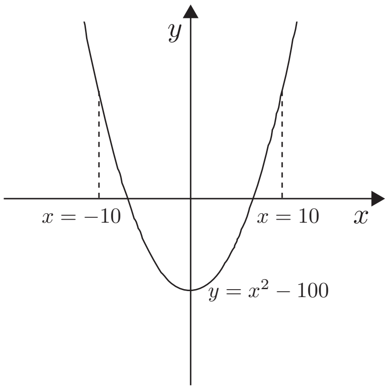
2. Our required graph is:



Here we have 2 regions in our solution: $y = x^2$ is above $y = 9$ if $x \leq -3$ *and* if $x \geq 3$.

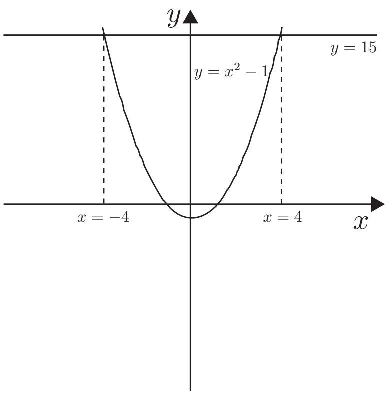
So, $x \leq -3$ and $x \geq 3$

3. Our required graph is:



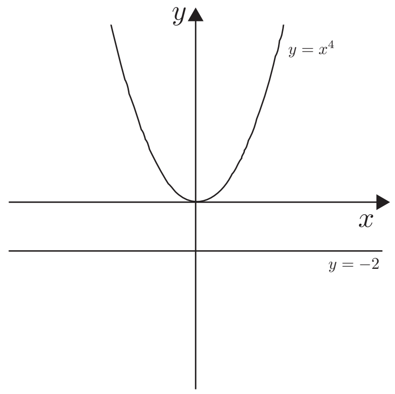
So, here we have just one region in our solution: $-10 < x < 10$.

4. Our required graph is:



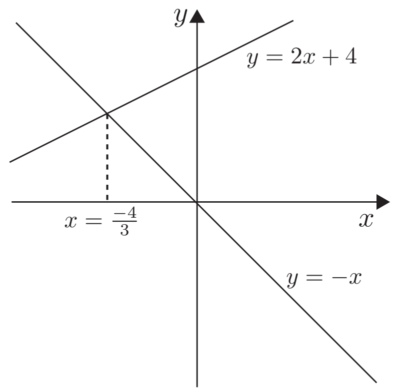
So $x < -4$ and $x > 4$.

5. Our required graph is:



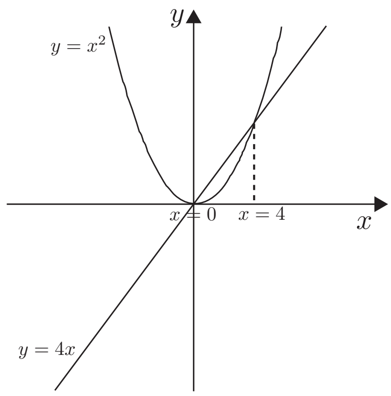
Here, we see that $y = x^4$ *never* lies below the line $y = -2$, so we have no solution to our inequality.

6. Our required graph is:



So $x < -\frac{4}{3}$.

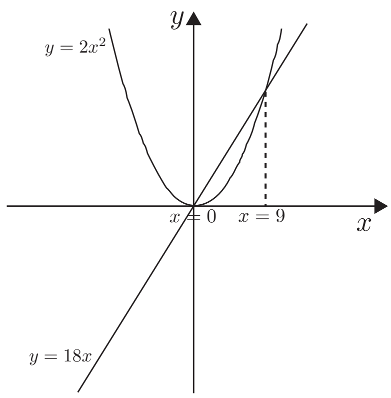
7. Our required graph is:



So $0 < x < 4$.

8. $2x^2 \geq 18x$.

   So our required graph is:



So $x \leq 0$ and $x \geq 9$.

1.3 Critical Values

1.

$$\begin{aligned} x^2 &> 4 \\ x^2 - 4 &> 0 \end{aligned}$$

Let's treat it like an equality to find critical values:

$$\begin{aligned} x^2 - 4 &= 0 \\ x &= \pm 2 \end{aligned}$$

So we have 3 regions to examine: $x < -2$, $-2 < x < 2$ and $x > 2$. We find which regions are correct through making a substitution:

$x = -10$ certainly satisfies $x < -2$. Here, $x^2 - 4 = 96$ and $96 > 0$. So $x < -2$ is in our solution.

$x = 0$ is a good value of x that lies in $-2 < x < 2$. At $x = 0$, $x^2 - 4 = -4$. But $-4 < 0$, so $-2 < x < 2$ is *not* in our solution.

Finally, taking $x = 10$ to test the region $x > 2$ gives us $x^2 - 4 = 96$, and $96 > 0$, so $x > 2$ is in our solution.

Hence $x < -2$ and $x > 2$.

2.

$$\begin{aligned} x^2 &> 2x \\ x^2 - 2x &> 0 \end{aligned}$$

$x(x - 2) = 0$ yields $x = 0$ and $x = 2$ as the critical values.

We can then take $x = -1$ to see that $x < 0$ is in the solution, take $x = 1$ to see that $0 < x < 2$ is not in the solution, and take $x = 3$ to see that $x > 2$ is in the solution (these are just "convenient" examples: any other suitable values would yield the same result).

So $x < 0$ and $x > 2$.

3.

$$\begin{aligned} x^2 &< 7x \\ x^2 - 7x &< 0 \end{aligned}$$

$x(x - 7) = 0$ yields $x = 0$ and $x = 7$ as the critical values.

Making appropriate substitutions ($x = -1$, $x = 1$ and $x = 8$ are good values to choose) shows that our solution is $0 < x < 7$.

4.

$$\begin{aligned} x^2 + 3 &> -4x \\ x^2 + 4x + 3 &> 0 \\ (x+3)(x+1) &> 0 \end{aligned}$$

So the critical values are $x = -1$ and $x = -3$.

The solution is $x < -3$ and $x > -1$.

5.

$$\begin{aligned} x^2 + 6x &< 3 \\ x^2 + 6x - 3 &< 0 \end{aligned}$$

Use the quadratic formula to find the critical values:

$$\begin{aligned} \frac{-6 \pm \sqrt{36+12}}{2} &= \frac{-6 \pm \sqrt{48}}{2} \\ &= -3 \pm \sqrt{12} \end{aligned}$$

By maing appropriate substitutions (-10, 0 and 10 are good candidates!) we see that the solution is:

$$-3 - \sqrt{12} < x < -3 + \sqrt{12}$$

6. From the numerator we see that $x = -8$ is a critical value, and from the denominator there is also one at $x = -4$. By appropriate substitutions, we see that solution is:

$$x < -8 \text{ and } x > -4$$

7. Our critical values are $x = -6$ and $x = 3$

The solution is $-6 < x < 3$

8.

$$\begin{aligned} \frac{x+5}{x-2} &> x+5 \\ \frac{x+5}{x-2} &> \frac{(x+5)(x-2)}{x-2} \end{aligned}$$

This last step is so that we have a common denominator.

$$\begin{aligned}\frac{x+5-(x+5)(x-2)}{x-2} &> 0\\ \frac{(x+5)(1-(x-2))}{x-2} &> 0\\ \frac{(x+5)(3-x)}{x-2} &> 0\\ \frac{(x+5)(x-3)}{x-2} &< 0\end{aligned}$$

(Here we multiplied through by -1, reversing the inequality.)

So our critical values are $x = -5$, $x = 2$ and $x = 3$.

By appropriate substitutions we find the solution to be $x < -5$ and $2 < x < 3$.

9.

$$\begin{aligned}\frac{x-2}{3-x} &< 3\\ \frac{x-2}{3-x} &< \frac{3(3-x)}{3-x}\\ \frac{x-2-3(3-x)}{3-x} &< 0\\ \frac{4x-11}{3-x} &< 0\end{aligned}$$

$x = \frac{11}{4}$ and $x = 3$ are the critical values, so: $x < \frac{11}{4}$ and $x > 3$

10.

$$\begin{aligned}\frac{2x+5}{x-3} &< \frac{(x+1)(x-3)}{x-3}\\ \frac{x^2-2x-3-2x-5}{x-3} &> 0\\ \frac{x^2-4x-8}{x-3} &> 0\end{aligned}$$

By the quadratic formula:

$$\begin{aligned}x &= \frac{4 \pm \sqrt{16+32}}{2}\\ &= \frac{4 \pm \sqrt{48}}{2}\\ &= 2 \pm \sqrt{12}\end{aligned}$$

$x = 3$ is the other critical value, from the denominator. By appropriate substitutions, $2-\sqrt{12} < x < 3$ and $x > 2+\sqrt{12}$ is our solution.

2
Trigonometry, Differentiation and Exponents

Test Yourself

1. We have that $\sin^2 x + \cos^2 x = 1$. If we then divide through by $\sin^2 x$, we get:
$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$
We can then write each of these expressions in their alternative forms:
$$1 + \cot^2 x = \mathrm{cosec}^2 x$$
If we then rearrange, we get:
$$\mathrm{cosec}^2 x - \cot^2 x = 1$$

2. We can make $\frac{7\pi}{12}$ by doing $\frac{\pi}{4} + \frac{\pi}{3}$. So we can then use our angle addition

formula:

$$\begin{aligned}\sin\frac{7\pi}{12} &= \sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\\ &= \sin\frac{\pi}{4}\cos\frac{\pi}{3}+\sin\frac{\pi}{3}\cos\frac{\pi}{4}\\ &= \frac{\sqrt{2}}{2}\cdot\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\\ &= \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

3. We can find $\frac{5\pi}{6}$ either by $\frac{\pi}{3}+\frac{\pi}{2}$ or by $\pi-\frac{\pi}{6}$. Let's use $\pi-\frac{\pi}{6}$, so:

$$\begin{aligned}\cos\frac{5\pi}{6} &= \cos\left(\pi-\frac{\pi}{6}\right)\\ &= \cos\pi\cos\frac{\pi}{6}+\sin\pi\sin\frac{\pi}{6}\\ &= -1\cdot\frac{\sqrt{3}}{2}+0\cdot\frac{1}{2}\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

4. Recall that:

$$\begin{aligned}\sin 2x &= 2\sin x\cos x\\ \cos 2x &= \cos^2 x-\sin^2 x\end{aligned}$$

Then using the identity $\cos^2 x+\sin^2 x = 1$, we have that $\cos^2 x = 1-\sin^2 x$. We can use this to give us the identity:

$$\cos 2x = 1-2\sin^2 x$$

Then we can find:

$$\begin{aligned}\tan 2x &= \frac{\sin 2x}{\cos 2x}\\ &= \frac{2\sin x\cos x}{1-2\sin^2 x}\end{aligned}$$

If we then divide both the numerator and denominator by $\cos^2 x$, we get

$$\begin{aligned}\tan 2x &= \frac{2\frac{\sin x}{\cos x}}{\frac{1}{\cos^2 x}-2\frac{\sin^2 x}{\cos^2 x}}\\ &= \frac{2\tan x}{\sec^2 x-2\tan^2 x}\end{aligned}$$

5. The quotient rule states that

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

We are required to differentiate $x^2\text{cosec}x$, which we can write in the form $\frac{x^2}{\sin x}$. So we will let $u = x^2$ and $v = \sin x$. This gives us that $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \cos x$. Hence:

$$\begin{aligned}\frac{d}{dx}\left(x^2\text{cosec}x\right) &= \frac{2x\sin x - x^2\cos x}{\sin^2 x} \\ &= \frac{2x}{\sin x} - \frac{x^2\cos x}{\sin x}\cdot\frac{1}{\sin x} \\ &= 2x\text{cosec}x - x^2\cot x\text{cosec}x\end{aligned}$$

6. The chain rule states that:

$$\frac{d}{dx}(M(N(x))) = M'(N(x))\cdot N'(x)$$

So we'll let $N(x) = \cos 2x$ and then $M(N(x)) = (N(x))^2$. We will require the chain rule again to find $N'(x)$. This will give us $N'(x) = -2\sin 2x$. So,

$$\begin{aligned}\frac{d}{dx}(\cos^2 2x) &= 2(\cos 2x)\cdot(-2\sin 2x) \\ &= -4\sin 2x\cos 2x\end{aligned}$$

7. Using the double angle formula we have $\cos 2x = \cos^2 x - \sin^2 x$. Then using the identity $\cos^2 x + \sin^2 x = 1$, we have that $\sin^2 x = 1 - \cos^2 x$. Hence we have $\cos 2x = 2\cos^2 x - 1$ which we can rearrange to $\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$. Hence

$$\begin{aligned}\int_0^{\frac{\pi}{3}}\left(\cos^2 x - \frac{1}{2}\right)dx &= \int_0^{\frac{\pi}{3}}\left(\frac{1}{2}\cos(2x) + \frac{1}{2} - \frac{1}{2}\right)dx \\ &= \int_0^{\frac{\pi}{3}}\left(\frac{1}{2}\cos(2x)\right)dx \\ &= \left[\frac{1}{4}\sin(2x)\right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{4}\cdot\frac{\sqrt{3}}{2} - 0 \\ &= \frac{\sqrt{3}}{8}\end{aligned}$$

8.
$$e^A \cdot e^B = e^{A+B}$$

So, plugging our numbers in gives us:

$$\begin{aligned} e^{7x} \cdot e^{3x} &= e^{7x+3x} \\ &= e^{10x} \end{aligned}$$

9. We require the chain *and* the product rule. Let $u = 4x$ and $v = e^{2x}$. Then $\frac{du}{dx} = 4$ and using the chain rule we get that $\frac{dv}{dx} = 2e^{2x}$. So,

$$\begin{aligned} \frac{d}{dx}(4xe^{2x}) &= 4e^{2x} + 4x \cdot 2e^{2x} \\ &= 4e^{2x} + 8xe^{2x} \end{aligned}$$

10. For this question, we will use the chain rule, with $N(x) = 3x^2$ and $M(N(x)) = \ln(N(x))$. Hence, $M'(N(x)) = \frac{1}{N(x)}$ and $N'(x) = 6x$. So we get:

$$\begin{aligned} \frac{d}{dx}(\ln(3x^2)) &= \frac{1}{3x^2} \cdot 6x \\ &= \frac{2}{x} \end{aligned}$$

2.1 Some Identities

1.

$$\begin{aligned} \text{LHS} &= \sec x(\sin^3 x + \sin x \cos^2 x) \\ &= \frac{\sin^3 x}{\cos x} + \sin x \cos x \end{aligned}$$

Then, rearranging the identity $\sin^2 x + \cos^2 x = 1$ we get $\sin^2 x = 1 - \cos^2 x$. Using this we have:

$$\begin{aligned} \text{LHS} &= \frac{\sin x(1 - \cos^2 x)}{\cos x} + \sin x \cos x \\ &= \frac{\sin x}{\cos x} - \frac{\sin x \cos^2 x}{\cos x} + \sin x \cos x \\ &= \tan x - \sin x \cos x + \sin x \cos x \text{ (using } \tan x = \frac{\sin x}{\cos x}) \\ &= \tan x \\ &= \text{RHS} \end{aligned}$$

2.

$$\begin{aligned}\text{LHS} &= \tan x \sin x \\ &= \frac{\sin^2 x}{\cos x}\text{ , using } \tan x = \frac{\sin x}{\cos x}.\end{aligned}$$

Then we once again use $\sin^2 x = 1 - \cos^2 x$. So

$$\begin{aligned}\text{LHS} &= \frac{1-\cos^2 x}{\cos x} \\ &= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}\end{aligned}$$

Using the fact that $\sec x = \frac{1}{\cos x}$, we end up with:

$$\begin{aligned}\text{LHS} &= \sec x - \cos x \\ &= \text{RHS}\end{aligned}$$

3. We can find $\frac{\pi}{12}$ by either using $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ or $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$. For this example we will use $\frac{\pi}{4} - \frac{\pi}{6}$.So, we have:

$$\begin{aligned}\sin\frac{\pi}{12} &= \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ &= \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{6}\cos\frac{\pi}{4} \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

4. $\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$. So:

$$\begin{aligned}\cos\frac{5\pi}{12} &= \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

It makes sense that this is the same value as in the previous question, due to the nature of sin and cos. We have $\sin\theta = \cos(\frac{\pi}{2} - \theta)$.

5. We can find $\frac{5\pi}{6}$ by either using $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$ or $\frac{5\pi}{6} = \frac{\pi}{3} + \frac{\pi}{2}$. We'll use $\frac{\pi}{3} + \frac{\pi}{2}$.
$$\begin{aligned}\sin\left(\frac{\pi}{3} + \frac{\pi}{2}\right) &= \sin\frac{\pi}{3}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\cos\frac{\pi}{3} \\ &= \frac{\sqrt{3}}{2}\cdot 0 + 1\cdot\frac{1}{2} \\ &= \frac{1}{2}\end{aligned}$$

6. $\frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$. So:
$$\begin{aligned}\cos\frac{7\pi}{12} &= \cos\frac{\pi}{4}\cos\frac{\pi}{3} - \sin\frac{\pi}{4}\sin\frac{\pi}{3} \\ &= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} \\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

7. From the double angle formula we have $\cos(2x) = \cos^2 x - \sin^2 x$ and $\cos^2 x + \sin^2 x = 1$, so $\cos^2 x = 1 - \sin^2 x$. Hence
$$\begin{aligned}\cos(2x) &= 1 - 2\sin^2 x \\ 2\sin^2 x &= 1 - \cos(2x) \\ \sin^2 x &= \frac{1}{2}(1 - \cos(2x))\end{aligned}$$

8. $\tan x = \frac{\sin x}{\cos x}$ and $\sin(2x) = 2\sin x\cos x$. So:
$$\begin{aligned}\text{LHS} &= 2\sin x\cos x\cdot\frac{\sin x}{\cos x} \\ &= 2\sin^2 x \\ &= 1 - \cos(2x) \text{ (from the previous question)} \\ &= \text{RHS}\end{aligned}$$

9. Using the double angle formula for cos we can show that $\cos(2x) = 2\cos^2 x - 1)$. So:
$$\begin{aligned}\text{LHS} &= \frac{\sin(2x)}{2\cos^2 x}\cdot(2\cos^2 x - 1) \\ &= \sin(2x) - \frac{\sin(2x)}{2\cos^2 x}\end{aligned}$$

The other double angle formula gets us $\sin(2x) = 2\sin x \cos x$. Using this we have:

$$\begin{aligned} \text{LHS} &= \sin(2x) - \frac{2\sin x \cos x}{2\cos^2 x} \\ &= \sin(2x) - \frac{\sin x}{\cos x} \\ &= \sin(2x) - \tan x \\ &= \text{RHS} \end{aligned}$$

10. From the definition of tan we have:

$$\begin{aligned} \tan(A \pm B) &= \frac{\sin(A \pm B)}{\cos(A \pm B)} \\ &= \frac{\sin A \cos B \pm \sin B cos A}{\cos A \cos B \mp \sin A \sin B} \end{aligned}$$

Dividing numerator and denominator by $\cos A \cos B$

$$\begin{aligned} \tan(A \pm B) &= \frac{\frac{\sin A}{\cos A} \cdot \frac{\cos B}{\cos B} \pm \frac{\sin B}{\cos B} \cdot \frac{\cos A}{\cos A}}{\frac{\cos A}{\cos A} \cdot \frac{\cos B}{\cos B} \mp \frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B}} \\ &= \frac{\frac{\sin A}{\cos A} \pm \frac{\sin B}{\cos B}}{1 \mp \frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B}} \\ &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \end{aligned}$$

2.2 Differentiating and Integrating

1. The product rule states:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

If we let $u = x^2$ and $v = \cos x$, then we have that $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = -\sin x$. Hence

$$\frac{d}{dx}(x^2 \cos x) = 2x \cos x - x^2 \sin x$$

2. This time let $u = \frac{1}{2}x^6$ and $v = \tan x$. Then $\frac{du}{dx} = 3x^5$ and $\frac{dv}{dx} = \sec^2 x$ (this is a result that we found in the chapter). So

$$\frac{d}{dx}(\frac{1}{2}x^6 \tan x) = 3x^5 \tan x + \frac{1}{2}x^6 \sec^2 x$$

3. The quotient rule states:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

So let's look at cosec$x$ which is the same as $\frac{1}{\sin x}$. Hence let $u = 1$ and $v = \sin x$. Then we have that $\frac{du}{dx} = 0$ and $\frac{dv}{dx} = \cos x$. So:

$$\begin{aligned}\frac{d}{dx}(\mathrm{cosec}x) &= \frac{0 \cdot \sin x - 1 \cdot \cos x}{\sin^2 x} \\ &= \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x} \\ &= -\cot x \mathrm{cosec} x\end{aligned}$$

4. First, write $\cot x$ as $\frac{\cos x}{\sin x}$. Then use the quotient rule with $u = \cos x$ and $v = \sin x$. This means that $\frac{du}{dx} = -\sin x$ and $\frac{dv}{dx} = \cos x$.

$$\begin{aligned}\frac{d}{dx}(\cot x) &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\ &= \frac{-1}{\sin^2 x} \\ &= -\mathrm{cosec}^2 x\end{aligned}$$

5. $x \sec x = \frac{x}{\cos x}$, So let $u = x$ and $v = \cos x$ in the quotient rule. We then have $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -\sin x$.

$$\begin{aligned}\frac{d}{dx}(x \sec x) &= \frac{1 \cdot \cos x + x \sin x}{\cos^2 x} \\ &= \frac{1}{\cos x} + \frac{x \sin x}{\cos x} \cdot \frac{1}{\cos x} \\ &= \sec x(1 + x \tan x)\end{aligned}$$

6. Recall the chain rule:

$$\frac{d}{dx}(M(N(x))) = M'(N(x)) \cdot N'(x)$$

Then let $N(x) = 4x$ so, $M(N(x)) = \frac{1}{2}\sin(N(x))$. Therefore we have $N'(x) = 4$ and $M'(N(x)) = \frac{1}{2}\cos(N(x))$

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}\sin(4x)\right) &= \frac{1}{2}\cos(4x) \cdot 4 \\ &= 2\cos(4x)\end{aligned}$$

7.
$$\begin{aligned}\frac{d}{dx}(\cos^6 x) &= \frac{d}{dx}((\cos x)^6)\\ &= 6(\cos x)^5 \cdot -\sin x\\ &= -6\sin x\cos^5 x\end{aligned}$$

8.
$$\begin{aligned}\frac{d}{dx}\left(\sin^2\left(\frac{x}{2}\right)\right) &= \frac{d}{dx}\left(\left(\sin\frac{x}{2}\right)^2\right)\\ &= 2\sin\left(\frac{x}{2}\right)\cdot\frac{d}{dx}\left(\sin\left(\frac{x}{2}\right)\right)\end{aligned}$$

We now need to use the chain rule again to find $\frac{d}{dx}(\sin\frac{x}{2})$. We get $\frac{1}{2}\cos(\frac{x}{2})$. So:

$$\begin{aligned}\frac{d}{dx}\left(\sin^2\left(\frac{x}{2}\right)\right) &= 2\cos\left(\frac{x}{2}\right)\cdot\frac{1}{2}\cos\left(\frac{x}{2}\right)\\ &= \sin\frac{x}{2}\cos\frac{x}{2}\end{aligned}$$

9. Let's look at what we have. Maybe $\sin(3x)$ is the correct answer? Using the chain rule we have that $\frac{d}{dx}(\sin 3x) = 3\cos(3x)$. However, we require a third of this so, we must have

$$\int \cos(3x)\ dx = \frac{1}{3}\sin(3x) + c$$

Hence:

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \cos(3x)\ dx &= \left[\frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{4}}\\ &= \frac{1}{3}\cdot\frac{\sqrt{2}}{2} - 0\\ &= \frac{\sqrt{2}}{6}\end{aligned}$$

10. For this question, we need to rearrange the double angle formula for cos to get $\sin^2 x = \frac{1}{2}(1-\cos(2x))$. Using this we get:

$$\int_0^{\frac{\pi}{3}} \sin^2 x\ dx = \frac{1}{2}\int_0^{\frac{\pi}{3}} (1-\cos(2x))dx$$

For the second part of this expression we will try $\sin(2x)$.

$$\frac{d}{dx}(\sin(2x)) = 2\cos(2x)$$

So, we must require $-\frac{1}{2}\sin(2x)$ and we have:

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sin^2 x \; dx &= \frac{1}{2}\int_0^{\frac{\pi}{3}} (1-\cos(2x)) \; dx \\ &= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{2}\left[\left(\frac{\pi}{3} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) - 0\right] \\ &= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) \\ &= \frac{\pi}{6} - \frac{\sqrt{3}}{8}\end{aligned}$$

2.3 Exponents and Logarithms

1. Recall:
$$e^A \cdot e^B = e^{A+B}$$
So, plugging our numbers in gives us:
$$\begin{aligned}e^{4x} \cdot e^{3x} &= e^{4x+3x} \\ &= e^{7x}\end{aligned}$$

2. $$\frac{e^A}{e^B} = e^{A-B}$$
So:
$$\begin{aligned}\frac{e^{6y}}{e^{3x}} &= e^{6y-3x} \\ &= e^{3(2y-x)}\end{aligned}$$

3. $\ln(e^x) = x$ as the logarithm is the inverse of the exponential function. Hence
$$\ln(e^{4x}) = 4x$$

4. $e^{\ln x} = x$ as the logarithm is the inverse of the exponential function. Hence
$$e^{\ln x^3} = x^3$$

5. We must use the chain rule from the previous section.

$$\begin{aligned}\frac{d}{dx}(e^{2x}) &= e^{2x} \cdot \frac{d}{dx}(2x) \\ &= 2e^{2x}\end{aligned}$$

6. Once again we use the chain rule:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{3}e^{x^3}\right) &= \frac{1}{3}e^{x^3} \cdot 3x^2 \\ &= x^2 e^{x^3}\end{aligned}$$

7. Use the product rule and let $u = e^x$ and $v = \sin x$. Then $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = \cos x$. So,

$$\frac{d}{dx}(e^x \sin x) = e^x \sin x + e^x \cos x$$

8. We wil use the chain rule:

$$\begin{aligned}\frac{d}{dx}(\ln x^3) &= \frac{1}{x^3} \cdot 3x^2 \\ &= \frac{3}{x}\end{aligned}$$

9. For this we will require both the product and chain rules. In the product rule we will let $u = e^{x^2}$ and $v = \ln(4x)$. To find $\frac{du}{dx}$ and $\frac{dv}{dx}$ we will use the chain rule:

$$\begin{aligned}\frac{du}{dx} &= e^{x^2} \cdot 2x \\ &= 2xe^{x^2}\end{aligned}$$

$$\begin{aligned}\frac{dv}{dx} &= \frac{1}{4x} \cdot 4 \\ &= \frac{1}{x}\end{aligned}$$

So using the product rule:

$$\begin{aligned}\frac{d}{dx}(e^{x^2} \ln(4x)) &= \ln(4x) \cdot 2xe^{x^2} + e^{x^2} \cdot \frac{1}{x} \\ &= 2xe^{x^2} \ln(4x) + \frac{e^{x^2}}{x}\end{aligned}$$

10. Once again we use both the product rule and the chain rule. Let $u = 4x^3$ and $v = \ln(x^2)$ then from the chain rule we get that $\frac{du}{dx} = 12x^2$ and $\frac{dv}{dx} = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$. So we get:

$$\begin{aligned} \frac{d}{dx}(4x^3 \ln(x^2)) &= 12x^2 \cdot \ln(x^2) + 4x^3 \cdot \frac{2}{x} \\ &= 12x^2 \ln(x^2) + 8x^2 \end{aligned}$$

3
Polar Co-Ordinates

Test Yourself

1. We are told that our point is 3 units away from the origin, so we know that $r = 3$.

   We are also told that our point lies on the negative x axis, which is an angle of π away from the positive x axis, which is where we start our measurements from.

   Hence, in polar co-ordinates, our point is the point $(3, \pi)$.

2. From the question we see that $r = 1$.

   We know that our point lies on the Cartesian line $y = x$, and so our value of θ could be either $\frac{\pi}{4}$ or $\frac{5\pi}{4}$, because these angles correspond to the half lines that form the Cartesian line $y = x$.

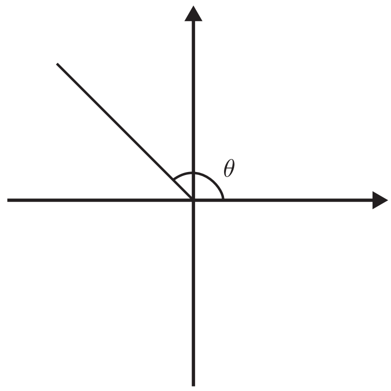
   Finally, we're told that our point lies in the first quadrant. From this we can deduce that our value of θ must satisfy $0 \leq \theta \leq \frac{\pi}{2}$, and so the answer is $(1, \frac{\pi}{4})$.

3. Firstly, let's use Pythagoras to find the distance of our point from the

origin.

$$\begin{aligned} r^3 &= (-2)^2 + 2^2 \\ &= 4 + 4 \\ &= 8 \\ \text{So } r &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

Now to find the angle:



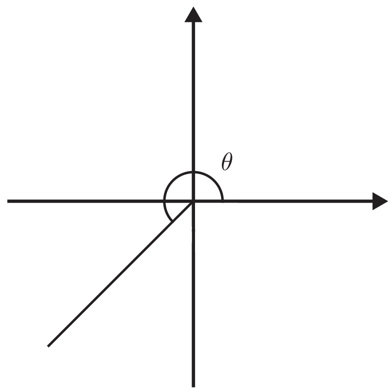
Here is the diagram of the angle that we require. We can use standard trigonometry to find θ, but here it should be clear that $\theta = \frac{3\pi}{4}$ (if it *isn't* clear, please work through the trigonometry yourself!).

So our point is $(2\sqrt{2}, \frac{3\pi}{4})$ in polar co-ordinates.

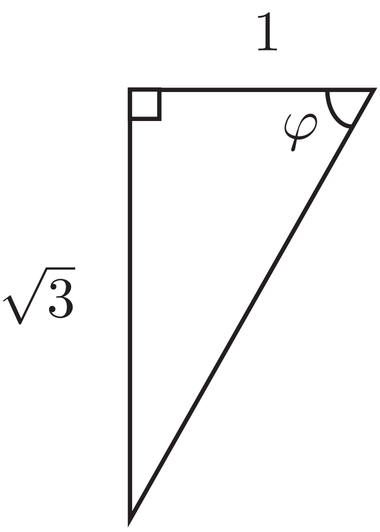
4. Again, finding r is relatively straightforward from Pythagoras:

$$\begin{aligned} r^2 &= (-1)^2 + (\sqrt{3})^2 \\ &= 1 + 3 \\ &= 4 \\ r &= 2 \end{aligned}$$

This time, we will need to do some trigonometry to find θ:



Note that we can consider the following triangle, with the hypotenuse being the same as the bold line in the diagram alone:



We can find ϕ using:

$$\begin{aligned} \tan\phi &= \frac{\sqrt{3}}{1} \\ &= \sqrt{3} \\ \phi &= \frac{\pi}{3} \end{aligned}$$

But then we see that $\theta = \phi + \pi$ (because θ measures the angle all the way around from the positive x axis, not just the negative x axis). Hence $\theta = \frac{4\pi}{3}$.

So our point is the point $(2, \frac{4\pi}{3})$.

5. The Cartesian line $y = x$ is comprised of the two half lines $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$.

   The third quadrant is bounded by the half lines $\theta = \pi$ and $\theta = \frac{3\pi}{2}$. Our value of θ must therefore satisfy $\pi \leq \theta \leq \frac{3\pi}{2}$.

   Hence our half line is $\theta = \frac{5\pi}{4}$.

6. Here, we're going to be using:

$$r^2 - 2rr_0\cos(\theta - \theta_0) + r_0^2 = a^2$$

   where the radius of the circle is a, and its centre lies at (r_0, θ_0).

   First and foremost, we're going to locate the centre of our circle. We're told that it passes through the Cartesian points $(1, 0)$ and $(3, 0)$, and so the centre must lie directly between these points, at $(2, 0)$.

   From this, we can see that the radius of our circle is 1. Now we need to express the centre of our circle in polar co-ordinates: this shouldn't prove too difficult.

   The Cartesian point $(2, 0)$ is clearly 2 away from the origin, and as it lies on the positive x axis we can immediately deduce that $\theta = 0$. Hence our (r_0, θ_0) is simply $(2, 0)$.

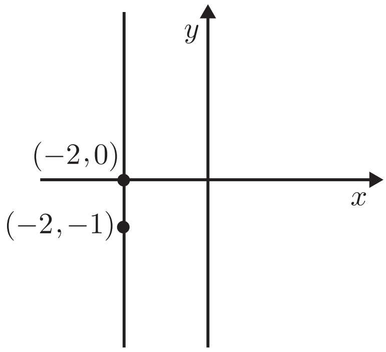
   Substituting everything into our formula yields:

$$\begin{aligned} r^2 - 2r \cdot 2\cos(\theta - 0) + 2^2 &= 1^2 \\ r^2 - 4r\cos\theta + 4 &= 1 \\ r^2 - 4r\cos\theta + 3 &= 0 \end{aligned}$$

7. Here we'll be using $r = \frac{r_0}{\cos(\theta - \theta_0)}$ where our line is perpendicular to the line $\theta = \theta_0$, and our line intersects the perpendicular line at (r_0, θ_0).

   Of course, we can choose any line perpendicular to our own, but taking a moment to choose a convenient one will save a lot of work in the long run.

   Here's a diagram of what our Cartesian line looks like:



A good choice of perpendicular line would be the x axis: that way, working out (r_0, θ_0) is very easy: our perpendicular line is $\theta_0 = \pi$, and our value of r_0 is simply 2, because the two lines intersect at the Cartesian point $(-2, 0)$, which is $(2, \pi)$ in polar co-ordinates.

Hence our line is the line:

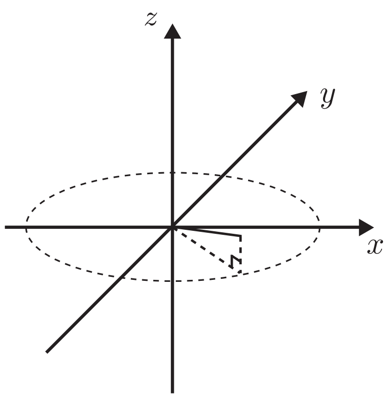
$$\begin{aligned} r &= \frac{2}{\cos(\theta - \pi)} \\ &= 2\sec(\theta - \pi) \end{aligned}$$

8. This is just a job for the memory cells! The answer is:

$$r = a + b\theta$$

where a varies the orientation and b varies the "tightness" of the spiral.

9. The most important thing here is to have a good mental picture of the point that we're dealing with. Here's the diagram:



The first job is to express the projection of the point onto the x, y plane in standard polar co-ordinates as this will give us our value of θ; given our 3 dimensial co-ordinates $(2, -2, 2)$, we find the projection simply by considering *only* the x and y co-ordinates. This gives us the point $(2, -2)$ in Cartesian co-ordinates, which we must convert into polar co-ordinates:

$$\begin{aligned} r_0^2 &= 2^2 + (-2)^2 \\ &= 4 + 4 \\ &= 8 \\ \theta_0 &= \frac{7\pi}{4} \end{aligned}$$

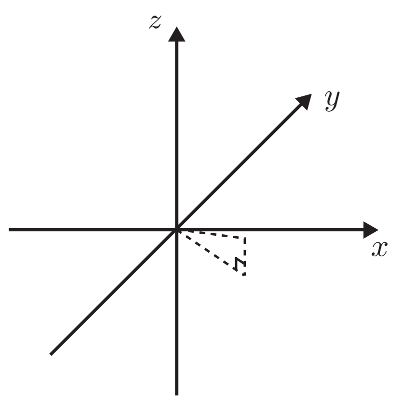
(This last statement can be easily seen from the diagram.) So, for the projection of the point, our values of r_0 and θ_0 are $r_0 = 2\sqrt{2}$ and $\theta_0 = \frac{7\pi}{4}$.

Now we use Pythagoras and some trigonometry to finish the job. r is the distance of our point from the origin, and we already know that the distance of the projection from the origin is $2\sqrt{2}$. Using Pythagoras once again, we have:

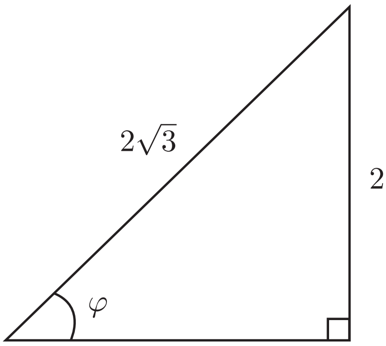
$$\begin{aligned} r^2 &= (2\sqrt{2})^2 + 2^2 \\ &= 8 + 4 \\ r &= \sqrt{12} \\ &= 2\sqrt{3} \end{aligned}$$

So we now have $r = 2\sqrt{3}$ and we already know θ, because θ is always exactly the same as θ_0, which we calculated earlier to be $\frac{7\pi}{4}$.

ϕ comes from using trigonometry on this triangle:



We know that the length of the hypotenuse of this triangle is our value of r, which is $2\sqrt{3}$. We also know that the length of the vertical component is 2, from inspecting the Cartesian co-ordinates given to us in the question. Here's the trigonometry to finish the question:



$$\begin{aligned}\sin\phi &= \frac{2}{2\sqrt{3}}\\ &= \frac{1}{\sqrt{3}}\\ \phi &= 0.6154794...\end{aligned}$$

$$\begin{aligned}\theta &= \frac{\pi}{2} - \phi\\ &= \frac{\pi}{2} - 0.6154794...\\ &= 0.955316618...\end{aligned}$$

So, we have $(2\sqrt{3}, 0.955, \frac{7\pi}{4})$ to 3d.p.

10. We're working in cylindrical polar co-ordinates here, and the helicopter maintains a constant height, so we need to be considering the projection of the helicopter's path onto a 2 dimensional plane first.

    If we consider the projection of the helicopter's path onto the ground, we'll see it actually just traces out a circle, centred at the origin and with radius 20. We know how to describe such a circle in polar co-ordinates: $r = 20$.

    Next, we use the fact that the helicopter's height is constant to see that our third parameter $t = 50$.

    Because θ will vary as the helicopter flies our final answer is $r = 20$, $t = 50$.

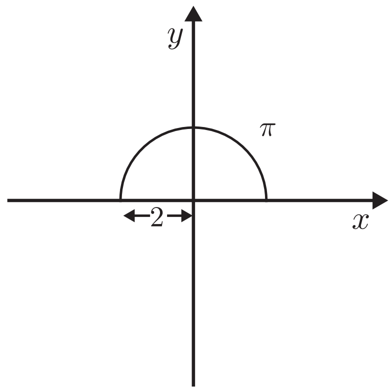
3.1 A Different Slant

1. We are told that our point is 2 units away from the origin, so $r = 2$. The positive x axis has $\theta = 0$, so our point is the point $(2, 0)$.

2. $r = 4$. The negative y axis is an angle of $\frac{3\pi}{2}$ away from the positive x axis, when measured anticlockwise.

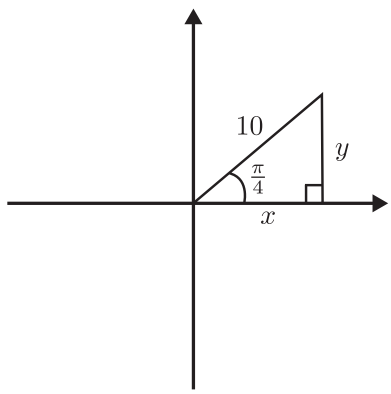
   Our point is therefore $(4, \frac{3\pi}{2})$.

3. We are told that our point is 2 units away from the origin, and at an angle of π from the positive x axis. Rather than getting bogged down in trigonometry, it is straightforward to visualise the point, like this:



   We can then use our knowledge of Cartesian co-ordinates to instantly recognise this as the point $(-2, 0)$.

4. Let's start with a diagram:



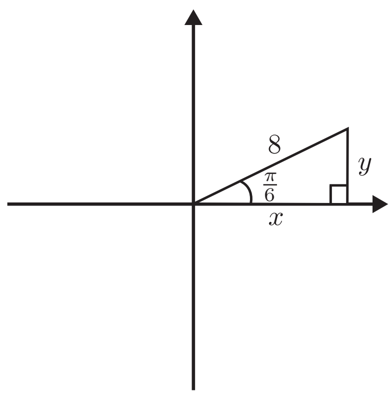
Now it's just trigonometry:

$$\begin{aligned} x &= 10\cos\frac{\pi}{4} \\ &= \frac{10}{\sqrt{2}} \\ &= 5\sqrt{2} \end{aligned}$$

$$\begin{aligned} y &= 10\sin\frac{\pi}{4} \\ &= \frac{10}{\sqrt{2}} \\ &= 5\sqrt{2} \end{aligned}$$

So our point is $(5\sqrt{2}, 5\sqrt{2})$.

5. Here's the diagram:

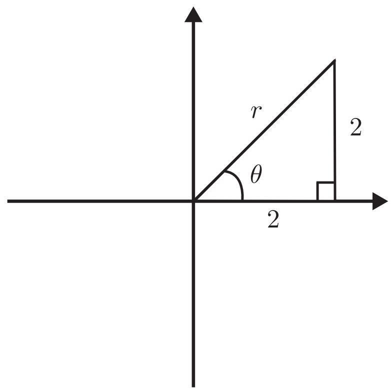


$$\begin{aligned} x &= 8\cos\frac{\pi}{6} \\ &= 4\sqrt{3} \end{aligned}$$

$$\begin{aligned} y &= 8\sin\frac{\pi}{6} \\ &= 4 \end{aligned}$$

So our point is $(4\sqrt{3}, 4)$.

6. Let's start with a diagram:



Pythagoras will give us r :

$$\begin{aligned} r^2 &= 2^2 + 2^2 \\ &= 4 + 4 \\ &= 8 \\ r &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

Trigonometry will give us θ:

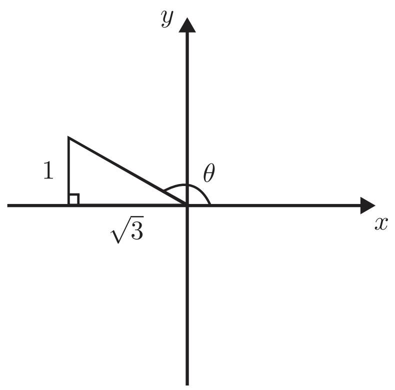
$$\begin{aligned} \tan\theta &= \frac{2}{2} \\ &= 1 \\ \theta &= \frac{\pi}{4} \end{aligned}$$

So, in polar co-ordinates the point is $(2\sqrt{2}, \frac{\pi}{4})$.

7. Drawing a diagram and using Pythagoras and trigonometry will certainly yield the right answer, but here it is quicker to note that the point lies directly on the positive y axis, so $\theta = \frac{\pi}{2}$, and the distance is simply 3, because the straight line from the origin to the point lies directly on the positive y axis.

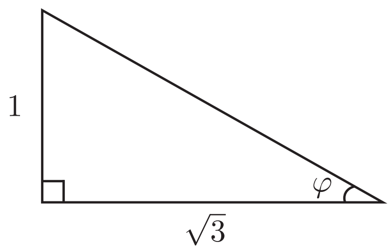
   So, in polar co-ordinates, the point is $(3, \frac{\pi}{2})$. (If you didn't follow the logic above, draw out the diagram and try the question using Pythagoras and trigonometry - you'll see how the shortcut works while you're working it out the long way!)

8. Here's the diagram:



$$
\begin{aligned}
r^2 &= (\sqrt{3})^2 + 1^2 \\
&= 3 + 1 \\
&= 4 \\
r &= 2
\end{aligned}
$$

To find θ, consider the triangle:



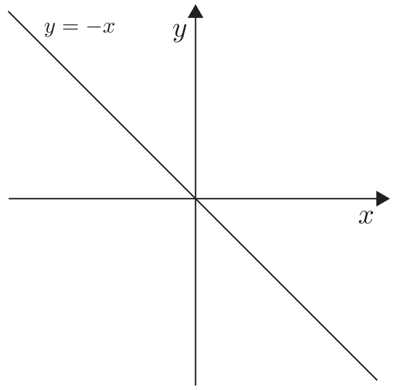
We can then see that $\theta = \pi - \phi$.

$$
\begin{aligned}
\tan\phi &= \frac{1}{\sqrt{3}} \\
\phi &= \frac{\pi}{6}
\end{aligned}
$$

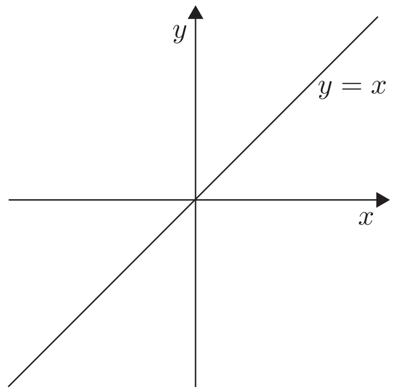
So, $\theta = \frac{5\pi}{6}$, and our point is $(2, \frac{5\pi}{6})$.

3.2 Lines and Circles

1. This line lies directly on the positive y axis. This means that it is an angle of $\frac{\pi}{2}$ away from the positive x axis, and so our line is the half line $\theta = \frac{\pi}{2}$.
2. Let's first examine a diagram:



   The half line that we want lies in the second quadrant, and so it is the half line which starts at the origin and runs diagonally up and to the left. This makes an angle of $\frac{3\pi}{4}$ when measured anticlockwise from the positive axis. Hence our required half line is $\theta = \frac{3\pi}{4}$.
3. Here's the diagram:



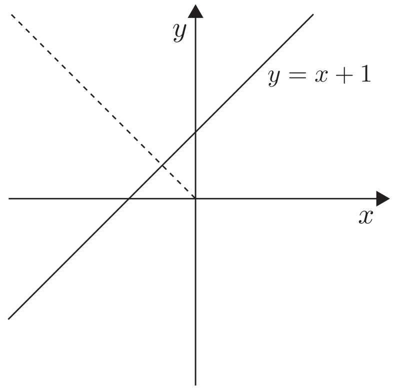
   We only want the half line in the third quadrant, which is $\theta = \frac{5\pi}{4}$.
4. When we're dealing with circles centred at the origin, remember that we only need to specify the radius, *and we're done*! Hence the answer here is $r = 4$.

5. We're going to have to call on:

$$r = \frac{r_0}{\cos(\theta - \theta_0)}$$

Where our line is perpendicular to the line $\theta = \theta_0$, and our line intersects the perpendicular line at (r_0, θ_0).

Let's examine the diagram of $y = x + 1$.



Dotted in is the line $y = x$, which our line is perpendicular to. The part of this line which intersects our line is the half line in the second quadrant, which is expressed in polar co-ordinates as $\theta_0 = \frac{3\pi}{4}$.

We need to express in polar co-ordinates the point at which these two lines intersect. We know the angle is $\theta_0 = \frac{3\pi}{4}$.

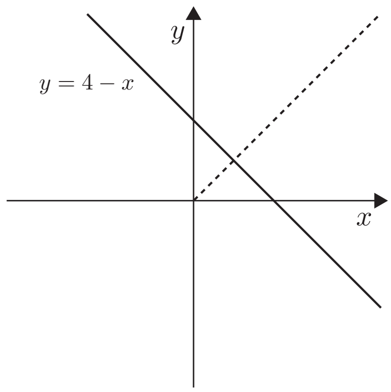
Clearly, in Cartesian co-ordinates the point of intersection is $(-\frac{1}{2}, \frac{1}{2})$. To find the distance of this point from the origin, we use Pythagoras:

$$\begin{aligned} r_0 &= \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2} \\ r_0 &= \frac{1}{\sqrt{2}} \end{aligned}$$

So substituting into our original formula yields:

$$\begin{aligned} r &= \frac{\frac{1}{\sqrt{2}}}{\cos(\theta - \frac{3\pi}{4})} \\ &= \frac{1}{\sqrt{2}} \sec\left(\theta - \frac{3\pi}{4}\right) \end{aligned}$$

6. Here's the diagram of our line and the perpendicular line:



The perpendicular half line that we require is $\theta_0 = \frac{\pi}{4}$

In Cartesian co-ordinates, the intersection occurs at $(2, 2)$. In polar co-ordinates, this is the point$(2\sqrt{2}, \frac{\pi}{4})$.

So, by substitution into the formula, we have:

$$\begin{aligned} r &= \frac{2\sqrt{2}}{\cos\left(\theta - \frac{\pi}{4}\right)} \\ &= 2\sqrt{2} \sec\left(\theta - \frac{\pi}{4}\right) \end{aligned}$$

7. We'll be needing the formula

$$r^2 - 2rr_0 \cos(\theta - \theta_0) + r_0^2 = a^2$$

where the radius of the circle is a, and its centre lies at (r_0, θ_0).

We're told in the question that $a = 1$, so the only thing we have to do is to express the Cartesian point $(1, 0)$ in polar co-ordinates. This is $(1, 0)$.

So, making the substitution:

$$\begin{aligned} r^2 - 2r \cdot 1\cos(\theta - 0) + 1^2 &= 1^2 \\ r^2 - 2r\cos\theta &= 0 \\ r^2 &= 2r\cos\theta \\ r &= 2\cos\theta \end{aligned}$$

For polar co-ordinates we require that $r \geq 0$ as it is a distance. Hence we include the constraint that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

8. We're told that the radius of our circle is 7. Its centre is at the Cartesian point $(1, 1)$, which is $(\sqrt{2}, \frac{\pi}{4})$ in polar co-ordinates. Substituting into the formula:

$$\begin{aligned} r^2 - 2r \cdot \sqrt{2}\cos(\theta - 0) + (\sqrt{2})^2 &= 7^2 \\ r^2 - 2\sqrt{2}r\cos\theta - 47 &= 0 \end{aligned}$$

3.3 Moving on Up

1. Firstly, let's consider the projection of the point onto the 2 dimensional plane. We do this by 'ignoring' the 3rd component of $(1, 1, 1)$, leaving us with $(1, 1)$ as the point of projection, in Cartesian co-ordinates. Converting this to polar co-ordinates yields $(\sqrt{2}, \frac{\pi}{4})$.

   The third componant t, of cyclindrical polar co-ordinates, is simply the height of our point above the plane. Finding that is easy here; it's just 1, the z component of our original Cartesian co-ordinates.

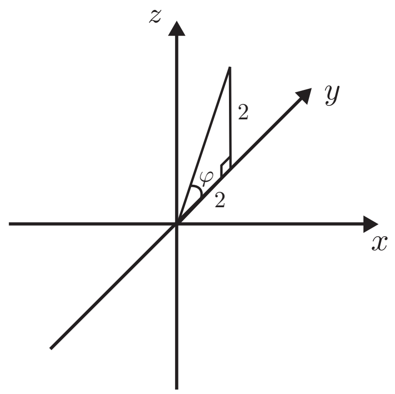
   Hence our answer is $(\sqrt{2}, \frac{\pi}{4}, 1)$.

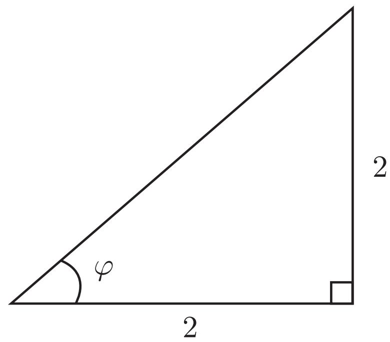
2. We need to find 3 things: r, θ and ϕ. r is found by Pythagoras. Because one of our components is 0, we can simply use Pythagoras in 2 dimensions, as usual:

$$\begin{aligned} r^2 &= 2^2 + 2^2 \\ &= 4 + 4 \\ &= 8 \\ r &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

To find θ, consider the projection of the point onto the x-y plane. This is ths Cartesian point $(0, 2)$, and so $\theta = \frac{\pi}{2}$.

Finally, to find ϕ, we use trigonometry on the vertical triangle, like this:

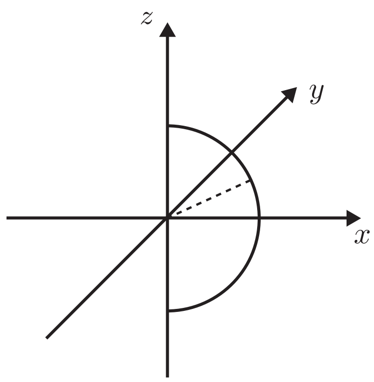




$$
\begin{aligned}
\tan\phi &= \frac{2}{2} \\
&= 1 \\
\phi &= \frac{\pi}{4}
\end{aligned}
$$

So our point is the point $(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2})$

3. The difficulty here is in visualising what's going on. Here's a diagram:



The dotted line shows the projection of the semi circle onto the line $y = x$, in the first quadrant as described in the question.

We need to first work out which of our variables are fixed, and which aren't. We're drawing a unit semi circle, so r is certainly fixed at 1.

Now, let's think about the projection of our semi ycircle onto the x, y plane. It always lies on the half line $\theta = \frac{\pi}{4}$, so we fix that too.

Finally, we let ϕ vary, so that we indeed, trace out a semicircle and don't just describe a point!

So our answer is $r = 1$, $\theta = \frac{\pi}{4}$ (remember, we only get a *semi* circle like this because, by definition ϕ only varies 0 to π, *not* 0 to 2π).

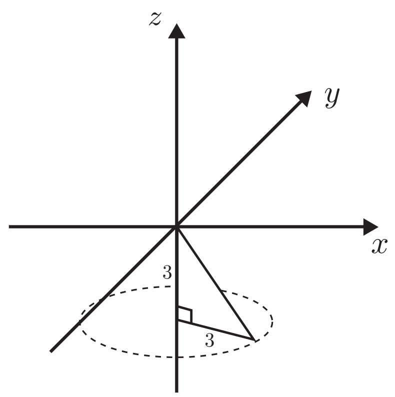
4. We need to work out which of our parameters are fixed, and which we will allow to vary.

   We're told that our horizontal circle has radius 4, and that its projection onto the x-y plane is centred at the origin. We certainly know how to describe a circle for this : fix r =4 and let θ vary.

   We're told that the circle has a constant height of 3, so we fix $t = 3$.

   That's everything accounted for: $r = 4$, $t = 3$.

5. A pitfall to avoid here is to think that $r = 3$. While the radius of the circle is indeed 3, this diagram illustrates how the distance from any point on the circle we require to the origin is actually *more* than 3:

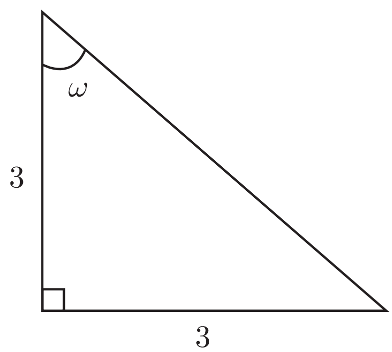


We use Pythagoras to find the correct value of r:

$$\begin{aligned} r^2 &= 3^2 + 3^2 \\ &= 9 + 9 \\ &= 18 \\ r &= 3\sqrt{2} \end{aligned}$$

We're going to need θ to vary, because we're dealing with a horizontal circle, but we do need to fix ϕ.

Using trigonometry on the same triangle as before



$$\begin{aligned} \tan\omega &= \frac{3}{3} \\ &= 1 \\ \omega &= \frac{\pi}{4} \end{aligned}$$

Noting that $\phi = \pi - \omega$ shows us that $\phi = \frac{3\pi}{4}$ (because θ is measured *downwards* from the positive z axis). Hence $r = 3\sqrt{2}$, $\theta = \frac{3\pi}{4}$.

6. This is just memory:
$$r^2 = a\cos 2\theta$$
7. This is just memory:
$$r^2 = a\sin 2\theta$$
8. This is just memory:
$$r = a + b\cos\theta \text{ or } r = a + b\sin\theta$$
9. This is just memory:
$$r = a + b\cos\theta \text{ or } r = a + b\sin\theta, \text{ with } a = b$$
10. This is just memory: It is an Archimedian Spiral.

4
Complex Numbers

Test Yourself

1. $\mathbb{N}$ is the set of "Natural Numbers". These are $\{1, 2, 3...\}$.

   $\mathbb{Z}$ is the set of "Integers". These are $\{... - 2, -1, 0, 1, 2, ...\}$.

   $\mathbb{Q}$ is the set of "Rational Numbers". These are any number that can be expressed as a fraction with both the numerator and denominator as integers.

   $\mathbb{R}$ is the set of "Real Numbers". Can include irrational numbers like π, e and stuff like 0.4623567....

   $\mathbb{C}$ is the set of "Complex Numbers". If you read the chapter, you might learn a thing or two about these babies.

2. $x^2 = -81$. No real number, when squared, will yield a negative. For this reason, we know that there must be an i in our solution. If we let x be the general complex number $(a + bi)$, we get:

$$\begin{aligned} (a + bi)^2 &= -81 \\ a^2 + 2abi + b^2 i^2 &= -81 \\ a^2 + 2abi - b^2 &= -81 \end{aligned}$$

   But there is no co-efficient in ıon the right hand side of the expression, so $2abi = 0$.

But now we can use the fact that both a and b are real numbers. If $2abi = 0$ then $ab = 0$, and so either $a = 0$ or $b = 0$.

Looking again at $a^2 + 2abi - b^2 = -81$ and setting $2abi = 0$, we see:

$$a^2 - b^2 = -81$$

Now, *again* using the fact that both a and b are real, we know that $a^2 \geq 0$ and $-b^2 \leq 0$.

Because the right hand side of our expression is negative, and because we know that either $a = 0$ or $b = 0$, we are forced to conclude that $a = 0$ and $b = \pm 9$. So $x = 0 \pm 9i = \pm 9i$.

3.

$$\begin{aligned} x^2 - 6x &= -10 \\ x^2 - 6x + 10 &= 0 \end{aligned}$$

Using the quadratic formula:

$$\begin{aligned} x &= \frac{6 \pm \sqrt{36 - 40}}{2} \\ &= \frac{6 \pm \sqrt{4}}{2} \\ &= \frac{6 \pm 2\sqrt{-1}}{2} \\ &= 3 \pm \sqrt{-1} \text{ But } \sqrt{-1} \text{ is } i\text{, so} \\ x &= 3 \pm i \end{aligned}$$

4.

$$\begin{aligned} (7 + 3i) - (6 - 2i) &= (7 - 6) + (3i + 2i) \\ &= 1 + 5i \end{aligned}$$

5.

$$\begin{aligned} (3 + 2i)(4 - 5i) &= 12 - 15i + 8i - 10i^2 \\ &= 12 - 7i + 10 \\ &= 22 - 7i \end{aligned}$$

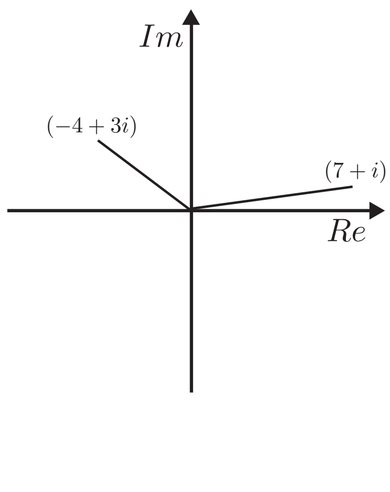
6.
$$\frac{6+2i}{3+i}$$

Remember that our strategy here is to eliminate i from the denominator, by multiplying top and bottom of our fraction by the complex conjugate of the denominator.

$$\begin{aligned}\frac{(6+2i)(3-5i)}{(3+5i)(3-5i)} &= \frac{18-30i+6i-10i^2}{9-15i+15i-25i^2}\\ &= \frac{28-24i}{34}\\ &= \frac{14-12i}{17}\\ &= \frac{14}{17}-\frac{12i}{17}\end{aligned}$$

7.
$$\begin{aligned}\frac{4+i}{-1-3i} &= \frac{(4+i)(-1+3i)}{(-1-3i)(-1+3i)}\\ &= \frac{-4+12i-i+3i^2}{1-3i+3i-9i^2}\\ &= \frac{-7+11i}{10}\\ &= \frac{-7}{10}+\frac{11}{10}i\end{aligned}$$

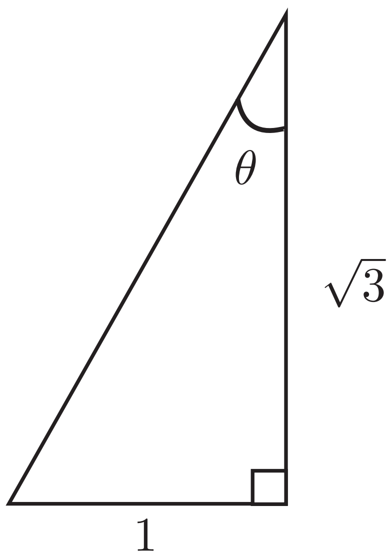
8. An Argand diagram uses the real part of a complex number as its x value, and the imaginary part as its y value. This means that the points in the question look like this:



9. Firstly, we use Pythagoras to find r:

$$\begin{aligned} r^2 &= 1^2 + \left(\sqrt{3}\right)^2 \\ &= 1 + 3 \\ &= 4 \\ r &= 2 \end{aligned}$$

Then we use trigonometry to find θ:

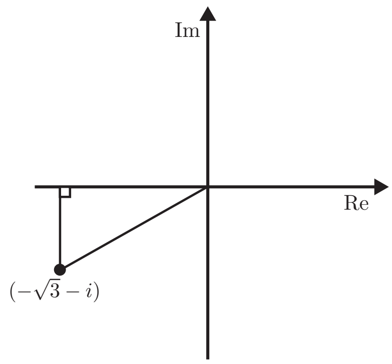


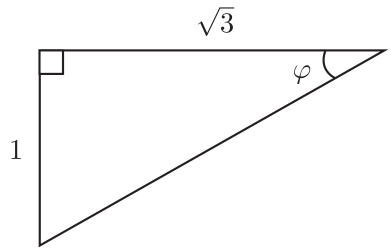
$$\begin{aligned} \tan\theta &= \frac{\sqrt{3}}{1} \\ &= \sqrt{3} \\ \theta &= \frac{\pi}{3} \end{aligned}$$

So we have $2e^{\frac{\pi}{3}i}$.

10. As the question suggests, it's easiest to proceed by first converting into the $re^{i\theta}$ form:

$$\begin{aligned} r^2 &= (-\sqrt{3})^2 + (-1)^2 \\ &= 3 + 1 \\ &= 4 \\ r &= 2 \end{aligned}$$





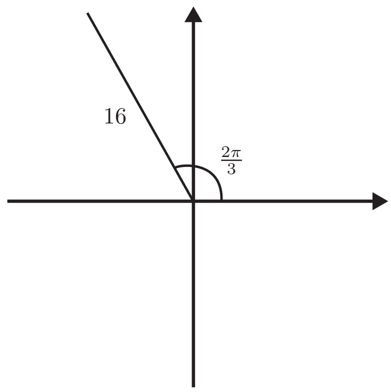
$$\tan\phi = \frac{1}{\sqrt{3}}$$
$$\phi = \frac{\pi}{6}$$

$$\begin{aligned}\theta &= \pi + \phi \\ &= \frac{7\pi}{6}\end{aligned}$$

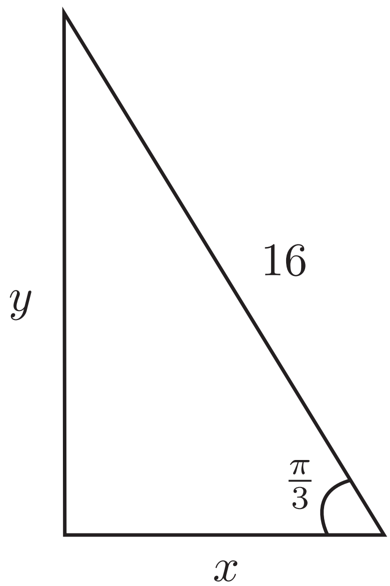
So we are trying to find $\left(2e^{\frac{7\pi}{6}i}\right)^4$. This breaks into:

$$\begin{aligned}2^4 \cdot (e^{\frac{7\pi}{6}i})^4 &= 16e^{\frac{28\pi}{6}i} \\ &= 16e^{\frac{14\pi}{3}i} \\ &= 16e^{\frac{2\pi}{3}i} \cdot e^{\frac{6\pi}{3}i} \cdot e^{\frac{6\pi}{3}i} \\ &= 16e^{\frac{2\pi}{3}i}\end{aligned}$$

This last step occurs because any positive whole number multiple of $e^{2\pi i}$ is 1. To convert back to the $a + bi$ form, it helps to draw another Argand diagram:



Finally, some trigonometry should finish the job:



$$\begin{aligned} x &= 16\cos\frac{\pi}{3} \\ &= 8 \\ y &= 16\sin\frac{\pi}{3} \\ &= 16\frac{\sqrt{3}}{2} \\ &= 8\sqrt{3} \end{aligned}$$

So in the $a+bi$ form, our answer is $-8+8\sqrt{3}i$.

4.1 Numbers

1. If we're working with natural numbers and we allow subtraction, we will definitely end up in the integers (but not necessarily in $\mathbb{N}$: consider $4-6=-2$. 4 and 6 were in $\mathbb{N}$, but -2 definitely isn't!)

   If we're working with integers and we allow division, we definitely end up in the rationals (but not necessarily in $\mathbb{Z}$)

   If we're working with rationals and we take the limit of a sequence (what we nickname "Analysis"), we will definitely end up in the reals (but not necessarily in $\mathbb{Q}$).

   If we're working with reals and we also define $\sqrt{-1}=i$, we arrive at the complex numbers.

$$\begin{aligned}\text{Hence } \alpha &= \text{“Subtraction”}\\ \beta &= \text{“Division”}\\ \gamma &= \text{“Analysis”}\\ \delta &= \text{“}\sqrt{-1}\text{”}\end{aligned}$$

2. $i^2=-1$. This is simply by definition of i.
3. A complex number is of the form $a+bi$, where a and b are real numbers and $i=\sqrt{-1}$.
4. $x^2=-100$. x is definitely not a real number, because any real number, when squared, would yield a positive result.

   If we notice that $(\pm10)^2=100$, and $i^2=-1$, then we see:

$$\begin{aligned}(i^2)(\pm10)^2 &= -100\\ (\pm10i)^2 &= -100\\ \text{So, } x &= \pm10i\end{aligned}$$

5. $x^2+64=0$, so we have that:

$$\begin{aligned}x^2 &= -64\\ &= (i^2)(\pm8)^2\\ x &= \pm8i\end{aligned}$$

6. $x^2 - 2x + 2 = 0$, so we must use the quadratic formula:

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4-8}}{2} \\ &= \frac{2 \pm \sqrt{-4}}{2} \\ &= \frac{2 \pm 2\sqrt{-1}}{2} \\ &= 1 \pm \sqrt{-1} \\ &= 1 \pm i \end{aligned}$$

7. $x^2 + 4x + 20 = 0$

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{16-80}}{2} \\ &= \frac{-4 \pm \sqrt{-64}}{2} \\ &= \frac{-4 \pm 8\sqrt{-1}}{2} \\ &= -2 \pm 4i \end{aligned}$$

8. $8x^2 - 4x + 1 = 0$

$$\begin{aligned} x &= \frac{4 \pm \sqrt{16-32}}{16} \\ &= \frac{4 \pm \sqrt{-16}}{16} \\ &= \frac{1}{4} \pm \frac{1}{4}i \end{aligned}$$

9. $x^2 - 2x + 3 = 0$

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4-12}}{2} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm 2\sqrt{2}i}{2} \\ &= 1 \pm \sqrt{2}i \end{aligned}$$

10. $3x^2 - 4x = -3$ so, $3x^2 - 4x + 3 = 0$

$$\begin{aligned} x &= \frac{4 \pm \sqrt{16 - 36}}{6} \\ &= \frac{4 \pm \sqrt{-20}}{6} \\ &= \frac{4 \pm 2\sqrt{5}i}{6} \\ &= \frac{2}{3} \pm \frac{\sqrt{5}}{3}i \end{aligned}$$

4.2 Working with Complex Numbers

1. We equate the real and imaginary parts seperately:

$$(6 + 3i) + (4 + 5i) = 10 + 8i$$

2.

$$(7 - 3i) + (2i) = 7 - i$$

3.

$$\begin{aligned} (3 - 2i) - (3 - 2i) &= 0 + 0i \\ &= 0 \end{aligned}$$

4.

$$(4 + i) - (7 - 11i) = -3 + 12i$$

5.

$$(12 + 3i) + (3 - 12i) = 15 - 9i$$

6.

$$\begin{aligned} (6 + 3i)(6 + 3i) &= 36 + 18i + 18i + 9i^2 \\ &= 36 + 36i + 9i^2 \\ \text{But } i^2 = -1, \text{ so} & \\ &= 36 + 36i - 9 \\ &= 27 + 36i \end{aligned}$$

7.

$$\begin{aligned}(7-i)(3+2i) &= 21+14i-3i-2i^2 \\ &= 21+11i-2i^2 \\ &= 21+11i+2 \\ &= 23+11i\end{aligned}$$

8.

$$\begin{aligned}(18+3i)(1-i) &= 18-18i+3i-3i^2 \\ &= 18-15i+3 \\ &= 21-15i\end{aligned}$$

9.

$$\begin{aligned}(12-i)(3+2i) &= 36+24i-3i-2i^2 \\ &= 36+21i+2 \\ &= 38+21i\end{aligned}$$

10.

$$\begin{aligned}(6+2i)(3+i)(4-i) &= (18+6i+6i+2i^2)(4-i) \\ &= (16+12i)(4-i) \\ &= (64-16i+48i-12i^2) \\ &= 76+32i\end{aligned}$$

11. We can't do anything here other than "break" the real and imaginary parts up:
$$\frac{6+2i}{2} = 3+i$$

12. Here, we need to get rid of the imaginary part of the denominator, which we achieve by multiplying top and bottom of the fraction by the complex conjugate of the denominator.

The complex conjuate of $1+i$ is $1-i$, so here goes:

$$\begin{aligned}\frac{(3+2i)(1-i)}{(1+i)(1-i)} &= \frac{3-3i+2i-2i^2}{1-i+i-i^2} \\ &= \frac{3-i+2}{1+1} \\ &= \frac{5-i}{2} \\ &= \frac{5}{2}-\frac{1}{2}i\end{aligned}$$

13. The complex conjugate of $2-2i$ is $2+2i$:

$$\begin{aligned}\frac{(3+5i)(2+2i)}{(2-2i)(2+2i)} &= \frac{6+6i+10i+10i^2}{4+4i-4i-4i^2} \\ &= \frac{6+16i-10}{4+4} \\ &= \frac{-4+16i}{8} \\ &= -\frac{1}{2}+2i\end{aligned}$$

14.

$$\begin{aligned}\frac{(10-i)(7-3i)}{(7+3i)(7-3i)} &= \frac{70-30i-7i+3i^2}{49-21i+21i-9i^2} \\ &= \frac{67-37i}{58} \\ &= \frac{67}{58}-\frac{37}{58}i\end{aligned}$$

15.

$$\begin{aligned}\frac{(-3-i)(-2-i)}{(-2+i)(-2-i)} &= \frac{6+3i+2i+i^2}{4+2i-2i-i^2} \\ &= \frac{5+5i}{5} \\ &= 1+i\end{aligned}$$

16. First, let's multiply the fractions together to get a single fraction:

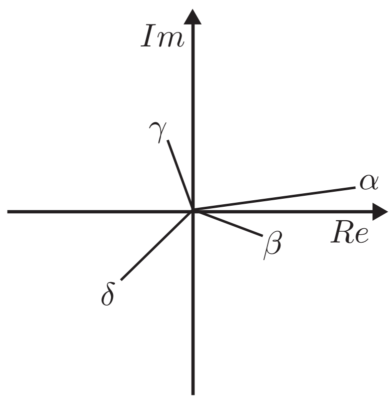
$$\begin{aligned}\frac{(-2+i)(4+i)}{(2-i)(-1+i)} &= \frac{-8-2i+4i+i^2}{-2+2i+i-i^2} \\ &= \frac{-9+2i}{-1+3i}\end{aligned}$$

Now, we proceed as before:

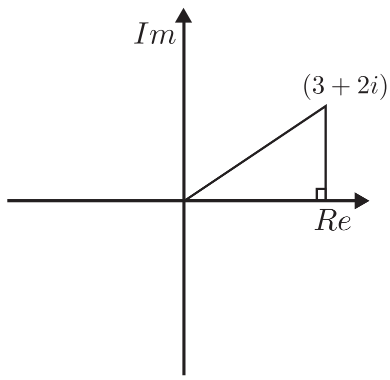
$$\begin{aligned}\frac{(-9+2i)(-1-3i)}{(-1+3i)(-1-3i)} &= \frac{9+27i-2i-6i^2}{1+3i-3i-9i^2}\\ &= \frac{15+25i}{10}\\ &= \frac{3}{2}+\frac{5}{2}i\end{aligned}$$

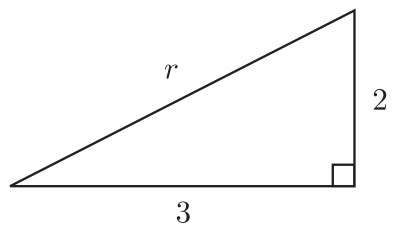
4.3 Tips and Tricks

1. An Argand diagram uses the real part of a complex number as its x value and the imaginary part as the y value. This means that the points in question look like this:



2. This is a question for Pythagoras:



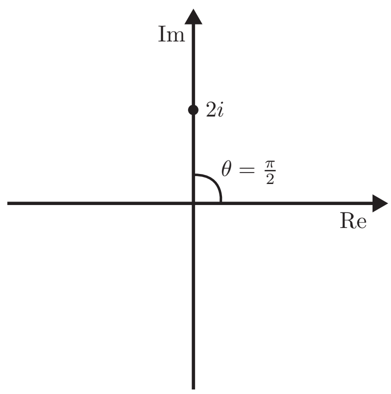


$$
\begin{aligned}
r^2 &= 3^2 + 2^2 \\
&= 9 + 4 \\
&= 13 \\
r &= \sqrt{13}
\end{aligned}
$$

3.

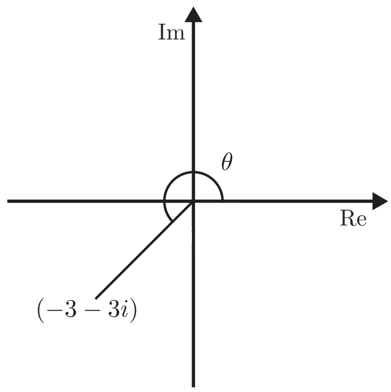
$$
\begin{aligned}
r^2 &= (-2)^2 + (-1)^2 \\
&= 4 + 1 \\
&= 5 \\
r &= \sqrt{5}
\end{aligned}
$$

4. Here's the argand diagram:

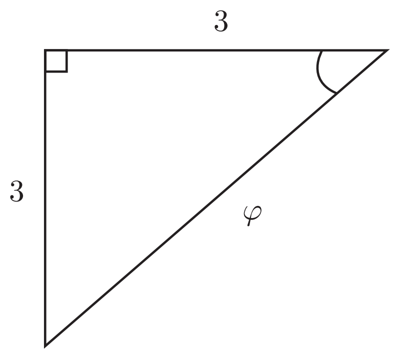


Hence $\theta = \frac{\pi}{2}$.

5. Here's the argand diagram:

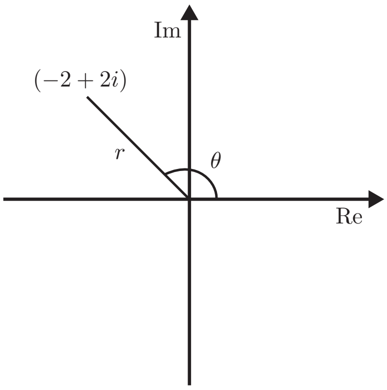


Remember that we measure θ anticlockwise from the positive x axis. Some quick trigonometry:



$$\begin{aligned}
\tan\phi &= \frac{3}{3} \\
&= 1 \\
\phi &= \frac{\pi}{4} \\
\text{So, } \theta &= \pi + \phi \\
&= \pi + \frac{\pi}{4} \\
&= \frac{5\pi}{4}
\end{aligned}$$

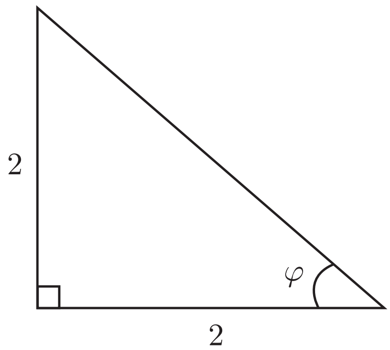
6. Let's first examine the Argand diagram:



We use Pythagoras to find r:

$$\begin{aligned} r^2 &= (-2)^2 + 2^2 \\ &= 4 + 4 \\ &= 8 \\ r &= 2\sqrt{2} \end{aligned}$$

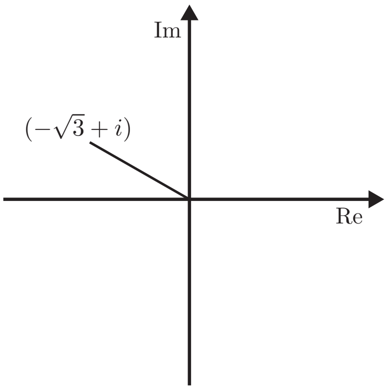
And trigonometry to find θ:



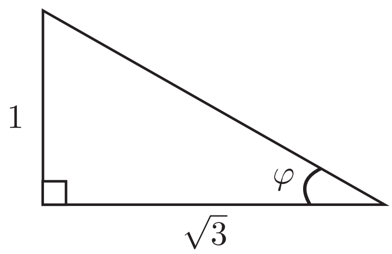
$$\begin{aligned}
\tan\phi &= \frac{2}{2}\\
&= 1\\
\phi &= \frac{\pi}{4}\\
\theta &= \pi - \phi\\
&= \pi - \frac{\pi}{4}\\
&= \frac{3\pi}{4}
\end{aligned}$$

So $-2+2$ is $2\sqrt{2}e^{\frac{3\pi}{4}i}$ in the $re^{i\theta}$

7. Let's first examine the Argand diagram:



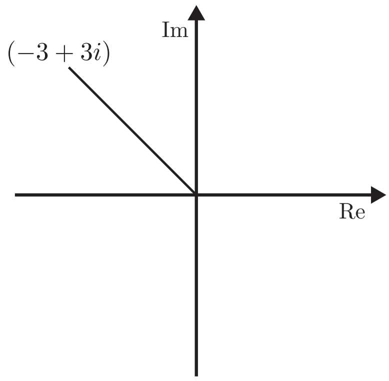
$$\begin{aligned}
r^2 &= (-\sqrt{3})^2 + 1^2\\
&= 3+1\\
&= 4\\
r &= 2
\end{aligned}$$



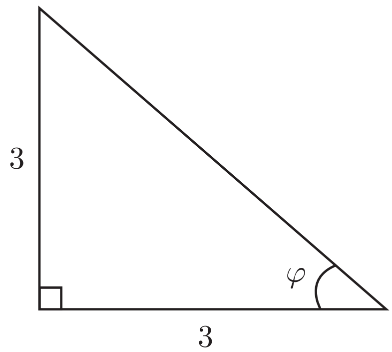
$$\begin{aligned}
\tan\phi &= \frac{1}{\sqrt{3}}\\
\phi &= \frac{\pi}{6}\\
\theta &= \pi - \phi\\
&= \pi - \frac{\pi}{6}\\
&= \frac{5\pi}{6}
\end{aligned}$$

So we have $2e^{\frac{5\pi}{6}i}$

8. Let's first examine the Argand diagram:



$$\begin{aligned}
r^2 &= (-3)^2 + 3^2\\
&= 9 + 9\\
&= 18\\
r &= 3\sqrt{2}
\end{aligned}$$



$$
\begin{aligned}
\tan\phi &= \frac{3}{3} \\
&= 1 \\
\phi &= \frac{\pi}{4} \\
\theta &= \pi - \phi \\
&= \pi - \frac{\pi}{4} \\
&= \frac{3\pi}{4}
\end{aligned}
$$

So we need to find:

$$
\begin{aligned}
(3\sqrt{2}e^{\frac{3\pi}{4}i})^4 &= (3\sqrt{2})^4 e^{4\cdot\frac{3\pi}{4}i} \\
&= 324e^{3\pi i} \\
&= -324
\end{aligned}
$$

(To see this convert $324e^{3\pi i}$ back to the $a + bi$ form).

5

Vectors

Test Yourself

1. The norm of a vector is found by squaring each component, adding them together, and finally square rooting the result.

$$\begin{aligned}\sqrt{3^2+8^2+3^2+3^2+3^2} &= \sqrt{9+64+9+9+9}\\ &= \sqrt{100}\\ &= 10\end{aligned}$$

2. A unit vector is a vector whoose norm is equal to 1, so we use this fact to solve the problem.

$$\begin{aligned}||\mathbf{u}|| &= \sqrt{\frac{1}{6}^2+\frac{1}{4}^2+x^2+\frac{1}{3}^2}\\ &= \sqrt{\frac{1}{36}+\frac{1}{16}+x^2+\frac{1}{9}}\\ &= \sqrt{\frac{4}{144}+\frac{9}{144}+x^2+\frac{16}{144}}\\ &= \sqrt{\frac{29}{144}+x^2} = 1\end{aligned}$$

(This last bit comes in because we are using a unit vector).

$$\begin{aligned} \frac{29}{144} + x^2 &= 1 \\ x^2 &= 1 - \frac{29}{144} \\ &= \frac{115}{144} \end{aligned}$$

So, $x = \pm\frac{\sqrt{115}}{12}$

3. Here, we work with each of the components seperately.

$$\begin{aligned} 3\mathbf{m} - 2\mathbf{n} &= 3(3,7,5) - 2(8,1,3) \\ &= (9,21,15) - (16,2,6) \\ &= (-7,19,9) \end{aligned}$$

4.

$$\begin{aligned} ||\mathbf{y}|| &= \sqrt{3^2 + 7^2 + 1^2 + 3^2} \\ &= \sqrt{9 + 49 + 1 + 9} \\ &= \sqrt{68} \\ &= 2\sqrt{17} \end{aligned}$$

Then we have:

$$\begin{aligned} 2\sqrt{17}\mathbf{y} &= 2\sqrt{17}(3,7,1,3) \\ &= (6\sqrt{17}, 14\sqrt{17}, 2\sqrt{17}, 6\sqrt{17}) \end{aligned}$$

5. This is just algebra: don't be put off by the vector notation!

$$\begin{aligned} 2\mathbf{u} + 3\mathbf{v} &= 2(8\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}) + 3(9\mathbf{i} + 5\mathbf{j} + \mathbf{k}) \\ &= 16\mathbf{i} + 4\mathbf{j} + 12\mathbf{k} + 27\mathbf{i} + 15\mathbf{j} + 3\mathbf{k} \\ &= 43\mathbf{i} + 19\mathbf{j} + 15\mathbf{k} \end{aligned}$$

6. To take the dot product of 2 vectors, we multiply the vectors component-wise, and sum the results.

$$\begin{aligned} \begin{pmatrix} 1 \\ 6 \\ 3 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 6 \\ 1 \end{pmatrix} &= (1 \times 4) + (6 \times 2) + (3 \times 6) + (8 \times 1) \\ &= 4 + 12 + 18 + 8 \\ &= 42 \end{aligned}$$

7. The brackets are crucial in telling us what to do first: the dot product.

$$\begin{aligned}\mathbf{b}\cdot\mathbf{c} &= (2\times 7)+(4\times 1)+(1\times 0)\\ &= 14+4+0\\ &= 18\end{aligned}$$

$$\begin{aligned}18\mathbf{a} &= 18(1,4,3)\\ &= (18,72,54)\end{aligned}$$

8. We're going to need the identity:

$$\mathbf{a}\cdot\mathbf{b} = ||\mathbf{a}||\cdot||\mathbf{b}||\cos\theta$$

We're looking for θ, so we need to equate $\mathbf{a}$, $\mathbf{b}$, $||\mathbf{a}||$ and $||\mathbf{b}||$.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (2\times 2)+(2\times 3)+(1\times 6)\\ &= 4+6+6\\ &= 16\\ ||\mathbf{a}|| &= \sqrt{2^2+2^2+1^2}\\ &= \sqrt{4+4+1}\\ &= \sqrt{9}\\ &= 3\end{aligned}$$

$$\begin{aligned}||\mathbf{b}|| &= \sqrt{2^2+3^2+6^2}\\ &= \sqrt{4+9+36}\\ &= \sqrt{49}\\ &= 7\end{aligned}$$

Now, substitutiong everything back into our identity:

$$\begin{aligned}16 &= 3\cdot 7\cos\theta\\ \frac{16}{21} &= \cos\theta\\ \theta &= \arccos\frac{16}{21}\end{aligned}$$

9. The formula for the cross product of vectors $(a,b,c)\times(d,e,f)$ is $(bf-ce,-(af-cd),ae-bd)$. Using this we get:

$$(15-12,-(10-4),6-3) = (3,-6,3)$$

10. As the brackets direct, we equate the cross product first:

$$\begin{aligned}(2,2,7)\times(3,7,8) &= (16-49,-(16-21),14-6)\\ &= (-33,5,8)\end{aligned}$$

Now we can equate the whole expression:

$$\begin{aligned}((2,2,7)\times(3,7,8))\cdot(5,6,7) &= (-33,5,8)\cdot(5,6,7)\\ &= (-33\times 5)+(5\times 6)+(8\times 7)\\ &= -165+30+56\\ &= -79\end{aligned}$$

5.1 Reinventing the Wheel

1. The norm of a vector is found by squaring each component, adding them together and finally squaring the result.

$$\begin{aligned}\sqrt{(-5)^2+6^2+(-2)^2+9^2+(-5)^2+5^2} &= \sqrt{25+36+4+81+25+25}\\ &= \sqrt{196}\\ &= 14\end{aligned}$$

2. A unit vector is a vector that has a norm equal to 1, so we use this to solve the problem:

$$\begin{aligned}||\mathbf{u}|| &= \sqrt{(\frac{2}{13})^2+(\frac{5}{13})^2+x^2+(\frac{2}{13})^2+(\frac{10}{13})^2}\\ &= \sqrt{\frac{4}{169}+\frac{25}{169}+x^2+\frac{4}{169}+\frac{100}{169}}\\ &= \sqrt{\frac{133}{169}+x^2}=1\end{aligned}$$

(This last step comes in because we have a unit vector).

$$\begin{aligned}\frac{133}{169}+x^2 &= 1\\ x^2 &= 1-\frac{133}{169}\\ &= \frac{36}{169}\\ \\ x &= \pm\frac{6}{13}\end{aligned}$$

3. Vector addition works componentwise:

$$\begin{aligned}(3,6,2,4)+(-2,8,3,-9) &= (3-2,6+8,2+3,4-9)\\ &= (1,14,5,-5)\end{aligned}$$

4.

$$\begin{aligned}(5,-2,-8,3,8)-(4,-3,5,-7,-5) &= (5-4,-2+3,-8-5,3+7,8+5)\\ &= (1,1,-13,10,13)\end{aligned}$$

5. Scalar-vector multiplication works componentwise:

$$\begin{aligned}5(4,7,3,8) &= (5\times 4,5\times 7,5\times 3,5\times 8)\\ &= (20,35,15,40)\end{aligned}$$

6.

$$\begin{aligned}3(2,5,6)-(8,2,9) &= (3\times 2,3\times 5,3\times 6)-(8,2,9)\\ &= (6,15,18)-(8,2,9)\\ &= (-2,13,9)\end{aligned}$$

7.

$$\begin{aligned}2(-5,8,4,2)+8(-8,4,6,-3) =&(2\times(-5),2\times 8,2\times 4,2\times 2)+\\ &(8\times(-8),8\times 4,8\times 6,8\times(-3))\\ =&(-10,16,8,4)+(-64,32,48,-24)\\ =&(-74,48,56,-20)\end{aligned}$$

8.

$$\begin{aligned}10(5,7,2)-3(-2,5,2)+(7,3,9) &= (50,70,20)-(-6,15,6)+(7,3,9)\\ &= (63,58,23)\end{aligned}$$

9.

$$\begin{aligned}||(6,2,-9)||(4,2,7) &= \sqrt{6^2+2^2+(-9)^2}\cdot(4,2,7)\\ &= \sqrt{121}(4,2,7)\\ &= (44,22,77)\end{aligned}$$

10.

$$\begin{aligned}||\mathbf{u}|| &= \sqrt{2^2 + 4^2 + (-4)^2 + 6^2} \\ &= \sqrt{72} \\ &= 6\sqrt{2}\end{aligned}$$

$$\begin{aligned}||\mathbf{v}|| &= \sqrt{2^2 + (-1)^2 + 3^2 + (-2)^2} \\ &= \sqrt{18} \\ &= 3\sqrt{2}\end{aligned}$$

So, we have that:

$$\begin{aligned}||\mathbf{u}||\mathbf{u} - ||\mathbf{v}||\mathbf{v} &= 6\sqrt{2}(2, 4, -4, 6) - 3\sqrt{2}(2, -1, 3, -2) \\ &= (12\sqrt{2} - 6\sqrt{2}, 24\sqrt{2} + 3\sqrt{2}, -24\sqrt{2} - 9\sqrt{2}, 36\sqrt{2} + 6\sqrt{2}) \\ &= (6\sqrt{2}, 27\sqrt{2}, -33\sqrt{2}, 42\sqrt{2})\end{aligned}$$

5.2 A Different Approach

1. The first component of the vector is the $\mathbf{i}$ component, the second is the $\mathbf{j}$ components and the third is the $\mathbf{k}$ component. So, we have $2\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}$.
2. This is just algebra.

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= 8\mathbf{i} - 2\mathbf{j} - 6\mathbf{k} + 3\mathbf{i} - 9\mathbf{j} + 4\mathbf{k} \\ &= 11\mathbf{i} - 11\mathbf{j} - 2\mathbf{k}\end{aligned}$$

$$\begin{aligned}5\mathbf{u} &= 5(8\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}) \\ &= 40\mathbf{i} - 10\mathbf{j} - 30\mathbf{k}\end{aligned}$$

3. To take the dot product of 2 vectors, we multiply the vectors component-

wise and sum the results.

$$\begin{aligned}
\begin{pmatrix}4\\2\\7\\2\\6\\4\end{pmatrix} \cdot \begin{pmatrix}9\\0\\2\\2\\1\\8\end{pmatrix} &= (4 \times 9) + (2 \times 0) + (7 \times 2) + (2 \times 2) + (6 \times 1) + (4 \times 8) \\
&= 36 + 0 + 14 + 4 + 6 + 32 \\
&= 92
\end{aligned}$$

4.

$$\begin{aligned}
\begin{pmatrix}8\\2\\6\end{pmatrix} \cdot \begin{pmatrix}9\\2\\5\end{pmatrix} &= (8 \times 9) + (2 \times 2) + (6 \times 5) \\
&= 72 + 4 + 30 \\
&= 106
\end{aligned}$$

5. The brackets tell us what to do first:

$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= (8 + 1, 2 + 5, 1 + 4) \\
&= (9, 7, 5)
\end{aligned}$$

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \begin{pmatrix}2\\7\\4\end{pmatrix} \cdot \begin{pmatrix}9\\7\\5\end{pmatrix} \\
&= 18 + 49 + 20 \\
&= 87
\end{aligned}$$

6.

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \begin{pmatrix}2\\1\\7\end{pmatrix} \cdot \begin{pmatrix}7\\2\\9\end{pmatrix} \\
&= 14 + 2 + 63 \\
&= 79
\end{aligned}$$

$$\begin{aligned}
(\mathbf{u} \cdot \mathbf{v})\mathbf{w} &= 79(2, 1, 4) \\
&= (158, 79, 316)
\end{aligned}$$

7. We're going to need the identity:

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \cos\theta$$

We're looking for θ, so we need to equate $\mathbf{a}$, $\mathbf{b}$, $||\mathbf{a}||$ and $||\mathbf{b}||$.

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= 15 + 48 \\
&= 63 \\
||\mathbf{a}|| &= \sqrt{3^2 + 4^2} \\
&= \sqrt{25} \\
&= 5 \\
\\
||\mathbf{b}|| &= \sqrt{5^2 + 12^2} \\
&= \sqrt{169} \\
&= 13
\end{aligned}$$

Now, substituting everything back into our identity:

$$\begin{aligned}
63 &= 5 \cdot 13 \cos\theta \\
\frac{63}{65} &= \cos\theta \\
\theta &= \arccos\frac{63}{65}
\end{aligned}$$

8.

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= 48 + 120 \\
&= 168 \\
\\
||\mathbf{a}|| &= \sqrt{6^2 + 8^2} \\
&= \sqrt{100} \\
&= 10 \\
\\
||\mathbf{b}|| &= \sqrt{8^2 + 15^2} \\
&= \sqrt{289} \\
&= 17
\end{aligned}$$

Now, substituting everything back into our identity:

$$\begin{aligned} 168 &= 10 \cdot 17 \cos\theta \\ \cos\theta &= \frac{168}{170} \\ &= \frac{84}{85} \\ \theta &= \arccos\frac{84}{85} \end{aligned}$$

5.3 The Cross Product

1. The formula for the cross product of vectors $(a, b, c) \times (d, e, f)$ is $(bf - ce, -(af - cd), ae - bd)$. Using this we get:

$$(2 \times 5 - 5 \times 7, -(3 \times 5 - 5 \times 2), 3 \times 7 - 2 \times 2) = (-25, -5, 17)$$

2. We find a vector that is orthogonal to two others by taking the cross product of these two vectors:

$$\begin{aligned} (2, 1, 5) \times (3, 2, 1) &= (1 \times 1 - 5 \times 2, -(2 \times 1 - 5 \times 3), 2 \times 2 - 1 \times 3) \\ &= (-9, 13, 1) \end{aligned}$$

3.

$$\begin{aligned} (1, 4, 6) \times (2, 2, 5) &= (20 - 12, -(5 - 12), 2 - 8) \\ &= (8, 7, -6) \end{aligned}$$

4.

$$\begin{aligned} (4, 6, 2) \times (5, 1, 3) &= (18 - 2, -(12 - 10), 4 - 30) \\ &= (16, -2, -26) \end{aligned}$$

6
Matrices

Test Yourself

1. Matrix addition is straightforward: it works componentwise.

$$\begin{pmatrix} 2 & 3 \\ 9 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 2+5 & 3+6 \\ 9-3 & 2-2 \end{pmatrix}$$
$$= \begin{pmatrix} 7 & 9 \\ 6 & 0 \end{pmatrix}$$

2.

$$3\begin{pmatrix} 8 & -2 \\ 9 & 5 \\ 4 & 6 \end{pmatrix} - \begin{pmatrix} -2 & 4 \\ 3 & -1 \\ 8 & 1 \end{pmatrix} = \begin{pmatrix} (3\times 8)+2 & (3\times(-2))-4 \\ (3\times 9)-3 & (3\times 5)+1 \\ (3\times 4)-8 & (3\times 6)-1 \end{pmatrix}$$
$$= \begin{pmatrix} 26 & -10 \\ 24 & 16 \\ 4 & 17 \end{pmatrix}$$

3.

$$4\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} + 6\begin{pmatrix} -1 & 6 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 8-6 & 4+36 \\ 12-24 & -4+12 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 40 \\ -12 & 8 \end{pmatrix}$$

4. If we multiply an $l \times m$ matrix with an $m \times n$ matrix, we get an $l \times n$ matrix. To find the (i, j)th entry in the solution matrix we find the dot product of the ith row of the first matrix with the jth column of the second.

   Here, we're multiplying a 2×2 matrix with another 2×2 matrix, so we get a 2×2 matrix as our result.

$$\begin{aligned}\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 5 & -6 \\ 1 & 2 \end{pmatrix} &= \begin{pmatrix} (2 \times 5) + (3 \times 1) & (2 \times (-6)) + (3 \times 2) \\ ((-1) \times 5) + (4 \times 1) & ((-1) \times (-6)) + (4 \times 2) \end{pmatrix} \\ &= \begin{pmatrix} 13 & -6 \\ -1 & 14 \end{pmatrix}\end{aligned}$$

5. We're multiplying a 3×2 matrix with a 2×2 matrix, so we get a 3×2 matrix as our result.

$$\begin{aligned}\begin{pmatrix} 2 & -1 \\ -3 & 4 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} 2 & -6 \\ 4 & 1 \end{pmatrix} &= \begin{pmatrix} (2 \times 2) + ((-1) \times 4) & (2 \times (-6)) + ((-1) \times 1) \\ ((-3) \times 2) + (4 \times 4) & ((-3) \times (-6)) + (4 \times 1) \\ ((-2) \times 2) + (9 \times 4) & ((-2) \times (-6)) + (9 \times 1) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -13 \\ 10 & 22 \\ 32 & 21 \end{pmatrix}\end{aligned}$$

6. "BoDMAS" applies to matrices too! We do the multiplication first and *then* the addition.

$$\begin{aligned}\begin{pmatrix} 9 & -5 & 3 \\ -2 & 1 & 6 \\ 5 & 4 & -1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -3 & 1 \\ 6 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ -4 & 1 \\ 5 & 9 \end{pmatrix} &= \begin{pmatrix} 51 & 37 \\ 29 & 5 \\ -8 & 22 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ -4 & 1 \\ 5 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 53 & 43 \\ 25 & 6 \\ -3 & 31 \end{pmatrix}\end{aligned}$$

7.

$$\begin{aligned}\begin{pmatrix} -2 & 1 \\ 9 & 3 \end{pmatrix}\begin{pmatrix} 4 & -1 & 9 \\ -3 & 2 & 3 \end{pmatrix} + \begin{pmatrix} -6 & 3 & 1 \\ 1 & 5 & 7 \end{pmatrix} &= \begin{pmatrix} -11 & 4 & -15 \\ 27 & -3 & 90 \end{pmatrix} + \\ &\quad \begin{pmatrix} -6 & 3 & 1 \\ 1 & 5 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -17 & 7 & -14 \\ 28 & 2 & 97 \end{pmatrix}\end{aligned}$$

8. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A)$ is defined as $ad - bc$. So here:

$$\begin{aligned} \det(A) &= (3 \times 1) - (4 \times (-2)) \\ &= 3 + 8 \\ &= 11 \end{aligned}$$

9. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the inverse of A is

$$\frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

First, let's find $\det(A)$:

$$\begin{aligned} \det(A) &= (4 \times (-8)) - (6 \times (-5)) \\ &= -32 + 30 \\ &= -2 \end{aligned}$$

So the inverse of A is:

$$\frac{1}{-2} \begin{pmatrix} -8 & -6 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -\frac{5}{2} & -2 \end{pmatrix}$$

10. A^{-1} is just shorthand for "A inverse":

$$\begin{aligned} \det(A) &= 54 - 57 \\ &= -3 \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{-3} \begin{pmatrix} 9 & -19 \\ -3 & 6 \end{pmatrix} \\ &= \begin{pmatrix} -3 & \frac{19}{3} \\ 1 & -2 \end{pmatrix} \end{aligned}$$

6.1 Enter the Matrix

1. This is just notation: the answer is $A = (\alpha_{ij})_{3\times 5}$.
2. The "order" of a matrix is just it's size, so the answer is 3×5.

3. Matrix addition works componentwise:

$$\begin{aligned}\begin{pmatrix}2 & 3\\ 1 & 5\end{pmatrix} + \begin{pmatrix}5 & 6\\ 2 & 1\end{pmatrix} &= \begin{pmatrix}2+5 & 3+6\\ 1+2 & 5+1\end{pmatrix}\\ &= \begin{pmatrix}7 & 9\\ 3 & 6\end{pmatrix}\end{aligned}$$

4.

$$\begin{aligned}\begin{pmatrix}-8 & 1\\ 7 & 4\end{pmatrix} + \begin{pmatrix}-2 & 6\\ -4 & 1\end{pmatrix} &= \begin{pmatrix}-8-2 & 1+6\\ 7-4 & 4+1\end{pmatrix}\\ &= \begin{pmatrix}-10 & 7\\ 3 & 5\end{pmatrix}\end{aligned}$$

5. To perform scalar-matrix multiplcation, we multiply every entry in the matrix by the scalar.

$$\begin{aligned}3\begin{pmatrix}8 & 1 & 4\\ -2 & 6 & 0\\ 5 & 1 & -2\end{pmatrix} &= \begin{pmatrix}(3\times 8) & (3\times 1) & (3\times 4)\\ (3\times -2) & (3\times 6) & (3\times 0)\\ (3\times 5) & (3\times 1) & (3\times -2)\end{pmatrix}\\ &= \begin{pmatrix}24 & 3 & 12\\ -6 & 18 & 0\\ 15 & 3 & -6\end{pmatrix}\end{aligned}$$

6.

$$\begin{aligned}\begin{pmatrix}4 & 12 & -2\\ 6 & 1 & 0\\ 4 & 12 & 7\end{pmatrix} - \begin{pmatrix}11 & 3 & 6\\ -4 & 1 & 12\\ 3 & -3 & 1\end{pmatrix} &= \begin{pmatrix}4-11 & 12-3 & -2-6\\ 6+4 & 1-1 & 0-12\\ 4-3 & 12+3 & 7-1\end{pmatrix}\\ &= \begin{pmatrix}-7 & 9 & -8\\ 10 & 0 & -12\\ 1 & 15 & 6\end{pmatrix}\end{aligned}$$

7. No it isn't possible to find either. Matrix addition and subtraction is only defined for matrices of the same order, but here B is of order 3×3, whereas C is of order 2×3.

8. "BoDMAS" applies to matrices too! We do the scalar multiplication first and *then* the addition.

$$\begin{aligned}8\begin{pmatrix}-1 & 3\\ 1 & 4\end{pmatrix} + \begin{pmatrix}9 & 0\\ 2 & 4\end{pmatrix} &= \begin{pmatrix}-8 & 24\\ 8 & 32\end{pmatrix} + \begin{pmatrix}9 & 0\\ 2 & 4\end{pmatrix}\\ &= \begin{pmatrix}1 & 24\\ 10 & 36\end{pmatrix}\end{aligned}$$

9.

$$6\begin{pmatrix}1 & 3 & 2\\ -1 & 4 & 9\end{pmatrix} - 3\begin{pmatrix}1 & 7 & 4\\ 4 & 2 & 1\end{pmatrix} = \begin{pmatrix}6 & 18 & 12\\ -6 & 24 & 54\end{pmatrix} - \begin{pmatrix}3 & 21 & 12\\ 12 & 6 & 3\end{pmatrix}$$
$$= \begin{pmatrix}3 & -3 & 0\\ -18 & 18 & 51\end{pmatrix}$$

10.

$$2\begin{pmatrix}1 & 8 & -4\\ -6 & -9 & 1\\ 2 & 5 & 6\end{pmatrix} + \begin{pmatrix}4 & -3 & 6\\ 2 & 5 & -9\\ -1 & 4 & 3\end{pmatrix} = \begin{pmatrix}2 & 16 & -8\\ -12 & -18 & 2\\ 4 & 10 & 12\end{pmatrix}$$
$$+ \begin{pmatrix}4 & -3 & 6\\ 2 & 5 & -9\\ -1 & 4 & 3\end{pmatrix}$$
$$= \begin{pmatrix}6 & 13 & -2\\ -10 & -13 & -7\\ 3 & 14 & 15\end{pmatrix}$$

6.2 Multiplication and More

1. If we multiply an $l \times m$ matrix with an $m \times n$ matrix, we get an $l \times n$ matrix. To find the (i, j)th entry in the solution matrix we find the dot product of the ith row of the first matrix with the jth column of the second.

   Here, we're multiplying a 2×2 matrix with another 2×2 matrix, so we get a 2×2 matrix as our result.

$$\begin{pmatrix}2 & 4\\ 3 & 4\end{pmatrix}\begin{pmatrix}1 & 9\\ 0 & 4\end{pmatrix} = \begin{pmatrix}(2\times 1)+(4\times 0) & (2\times 9)+(4\times 4)\\ (3\times 1)+(1\times 0) & (3\times 9)+(1\times 4)\end{pmatrix}$$
$$= \begin{pmatrix}2 & 34\\ 3 & 31\end{pmatrix}$$

2.

$$\begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}\begin{pmatrix}4 & 3\\ 0 & 2\end{pmatrix} = \begin{pmatrix}(3\times 4)+(1\times 0) & (3\times 3)+(1\times 2)\\ (5\times 4)+(2\times 0) & (5\times 3)+(2\times 2)\end{pmatrix}$$
$$= \begin{pmatrix}12 & 11\\ 20 & 19\end{pmatrix}$$

3. Here, we're multiplying a 3×2 matrix with another 2×2 matrix, so we get a 3×2 matrix as our result.
$$\begin{aligned}\begin{pmatrix}4 & 2\\3 & 6\\1 & 4\end{pmatrix}\begin{pmatrix}2 & 1\\3 & 6\end{pmatrix} &= \begin{pmatrix}(4\times 2)+(2\times 3) & (4\times 1)+(2\times 6)\\(3\times 2)+(6\times 3) & (3\times 1)+(6\times 6)\\(1\times 2)+(4\times 3) & (1\times 1)+(4\times 6)\end{pmatrix}\\ &= \begin{pmatrix}14 & 16\\24 & 39\\14 & 25\end{pmatrix}\end{aligned}$$

4. Here, we're multiplying a 2×3 matrix with another 3×3 matrix, so we get a 2×3 matrix as our result.
$$\begin{pmatrix}-8 & 3 & 9\\2 & 4 & -6\end{pmatrix}\begin{pmatrix}3 & -6 & 2\\4 & 6 & -1\\1 & -9 & 3\end{pmatrix} = \begin{pmatrix}-3 & -15 & 8\\16 & 66 & -18\end{pmatrix}$$

5. $$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}3 & 4\\2 & 5\end{pmatrix} = \begin{pmatrix}3 & 4\\2 & 5\end{pmatrix}$$
$$\begin{pmatrix}3 & 4\\2 & 5\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}3 & 4\\2 & 5\end{pmatrix}$$
When multiplying by the identity matrix, it makes no difference whether we "premultiply" or "postmultiply" by it. We get the original matrix as our answer in both cases!

6. $$\begin{pmatrix}2 & 0\\0 & 4\end{pmatrix}\begin{pmatrix}-8 & 3\\0 & 2\end{pmatrix} = \begin{pmatrix}-16 & 6\\0 & 8\end{pmatrix}$$

7. Remember: multiplication happens before addition.
$$\begin{aligned}\begin{pmatrix}-2 & 1\\5 & 0\end{pmatrix}\begin{pmatrix}3 & 8\\5 & 7\end{pmatrix} + \begin{pmatrix}2 & 4\\1 & 3\end{pmatrix} &= \begin{pmatrix}-1 & -9\\15 & 40\end{pmatrix} + \begin{pmatrix}2 & 4\\1 & 3\end{pmatrix}\\ &= \begin{pmatrix}1 & -5\\16 & 43\end{pmatrix}\end{aligned}$$

8. $$\begin{aligned}\begin{pmatrix}1 & -6 & 5\\0 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\8 & 4\\9 & 0\end{pmatrix} + \begin{pmatrix}-4 & 0\\2 & -3\end{pmatrix} &= \begin{pmatrix}-2 & -24\\26 & 4\end{pmatrix} + \begin{pmatrix}-4 & 0\\2 & -3\end{pmatrix}\\ &= \begin{pmatrix}-6 & -24\\28 & 1\end{pmatrix}\end{aligned}$$

9. I_2 is just a shorthand way of writing "the 2×2 identity matrix", which is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\begin{aligned} \begin{pmatrix} 2 & 6 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 20 & 4 \\ 5 & 10 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 21 & 4 \\ 5 & 11 \end{pmatrix} \end{aligned}$$

10. $diag(1,2,3)$ is just a shorthand way of writing $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

$$\begin{aligned} \begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 8 & 1 \\ 6 & 2 & 5 \\ 0 & 3 & 8 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} &= \begin{pmatrix} 26 & 25 & 25 \\ 0 & 3 & 8 \\ 12 & 13 & 34 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 25 & 25 & 25 \\ 0 & 1 & 8 \\ 12 & 13 & 31 \end{pmatrix} \end{aligned}$$

6.3 Determinants and Inverse

1. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A)$ is defined as $ad - bc$. So here,

$$\begin{aligned} \det(A) &= (2 \times 1) - (3 \times 4) \\ &= 2 - 12 \\ &= -10 \end{aligned}$$

2.

$$\begin{aligned} \det(B) &= ((-4) \times 1) - (2 \times 3) \\ &= -10 \end{aligned}$$

3. Putting square brackets around a matrix is the notation for "determinant":

$$\begin{aligned} \begin{vmatrix} -2 & 3 \\ -1 & 6 \end{vmatrix} &= ((-2) \times 6) - (3 \times (-1)) \\ &= -9 \end{aligned}$$

4.

$$\begin{vmatrix} 4 & -9 \\ -6 & 2 \end{vmatrix} = (4 \times 2) - ((-9) \times (-6))$$
$$= -46$$

5. A 2×2 matrix is inverible if *and only if* its determinant is non-zero, so what this question is really asking is to find the determinants:

   $\det(A) = 1 - 0 = 1$, which is non-zero. So matrix A is invertible.

   $\det(B) = -10 + 18 = 8$, which is non-zero. So matrix B is invertible.

   $\det(C) = 18 - 18 = 0$. So matrix C is *not* invertible.

6. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the inverse of A is

$$\frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

   First, let's find $\det(A)$:

$$\begin{vmatrix} -3 & -13 \\ 6 & 10 \end{vmatrix} = -30 + 78$$
$$= 48$$

   So, the inverse of A is

$$\frac{1}{48} \begin{pmatrix} 10 & 13 \\ -6 & -3 \end{pmatrix} = \begin{pmatrix} \frac{5}{24} & \frac{13}{48} \\ -\frac{1}{8} & -\frac{1}{16} \end{pmatrix}$$

7.

$$\begin{vmatrix} -3 & 7 \\ 3 & -6 \end{vmatrix} = 18 - 21$$
$$= -3$$

   So, the inverse of our matrix is

$$\frac{1}{-3} \begin{pmatrix} -6 & -7 \\ -3 & -3 \end{pmatrix} = \begin{pmatrix} 2 & \frac{7}{3} \\ 1 & 1 \end{pmatrix}$$

8. A^{-1} is just shorthand for "A inverse":

$$\begin{aligned} \det(A) &= 6 + 36 \\ &= 42 \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{42}\begin{pmatrix} -3 & -9 \\ 4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{14} & -\frac{3}{14} \\ \frac{2}{21} & -\frac{1}{21} \end{pmatrix} \end{aligned}$$

7
Matrices as Maps

Test Yourself

1. Recall that we can write this in vector form:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

   So the corresponding matrix is $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$

2. Recall that we can write this in vector form:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

   So the corresponding matrix is $\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$

3. The vector equation with this matrix is

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

   If we multiply this out we get:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 3x_0 - 2y_0 \\ 0x_0 + 1y_0 \end{pmatrix}$$

We can then look at each component seperately to give us:

$$\begin{array}{rcrcr} x_1 & = & 3x_0 & - & 2y_0 \\ y_1 & = & & & y_0 \end{array}$$

4. For a reflection about the line that is an angle of α to the positive x axis, we have that the corresponding matrix is:

$$\begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix}$$

In this question we have an angle of 0. So our matrix is:

$$\begin{pmatrix} \cos(0) & \sin(0) \\ \sin(0) & -\cos(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

5. For a rotation of angle θ anticlockwise about the origin, we get the matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

In this question we have $\theta = \frac{\pi}{4}$, so our matrix is:

$$\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4}) \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

6. For the rotation we have $\theta = \pi$, so our matrix is:

$$A = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

When we enlarge by a factor of 3 in both directions we will require the matrix:

$$B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Let's let $\mathbf{v}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. Then we have the first linear map as

$$\mathbf{v}_1 = A\mathbf{v}_0$$

and the second one as

$$\mathbf{v}_2 = B\mathbf{v}_1$$

Hence to do both in a single step we have

$$\mathbf{v}_2 = BA\mathbf{v}_1$$

So we require the matrix BA, which is:

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$$

7. We will use the equation $A\mathbf{v} = \lambda\mathbf{v}$ with $A = \begin{pmatrix} 4 & 3 \\ 8 & 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. So we have that:

$$\begin{aligned} A\mathbf{v} = \begin{pmatrix} 4 & 3 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 12+12 \\ 24+8 \end{pmatrix} \\ &= \begin{pmatrix} 24 \\ 32 \end{pmatrix} \\ &= 8\begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ &= 8\mathbf{v} \end{aligned}$$

So $\lambda = 8$ hence the eigenvector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ has a corresponding eigenvalue of 8.

8. We will use the equation $A\mathbf{v} = \lambda\mathbf{v}$ with $A = \begin{pmatrix} 4 & 3 \\ 8 & 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. So we have that:

$$\begin{aligned} A\mathbf{v} &= \begin{pmatrix} 4 & 3 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 4 \end{pmatrix} \\ &= -2\begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= -2\mathbf{v} \end{aligned}$$

So $\lambda = -2$ hence the eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ has a corresponding eigenvalue of -2.

9. Remember that to find eigenvalues we must solve the equation $\det(A -$

$\lambda I_n) = 0$.

$$\begin{aligned}
A - \lambda I_n &= \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\
&= \begin{pmatrix} 2-\lambda & 1 \\ 4 & -1-\lambda \end{pmatrix} \\
\det(A - \lambda I_n) &= (2-\lambda)(-1-\lambda) - 4 \\
&= \lambda^2 - \lambda - 2 - 4 \\
&= \lambda^2 - \lambda - 6 \\
&= (\lambda + 2)(\lambda - 3) = 0
\end{aligned}$$

Hence our eigenvalues are -2 and 3.

Then using $A\mathbf{v} = \lambda\mathbf{v}$

$$\begin{aligned}
\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\begin{pmatrix} 2v_1 + v_2 \\ 4v_1 - v_2 \end{pmatrix} &= \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}
\end{aligned}$$

So for $\lambda = 3$ we have:

$$\begin{aligned}
2v_1 + v_2 &= 3v_1 \\
4v_1 - v_2 &= 3v_2
\end{aligned}$$

Therefore:

$$\begin{aligned}
v_2 - v_1 &= 0 \\
4v_1 - 4v_2 &= 0
\end{aligned}$$

So we simply require that $v_1 = v_2$. So for the eigenvalue of 3 we can choose the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Similarly for the eigenvalue -2 we have that:

$$\begin{aligned}
2v_1 + v_2 &= -2v_1 \\
4v_1 - v_2 &= -2v_2
\end{aligned}$$

Therefore:

$$\begin{aligned}
4v_1 + v_2 &= 0 \\
4v_1 + v_2 &= 0
\end{aligned}$$

Hence we require that $4v_1 = -v_2$. So for the eigenvalue of 3 we can choose the eigenvector $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

10. Remember that to find eigenvalues we must solve the equation $\det(A - \lambda I_n) = 0$.

$$
\begin{aligned}
A - \lambda I_n &= \begin{pmatrix} -6 & 2 \\ -7 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\
&= \begin{pmatrix} -6-\lambda & 2 \\ -7 & 3-\lambda \end{pmatrix} \\
\det(A - \lambda I_n) &= (-6-\lambda)(3-\lambda) + 14 \\
&= \lambda^2 + 3\lambda - 18 + 14 \\
&= \lambda^2 + 3\lambda - 4 \\
&= (\lambda - 1)(\lambda + 4) = 0
\end{aligned}
$$

Hence our eigenvalues are 1 and -4.

Then using $A\mathbf{v} = \lambda\mathbf{v}$:

$$
\begin{aligned}
\begin{pmatrix} -6 & 2 \\ -7 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\begin{pmatrix} -6v_1 + 2v_2 \\ -7v_1 + 3v_2 \end{pmatrix} &= \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}
\end{aligned}
$$

So for the eigenvalue 1 we have:

$$
\begin{aligned}
-6v_1 + 2v_2 &= v_1 \\
-6v_1 + 2v_2 &= v_2
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
-7v_1 + 2v_2 &= 0 \\
-7v_1 + 2v_2 &= 0
\end{aligned}
$$

So we require that $7v_1 = 2v_2$. So for the eigenvalue of 1 we can choose the eigenvector $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

Similarly for the eigenvalue -4 we have that:

$$
\begin{aligned}
-6v_1 + 2v_2 &= -4v_1 \\
-6v_1 + 2v_2 &= -4v_2
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
-2v_1 + 2v_2 &= 0 \\
-7v_1 + 7v_2 &= 0
\end{aligned}
$$

Hence we simply require that $v_1 = v_2$. So for the eigenvalue of -4 we can choose the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

7.1 Over and Over

1. We this map takes any point to itself as:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

   Multiplying this out we get:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

   So we call it the identity map.

2. Recall that we can write this in vector form:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

   So the corresponding matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

3. Recall that we can write this in vector form:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

   So the corresponding matrix is $\begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}$

4. Recall that we can write this in vector form:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

   So the corresponding matrix is $\begin{pmatrix} -3 & 0 \\ 1 & -2 \end{pmatrix}$

5. The vector equation with this matrix is

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

   If we multiply this out we get:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 + 2y_0 \\ 5x_0 + 3y_0 \end{pmatrix}$$

   We can then look at each component seperately to give us:

$$\begin{array}{rcrcr} x_1 & = & x_0 & + & 2y_0 \\ y_1 & = & 5x_0 & + & 3y_0 \end{array}$$

6. The vector equation with this matrix is

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

If we multiply this out we get:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2x_0 - 3y_0 \\ 6x_0 + 2y_0 \end{pmatrix}$$

We can then look at each component seperately to give us:

$$\begin{array}{rcrcr} x_1 & = & 2x_0 & - & 3y_0 \\ y_1 & = & 6x_0 & + & 2y_0 \end{array}$$

7. The vector equation with this matrix is

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

If we multiply this out we get:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 4x_0 + 0y_0 \\ 2x_0 + 1y_0 \end{pmatrix}$$

We can then look at each component seperately to give us:

$$\begin{array}{rcrcr} x_1 & = & 4x_0 & & \\ y_1 & = & 2x_0 & + & y_0 \end{array}$$

8. If we look at the vecor equation:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence the point $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ is mapped to the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. In fact, if we take any point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ then

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So every point is mapped to the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

7.2 Old Friends

1. For a reflection about the line that is an angle of α to the positive x axis, we have that the corresponding matrix is:

$$\begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix}$$

   In this question we have an angle of $\frac{\pi}{6}$. So our matrix is:

$$\begin{pmatrix} \cos(\frac{\pi}{3}) & \sin(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) & -\cos(\frac{\pi}{3}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

2. Recall that $\tan\theta = \frac{opposite}{adjacent}$. So we can use this to say that $\tan\alpha = \frac{y}{x}$. So we have that $\tan\alpha = \frac{x\sqrt{3}}{x} = \sqrt{3}$, hence $\alpha = \frac{\pi}{3}$. So the matrix is:

$$\begin{pmatrix} \cos(\frac{2\pi}{3}) & \sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & -\cos(\frac{2\pi}{3}) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

3. For a rotation of angle θ anticlockwise about the origin, we get the matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   In this question we have $\theta = \frac{\pi}{4}$, so our matrix is:

$$\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4}) \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

4. For a rotation of angle θ anticlockwise about the origin, we get the matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   In this question we have $\theta = \frac{\pi}{3}$, so our matrix is:

$$\begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3}) \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

5. For an enlargement by m in the x direction and n in the y direction we require the matrix:

$$\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$$

In this question we are enlarging by a factor of $\frac{1}{2}$ so $m = n = \frac{1}{2}$. Therefore our matrix is:

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

.

6. For an enlargement by m in the x direction and n in the y direction we require the matrix:

$$\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$$

In this question we are enlarging by a factor of 3 in the x direction and 6 in the y direction, so $m = 3$ and $n = 6$. Therefore our matrix is:

$$\begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$$

.

7. For the rotation we have $\theta = \frac{\pi}{6}$, so our matrix is:

$$A = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6}) \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

For the enlargement we have that $m = n = 3$ so we have the matrix:

$$B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Let's let $\mathbf{v}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$. Then we have the first linear map as

$$\mathbf{v}_1 = A\mathbf{v}_0$$

and the second one as

$$\mathbf{v}_2 = B\mathbf{v}_1$$

Hence to do both in a single step we have

$$\mathbf{v}_2 = BA\mathbf{v}_1$$

So we require the matrix BA, which is:

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{3}}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{3\sqrt{3}}{2} \end{pmatrix}$$

8. For the reflection we have $\alpha = \frac{3\pi}{4}$, so our matrix is:

$$A = \begin{pmatrix} \cos\frac{3\pi}{2} & \sin\frac{3\pi}{2}) \\ \sin\frac{3\pi}{2} & -\cos\frac{3\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

For the enlargement we have that $m = 2$ and $n = 4$ so we have the matrix:

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

Once again we require the matrix BA, which is:

$$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -4 & 0 \end{pmatrix}$$

9. For the rotation we have $\theta = \frac{\pi}{2}$, so our matrix is:

$$A = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2}) \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Recall that $\tan\theta = \frac{opposite}{adjacent}$. So we can use this to say that $\tan\alpha = \frac{y}{x}$. So we have that $\tan\alpha = \frac{\frac{x}{\sqrt{3}}}{x} = \frac{1}{\sqrt{3}}$, hence $\alpha = \frac{\pi}{6}$. So for the reflection we have the matrix:

$$B = \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & -\cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Once again we require the matrix BA, which is:

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

10. A reflection in the x axis is the same as a reflection that makes an angle of 0 with the x axis, so we have the matrix:

$$A = \begin{pmatrix} \cos 0 & \sin 0 \\ \sin 0 & -\cos 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For the rotation we have $\theta = \frac{\pi}{3}$, so our matrix is:

$$B = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3}) \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

And for the enlargement we have that $m = 2$ and $n = 1$ so we have the matrix:

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Using the notation $\mathbf{v}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ for any i. Then we have the following linear maps:

$$\begin{aligned} \mathbf{v}_1 &= A\mathbf{v}_0 \\ \mathbf{v}_2 &= B\mathbf{v}_1 \\ \mathbf{v}_3 &= C\mathbf{v}_2 \end{aligned}$$

So we can write $\mathbf{v}_3 = CB\mathbf{v}_1$, therefore:

$$\mathbf{v}_3 = CBA\mathbf{v}_0$$

Hence the single matrix that we require is the matrix CBA. Note: $(CB)A = C(BA)$ so we can do the multiplication in any way that we like.

$$\begin{aligned} BA &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \\ CBA &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \end{aligned}$$

So our single matrix for these three transformations is $\begin{pmatrix} 1 & \sqrt{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$.

7.3 Eigenvalues and Eigenvectors

1. We will use the equation $A\mathbf{v} = \lambda\mathbf{v}$ with $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. So we have that:

$$\begin{aligned} A\mathbf{v} &= \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 3-2 \\ 2-4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= 1\mathbf{v} \end{aligned}$$

So $\lambda = 1$, hence the eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ has a corresponding eigenvalue of 1.

2. We will use the equation $A\mathbf{v} = \lambda\mathbf{v}$ with $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So we have that:

$$\begin{aligned} A\mathbf{v} &= \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 3+1 \\ 2+2 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ &= 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 4\mathbf{v} \end{aligned}$$

So $\lambda = 4$, hence the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has a corresponding eigenvalue of 4.

3. We will use the equation $A\mathbf{v} = \lambda\mathbf{v}$ with $A = \begin{pmatrix} 6 & 2 \\ -8 & -4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$. So we have that:

$$\begin{aligned} A\mathbf{v} &= \begin{pmatrix} 6 & 2 \\ -8 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 6-8 \\ -8+16 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 8 \end{pmatrix} \\ &= -2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} \\ &= -2\mathbf{v} \end{aligned}$$

So $\lambda = -2$, hence the eigenvector $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ has a corresponding eigenvalue of -2.

4. We will use the equation $A\mathbf{v} = \lambda\mathbf{v}$ with $A = \begin{pmatrix} 6 & 2 \\ -8 & -4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

So we have that:

$$\begin{aligned} A\mathbf{v} &= \begin{pmatrix} 6 & 2 \\ -8 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 6-2 \\ -8+4 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -4 \end{pmatrix} \\ &= 4\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= 4\mathbf{v} \end{aligned}$$

So $\lambda = 4$, hence the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ has a corresponding eigenvalue of 4.

5. Remember that to find eigenvalues we must solve the equation $\det(A - \lambda I_n) = 0$.

$$\begin{aligned} A - \lambda I_n &= \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 4-\lambda & 3 \\ 2 & -1-\lambda \end{pmatrix} \\ \det(A - \lambda I_n) &= (4-\lambda)(-1-\lambda) - 6 \\ &= \lambda^2 - 3\lambda - 4 - 6 \\ &= \lambda^2 - 3\lambda - 10 \\ &= (\lambda - 5)(\lambda + 2) = 0 \end{aligned}$$

Hence our eigenvalues are 5 and -2.

Then using $A\mathbf{v} = \lambda\mathbf{v}$

$$\begin{aligned} \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ \begin{pmatrix} 4v_1 + 3v_2 \\ 2v_1 - v_2 \end{pmatrix} &= \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix} \end{aligned}$$

So for the eigenvalue -2 we have:

$$\begin{aligned} 4v_1 + 3v_2 &= -2v_1 \\ 2v_1 - v_2 &= -2v_2 \end{aligned}$$

Therefore:

$$\begin{aligned} 6v_1 + 3v_2 &= 0 \\ 2v_1 + v_2 &= 0 \end{aligned}$$

So we simply require that $2v_1 = -v_2$. So for the eigenvalue of -2 we can choose the eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Similarly for the eigenvalue -5 we have that:

$$\begin{aligned} 4v_1 + 3v_2 &= 5v_1 \\ 2v_1 - v_2 &= 5v_2 \end{aligned}$$

Therefore:

$$\begin{aligned} -v_1 + 3v_2 &= 0 \\ 2v_1 - 6v_2 &= 0 \end{aligned}$$

Hence we require that $v_1 = 3v_2$. So for the eigenvalue of -5 we can choose the eigenvector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

6. Remember that to find eigenvalues we must solve the equation $\det(A - \lambda I_n) = 0$.

$$\begin{aligned} A - \lambda I_n &= \begin{pmatrix} -2 & -1 \\ 7 & 6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} -2-\lambda & -2 \\ 7 & 6-\lambda \end{pmatrix} \\ \det(A - \lambda I_n) &= (-2-\lambda)(6-\lambda) + 7 \\ &= \lambda^2 - 4\lambda - 12 + 7 \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda + 1)(\lambda - 5) = 0 \end{aligned}$$

Hence our eigenvalues are 5 and -1.

Then using $A\mathbf{v} = \lambda\mathbf{v}$

$$\begin{aligned} \begin{pmatrix} -2 & -1 \\ 7 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ \begin{pmatrix} -2v_1 - v_2 \\ 7v_1 + 6v_2 \end{pmatrix} &= \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix} \end{aligned}$$

So for the eigenvalue 5 we have:

$$\begin{aligned} -2v_1 - v_2 &= 5v_1 \\ 7v_1 + 6v_2 &= 5v_2 \end{aligned}$$

Therefore:

$$\begin{aligned} -7v_1 - v_2 &= 0 \\ 7v_1 + v_2 &= 0 \end{aligned}$$

So we simply require that $7v_1 = -v_2$. So for the eigenvalue of 5 we can choose the eigenvector $\begin{pmatrix} -1 \\ 7 \end{pmatrix}$.

Similarly for the eigenvalue -1 we have that:

$$\begin{aligned} -2v_1 - v_2 &= -v_1 \\ 7v_1 + 6v_2 &= -v_2 \end{aligned}$$

Therefore:

$$\begin{aligned} -v_1 - v_2 &= 0 \\ 7v_1 + 7v_2 &= 0 \end{aligned}$$

Hence we require that $v_1 = -v_2$. So for the eigenvalue of -1 we can choose the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

7. Remember that to find eigenvalues we must solve the equation $\det(A - \lambda I_n) = 0$.

$$\begin{aligned} A - \lambda I_n &= \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 1-\lambda & 5 \\ 1 & -3-\lambda \end{pmatrix} \\ \det(A - \lambda I_n) &= (1-\lambda)(-3-\lambda) - 5 \\ &= \lambda^2 + 2\lambda - 3 - 5 \\ &= \lambda^2 + 2\lambda - 8 \\ &= (\lambda - 2)(\lambda + 4) = 0 \end{aligned}$$

Hence our eigenvalues are 2 and -4.

Then using $A\mathbf{v} = \lambda\mathbf{v}$

$$\begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\begin{pmatrix} v_1 + 5v_2 \\ v_1 - 3v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

So for the eigenvalue 2 we have:

$$v_1 + 5v_2 = 2v_1$$
$$v_1 - 3v_2 = 2v_2$$

Therefore:

$$-v_1 + 5v_2 = 0$$
$$v_1 - 5v_2 = 0$$

So we simply require that $v_1 = 5v_2$. So for the eigenvalue of 2 we can choose the eigenvector $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

Similarly for the eigenvalue -4 we have that:

$$v_1 + 5v_2 = -4v_1$$
$$v_1 - 3v_2 = -4v_2$$

Therefore:

$$5v_1 + 5v_2 = 0$$
$$v_1 + v_2 = 0$$

Hence we require that $v_1 = -v_2$. So for the eigenvalue of -4 we can choose the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

8. Remember that to find eigenvalues we must solve the equation $\det(A - \lambda I_n) = 0$.

$$\begin{aligned} A - \lambda I_n &= \begin{pmatrix} -5 & 3 \\ 3 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} -5-\lambda & 3 \\ 3 & 3-\lambda \end{pmatrix} \\ \det(A - \lambda I_n) &= (-5-\lambda)(3-\lambda) - 9 \\ &= \lambda^2 + 2\lambda - 15 - 9 \\ &= \lambda^2 + 2\lambda - 24 \\ &= (\lambda + 6)(\lambda - 4) = 0 \end{aligned}$$

Hence our eigenvalues are -6 and 4.

Then using $A\mathbf{v} = \lambda\mathbf{v}$

$$\begin{pmatrix} -5 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\begin{pmatrix} -5v_1 + 3v_2 \\ 3v_1 + 3v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

So for the eigenvalue -6 we have:

$$\begin{aligned} -5v_1 + 3v_2 &= -6v_1 \\ 3v_1 + 3v_2 &= -6v_2 \end{aligned}$$

Therefore:

$$\begin{aligned} v_1 + 3v_2 &= 0 \\ 3v_1 + 9v_2 &= 0 \end{aligned}$$

So we simply require that $v_1 = -3v_2$. So for the eigenvalue of -6 we can choose the eigenvector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

Similarly for the eigenvalue 4 we have that:

$$\begin{aligned} -5v_1 + 3v_2 &= 4v_1 \\ 3v_1 + 3v_2 &= 4v_2 \end{aligned}$$

Therefore:

$$\begin{aligned} -9v_1 + 3v_2 &= 0 \\ 3v_1 - v_2 &= 0 \end{aligned}$$

Hence we require that $3v_1 = v_2$. So for the eigenvalue of 4 we can choose the eigenvector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

9. Remember that to find eigenvalues we must solve the equation $\det(A - \lambda I_n) = 0$.

$$\begin{aligned} A - \lambda I_n &= \begin{pmatrix} 7 & -1 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 7-\lambda & -1 \\ 4 & 2-\lambda \end{pmatrix} \\ \det(A - \lambda I_n) &= (7-\lambda)(2-\lambda) + 4 \\ &= \lambda^2 - 9\lambda + 14 + 4 \\ &= \lambda^2 - 9\lambda + 18 \\ &= (\lambda - 6)(\lambda - 3) = 0 \end{aligned}$$

Hence our eigenvalues are 6 and 3.

Then using $A\mathbf{v} = \lambda\mathbf{v}$

$$\begin{pmatrix} 7 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\begin{pmatrix} 7v_1 - v_2 \\ 4v_1 + 2v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

So for the eigenvalue 3 we have:

$$7v_1 - v_2 = 3v_1$$
$$4v_1 + 2v_2 = 3v_2$$

Therefore:

$$4v_1 - v_2 = 0$$
$$4v_1 - v_2 = 0$$

So we simply require that $4v_1 = v_2$. So for the eigenvalue of 3 we can choose the eigenvector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Similarly for the eigenvalue 6 we have that:

$$7v_1 - v_2 = 6v_1$$
$$4v_1 + 2v_2 = 6v_2$$

Therefore:

$$v_1 - v_2 = 0$$
$$4v_1 - 4v_2 = 0$$

Hence we require that $v_1 = v_2$. So for the eigenvalue of 6 we can choose the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

10. Remember that to find eigenvalues we must solve the equation $\det(A - \lambda I_n) = 0$.

$$\begin{aligned} A - \lambda I_n &= \begin{pmatrix} -9 & -10 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} -9-\lambda & -10 \\ 3 & 2-\lambda \end{pmatrix} \\ \det(A - \lambda I_n) &= (-9-\lambda)(2-\lambda) + 30 \\ &= \lambda^2 + 7\lambda - 18 + 30 \\ &= \lambda^2 + 7\lambda + 12 \\ &= (\lambda + 4)(\lambda + 3) = 0 \end{aligned}$$

Hence our eigenvalues are -4 and -3.

Then using $A\mathbf{v} = \lambda\mathbf{v}$

$$\begin{pmatrix} -9 & -10 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$\begin{pmatrix} -9v_1 - 10v_2 \\ 3v_1 + 2v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

So for the eigenvalue -3 we have:

$$-9v_1 - 10v_2 = -3v_1$$
$$3v_1 + 2v_2 = -3v_2$$

Therefore:

$$-6v_1 - 10v_2 = 0$$
$$3v_1 + 5v_2 = 0$$

So we simply require that $3v_1 = -5v_2$. So for the eigenvalue of -3 we can choose the eigenvector $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$.

Similarly for the eigenvalue -4 we have that:

$$-9v_1 - 10v_2 = -4v_1$$
$$3v_1 + 2v_2 = -4v_2$$

Therefore:

$$5v_1 - 10v_2 = 0$$
$$3v_1 + 6v_2 = 0$$

Hence we require that $v_1 = -2v_2$. So for the eigenvalue of -4 we can choose the eigenvector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

8
Seperable Differential Equations

Test Yourself

1. First make both fractions have a common denominator:

$$\frac{3}{x+2} + \frac{x+5}{x^2} = \frac{3x^2}{(x+2)x^2} + \frac{(x+5)(x+2)}{(x+2)x^2}$$

   Now just simplify:

$$\begin{aligned} &= \frac{3x^2 + (x+5)(x+2)}{(x+2)x^2} \\ &= \frac{3x^2 + x^2 + 7x + 10}{(x+2)x^2} \\ &= \frac{4x^2 + 7x + 10}{(x+2)x^2} \end{aligned}$$

2. We first need to split the denominator up into a product:

$$\frac{7}{x^2-4} = \frac{7}{(x+2)(x-2)}$$

$$\text{Then we solve : } \frac{A}{x+2} + \frac{B}{x-2} = \frac{7}{(x+2)(x-2)} \text{ for } A \text{ and } B$$

   Multiply through by $x+2$:

$$A + \frac{B(x+2)}{x-2} = \frac{7}{x-2}$$

Multiply through by $x-2$:

$$\begin{aligned} A(x-2)+B(x+2) &= 7 \\ Ax-2A+Bx+2B &= 7 \end{aligned}$$

Equate the coefficients in x, and the constant coefficient:

$$(A+B)x+(2B-2A)=7$$

Equating the coefficients of x we get that $A+B=0$ so $A=-B$. Then equating the constant coefficients gives us that $2A-2B=7$. Putting the first equation into the second we get:

$$\begin{aligned} 2B+2B &= 7 \\ 4B &= 7 \\ B &= \frac{7}{4} \\ A &= \frac{-7}{4} \end{aligned}$$

So

$$\begin{aligned} \frac{7}{x^2-4} &= \frac{7}{4(x-2)}-\frac{7}{4(x+2)} \\ &= \frac{7}{4}\left(\frac{1}{x-2}-\frac{1}{x+2}\right) \end{aligned}$$

3. The best way to go about this is to use partial fractions:

$$\frac{3}{x^2+x}=\frac{3}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}$$

So, we have that:

$$\begin{aligned} A(x+1)+Bx &= 3 \\ Ax+A+Bx &= 3 \\ (A+B)x+A &= 3 \end{aligned}$$

Equating the coefficients gives us that:

$$\begin{aligned} A &= 3 \\ A+B &= 0 \\ B &= -3 \end{aligned}$$

Therefore our integral is:

$$\begin{aligned}\int \frac{3}{x^2+x}\, dx &= \int \left(\frac{3}{x} - \frac{3}{x+1}\right)\, dx \\ &= 3\ln|x| - 3\ln|x+1| + k \\ &= 3(\ln|x| - \ln|x+1|) + k \\ &= 3\ln\left|\frac{x}{x+1}\right| + k\end{aligned}$$

4.
$$\frac{dy}{dx} = \frac{y^2}{x^2}$$

This problem is in the form $\frac{dy}{dx}P(x)Q(y) = R(x)S(y)$, where $P(x) = 1$, $Q(y) = 1$, $R(x) = \frac{1}{x^2}$ and $S(y) = y^2$. We re-arrange this and then "multiply through by dx" and integrate:

$$\begin{aligned}\frac{dy}{dx}\frac{1}{y^2} &= \frac{1}{x^2} \\ \int \frac{1}{y^2}\, dy &= \int \frac{1}{x^2}\, dx \\ \text{So } -\frac{1}{y} &= -\frac{1}{x} + c \\ &= \frac{-1+cx}{x} \\ \text{Hence } -y &= \frac{x}{-1+cx} \\ y &= \frac{x}{1-cx}\end{aligned}$$

(We could also have chosen a negative c, say $c = -k$, which yields the potentially "nicer" $y = \frac{x}{1+kx}$).

5.

$$\begin{aligned}\frac{dy}{dx}x &= yx^2 \\ \frac{dy}{dx} &= yx \\ \frac{dy}{dx}\frac{1}{y} &= x\end{aligned}$$

"Multiplying through by dx" and integrating:

$$\begin{aligned}
\int \frac{1}{y}\ dx &= \int x\ dx \\
\ln|y| &= \frac{x^2}{2} + c \\
y &= e^{\frac{x^2}{2}+c} \\
&= e^{\frac{x^2}{2}} e^c \\
&= Ae^{\frac{x^2}{2}}
\end{aligned}$$

For this last step we set $A = e^c$ (which usually makes for a more convenient calculation later on!)

6.

$$\begin{aligned}
\frac{dy}{dx}4y^2 &= \frac{y}{x} \\
\frac{dy}{dx}4y &= \frac{1}{x} \\
\int 4y\ dy &= \int \frac{1}{x}\ dx \\
2y^2 &= \ln|x| + c
\end{aligned}$$

We then set $c = \ln|k|$:

$$\begin{aligned}
2y^2 &= \ln|x| + \ln|k| \\
&= \ln|xk| \\
y^2 &= \frac{\ln|xk|}{2} \\
y &= \sqrt{\frac{\ln|xk|}{2}}
\end{aligned}$$

7.

$$\begin{aligned}\frac{dy}{dx}3x^2 &= \frac{y}{x^2}\\ \frac{dy}{dx}\frac{3}{y} &= \frac{1}{x^4}\\ \int \frac{3}{y}\ dy &= \int \frac{1}{x^4}\ dx\\ 3\ln|y| &= -\frac{1}{3x^3}+c\\ \ln|y| &= -\frac{1}{9x^3}+\frac{c}{3}\\ y &= e^{-\frac{1}{9x^3}+\frac{c}{3}}\\ &= Ae^{-\frac{1}{9x^3}}\end{aligned}$$

8.

$$\begin{aligned}\frac{dy}{dx}(x^2+3x)y = 11&\\ \frac{dy}{dx}y &= \frac{11}{x^2+3x}\\ \int y\ dy &= \int \frac{11}{x^2+3x}\ dx\end{aligned}$$

We need to put the right hand side of this equation into partial fractions before we can perform the integration:

$$\begin{aligned}\frac{11}{x^2+3x} &= \frac{11}{x(x+3)}\\ &= \frac{A}{x}+\frac{B}{x+3}\end{aligned}$$

So we have that:

$$\begin{aligned}A(x+3)+Bx &= 11\\ 3A+(A+B)x &= 11\\ A &= \frac{11}{3}\\ A+B &= 0\\ B &= -\frac{11}{3}\end{aligned}$$

So our integral is:

$$\begin{aligned}
\int \frac{11}{x^2+3x}\ dx &= \frac{11}{3}\int\left(\frac{1}{x}-\frac{1}{x+3}\right)\ dx \\
&= \frac{11}{3}(\ln|x| - \ln|x+3|) + c \\
&= \frac{11}{3}\ln\left|\frac{x}{x+3}\right| + c \\
\text{So, } \frac{y^2}{2} &= \frac{11}{3}\ln\left|\frac{x}{x+3}\right| + c \\
y &= \sqrt{\frac{22}{3}\ln\left|\frac{x}{x+3}\right| + 2c}
\end{aligned}$$

Letting $k = 2c$ we have:

$$y = \sqrt{\frac{22}{3}\ln\left|\frac{x}{x+3}\right| + k}$$

9.

$$\begin{aligned}
6\frac{dy}{dx} &= 2xy^2 - 2xy \\
&= 2xy(y-1) \\
\frac{6}{y(y-1)}\frac{dy}{dx} &= 2x \\
\frac{3}{y(y-1)}\frac{dy}{dx} &= x \\
3\int \frac{3}{y(y-1)}\ dy &= \int x\ dx
\end{aligned}$$

Expressing the left hand side as partial fractions:

$$\begin{aligned}
\frac{3}{y(y-1)} &= \frac{A}{y} + \frac{B}{y-1} \\
A(y-1) + By &= 1 \\
(A+B)y - A &= 1 \\
A &= -1 \\
A + B &= 0 \\
B &= 1
\end{aligned}$$

So we have that the integral is:

$$\begin{aligned}
3\int\left(\frac{1}{y-1}-\frac{1}{y}\right)\,dy &= \int x\,dx \\
3(\ln|y-1|-\ln|y|) &= \frac{x^2}{2}+c \\
3\ln\left|\frac{y-1}{y}\right| &= \frac{x^2}{2}+c \\
6\ln\left|\frac{y-1}{y}\right| &= x^2+2c
\end{aligned}$$

Setting $k=2c$ we have that:

$$6\ln\left|\frac{y-1}{y}\right| = x^2+k$$

10.

$$\begin{aligned}
\frac{dy}{dx}3(x^2-1) &= 6(y^2+2y) \\
\frac{dy}{dx}(x^2-1) &= 2(y^2+2y) \\
\frac{dy}{dx}\frac{1}{y^2+2y} &= \frac{2}{x^2-1} \\
\int\frac{1}{y^2+2y}\,dy &= \int\frac{2}{x^2-1}\,dx
\end{aligned}$$

For the partial fractions on the left:

$$\begin{aligned}
\frac{1}{y^2+2y} &= \frac{1}{y(y+2)} \\
&= \frac{A}{y}+\frac{B}{y+2}
\end{aligned}$$

So this means that:

$$\begin{aligned}
A(y+2)+By &= 1 \\
2A+(A+B)y &= 1
\end{aligned}$$

Equating the coefficients:

$$\begin{aligned}
A &= \frac{1}{2} \\
A+B &= 0 \\
B &= -\frac{1}{2}
\end{aligned}$$

For the partial fractions on the right:

$$\begin{aligned}\frac{1}{x^2-1} &= \frac{1}{(x+1)(x-1)} \\ &= \frac{C}{x+1} + \frac{D}{x-1}\end{aligned}$$

This gives us:

$$\begin{aligned}C(x-1) + D(x+1) &= 1 \\ (D-C) + (C+D)x &= 1\end{aligned}$$

Therefore we get:

$$\begin{aligned}D - C &= 1 \\ C &= -D \\ D &= \frac{1}{2} \\ C &= -\frac{1}{2}\end{aligned}$$

Using what we've found we have the expressions:

$$\begin{aligned}\frac{1}{2}\int\left(\frac{1}{y} - \frac{1}{y+2}\right)\, dy &= 2 \cdot \frac{1}{2}\int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)\, dx \\ \frac{1}{2}(\ln|y| - \ln|y+2|) &= \ln|x-1| - \ln|x+1| + k \\ \frac{1}{2}\ln\left|\frac{y}{y+2}\right| &= \ln\left|\frac{x-1}{x+1}\right| + k\end{aligned}$$

8.1 Repetition is the Key to Success

1. Put both fractions over a common denominator:

$$\begin{aligned}\frac{3}{x+2}+\frac{2}{5} &= \frac{2\times 5}{5(x+2)}+\frac{2(x+2)}{5(x+2)}\\ &= \frac{15+2(x+2)}{5(x+2)}\\ &= \frac{2x+19}{5(x+2)}\\ &= \frac{2x+19}{5x+10}\end{aligned}$$

2.

$$\begin{aligned}\frac{x+3}{5}+\frac{2x-1}{6} &= \frac{6(x+3)}{5\times 6}+\frac{5(2x-1)}{5\times 6}\\ &= \frac{6(x+3)+5(2x-1)}{30}\\ &= \frac{6x+18+10x-5}{30}\\ &= \frac{16x+13}{30}\end{aligned}$$

3.

$$\begin{aligned}\frac{x+3}{2x-1}+\frac{x^2}{x-3} &= \frac{(x+3)(x-3)}{(2x-1)(x-3)}+\frac{x^2(2x-1)}{(2x-1)(x-3)}\\ &= \frac{(x+3)(x-3)+x^2(2x-1)}{(2x-1)(x-3)}\\ &= \frac{x^2-9+2x^3-x^2}{2x^2-6x-x+3}\\ &= \frac{2x^3-9}{2x^2-7x+3}\end{aligned}$$

4.

$$\begin{aligned}\frac{x}{5}-\frac{2x}{6} &= \frac{x}{5}-\frac{x}{3}\\ &= \frac{x}{5}+\frac{-x}{3}\\ &= \frac{3x}{5\times 3}+\frac{-5x}{5\times 3}\\ &= \frac{-2x}{15}\end{aligned}$$

5. We need to first split the denominator up ito a product:

$$\frac{4x-10}{x^2-2x-8} = \frac{4x-10}{(x-4)(x+2)}$$

Now we solve the following for A and B:

$$\frac{A}{x-4} + \frac{B}{x+2} = \frac{4x-10}{(x-4)(x-2)}$$

Multiply through by $x-4$:

$$A + \frac{B(x-4)}{x+2} = \frac{4x-10}{x+2}$$

Multiply through by $x+2$:

$$A(x+2) + B(x-4) = 4x-10$$

Equate the coefficient in x, and the constant coefficient:

$$\begin{aligned}(A+B)x + (2A-4B) &= 4x-10\\ A+B &= 4\\ A &= 4-B\\ 2A-4B &= -10\\ 2(4-B)-4B &= -10\\ 6B &= 18\\ B &= 3\\ A &= 1\end{aligned}$$

So we have that:

$$\frac{4x-10}{x^2-2x-8} = \frac{1}{x-4} + \frac{3}{x+2}$$

6. Split the denominator into a product:

$$\frac{12x+16}{x^2+2x-3} = \frac{12x+16}{(x+3)(x-1)}$$

Solve for A and B:

$$\frac{12x+16}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

Then we have:

$$\begin{aligned}A(x-1) + B(x+3) &= 12x+16\\ (A+B)x + (3B-A) &= 12x+16\\ A+B &= 12\\ 3B-A &= 16\end{aligned}$$

Solving these simultaneously:

$$\begin{aligned} A &= 12 - B \\ 3B - (12 - B) &= 16 \\ 4B - 12 &= 16 \\ 4B &= 28 \\ B &= 7 \\ A &= 5 \end{aligned}$$

So, we have that:

$$\frac{12x+16}{x^2+2x-3} = \frac{5}{x+3} + \frac{7}{x-1}$$

7.

$$\begin{aligned} \frac{x+17}{x^2+x-2} &= \frac{x+17}{(x+2)(x-1)} \\ &= \frac{A}{x+2} + \frac{B}{x-1} \end{aligned}$$

Then we have that:

$$\begin{aligned} A(x-1) + B(x+2) &= x+17 \\ (A+B)x + (2B-A) &= x+17 \\ A+B &= 1 \\ 2B - A &= 17 \end{aligned}$$

Solve simultaneously:

$$\begin{aligned} A &= 1 - B \\ 2B - (1 - B) &= 17 \\ 3B - 1 &= 17 \\ 3B &= 18 \\ B &= 6 \\ A &= -5 \end{aligned}$$

So:

$$\frac{x+17}{x^2+x-2} = \frac{6}{x-1} - \frac{5}{x+2}$$

8. To split the denominator into a product we first notice that it is a difference of 2 squares. $x^2 - 4 = (x+2)(x-2)$:

$$\begin{aligned}\frac{2x+8}{x^2-4} &= \frac{2x+8}{(x+2)(x-2)} \\ &= \frac{A}{x+2} + \frac{B}{x-2}\end{aligned}$$

Then we have:

$$\begin{aligned}A(x-2) + B(x+2) &= 2x+8 \\ (A+B)x + (2B-2A) &= 2x+8 \\ A+B &= 2 \\ 2B-2A &= 8 \\ B-A &= 4 \\ B &= 4+A \\ A+(4+A) &= 2 \\ 2A &= -2 \\ A &= -1 \\ B &= 3\end{aligned}$$

So we have:

$$\frac{2x+8}{x^2-4} = \frac{3}{x-2} - \frac{1}{x+2}$$

9. Before we integrate, we need to split this up into partial fractions:

$$\frac{2x-2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

Then we have:

$$\begin{aligned}A(x-2) + Bx &= 2x-2 \\ (A+B)x - 2A &= 2x-2\end{aligned}$$

This means that we have:

$$\begin{aligned}A &= 1 \\ B &= 1\end{aligned}$$

So back to the integration:

$$\begin{aligned}\int \frac{2x-2}{x(x-2)}\,dx &= \int \left(\frac{1}{x} + \frac{1}{x-2}\right)\,dx \\ &= \ln|x| + \ln|x-2| + k \\ &= \ln|x^2-2x| + k\end{aligned}$$

10. Split into partial fractions:

$$\begin{aligned}\frac{7x-5}{x^2-2x-3} &= \frac{7x-5}{(x-3)(x+1)} \\ &= \frac{A}{x-3}+\frac{B}{x+1}\end{aligned}$$

So we have that:

$$A(x+1)+B(x-3)=7x-5$$

This means that:

$$\begin{aligned}A+B&=7\\ A-3B&=-5\\ A&=7-B\\ (7-B)-3B&=-5\\ 12&=4B\\ B&=3\\ A&=4\end{aligned}$$

So our integral is:

$$\begin{aligned}\int \frac{7x-5}{x^2-2x-3}\,dx &= \int \left(\frac{3}{x-3}+\frac{3}{x+1}\right)\,dx \\ &= 4\ln|x-3|+3\ln|x+1|+k\end{aligned}$$

11.

$$\begin{aligned}\frac{6x+11}{x^2-3x-4} &= \frac{6x+11}{(x-4)(x+1)} \\ &= \frac{A}{x-4}+\frac{B}{x+1}\end{aligned}$$

So we have:

$$A(x+1)+B(x-4)=6x+11$$

This gives us:

$$\begin{aligned}A+B&=6\\ A-4B&=11\\ A&=6-B\\ (6-B)-4B&=11\\ 6-5B&=11\\ -5B&=5\\ B&=-1\\ A&=7\end{aligned}$$

So our integral is:

$$\begin{aligned}\int \frac{6x+11}{x^2-3x-4}\,dx &= \int \left(\frac{7}{x-4} - \frac{1}{x+1}\right)\,dx\\ &= 7\ln|x-4| - \ln|x+1| + k\end{aligned}$$

12.

$$\begin{aligned}\frac{13x-55}{x^2-9x+20} &= \frac{13x-55}{(x-4)(x-5)}\\ &= \frac{A}{x-4} + \frac{B}{x-5}\end{aligned}$$

So we have:

$$A(x-5) + B(x-4) = 13x - 55$$

This gives us:

$$\begin{aligned}A + B &= 13\\ -5A - 4B &= -55\\ A &= 13 - B\\ 5A + 4B &= 55\\ 5(13-B) + 4B &= 55\\ 65 - 5B + 4B &= 55\\ B &= 10\\ A &= 3\end{aligned}$$

So our integral is:

$$\begin{aligned}\int \frac{13x-55}{x^2-9x+20}\,dx &= \int \left(\frac{3}{x-4} + \frac{10}{x-5}\right)\,dx\\ &= 3\ln|x-4| + 10\ln|x-5| + k\end{aligned}$$

8.2 Seperation of Variables

1.

$$\frac{dy}{dx} = 2x + 3$$

This problem is of the form $\frac{dy}{dx}P(x)Q(y) = R(x)S(y)$, where $P(x) = 1$, $Q(y) = 1$, $R(x) = 2x + 3$ and $S(y) = 1$.

"Multiplying through by dx and integrating"

$$\int 1\ dy = \int 2x + 3\ dx$$
$$y = x^2 + 3x + k$$

2.
$$\frac{dy}{dx} = \frac{x}{y}$$

This problem is of the form $\frac{dy}{dx}P(x)Q(y) = R(x)S(y)$, where $P(x) = 1$, $Q(y) = 1$, $R(x) = x$ and $S(y) = \frac{1}{y}$.

Re-arranging and then "multiplying through by dx and integrating":

$$\frac{dy}{dx} = \frac{x}{y}$$
$$\int y\ dx = \int x\ dx$$
$$\frac{y^2}{2} = \frac{x^2}{2} + c$$
$$y^2 = x^2 + 2c$$
$$y = \sqrt{x^2 + 2c}$$

Setting $k = 2c$ we have:

$$y = \sqrt{x^2 + k}$$

3.
$$\frac{dy}{dx} = \frac{y}{x}$$

Re-arranging and then "multiplying through by dx and integrating":

$$\int \frac{1}{y}\ dx = \int \frac{1}{x}\ dx$$
$$\ln|y| = \ln|x| + \ln|k|$$
$$= \ln|kx|$$
$$y = kx$$

4.
$$\frac{dy}{dx} = xy$$

Re-arranging and then "multiplying through by dx and integrating":

$$\begin{aligned}\int \frac{1}{y}\ dy &= \int x\ dx \\ \ln|y| &= \frac{x^2}{2} + c \\ y &= e^{\frac{x^2}{2}+c} \\ &= e^{\frac{x^2}{2}}e^c \\ &= Ae^{\frac{x^2}{2}}\end{aligned}$$

In this last step we let $A = e^c$ (which usually gives for a more convenient calculation later on!)

5.

$$\begin{aligned}\frac{dy}{dx}x &= \frac{1}{y} \\ \int y\ dy &= \int \frac{1}{x}\ dx \\ \frac{y^2}{2} &= \ln|x| + c \\ y &= \sqrt{2(\ln|x| + c)}\end{aligned}$$

If we then let $k = 2c$ we have that:

$$y = \sqrt{2\ln|x| + k}$$

6.

$$\begin{aligned}\frac{dy}{dx}y^2 &= 2xy \\ \int y\ dy &= 2\int x\ dx \\ \frac{y^2}{2} &= x^2 + c \\ y^2 &= 2x^2 + 2c \\ y &= \sqrt{2(x^2 + c)}\end{aligned}$$

If we then let $k = 2c$ we have that:

$$y = \sqrt{2x^2 + k}$$

8.3 Combining the Tools

1. In its current form, our equation is *not* of the type $\frac{dy}{dx}P(x)Q(y) = R(x)S(y)$, but if we subtract 1 from each side we arrive at:

$$\frac{dy}{dx} = y^2 - 1$$

Now we *are* in the correct form: $P(x) = 1$, $Q(y) = 1$, $R(x) = 1$ and $S(y) = y^2 - 1$. Re-arranging and then "multiplying through by dx and integrating":

$$\int \frac{1}{y^2 - 1}\ dy = \int 1\ dx$$

On the left side we're going to need partial fractions in order to perform the integration:

$$\begin{aligned}\frac{1}{y^2-1} &= \frac{1}{(y+1)(y-1)}\\ &= \frac{A}{y+1} + \frac{B}{y-1}\end{aligned}$$

Then we have:

$$A(y-1) + B(y+1) = 1$$

Which gives us:

$$\begin{aligned}A + B &= 0\\ B - A &= 1\\ B &= 1 + A\\ A + (1 + A) &= 0\\ 2A &= -1\\ A &= -\frac{1}{2}\\ B &= \frac{1}{2}\end{aligned}$$

Back to the integral:

$$\begin{aligned}\int \frac{1}{y^2-1}\ dy &= \frac{1}{2}\int (\frac{1}{y-1} - \frac{1}{y+1})\ dy\\ \frac{1}{2}(\ln|y-1| - \ln|y+1|) &= x + k\\ \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| &= x + k\end{aligned}$$

2. Firstly, re-arrange the equation so that it's in the form we want to work with:

$$\begin{aligned}\frac{dy}{dx} + y &= y^2 \\ \frac{dy}{dx} &= y^2 - y \\ &= y(y-1) \\ \int \frac{1}{y(y-1)}\ dy &= \int 1\ dx\end{aligned}$$

Now, splitting the left hand side into partial fractions:

$$\begin{aligned}\frac{1}{y(y-1)} &= \frac{A}{y} + \frac{B}{y-1} \\ A(y-1) + By &= 1 \\ A &= -1 \\ B &= 1 \\ \int \left(\frac{1}{y-1} - \frac{1}{y}\right)\ dy &= \int 1 dx \\ \ln|y-1| - \ln|y| &= x + k \\ \ln\left|\frac{y-1}{y}\right| &= x + k\end{aligned}$$

3.

$$\begin{aligned}\frac{dy}{dx}(x^2 - 3x) &= \frac{6}{y} \\ \int y\ dy &= 6\int \frac{1}{x(x-3)}\ dx\end{aligned}$$

So, splitting into partial fractions:

$$\begin{aligned}\frac{1}{x(x-3)} &= \frac{A}{x} + \frac{B}{x-3} \\ A(x-3) + Bx &= 1 \\ A &= -\frac{1}{3} \\ B &= \frac{1}{3}\end{aligned}$$

Back to the integral:

$$\begin{aligned}
\int y\ dy &= 2\int\left(\frac{1}{x-3}-\frac{1}{x}\right)\ dx \\
\frac{y^2}{2} &= 2(\ln|x-3|-\ln|x|)+c \\
y^2 &= 4\ln\left|\frac{x-3}{x}\right|+2c
\end{aligned}$$

Letting $k=\frac{c}{2}$ we have:

$$y = 2\sqrt{\ln\left|\frac{x-3}{x}\right|+k}$$

4.

$$\begin{aligned}
\frac{dy}{dx}(x^2-1) &= \frac{7}{2y} \\
2\int y\ dy &= 7\int\frac{1}{(x+1)(x-1)}\ dx
\end{aligned}$$

So, splitting into partial fractions:

$$\begin{aligned}
\frac{1}{(x+1)(x-1)} &= \frac{A}{x+1}+\frac{B}{x-1} \\
A(x-1)+B(x+1) &= 1 \\
B &= \frac{1}{2} \\
A &= -\frac{1}{2}
\end{aligned}$$

Then back to the integral:

$$\begin{aligned}
2\int y\ dy &= \frac{7}{2}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)\ dx \\
y^2 &= \frac{7}{2}\ln\left|\frac{x-1}{x+1}\right|+k \\
y &= \sqrt{\frac{7}{2}\ln\left|\frac{x-1}{x+1}\right|+k}
\end{aligned}$$

5.

$$\begin{aligned}\frac{dy}{dx}5(x^2-3x) &= 6(y^2+y)\\ 5\int \frac{1}{y(y+1)}\ dy &= 6\int \frac{1}{x(x-3)}\ dx\end{aligned}$$

This time we're going to need partial fractions on both sides of the equality. So, for the left:

$$\begin{aligned}\frac{1}{y(y+1)} &= \frac{A}{y}+\frac{B}{y+1}\\ A(y+1)+By &= 1\\ A &= 1\\ B &= -1\\ \frac{1}{y(y+1)} &= \frac{1}{y}-\frac{1}{y+1}\end{aligned}$$

And on the right:

$$\begin{aligned}\frac{1}{x(x-3)} &= \frac{C}{x}+\frac{D}{x-3}\\ C(x-3)+Dx &= 1\\ C &= -\frac{1}{3}\\ D &= \frac{1}{3}\\ \frac{1}{x(x-3)} &= \frac{1}{3}\left(\frac{1}{x-3}-\frac{1}{x}\right)\end{aligned}$$

Substituting this back into the original problem:

$$\begin{aligned}5\int\left(\frac{1}{y}-\frac{1}{y+1}\right)\ dy &= 2\int\left(\frac{1}{x-3}-\frac{1}{x}\right)\ dx\\ 5\ln|\frac{y}{y+1}| &= 2\ln|\frac{x-3}{x}|+k\end{aligned}$$

6.

$$\begin{aligned}\frac{dy}{dx}4(2x^2+2x) &= 9(y^2-1)\\ 4\int\frac{1}{(y+1)(y-1)}\ dy &= \frac{9}{2}\int\frac{1}{x(x+1)}\ dx\end{aligned}$$

Partial fractions on the left:

$$\begin{aligned}
\frac{1}{(y+1)(y-1)} &= \frac{A}{y+1} + \frac{B}{y-1} \\
A(y-1) + B(y+1) &= 1 \\
A &= -\frac{1}{2} \\
B &= \frac{1}{2}
\end{aligned}$$

Partial fractions on the right:

$$\begin{aligned}
\frac{1}{x(x+1)} &= \frac{C}{x} + \frac{D}{x+1} \\
C(x+1) + Dx &= 1 \\
C &= 1 \\
D &= -1
\end{aligned}$$

Hence:

$$\begin{aligned}
2\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right)\, dy &= \frac{9}{2}\int \left(\frac{1}{x} - \frac{1}{x+1}\right)\, dx \\
4\ln\left|\frac{y-1}{y+1}\right| &= 9\ln\left|\frac{x}{x+1}\right| + k
\end{aligned}$$

9
Integrating Factors

Test Yourself

1. To tackle a problem ising integrating factors we require it to be in the form $\frac{dy}{dx} + P(x)y = Q(x)$.
2. The integrating factor for an equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ is always $e^{\int P(x)\ dx}$.

   Here, $P(x) = 4x^3$ so $\int P(x)\ dx = x^4$, and hence the integrating factor is e^{x^4}.
3. $P(x) = \frac{2}{x}$, and the integrating factor is $e^{\int P(x)\ dx}$, which is:

$$e^{\int \frac{2}{x}\ dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

4. We need our equation to be in the form $\frac{dy}{dx} + P(x)y = Q(x)$, so before we do anything let's sort that out first.

$$\begin{aligned} 2\frac{dy}{dx} + 10x^4y &= 6x^2 \\ \frac{dy}{dx} + 5x^4y &= 3x^2 \end{aligned}$$

   Now the integrating factor:

$$e^{\int 5x^4\ dx} = e^{x^5}$$

Finally, multiplying through by it:

$$e^{x^5}\frac{dy}{dx} + 5x^4 y e^{x^5} = 3x^2 e^{x^5}$$

5. First we need to find the integrating factor, and then to do that, we need to rearrange the equation into the required form:

$$\begin{aligned} 2x\frac{dy}{dx} + 2y &= 17x^2 \\ \frac{dy}{dx} + \frac{y}{x} &= \frac{17}{2}x \end{aligned}$$

Finding the integrating factor:

$$e^{\int \frac{1}{x}\ dx} = e^{\ln x} = x$$

Multiplying through by it:

$$x\frac{dy}{dx} + y = \frac{17}{2}x^2$$

Writing the equation in the requested form (using the product rule):

$$\frac{d}{dx}(xy) = \frac{17}{2}x^2 \text{ or } \frac{d}{dx}(2xy) = 17x^2$$

6. Rearranging:

$$\frac{dy}{dx} + 6x^2 y = 12e^{-2x^3}$$

Finding the integrating factor:

$$e^{\int 6x^2\ dx} = e^{2x^3}$$

Multiplying through by it:

$$e^{2x^3}\frac{dy}{dx} + 6x^2 y e^{2x^3} = 12$$

Finally, by the product rule:

$$\frac{d}{dx}(e^{2x^3}) = 12$$

7.

$$\frac{dy}{dx} + 2xy = 6x^3$$

Integrating factor is $e^{\int 2x\ dx} = e^{x^2}$.

$$\begin{aligned} e^{x^2}\frac{dy}{dx} + 2xye^{x^2} &= 4x^3 e^{x^2} \\ \frac{d}{dx}(e^{x^2}y) &= 4x^3 e^{x^2} \end{aligned}$$

8. First, finding the integrating factor:

$$e^{\int \frac{1}{x}\ dx} = e^{\ln x} = x$$

Multiplying through by it:

$$\begin{aligned} x\frac{dy}{dx} + y &= 8x \\ \frac{d}{dx}(xy) &= 8x \end{aligned}$$

Integrate both sides with respect to x:

$$\begin{aligned} xy &= \int 8x\ dx \\ &= 4x^2 + k \\ y &= 4x + \frac{k}{x} \end{aligned}$$

9.

$$\frac{dy}{dx} + 2xy = 10x$$

The integrating factor is $e^{\int 2x\ dx} = e^{x^2}$.

$$\begin{aligned} e^{x^2}\frac{dy}{dx} + 2xye^{x^2} &= 10xe^{x^2} \\ \frac{d}{dx}(e^{x^2}y) &= 10xe^{x^2} \end{aligned}$$

Integrating both sides with respect to x:

$$\begin{aligned} e^{x^2}y &= \int 10xe^{x^2}\ dx \\ e^{x^2}y &= 5e^{x^2} + k \\ y &= 5 + ke^{-x^2} \end{aligned}$$

10.

$$\frac{dy}{dx} + \frac{y}{x} = e^x$$

The integrating factor is $e^{\int \frac{1}{x}\ dx} = e^{\ln x} = x$.

$$\begin{aligned} x\frac{dy}{dx} + y &= xe^x \\ \frac{d}{dx}(xy) &= xe^x \end{aligned}$$

Integrating:

$$
\begin{aligned}
xy &= \int xe^x \ dx \\
xy &= xe^x - e^x + k \\
y &= e^x - \frac{e^x}{x} + \frac{k}{x}
\end{aligned}
$$

9.1 Troubling forms

1. If $\frac{dy}{dx} + P(x)y = Q(x)$, then the integrating factor is $e^{\int P(x) \ dx}$.
2. For an expression of the form $\frac{dy}{dx} + P(x)y = Q(x)$, the integrating factor is $e^{\int P(x) \ dx}$. Here, $P(x) = 3$, so the integrating factor is $e^{\int 3 \ dx} = e^{3x}$.
3. $P(x) = 4x$, so the integrating factor is $e^{\int 4x \ dx} = e^{2x^2}$
4. First we need to rearrange the equation so that it is in the required form, which we do here by dividing through by $3x$:
$$
\begin{aligned}
3x\frac{dy}{dx} + 9x^2y &= 1 \\
\frac{dy}{dx} + 3xy &= \frac{1}{3x}
\end{aligned}
$$
Then the integrating factor is $e^{\int 3x \ dx} = e^{\frac{3}{2}x^2}$.
5. $$e^{\int \frac{1}{x^3} \ dx} = e^{-\frac{1}{2x^2}}$$
6. Rearranging:
$$
\begin{aligned}
4x\frac{dy}{dx} + 8y &= 16x^2 \\
\frac{dy}{dx} + \frac{2y}{x} &= 4x
\end{aligned}
$$
So the integrating factor is $e^{\int \frac{2}{x} \ dx} = e^{2\ln x} = e^{\ln x^2} = x^2$.
7. Rearranging:
$$\frac{dy}{dx} + \frac{y}{x} = 6$$
Integrating factor: $e^{\int \frac{1}{x} \ dx} = e^{\ln x} = x$. Multiplying through by the integrating factor:
$$x\frac{dy}{dx} + y = 6x$$

8. Integrating factor: $e^{\int 3x^2\ dx} = e^{x^3}$. Multiplying through by the integrating factor:
$$e^{x^3}\frac{dy}{dx} + 3x^2ye^{x^3} = 4xe^{x^3}$$

9.2 Productivity

1. Remember that the product rule for differentiation says that:
$$\frac{d}{dx}(Q(x)R(x)) = Q(x)\frac{d}{dx}(R(x)) + R(x)\frac{d}{dx}(Q(x))$$
From this, our answer is:
$$\frac{d}{dx}(3xy) = 12x$$
because
$$\begin{aligned}\frac{d}{dx}(3xy) &= 3x\frac{d}{dx}(y) + y\frac{d}{dx}(3x)\\ &= 3x\frac{dy}{dx} + 3y\end{aligned}$$

2. Using the product rule:
$$\frac{d}{dx}(4x^2y) = e^{x^2}$$

3. Using the product rule:
$$\frac{d}{dx}(e^{3x}y) = 16$$

4. Using the product rule:
$$\frac{d}{dx}(e^{x^2}y) = 8x$$

5. Finding the integrating factor: $e^{\int 6x\ dx} = e^{3x^2}$. Multiplying through by the integrating factor:
$$e^{3x^2}\frac{dy}{dx} + 6xye^{3x^2} = 4xe^{3x^2}$$
Using the product rule:
$$\frac{d}{dx}(e^{3x^2}y) = 4xe^{3x^2}$$

6. Rearranging:

$$\frac{dy}{dx} + 2\frac{y}{x} = 2x^2$$

Integrating factor: $e^{\int \frac{2}{x}\ dx} = e^{2\ln x} = e^{\ln x^2} = x^2$. Multiplying through by the integrating factor:

$$x^2\frac{dy}{dx} + 2xy = x^4$$

By the product rule:

$$\frac{d}{dx}(x^2y) = x^4$$

9.3 The Finishing Line

1. First, let's find the integrating factor:

$$e^{\int \frac{1}{x}\ dx} = e^{\ln x} = x$$

Multiplying through by it:

$$x\frac{dy}{dx} + y = 2x$$

Rewrite the left side making use of the product rule for differentiation:

$$\frac{d}{dx}(xy) = 2x$$

Integrate both sides:

$$\begin{aligned} xy &= \int 2x\ dx \\ &= x^2 + k \end{aligned}$$

And rearrange to give an answer explicitly in terms of y:

$$y = x + \frac{k}{x}$$

2.

$$\begin{aligned} x^2\frac{dy}{dx} + xy &= x \\ \frac{dy}{dx} + \frac{y}{x} &= \frac{1}{x} \end{aligned}$$

Integrating factor: $e^{\int \frac{1}{x} dx} = x$

$$\begin{aligned} x\frac{dy}{dx} + y &= 1 \\ \frac{d}{dx}(xy) &= 1 \\ xy &= x + k \\ y &= 1 + \frac{k}{x} \end{aligned}$$

3.

$$\frac{dy}{dx} + 3y = 4$$

Integrating factor: $e^{\int 3 \ dx} = e^{3x}$.

$$\begin{aligned} e^{3x}\frac{dy}{dx} + 3ye^{3x} &= 4e^{3x} \\ \frac{d}{dx}(e^{3x}y) &= 4e^3x \\ e^{3x}y &= \frac{4}{3}e^{3x} + k \\ y &= \frac{4}{3} + ke^{-3x} \end{aligned}$$

4.

$$\frac{dy}{dx} + 2xy = 6x$$

Integrating factor: $e^{\int 2x \ dx} = e^{x^2}$

$$\begin{aligned} \frac{d}{dx}(e^{x^2}y) &= 6xe^{x^2} \\ e^{x^2}y &= 3e^{x^2} + k \\ y &= 3 + ke^{-x^2} \end{aligned}$$

5.

$$\frac{dy}{dx} + 4x^3y = 4x^3$$

Integrating factor: $e^{\int 4x^3 \ dx} = e^{x^4}$

$$\begin{aligned} \frac{d}{dx}(e^{x^4}y) &= 4x^3e^{x^4} \\ e^{x^4}y &= e^{x^4} + k \\ y &= 1 + ke^{-x^4} \end{aligned}$$

6.
$$\frac{dy}{dx} + 3x^2y = 9x^2$$

Integrating factor: $e^{\int 3x^2\ dx} = e^{x^3}$.

$$\begin{aligned}\frac{d}{dx}(e^{x^3}y) &= 9x^2e^{x^3}\\ e^{x^3}y &= 3e^{x^3} + k\\ y &= 3 + ke^{-x^3}\end{aligned}$$

We have that when $x = 0$, $y = 0$. Hence $e^{-x^3} = e^0 = 1$. So:

$$\begin{aligned}0 &= 3 + k\\ k &= -3\end{aligned}$$

Therefore, we have:
$$y = 3 - 3e^{-x^3}$$

10
Mechanics

Test Yourself

1. Let's decipher what we're given: $t_0 = 0$, $t_1 = 10$, $v_0 = 4$ and $a = 2$. Also, we can assume that $x_0 = 0$. Then, to find the velocity we use:

$$\begin{aligned} v_1 &= v_0 + \int_{t_0}^{t_1} a \; dt \\ &= 4 + \int_0^{10} 2 \; dt \\ &= 4 + [2t]_0^{10} \\ &= 4 + (20 - 0) \\ &= 24\text{ms}^{-1} \end{aligned}$$

Then, to find the displacement we use:

$$\begin{aligned}
x_1 &= x_0 + v_0(t_1 - t_0) + \int_{t_0}^{t_1} \left(\int_{t_0}^{t_n} a\ dt \right)\ dt_n \\
&= 0 + 4(10 - 0) + \int_0^{10} \left(\int_0^{t_n} 2\ dt \right)\ dt_n \\
&= 4 \cdot 10 + \int_0^{10} \left([2t]_0^{t_n} \right)\ dt_n \\
&= 40 + \int_0^{10} (2t_n)\ dt_n \\
&= 40 + [t_n^2]_0^{10} \\
&= 40 + (100 - 0) \\
&= 140\text{m}
\end{aligned}$$

2. Let's decipher what we're given: $t_0 = 2$, $t_1 = 4$, $v_0 = 4$, $x_0 = 2$ and $a = \frac{2}{t^2} = 2t^{-2}$. Then, to find the velocity we use:

$$\begin{aligned}
v_1 &= v_0 + \int_{t_0}^{t_1} a\ dt \\
&= 4 + \int_2^4 2t^{-2}\ dt \\
&= 4 + \left[-2t^{-1} \right]_2^4 \\
&= 4 + \left(-\frac{2}{4} + \frac{2}{2} \right) \\
&= 4 + \left(1 - \frac{1}{2} \right) \\
&= \frac{9}{2}\text{ms}^{-1}
\end{aligned}$$

Then, to find the displacement we use:

$$
\begin{aligned}
x_1 &= x_0 + v_0(t_1 - t_0) + \int_{t_0}^{t_1} \left(\int_{t_0}^{t_n} a \ dt \right) \ dt_n \\
&= 2 + 4(4-2) + \int_2^4 \left(\int_2^{t_n} 2t^{-2} \ dt \right) \ dt_n \\
&= 2 + 4 \cdot 2 + \int_2^4 \left(\left[-2t^{-1} \right]_2^{t_n} \right) \ dt_n \\
&= 2 + 8 + \int_2^4 \left(1 - \frac{2}{t_n} \right) \ dt_n \\
&= 10 + [t_n - 2\ln(t_n)]_2^4 \\
&= 10 + (4 - 2\ln 4) - (2 - 2\ln 2) \\
&= 12 + 2(\ln 2 - \ln 4) \\
&= 12 + 2\ln\left(\frac{1}{2}\right) \\
&= 12 + \ln\left(\frac{1}{2}\right)^2 \\
&= 12 + \ln\left(\frac{1}{4}\right) \text{ m}
\end{aligned}
$$

3. This is purely memory:

   A body will continue in uniform motion unless acted upon by a force.

   Force equals rate of change of momentum.

   For every action, there is an equal and opposite reaction.

4. We have constant mass so we can use:

$$
\begin{aligned}
F &= ma \\
&= 5 \cdot 3 \\
&= 15\text{N}
\end{aligned}
$$

5. We do not have a constant mass here so we will use:

$$
F = m\frac{dv}{dt} + v\frac{dm}{dt}
$$

We have $m = (20 - t)$ and $v = (3t^2 + 2)$. So $\frac{dm}{dt} = -1$ and $\frac{dv}{dt} = t$. Hence:

$$\begin{aligned}
F &= (20 - t)6t - 1 \cdot (3t^2 + 2) \\
&= 120t - 6t^2 - 3t^2 - 2 \\
&= 120t - 9t^2 - 2 \\
\text{When } t = 5 \\
F &= 600 - 225 - 2 \\
&= 373\text{N}
\end{aligned}$$

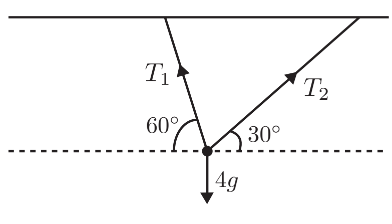
6. We have that $D = 8$ and $m = 40$. We find the terminal velocity by:

$$\begin{aligned}
v &= \sqrt{\frac{mg}{D}} \\
&= \sqrt{\frac{40 \cdot 9.8}{8}} \\
&= \sqrt{49} \\
&= 7\text{ms}^{-1}
\end{aligned}$$

7. We have that $D = 20$ and $v = 15$. Rearranging $\sqrt{\frac{mg}{D}}$, we get:

$$\begin{aligned}
m &= \frac{Dv^2}{g} \\
&= \frac{20 \cdot (15^2)}{g} \\
&= \frac{20 \cdot 225}{g} \\
&= \frac{4500}{g}
\end{aligned}$$

8. We will require the following diagram:



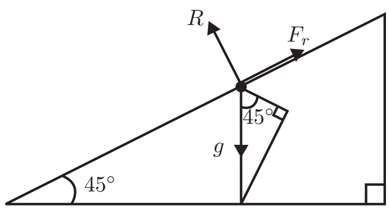
If we resolve horiontally:

$$
\begin{aligned}
T_1 \cos 60 &= T_2 \cos 30 \\
T_1 \frac{1}{2} &= T_2 \frac{\sqrt{3}}{2} \\
T_1 &= T_2 \sqrt{3}
\end{aligned}
$$

If we resolve vertically:

$$
\begin{aligned}
T_1 \sin 60 + T_2 \sin 30 &= 4g \\
T_1 \frac{\sqrt{3}}{2} + T_2 \frac{1}{2} &= 4g \\
T_1 \sqrt{3} + T_2 &= 8g \\
\left(T_2 \sqrt{3}\right) \sqrt{3} + T_2 &= 8g \text{ (from previous result)} \\
3T_2 + T_2 &= 8g \\
4T_2 &= 8g \\
T_2 &= 2g \\
\text{So, } T_1 &= 2\sqrt{3}g
\end{aligned}
$$

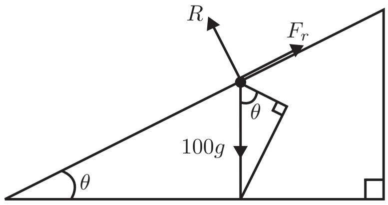
9. The diagram looks as follows:



If we resolve parallel to the plane:

$$
\begin{aligned}
F_r &= g \sin 45 \\
&= g \cdot \frac{1}{\sqrt{2}} \\
&= \frac{g}{\sqrt{2}}
\end{aligned}
$$

10. The diagram looks as follows:



We need to find the point where the frictional force becomes equal to the force down the slope as this is the point when the box begins to move.

If we resolve perpendicular to the plane:

$$R = 100g\cos\theta$$

Then we use:

$$\begin{aligned} F_r &= \mu R \\ &= \frac{1}{\sqrt{3}} \cdot 100g\cos\theta \end{aligned}$$

If we resolve parallel to the plane:

$$\begin{aligned} 100g\sin\theta &= F_r \\ 100g\sin\theta &= \frac{1}{\sqrt{3}} \cdot 100g\cos\theta \\ sin\theta &= \frac{1}{\sqrt{3}} \cdot \cos\theta \\ \frac{\sin\theta}{\cos\theta} &= \frac{1}{\sqrt{3}} \\ \tan\theta &= \frac{1}{\sqrt{3}} \\ \theta &= 30^\circ \end{aligned}$$

As it moves one degree per second, it will take 30 seconds to slide.

10.1 Where You Want to Be

1. Let's decipher what we're given: $t_0 = 0$, $t_1 = 5$, $v_0 = 3$ and $a = 5$. Then, to find the velocity we use:

$$\begin{aligned} v_1 &= v_0 + \int_{t_0}^{t_1} a\ dt \\ &= 3 + \int_0^5 5\ dt \\ &= 3 + [5t]_0^5 \\ &= 3 + (25 - 0) \\ &= 28\text{ms}^{-1} \end{aligned}$$

2. Note: now we have $t_1 = 10$. Also, we can assume that $x_0 = 0$. Then, to find the displacement we use:

$$\begin{aligned} x_1 &= x_0 + v_0(t_1 - t_0) + \int_{t_0}^{t_1} \left(\int_{t_0}^{t_n} a\ dt \right)\ dt_n \\ &= 0 + 3(10 - 0) + \int_0^{10} \left(\int_0^{t_n} 5\ dt \right)\ dt_n \\ &= 3 \cdot 10 + \int_0^{10} \left([5t]_0^{t_n} \right)\ dt_n \\ &= 30 + \int_0^{10} (5t_n)\ dt_n \\ &= 30 + \left[\frac{5}{2} t_n^2 \right]_0^{10} \\ &= 30 + (250 - 0) \\ &= 280\text{m} \end{aligned}$$

3. Let's decipher what we're given: $t_0 = 0$, $t_1 = 3$, $v_0 = 20$ and $a = 4$. Then, to find the velocity we use:

$$\begin{aligned} v_1 &= v_0 + \int_{t_0}^{t_1} a\ dt \\ &= 20 + \int_0^3 4\ dt \\ &= 4 + [4t]_0^3 \\ &= 4 + (12 - 0) \\ &= 32\text{ms}^{-1} \end{aligned}$$

4. Let's decipher what we're given: $t_0 = 0$, $t_1 = 2$, $v_0 = 26$ and $a = -4$. Also, we can assume that $x_0 = 0$. Then, to find the displacement we use:

$$\begin{aligned}
x_1 &= x_0 + v_0(t_1 - t_0) + \int_{t_0}^{t_1} \left(\int_{t_0}^{t_n} a \ dt \right) \ dt_n \\
&= 0 + 26(2-0) + \int_0^2 \left(\int_0^{t_n} -4 \ dt \right) \ dt_n \\
&= 26 \cdot 2 + \int_0^2 \left([-4t]_0^{t_n} \right) \ dt_n \\
&= 52 + \int_0^2 (-4t_n) \ dt_n \\
&= 52 + \left[-2t_n^2\right]_0^2 \\
&= 52 + (-8 - 0) \\
&= 44\text{m}
\end{aligned}$$

5. This time we have an accelleration that varies with time, and we do not begin at 0 seconds. Let's decipher what we're given: $t_0 = 2$, $t_1 = 10$, $v_0 = 2$, $x_0 = 40$ and $a = 6t$. Then, to find the velocity we use:

$$\begin{aligned}
v_1 &= v_0 + \int_{t_0}^{t_1} a \ dt \\
&= 2 + \int_2^{10} 6t \ dt \\
&= 2 + \left[3t^2\right]_2^{10} \\
&= 2 + (300 - 12) \\
&= 290\text{ms}^{-1}
\end{aligned}$$

Then, to find the displacement we use:

$$\begin{aligned}
x_1 &= x_0 + v_0(t_1 - t_0) + \int_{t_0}^{t_1} \left(\int_{t_0}^{t_n} a \; dt \right) \; dt_n \\
&= 40 + 2(10-2) + \int_{2}^{10} \left(\int_{2}^{t_n} 6t \; dt \right) \; dt_n \\
&= 40 + 2 \cdot 8 + \int_{2}^{10} (\left[3t^2\right]_2^{t_n}) \; dt_n \\
&= 40 + 16 \int_{2}^{10} (3t_n^2 - 12) \; dt_n \\
&= 56 + [t_n^3 - 12t_n]_2^{10} \\
&= 56 + (1000 - 120) - (8 - 24) \\
&= 952\text{m}
\end{aligned}$$

6. Let's decipher what we're given: $t_0 = 5$, $t_1 = 5$, $v_0 = 8$ and $a = \frac{1}{t^2} = t^{-2}$. Also, we can assume that $x_0 = 0$. Then, to find the velocity we use:

$$\begin{aligned}
v_1 &= v_0 + \int_{t_0}^{t_1} a \; dt \\
&= 8 + \int_{5}^{7} t^{-2} \; dt \\
&= 8 + [-t^{-1}]_5^7 \\
&= 8 + (-\frac{1}{7} + \frac{1}{5}) \\
&= 8\frac{2}{35}\text{ms}^{-1}
\end{aligned}$$

Note: we now have that $t_1 = 14$. Then, to find the displacement we use:

$$\begin{aligned}
x_1 &= x_0 + v_0(t_1 - t_0) + \int_{t_0}^{t_1} \left(\int_{t_0}^{t_n} a \; dt \right) \; dt_n \\
&= 0 + 8(14-5) + \int_5^{14} \left(\int_5^{t_n} t^{-2} \; dt \right) \; dt_n \\
&= 8 \cdot 9 + \int_5^{14} \left(\left[-t^{-1} \right]_5^{t_n} \right) \; dt_n \\
&= 72 + \int_5^{14} \left(\frac{1}{5} - \frac{1}{t_n} \right) \; dt_n \\
&= 72 + \left[\frac{1}{5} t_n - \ln t_n \right]_5^{14} \\
&= 72 + \left(\frac{14}{5} - \ln 14 \right) - (1 - \ln 5) \\
&= 73\frac{4}{5} + \ln 5 - \ln 14 \\
&= \left(73\frac{4}{5} + \ln \frac{5}{14} \right) \text{m}
\end{aligned}$$

10.2 Faster! Faster!

1. We have a constant mass, so $\frac{dm}{dt} = 0$. Therefore:

$$\begin{aligned}
F &= m\frac{dv}{dt} + v\frac{dm}{dt} \\
&= m\frac{dv}{dt} \\
&= ma
\end{aligned}$$

$$\begin{aligned}
\text{So, } F &= 6 \cdot 5 \\
&= 30\text{N}
\end{aligned}$$

2. Once again we can use:

$$\begin{aligned}
F &= ma \\
100 &= 4a \\
a &= \frac{100}{4} \\
&= 25\text{ms}^{-1}
\end{aligned}$$

3. Note: mass is not constant here, so we cannot use $F = ma$. But we do have constant velocity, $\frac{dv}{dt} = 0$ So,

$$\begin{aligned} F &= m\frac{dv}{dt} + v\frac{dm}{dt} \\ &= v\frac{dm}{dt} \\ &= 100 \cdot \frac{d}{dt}(100 - 2t) \\ &= 100 \cdot -2 \\ &= -200\text{N} \end{aligned}$$

4. We have that $m = 4t^2$ and $v = 200t$, so $\frac{dm}{dt} = 8t$ and $\frac{dv}{dt} = 200$. Hence:

$$\begin{aligned} F &= m\frac{dv}{dt} + v\frac{dm}{dt} \\ &= 4t^2 \cdot 200 + 200t \cdot 8t \\ &= 800t^2 + 1600t^2 \\ &= 2400t^2 \end{aligned}$$

When $t = 2$ we have:

$$\begin{aligned} F &= 2400 \cdot 4 \\ &= 9600\text{N} \end{aligned}$$

5. We have that $D = 24.5$ and $m = 20$. We find the terminal velocity by:

$$\begin{aligned} v &= \sqrt{\frac{mg}{D}} \\ &= \sqrt{\frac{20 \cdot 9.8}{24.5}} \\ &= \sqrt{8} \\ &= 2\sqrt{2}\text{ms}^{-1} \end{aligned}$$

6. We have that $m = 80$ and $v = 2$. Rearranging $\sqrt{\frac{mg}{D}}$, we get:

$$\begin{aligned} D &= \frac{mg}{v^2} \\ &= \frac{80g}{2^2} \\ &= \frac{80g}{4} \\ &= 20g \end{aligned}$$

7. We have that $D = 0.5$ and $v = 10$. Rearranging $\sqrt{\frac{mg}{D}}$, we get:

$$\begin{aligned} m &= \frac{Dv^2}{g} \\ &= \frac{0.5 \cdot (10^2)}{g} \\ &= \frac{0.5 \cdot 100}{g} \\ &= \frac{50}{g} \end{aligned}$$

10.3 Resolving Forces

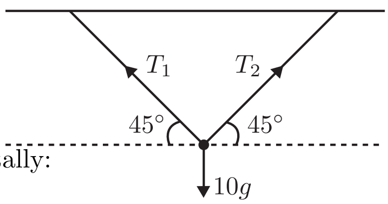
1. We will require the following diagram:

If we resolve horiontally:

$$x = T$$

So the tension in the other rope is also T.

2. We will require the following diagram:



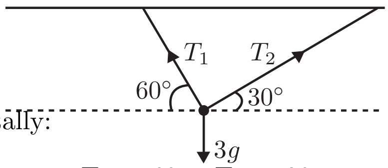
If we resolve horiontally:

$$\begin{aligned} T_1 \cos 45 &= T_2 \cos 45 \\ T_1 \frac{\sqrt{2}}{2} &= T_2 \frac{\sqrt{2}}{2} \\ T_1 &= T_2 \end{aligned}$$

If we resolve vertically:

$$\begin{aligned} T_1 \sin 45 + T_2 \sin 45 &= 10g \\ T_1 \frac{\sqrt{2}}{2} + T_2 \frac{\sqrt{2}}{2} &= 10g \\ 2T_1 \frac{\sqrt{2}}{2} &= 10g \text{ (From previous result)} \\ T_1\sqrt{2} &= 10g \\ T_1 = T_2 &= \frac{10}{\sqrt{2}} g \\ &= 5g\sqrt{2} \end{aligned}$$

3. We will require the following diagram:



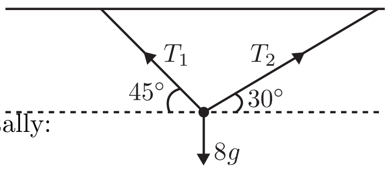
If we resolve horiontally:

$$\begin{aligned} T_1 \cos 60 &= T_2 \cos 30 \\ T_1 \frac{1}{2} &= T_2 \frac{\sqrt{3}}{2} \\ T_1 &= T_2\sqrt{3} \end{aligned}$$

If we resolve vertically:

$$\begin{aligned} T_1 \sin 60 + T_2 \sin 30 &= 3g \\ T_1 \frac{\sqrt{3}}{2} + T_2 \frac{1}{2} &= 3g \\ T_1\sqrt{3} + T_2 &= 6g \\ (T_2\sqrt{3})\sqrt{3} + T_2 &= 6g \text{ (From previous result)} \\ 3T_2 + T_2 &= 6g \\ 4T_2 &= 6g \\ T_2 &= \frac{3}{2} g \\ \text{So, } T_1 &= \frac{3\sqrt{3}}{2} g \end{aligned}$$

4. We will require the following diagram:



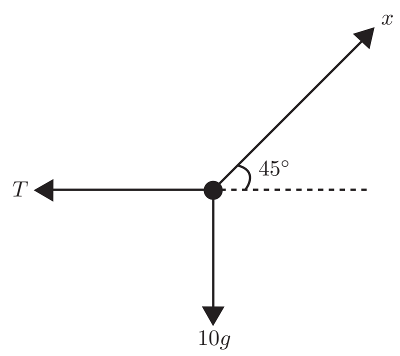
If we resolve horiontally:

$$T_1 \cos 45 = T_2 \cos 30$$
$$T_1 \frac{\sqrt{2}}{2} = T_2 \frac{\sqrt{3}}{2}$$
$$T_1 \sqrt{2} = T_2 \sqrt{3}$$

If we resolve vertically:

$$\begin{aligned}
T_1 \sin 45 + T_2 \sin 30 &= 8g \\
T_1 \frac{\sqrt{2}}{2} + T_2 \frac{1}{2} &= 8g \\
T_1 \sqrt{2} + T_2 &= 16g \\
T_2 \sqrt{3} + T_2 &= 16g \text{ (From previous result)} \\
T_2 (1 + \sqrt{3}) &= 16g \\
T_2 &= \frac{16}{(1 + \sqrt{3})} g \\
T_1 &= \frac{\sqrt{3}}{2} \cdot \frac{16}{(1 + \sqrt{3})} g \\
&= \frac{8\sqrt{6}}{(1 + \sqrt{3})} g
\end{aligned}$$

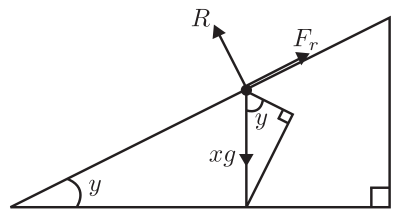
5. We will require the following diagram:



If we resolve horiontally:

$$\begin{aligned} T &= x\cos 45 \\ &= x\frac{1}{\sqrt{2}} \\ x &= T\sqrt{2} \end{aligned}$$

6. We will require the following diagram:



If we resolve up the slope:

$$F_r = xg\sin y$$

11
Sets and Functions

Test Yourself

1. This one's just memory: $\notin$ means "is not an element of".
2. $A = \{6, 12, 18 \ldots\}$ and $B = \{\ldots - 16, -8, 0, 8, 16 \ldots\}$

   $A \cup B$ is the *union* of A and B: it's made up of all those numbers in A *or* B *or both*.

   So $A \cup B = \{\ldots - 32, -24, -16, -8, 0, 6, 8, 12, 16, 18 \ldots\}$

   Hence we can see that 24 is indeed an element of $A \cup B$.
3. $C = \{5, 10, 15 \ldots\}$ and $D = \{15, 30, 45 \ldots\}$

   $C \cap D$ is the *intersection* of C and D: it's made up of all those numbers in *both* C *and* D.

   So $C \cap D = \{15, 30, 45 \ldots\}$

   Hence we can see that 20 is *not* an element of $C \cap D$.
4. $E = \{10, 20, 30 \ldots\}$ and $F = \{5, 10, 15, 20 \ldots\}$

   $F \setminus E$ is the set of all those elements of F which are not in E.

   So $F \setminus E = \{5, 15, 25, 35, 45 \ldots\}$

   Hence the 3 smallest strictly positive members of $F \setminus E$ are 5, 15 and 25.
5. $\neg p$ is simply "not p", so applying this just negates p. Therefore:

$\neg$	p
T	F
F	T

6. $p \vee q$ returns true if: p is true, q is true or both are true. So:

p	$\vee$	q
T	T	T
T	T	F
F	T	T
F	F	F
0	1	0

7. We have:

$\neg$	$(p$	$\vee$	$q)$	$\Leftrightarrow$	$(\neg$	$p)$	$\wedge$	$(\neg$	$q)$
F	T	T	T	T	F	T	F	F	T
F	T	T	F	T	F	T	F	T	F
F	F	T	T	T	T	F	F	F	T
T	F	F	F	T	T	F	T	T	F
2	0	1	0	3	1	0	2	1	0

So, yes the statements are logically equivalent.

8. We have:

$(p$	$\vee$	$q)$	$\wedge$	r	$\Leftrightarrow$	$(p$	$\wedge$	$r)$	$\wedge$	$(q$	$\wedge$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	F	T	T	T	T	F	T	F	F
T	T	F	T	T	F	T	F	F	F	F	F	T
T	T	F	F	F	T	T	F	F	F	F	F	F
F	T	T	T	T	F	F	F	T	F	T	T	T
F	T	T	F	F	T	F	F	T	F	T	F	F
F	F	F	F	T	T	F	F	F	F	F	F	T
F	F	F	F	F	T	F	F	F	F	F	F	F
0	1	0	2	0	3	0	1	0	2	0	1	0

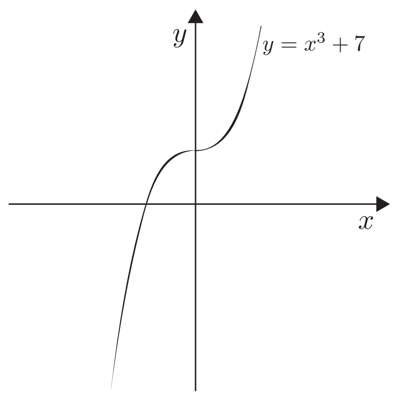
So no, the statements are not logically equivalent.

9. First, let's test injectivity: $f(0) = 0$, and $f(\pi) = 0$, so injectivity fails (there are different values in the domain which are mapped to the same value in the range).

How about surjectivity? 2 is certainly a real number but there is no real value of x which would satisfy $\sin(x) = 2$, so surjectivity fails as well.

So our function has none of the properties listed.

10. If we sketch the graph of $y = x^3 + 7$, we see:



We definitely have injectivity: for every value that the function takes in the range, there is only one value in the domain which is mapped to this point. (We can prove this formally by using calculus: there are no turning points in the function).

We also have surjectivity: given any real number, we can find a point in the domain which is mapped to it (specifically, given any $b \in \mathbb{R}$, we can take $a = \sqrt[3]{b - 7}$, and be certain that $f(a) = b$.

So our function is both injective and surjective, hence it is bijective.

11.1 Set Notation

1. This one's just memory: $\in$ means "is an element of".
2. We can most easily define $\setminus$ by defining $A \setminus B$ to be the elements in A that are not in B.
3. $\{4x|x \in \mathbb{Z}\}$ is simply the set of numbers which can be obtained by multiplying the number 4 by an integer. In other words, it is the set of integer multiples of 4.
4. $A = \{3, 6, 9, 12 \ldots\}$ and $B = \{\ldots - 10, -5, 0, 5, 10 \ldots\}$

   $A \cup B$ is the *union* of A and B: it's made up of all those numbers in A *or* B *or both*.

   So $A \cup B = \{\ldots, -10, -5, 0, 3, 5, 6, 9, 10, 12, 15 \ldots\}$

   Hence we can see that 10 is indeed an element of $A \cup B$.
5. $C = \{3, 6, 9, 12, 15 \ldots\}$ and $D = \{\ldots, -20, -10, 0, 10, 20 \ldots\}$

   $A \cap B$ is the *intersection* of C and D: it's made up of all those numbers in *both* C *and* D.

   So $C \cap D = 30, 60, 90 \ldots$

   Hence we can see that 20 is *not* an element of $C \cap D$.
6. $E = \{6, 12, 18, 24 \ldots\}$ and $F = \{3, 6, 9, 12 \ldots\}$

   $E \setminus F$ is the set of all those elements of E which are not in F.

   So $E \setminus F = \emptyset$

   So there are no elements of $E \setminus F$ which are less than 20.
7. $G = \{3, 5, 7, 9 \ldots\}$ and $H = \{3, 6, 9 \ldots\}$

   $G \setminus H = \{5, 7, 11, 13, 17 \ldots\}$

   So the smallest strictly positive member of $G \setminus H$ is 5.
8. $J = \{4, 8, 12, 16 \ldots\}$ and $K = \{\ldots - 14, -7, 0, 7, 14 \ldots\}$

   $J \cap K = \{28, 56, 84 \ldots\}$

   So the smallest strictly positive member of $J \cap K$ is 28.

11.2 Logical Equivalence

1. "If and only if" only returns true if both p and q are true, or both p and q are false. So the truth table is:

p	$\Leftrightarrow$	q
T	T	T
T	F	F
F	F	T
F	T	F
0	1	0

2. $p \wedge q$ returns true if both p and q are true, otherwis it returns false. So:

p	$\wedge$	q
T	T	T
T	F	F
F	F	T
F	F	F
0	1	0

3. We have:

p	$\wedge$	$(\neg$	$q)$	$\Leftrightarrow$	$\neg$	$(p$	$\wedge$	$q)$
T	F	F	T	T	F	T	T	T
T	T	T	F	T	T	T	F	F
F	F	F	T	F	T	F	F	T
F	F	T	F	F	T	F	F	F
0	2	1	0	3	2	0	1	0

So no, the statements are not logically equivalent.

4. Our table is:

p	$\vee$	$(q$	$\wedge$	$r)$	$\Leftrightarrow$	$(p$	$\vee$	$q)$	$\wedge$	$(q$	$\vee$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	F	T	T	T	T	T	T	T	F
T	T	F	F	T	T	T	T	F	T	T	T	T
T	T	F	F	F	T	T	T	F	T	T	T	F
F	T	T	T	T	T	F	T	T	T	F	T	T
F	F	T	F	F	T	F	T	T	F	F	F	F
F	F	F	F	T	T	F	F	F	F	F	T	T
F	F	F	F	F	T	F	F	F	F	F	F	F
0	2	0	1	0	3	0	1	0	2	0	1	0

So yes, the statements are logically equivalent.

5. Our table is:

$\neg$	$(p$	$\vee$	$q)$	$\Leftrightarrow$	$(\neg$	$p)$	$\vee$	$(\neg$	$q)$
F	T	T	T	T	F	T	F	F	T
F	T	T	F	F	F	T	T	T	F
F	F	T	T	F	T	F	T	F	T
T	F	F	F	T	T	F	T	T	F
2	0	1	0	3	1	0	2	1	0

So no, the statements are not logically equivalent.

6. Our table is:

$\neg$	$(p$	$\wedge$	$q)$	$\Leftrightarrow$	$(\neg$	$p)$	$\vee$	$(\neg$	$q)$
F	T	T	T	T	F	T	F	F	T
T	T	F	F	T	F	T	T	T	F
T	F	F	T	T	T	F	T	F	T
T	F	F	F	T	T	F	T	T	F
2	0	1	0	3	1	0	2	1	0

So yes, the statements are logically equivalent.

11.3 Functions

1. This is just memory: for every element in the range, there is at least one element in the domain that is mapped to it.
2. The definition of injective is : no two distinct elements in the domain are mapped to the same element in the range. This is an equivalent definition to the one given in the chapter which uses x and y.
3. To check for bijectivity, we need to check for both injectivity and surjectivity.

   The first fact that we are given in the statement is the definition of surjectivity, but we look closely at the second fact: it is *not* the definition of injectivity. It is actually just a property that every function has.

   So we don't have injectivity and we therefore don't have bijectivity.
4. No it isn't: the number -3 is an integer so it certainly lies in the range, but we can find no element of the domain (the natural numbers) which would be mapped to -3.

5. If we pick any element of the range which *is* mapped to, we would only be able to find *one* element of the domain which was mapped there. So yes, the function *is* injective.

6. Although we do have injectivity here, we don't have surjectivity (for example -4 is never mapped to, yet it certainly lies in the range). So no, the function *is not* bijective.

7. The function has a problem whenever an odd number is chosen in the domain: it doesn't end up being mapped to an integer, which is what we claim the range to be!

8. Neither of $\sin(x)$ or $\cos(x)$ are: there is no real number x which can satisfy $\sin(x) = 2$ or $\cos(x) = 2$.

   $\tan(x)$ is surjective though: it maps to every real number.

12
Formal Logic

Test Yourself

1. This one's just memory:

   Every non-empty set of $\mathbb{N}$ has a least element.

2. This is the set $\{1, 2, 3...\}$. So the least element is simply 1.

3. Let us try some entries. Let $f(x) = x^2 - 8x + 30$. Then $f(1) = 23$, $f(2) = 18$, $f(3) = 15$, $f(4) = 14$, $f(5) = 15$,... Because of the nature of x^2 $f(x)$ only has one turning point. Which means our least element must be 14.

4. Let $f(x) = x^2 - 9x + 42$. Then $f(1) = 34$, $f(2) = 28$, $f(3) = 24$, $f(4) = 22$, $f(5) = 22$, $f(6) = 34$,... Because of the nature of x^2 $f(x)$ only has one turning point. Which means our least element must be 22.

5. This one's just memory:

   It works for the first value of n.

   In our rule, if it's true for n, it's true for $n + 1$.

6. First, let's test it for $n = 1$:

   $$\frac{2 \cdot (1^2) + 3 \cdot 1}{2} = 3$$

   So, it holds for $n = 1$. Now, assume that it holds for n:

   $$3 + \ldots + 3n = \frac{(3n^2 + 3n)}{2}$$

Then adding on the next term gives us:

$$\begin{aligned}
3 + \ldots + 3n + 3(n+1) &= \frac{3n^2 + 3n}{2} + 3(n+1) \\
&= \frac{3n^2 + 3n + 6n + 6}{2} \\
&= \frac{3n^2 + 6n + 3 + 3n + 3}{2} \\
&= \frac{3(n^2 + 2n + 1) + 3(n+1)}{2} \\
&= \frac{3(n+1)^2 + 3(n+1)}{2}
\end{aligned}$$

So, if it holds for n then it holds for $n+1$.

7. We must "Guess" the rule $4n - \frac{n}{2}(n+1)$. First, let's test it for $n = 1$:

$$4 \cdot 1 - \frac{1}{2}(1+1) = 3$$

So, it holds for $n = 1$. Now, assume that it holds for n:

$$3 + \ldots + (4-n) = 4n - \frac{n}{2}(n+1)$$

Then adding on the next term gives us:

$$\begin{aligned}
3 + \ldots + (4-n) + (3-n) &= 4n - \frac{n}{2}(n+1) + (3-n) \\
&= 4n + 4 - \frac{n}{2}(n+1) - 1 - n \\
&= 4(n+1) - (n+1)(\frac{n}{2} + 1) \\
&= 4(n+1) - \frac{1}{2}(n+1)(n+2) \\
&= 4(n+1) - \frac{(n+1)}{2}((n+1)+1)
\end{aligned}$$

So, if it holds for n then it holds for $n+1$.

8. I am drinking coffee $\Rightarrow$ I am hot. (Note: I am hot $\Rightarrow$ I am drinking coffee is wrong. Nowhere does it say that the only time I get hot is when drinking coffee).

9. The water is not deep here $\Rightarrow$ Diving is not permitted here (Diving is not permitted here $\Rightarrow$ the water is not deep here is wrong).

10. $x > 7 \Rightarrow x \geq 7$. The contrapositive is $x < 7 \Rightarrow x \leq 7$ (You guessed it: any other order is wrong). So we note that "x is *not* greater than or equal to 7", is equivalent to saying "x is less than 7".

12.1 Proof by Induction

1. The smallest element is simply 1.
2. This set is $\{2, 4, 6, ...\}$, the set of positive even numbers, so the smallest element is 2.
3. Let $f(n) = n^2 + 4n + 6$. Then $f(1) = 3$, $f(2) = 2$, $f(3) = 3$, $f(4) = 6$,... Because of the nature of n^2, $f(n)$ only has one turning point. Which means our least element must be 2.
4. This isn't true . Take the set $\{n|n = 1, 2, 3...\}$. It has no greatest element. It *is* true that every finite, non-empty set has a greatest element.
5. This isn't true . Similarly to the previous question the set $\{-n|n = 1, 2, 3...\} = \{-1, -2, -3..\}$. It has no greatest element.

12.2 The Principle of Induction

1. For $n = 1$ we have $1 = 1$. So, it holds for $n = 1$. Now, assume that it holds for n:
$$\underbrace{1 + 1 + \ldots + 1}_{n \text{ times}} = n$$
Then adding on the next term gives us:
$$\underbrace{1 + 1 + \ldots + 1}_{n \text{ times}} + 1 = n + 1$$
So we have:
$$\underbrace{1 + 1 + \ldots + 1}_{n+1 \text{ times}} = n + 1$$
So if it holds for n then it holds for $n + 1$. So, by induction the rule is true.
2. For $n = 1$ we have $1 + 1 = 2$. So, it holds for $n = 1$. Now, assume that it holds for n:
$$2 + \ldots + 2n = n^2 + n$$

Then adding on the next term gives us:

$$\begin{aligned}2 + \ldots + 2n + 2(n+1) &= n^2 + n + 2(n+1) \\ &= n^2 + n + 2 + 1 + (n+1) \\ &= (n+1)^2 + (n+1)\end{aligned}$$

So we have:

$$2 + \ldots + 2n + 2(n+1) = (n+1)^2 + (n+1)$$

So if it holds for n then it holds for $n+1$. So, by induction the rule is true.

3. For $n = 1$ we have $(\frac{1}{2}(1+1))^2 = 1$. So, it holds for $n = 1$. Now, assume that it holds for n:

$$1^3 + \ldots + n^3 = (\frac{n}{2}(n+1))^2$$

Then adding on the next term gives us:

$$\begin{aligned}1^3 + \ldots + n^3 + (n+1)^3 &= (\frac{n}{2}(n+1))^2 + (n+1)^3 \\ &= (n+1)^2(\frac{n^2}{4} + (n+1)) \\ &= (n+1)^2(\frac{n^2+4n+4}{4}) \\ &= \frac{(n+1)^2}{4}(n+2)^2 \\ &= (\frac{(n+1)}{2}((n+1)+1))^2\end{aligned}$$

So we have:

$$1^3 + \ldots + n^3 + (n+1)^3 = (\frac{(n+1)}{2}((n+1)+1))^2$$

So if it holds for n then it holds for $n+1$. So, by induction the rule is true.

4. We will "Guess" the rule n^2. For $n = 1$ we have $1 = 1$. So, it holds for $n = 1$. Now, assume that it holds for n:

$$1 + \ldots + (2n-1) = n^2$$

Then adding on the next term gives us:

$$\begin{aligned}1+\ldots+(2n-1)+(2(n+1)-1) &= n^2+(2(n+1)-1)\\ &= n^2+2n+1\\ &= (n+1)^2\end{aligned}$$

So we have:

$$1+\ldots+(2n-1)+(2(n+1)-1)=(n+1)^2$$

So if it holds for n then it holds for $n+1$. So, by induction the rule is true.

5. We will "Guess" the rule $2n(n+1)$. For $n=1$ we have $2(1+1)=4$. So, it holds for $n=1$. Now, assume that it holds for n:

$$4+\ldots+4n=2n(n+1)$$

Then adding on the next term gives us:

$$\begin{aligned}4+\ldots+4n+4(n+1) &= 2n(n+1)+4(n+1)\\ &= 2(n+1)(n+2)\\ &= 2(n+1)((n+1)+1)\end{aligned}$$

So we have:

$$4+\ldots+4n+4(n+1)=2(n+1)((n+1)+1)$$

So if it holds for n then it holds for $n+1$. So, by induction the rule is true.

12.3 Contrapositive Statements

1. Remember: negate the statements and reverse the implication
   - I have been up the Eiffel tower at least once $\Rightarrow$ I don't have a phobia of heights.
   - x is not odd $\Rightarrow$ either x is 2 or x is not prime.
   - The positive number is not divisible by 3 $\Rightarrow$ The sum of the digits is not divisible by 3.
   - $x \le 0 \Rightarrow x$ is not positive.

2. If the number is a positive whole number, we see from *iv* that $x > 0$. It is therefore either odd or even. Now, the contrapositive of *ii* is helpful: if the number we choose is not odd, and is not 2, then we see for sure that the number is not prime.

13
Probability

Test Yourself

1.

$$\begin{aligned}9! &= 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 362880\end{aligned}$$

2. This one's just memory: $0! = 1$.
3. We do "10 choose 4", which is:

$$\begin{aligned}\binom{10}{4} &= \frac{10!}{4!6!} \\ &= 210\end{aligned}$$

4. This one's just memory:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

5. For one person to not be chosen they must be "not picked" by every computer. So, the probability of one person *not* being picked by one computer is $\frac{24}{25}$ (because the computer chooses randomly between 25 people). Then the probability of one person not being chosen by 20 computers is $\left(\frac{24}{25}\right)^{20}$.

So to find the total number of people that aren't chosen, we simply multiply this by the number of people, which is 25. So the answer is :

$$25\left(\frac{24}{25}\right)^{20}$$

6. We have to choose 1 heart in 13, 2 clubs in 13, 3 diamonds in 13 and 4 spades in 13. We then have only one way of getting this where there was 10 in 52. So we have:

$$\frac{\binom{13}{1}\binom{13}{2}\binom{13}{3}\binom{13}{4}}{\binom{52}{10}}$$

7. The mobile phone will not be faulty if it is "from Wales and not faulty" or "from Scotland and not faulty". Remember that in probability "and" means we multiply and "or" means we add. So the answer is:

$$\frac{20}{100}\cdot\frac{90}{100}+\frac{80}{100}\cdot\frac{95}{100}=\frac{47}{50}=0.94$$

8. The probability that we rolled at least a 4 means that we could have got a 4, 5 or 6. So we now only have 3 options and we want to know the probability that we got a 4. So, it is simply $\frac{1}{3}$.

9. As the first tile was above 45 it was definitely above 40 so it's one of the ones that we require for the second pick. As we didn't replace the previous tile, there are now only 19 tiles above 40 (were 20 before) and there are 99 tiles altogether (were 100). So the probability of now picking a tile over 40 is $\frac{19}{99}$.

10. We find the total fraction of cars produced that aren't red by:

$$\left(\frac{30}{100}\cdot\frac{80}{100}\right)+\left(\frac{70}{100}\cdot\frac{60}{100}\right)=\frac{33}{50}$$

The fraction of non-red cars that were produced on the north side of town is:

$$\frac{30}{100}\cdot\frac{80}{100}=\frac{6}{25}$$

Then we simply find the fraction of the non-red cars that were produced on the north side of town:

$$\frac{\frac{6}{25}}{\frac{33}{50}}=\frac{4}{11}$$

13.1 Turn that Frown Upside Down

1. This is simply:

$$\begin{aligned} 5! &= 5 \times 4 \times 3 \times 2 \times 1 \\ &= 120 \end{aligned}$$

2. This is simply:

$$\begin{aligned} 8! &= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 40320 \end{aligned}$$

3. We do "5 choose 3", which is:

$$\begin{aligned} \binom{5}{3} &= \frac{5!}{3!2!} \\ &= 10 \end{aligned}$$

4. This is "6 choose 2", which is:

$$\begin{aligned} \binom{6}{2} &= \frac{6!}{2!4!} \\ &= 15 \end{aligned}$$

5. This is "8 choose 7", which is:

$$\begin{aligned} \binom{8}{7} &= \frac{8!}{7!1!} \\ &= 8 \end{aligned}$$

6. This one's just memory:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

7. The probability that we roll a 3 is $\frac{1}{6}$ and the probability that we roll a 4 is $\frac{1}{6}$. Remember that "or" means that we add the probabilities because either can happen for a success. So the probability of rolling a 3 *or* a 4 is:

$$\begin{aligned} \frac{1}{6} + \frac{1}{6} &= \frac{2}{6} \\ &= \frac{1}{3} \end{aligned}$$

8. We have 20 balls altogether, 10 that aren't orange and would therefore result in a success. So our probability is "10 in 20", which is:

$$\frac{10}{20} = \frac{1}{2}$$

9. Because we reset the pack by replacing the chosen card, the outcome of the first choice does not affect the outcome of the second choice. This means that they *are* independent.

10. If we do not replace the card, then the card chosen by the first person is not available for the second to choose. This means that the probabilitis have changed for the second person depending on the first person's decision. Therefore they are not independent events.

13.2 Solving Probability Problems

1. For one candidate to survive they must not be picked by every computer. So, the probability of one constestant not being picked by one computer is $\frac{19}{20}$ (because the computer chooses randomly between 20 contestants). Then the probability of one contestant not being chosen by 20 computers is $\left(\frac{19}{20}\right)^{20}$. So to find the total number of contestants that survive, we simply multiply this by the number of contestants, which is 20. So the answer is :

$$20\left(\frac{19}{20}\right)^{20}$$

2. For one candidate to survive they must not be picked by every computer. So, the probability of one contestant not being picked by one computer is $\frac{9}{10}$ (because the computer chooses randomly between 10 contestants). Then the probability of one contestant not being chosen by 20 computers is $\left(\frac{9}{10}\right)^{20}$. So to find the total number of contestants that survive, we simply multiply this by the number of contestants, which is 10. So the answer is :

$$10\left(\frac{9}{10}\right)^{20}$$

3. The probability of rolling a 2, eight times is $\left(\frac{1}{6}\right)^{8}$. The probability of rolling a 6, twice is $\left(\frac{1}{6}\right)^{2}$. To find the total probability we simply multiply these together:

$$\left(\frac{1}{6}\right)^{8}\left(\frac{1}{6}\right)^{2} = \left(\frac{1}{6}\right)^{10}$$

(because they are ordered we only want 1 times this, as we only have 1 possible arrangement).

4. The probability of rolling a 2, eight times is $\left(\frac{1}{6}\right)^8$. The probability of rolling a 6, twice is $\left(\frac{1}{6}\right)^2$. To find the total probability we simply multiply these together:
$$\left(\frac{1}{6}\right)^8\left(\frac{1}{6}\right)^2=\left(\frac{1}{6}\right)^{10}$$
However, this time we can have them in any order. So, we must find all the possible ways to split 10 into 8 and 2. This is just "10 choose 8" (or "10 choose 2", they're the same). So we have:
$$\binom{10}{8}\left(\frac{1}{6}\right)^{10}$$

5. The two ways that they could enjoy their CD is to either get the rock hits and enjoy rock, or get the pop hits and enjoy pop. Remember, in probability if we say "and" we multiply and if we say "or" we add. So the probability of getting the rock hits *and* enjoying rock is $\frac{3}{10}\cdot\frac{3}{10}=\frac{9}{100}$. The probability of getting the pop hits *and* enjoying pop is $\frac{7}{10}\cdot\frac{9}{10}=\frac{63}{100}$. Then they enjoyed their CD if they if they got what they wanted so:
$$\begin{aligned}\frac{9}{100}+\frac{63}{100}&=\frac{72}{100}\\&=0.72\end{aligned}$$

6. The probability that our ornament is glass is $\frac{350}{1000}$ and the probability that a glass ornament didn't break is $\frac{1}{10}$. So, the probability that both of these events occured is:
$$\frac{350}{1000}\cdot\frac{1}{10}=\frac{35}{1000}$$
The probability that our ornament is clay is $\frac{650}{1000}$ and the probability that a clay ornament didn't break is $\frac{8}{10}$.
$$\frac{650}{1000}\cdot\frac{8}{10}=\frac{65}{125}$$
To find the amount of money that we are expected to have, we multiply the cost of each item by the likelihood of us having an intact version of that item and add them:
$$100\frac{35}{1000}+10\frac{65}{125}=£8.70$$

13.3 Conditioning

1. The probability that we rolled at least a 3 means that we could have got a 3, 4, 5 or 6. So we now only have 4 options and we want to know the probability that we got a 6. So, it is simply $\frac{1}{4}$.
2. If we already know that we rolled an odd number then we can't have got a 2. Therefore the answer is 0.
3. If we didn't spin yellow, then we could have got red, green or blue. So there are now only 3 possibilities. Hence the probability of getting green given we didn't get yellow is $\frac{1}{3}$.
4. If we have definitely chosen a multiple of 12, we could have picked: 12, 24, 36, 48, 60, 72, 84 or 96, which is 8 options. The probability that we have 60 is therefore $\frac{1}{8}$.
5. Using similar logic to earlier questions we find the total fraction CD's produced that are faulty by:

   $$\left(\frac{40}{100} \cdot \frac{5}{100}\right) + \left(\frac{60}{100} \cdot \frac{10}{100}\right) = \frac{2}{25}$$

   The fraction of faulty CD's that were produced in Newcastle is:

   $$\frac{40}{100} \cdot \frac{5}{100} = \frac{1}{50}$$

   Then we simply find the fraction of the faulty CD's that were produced in Newcastle:

   $$\frac{\frac{1}{50}}{\frac{2}{25}} = \frac{1}{4}$$

14
Distributions

Test Yourself

1. Recall that when we only know an average rate, we should use the Poisson distribution with λ equal to our average rate of occurance.

2. A single trial with a probability of success is called a Bernoulli trial. If we have multiple repetitions of a Bernoulli trial with a fixed success probability we use the Binomial distribution with the parameters: n =number of trials and p=probability of a success.

3. We are looking at a series of repetitions of a fixed trial with a fixed probability of success, so we require the Binomial distribution. We are doing 10 trials so $n = 10$ and each trial has a probability of success of $\frac{4}{52}$, so $p = \frac{1}{13}$ (success is getting an ace). Hence we want to find the probability that we draw 4 aces in 10 trials. This means we need 4 successes and 6 failures. So, for one possible ordering of our required outcome, we need:

   $$\left(\frac{1}{13}\right)^4 \left(\frac{12}{13}\right)^6$$

   But we can have our answer in any order (and we are choosing 4 aces out of the 10 cards that we pick), so we have:

   $$\binom{10}{4} \left(\frac{1}{13}\right)^4 \left(\frac{12}{13}\right)^6$$

4. We are looking at a series of repetitions of a fixed trial with a fixed probability of success, so we require the Binomial distribution. If we call it a success when we choose a yellow ball then the probability of a success is the number of yellow balls in the total number of balls. This is $\frac{2}{9}$. Hence the probabilty of a failure is $\frac{7}{9}$.We take the trial 5 times so $n = 5$. We want the probability of 2 successes which, in a fixed order is:

$$\left(\frac{2}{9}\right)^2 \left(\frac{7}{9}\right)^3$$

We can have this in any order (and we are choosing 2 yellow balls out of the 5 balls that we pick):

$$\binom{5}{2} \left(\frac{2}{9}\right)^2 \left(\frac{7}{9}\right)^3$$

5. We have an average probability so we will need the Poisson distribution with $\lambda = 250$. We require the probability that 250 people get on:

$$P(X = 250) = \frac{e^{-250} 250^{250}}{250!}$$

6. We have an average probability so we will need the Poisson distribution. He normally finds litter in 1% of 1000 rooms so $\lambda = \frac{1}{100} \cdot 1000 = 10$. We require the probability that 5 rooms have litter in:

$$P(X = 5) = \frac{e^{-10} 10^5}{5!}$$

7. We have a Binomial distribution with $n = 10$ and $p = 0.2$. Then the mean is:

$$\begin{aligned} \mu &= np \\ &= 10 \cdot 0.2 \\ &= 2 \end{aligned}$$

Then the variance is:

$$\begin{aligned} \sigma^2 &= np(1 - p) \\ &= 10 \cdot 0.2 \cdot 0.8 \\ &= 1.6 \end{aligned}$$

8. We have a Poisson distribution with $\lambda = 5$. Then the mean is:

$$\begin{aligned} \mu &= \lambda \\ &= 5 \end{aligned}$$

Then the variance is:

$$\begin{aligned} \sigma^2 &= \lambda \\ &= 5 \end{aligned}$$

9. We have a Binomial distribution with $n = 5000$ and $p = 0.001$. Recall that if we have a Binomial distribution with "large" n and "small" p then we can approximate it by the Poisson distribution by setting $\lambda = np = 5$. So we then want:

$$P(X = 6) = \frac{e^{-5}5^6}{6!}$$

10. We have a Binomial distribution with $n = 1000$ and $p = 0.003$. Recall that if we have a Binomial distribution with "large" n and "small" p then we can approximate it by the Poisson distribution by setting $\lambda = np = 3$. We cannot meet the demand if the demand is greater than 2. We can find this by:

$$\begin{aligned} 1 - P(X \leq 2) &= 1 - \{P(X = 0) + P(X = 1) + P(X = 2)\} \\ &= 1 - \left(\frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!}\right) \\ &= 1 - \left(e^{-3} + 3e^{-3} + \frac{9e^{-3}}{2}\right) \\ &= 1 - \left(4e^{-3} + \frac{9e^{-3}}{2}\right) \end{aligned}$$

14.1 Binomial Events

1. A single trial with two possible outcomes and a fixed probability of success is called a Bernoulli trial.
2. If we have multiple repetitions of a Bernoulli trial with a fixed success probability we use the Binomial distribution.
3. There is only one possible way of doing this: getting 10 in a row. The probability of a head is $\frac{1}{2}$. So the answer is:

$$\left(\frac{1}{2}\right)^{10}$$

4. Once again the probability of a head is $\frac{1}{2}$. So, for one possible ordering of our required outcome, we need:

$$\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{10}=\left(\frac{1}{2}\right)^{20}$$

But we can have our answer in any order (and we are choosing 10 heads out of the 20 flips of the coin), so we have:

$$\binom{20}{10}\left(\frac{1}{2}\right)^{20}$$

5. We are doing 25 trials so $n = 25$ and each trial has a probability of success of $\frac{13}{52}$, so $p = \frac{1}{4}$ (success is getting an heart). Hence we want to find the probability that we draw 0 or 1 in 25 trials.

For 0, we need 0 successes and 25 failures. So, for one possible ordering of our required outcome, we need:

$$\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{25}$$

But we can have our answer in any order (and we are choosing 0 hearts out of the 25 cards that we pick), so we have:

$$\binom{25}{0}\left(\frac{3}{4}\right)^{25}$$

For 1, we need 1 success and 24 failures. So, for one possible ordering of our required outcome, we need:

$$\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{24}$$

But we can have our answer in any order (and we are choosing 1 heart out of the 25 cards that we pick), so we have:

$$\binom{25}{1}\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{24}$$

We can have either of these outcomes for less than 2 hearts, so the answer is:

$$\left(\frac{3}{4}\right)^{25}+25\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{24}$$

6. The probability of choosing a particular colour in a specific trial changes because it is dependent on the outcomes of the previous trials. The trials are therefore not independant, and the success probability is not fixed.

14.2 Poisson Events

1. Every event must be independent and the average rate must not change over the interval we examine. The number of events that occur in a given time period must depend only on the length of the time period and the average rate of occurrances. For a tiny time interval, either 0 or 1 events will occur (i.e. 2 or more events cannot occur at exactly the same instant).
2. We denote this: $X \sim Po(7)$
3. We have that $\lambda = 5$ and we require:
$$\begin{aligned} P(X=2) &= \frac{e^{-5}5^2}{2!} \\ &= \frac{25}{2}e^{-5} \end{aligned}$$
4. We have that $\lambda = 45$ and we require:
$$P(X=60) = \frac{e^{-45}45^{60}}{60!}$$
5. If we have a rate of 12 every 20 minutes, this is the same as a rae of 36 every hour.So $\lambda = 36$ and we require:
$$P(X=40) = \frac{e^{-36}36^{40}}{40!}$$
6. We expect 2 dents every 10 metres, so we should expect $\frac{1}{5}$ dents per metre. Hence for 1m we have $\lambda = \frac{1}{5}$. So we require:
$$\begin{aligned} P(X=0) &= \frac{e^{-\frac{1}{5}}(\frac{1}{5})^0}{0!} \\ &= e^{-\frac{1}{5}} \end{aligned}$$
7. There are 30 children in a class and we expect 10% of children to be absent a day, so we should expect 3 children to be absent from the class. Hence for 1 class we have $\lambda = 3$. So we require:
$$\begin{aligned} P(X=1) &= \frac{e^{-3}(3^1}{1!} \\ &= 3e^{-3} \end{aligned}$$
8. There are 1000 calls per day and we expect 1% to be terminating the contract, so we should expect 10 contracts to be cancelling contracts per day. Hence for 1 day we have $\lambda = 10$. So we require:
$$P(X=15) = \frac{e^{-10}10^{15}}{15!}$$

14.3 Using Binomial and Poisson

1. We have a Binomial distribution with $n = 3$ and $p = 0.2$. Then the mean is:

$$\begin{aligned} \mu &= np \\ &= 3 \cdot 0.2 \\ &= 0.6 \end{aligned}$$

2. We have a Binomial distribution with $n = 5$ and $p = 0.4$. Then the variance is:

$$\begin{aligned} \sigma &= np(1-p) \\ &= 5 \cdot 0.4 \cdot 0.6 \\ &= 1.2 \end{aligned}$$

3. We have a Poisson distribution with $\lambda = 5$. Then the mean is:

$$\begin{aligned} \mu &= \lambda \\ &= 5 \end{aligned}$$

4. We have a Poisson distribution with $\lambda = 5$. Then the variance is:

$$\begin{aligned} \sigma &= \lambda \\ &= 12 \end{aligned}$$

5. We require n to be "suffiently large" and p to be "sufficiently small".
6. It is difficult to calculate the value of the "choose" term with very large numbers, and this isn't necessary when working with Poisson.
7. We have a Binomial distribution with $n = 50000$ and $p = 0.00008$. Recall that if we have a Binomial distribution with "large" n and "small" p then we can approximate it by the Poisson distribution by setting $\lambda = np = 4$. So we then want:
$$P(X = 6) = \frac{e^{-4}4^6}{6!}$$
8. We have a Binomial distribution with $n = 1000$ and $p = 0.002$. Recall that if we have a Binomial distribution with "large" n and "small" p then we can approximate it by the Poisson distribution by setting $\lambda = np = 2$. So for the dentist not to buy more treatment he will have to have only 0 or 1

patient requiring treatment. So the probability of him needing to buy more is 1 minus these two probabilities:

$$
\begin{aligned}
1 - P(X = 0) - (X = 1) &= 1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2^1}{1!} \\
&= 1 - e^{-2} - 2e^{-2} \\
&= 1 - 3e^{-2}
\end{aligned}
$$

15
Making Decisions

Test Yourself

1. The behavioural approach to probability asks an individual which of 2 bets they prefer, then changes one of the bets to make it more or less favourable. This process is repeated until the individual being asked is indifferent between the 2 bets, at which point the individual's personal probability elicitation can by found by finding the probability of success of the bet that was being modified. This is a better approach than the frequency approach because it does not require that we repeat an experiment "many times", as we do not know the answer to the question "how many is many?".
2. The probability of getting one head is $\frac{1}{2}$, so the probabiliy of getting two heads is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. The spinner that corresbond to this would have $\frac{1}{4}$ (i.e. 90 degrees) coloured the "success" colour, and $\frac{3}{4}$ (i.e. 270 degrees) coloured the "failure" colour.
3. There are 7 blue balls and 10 balls altogether so the probability of getting a blue ball is $\frac{7}{10}$. So we would would have a spinner 70% coloured in the "success" colour, and 30% coloured in the "failure" colour.
4. The probability of getting a 3 on a fair 6 sided die is $\frac{1}{6}$. Hence we would use a spinner which is coloured $\frac{360}{6} = 60$ degrees in the "success" colour.
5. They want to win exactly the same amount that they lose, so they value reward at its exact monetary value (so they are adopting a "1 for 1" utility function).

6. For the EMV strategy we require the game to be fair depending only on monetary value. As the probability that we win is $\frac{1}{10}$ we will expect to win 1 in every 10 games. So, to be fair the game has to cost us exactly £20 in 10 games, which is £2 a game.

7. As the game is fair our expected winnings is equal to the price we pay. So, letting s be the amount of the small prize:

$$\begin{aligned} 1 &= \frac{1}{4}s + \frac{1}{8}5 \\ 8 &= 2s + 5 \\ 2s &= 3 \\ s &= £1.50 \end{aligned}$$

8. As the game is fair our expected winnings is equal to the price we pay. So, letting p be the probability of a win:

$$\begin{aligned} 5 &= 42p \\ p &= \frac{5}{42} \end{aligned}$$

9. We find the amount of profit in the following table

	500	200
Sunny	$2 \times 480 - 250$	$2 \times 200 - 150$
Not Sunny	$2 \times 100 - 250$	$2 \times 100 - 150$

Which is the table:

	500	200
Sunny	710	250
Not Sunny	-50	50

This means that the points $(0, -50)$ and $(1, 710)$ are on the line for the 500 plants. So the equation is $y_1 = 760x_1 - 50$. Also the points $(0, 50)$ and $(1, 250)$ are on the line for the 200 plants. So the equation is $y_2 = 200x_1 + 50$.

When $y_1 = y_2$:

$$\begin{aligned} 760x_1 - 50 &= 200x_1 + 50 \\ 560x_1 &= 100 \\ x_1 &= \frac{5}{28} \end{aligned}$$

If the probability of it being sunny is greater than $\frac{5}{28}$ then the shop should choose the 500 plants, otherwise choose the smaller one. There is a 90% chance that it will be sunny, which *is* greater than our requirement. So they should get 500 plants.

10. We find the amount of profit in the following table

	A	B
Good Business	-10000	$-4000 - 7 \times 2000$
Bad Business	-10000	-4000

Which is the table:

	A	B
Good Business	-10000	-18000
Bad Business	-10000	-4000

This means that the points $(0, -10000)$ and $(1, -10000)$ are on the line for company A. So the equation is $y_1 = -10000$. Also the points $(0, -4000)$ and $(1, -18000)$ are on the line for company B. So the equation is $y_2 = -14000x_1 - 4000$.

When $y_1 = y_2$:

$$\begin{aligned} -10000 &= -14000x_1 - 4000 \\ 14000x_1 &= 6000 \\ x_1 &= \frac{3}{7} \end{aligned}$$

If the probability of being getting good business is greater than $\frac{3}{7}$ then the shop should choose company A, otherwise choose company B.

15.1 A Whole New Probability

1. The frequency approach is where we use the fact that the probability of an event happening is equal to the number of times that it happened, divided by the number of trials.
2. The frequency approach is only valid if we repeat the trial "many times" — but how many is many? It isn't always perfect for modelling real world problems as we can only use if we know exact probabilities. What is we want to find a probability for someone's opinion on something?

3. This is just memory: When the number of trials tends to infinity, the number of successes divided by the total number of trials will tend towards the probability of a single success, with probability 1.
4. The probability of getting a head is $\frac{1}{2}$. The spinner that corresbond to this would have $\frac{1}{2}$ (i.e. 180 degrees) coloured the "success" colour, and $\frac{1}{2}$ (i.e. 180 degrees) coloured the "failure" colour.
5. There are 3 white balls and 10 balls altogether so the probability of getting a white ball is $\frac{3}{10}$. So we would would have a spinner $\frac{3}{10}$ coloured in the "success" colour, and $\frac{7}{10}$ coloured in the "failure" colour. This is the same as having 108 degrees of the spinner coloured in the "success" colour, and 252 degrees of the spinner coloured in the "failure" colour.
6. $\frac{54}{360} = \frac{3}{20}$. So the persons personal elicitation is that ther is a probability of $\frac{3}{20}$ that team A wins the cricket match.

15.2 Story Time

1. This is just memory: A utility function is a function that assigns a value to a reward.
2. There is a probability of winning of $\frac{4}{10}$. So, we expect to win 4 in every 10 games. Let's let p stand for the value of our prize. For the game to be fair we need our cost per game to equal our expected winnings, so:
$$\begin{aligned} 1 &= p\frac{4}{10} \\ p &= \frac{10}{4} \\ &= £2.50 \end{aligned}$$
3. There is a probability of winning of $\frac{1}{50}$. So, we expect to win 1 in every 50 games. Let's let x stand for the price of the game. For the game to be fair we need our cost per game to equal our expected winnings, so:
$$\begin{aligned} x &= 10\frac{1}{50} \\ x &= \frac{1}{5} \\ &= 20p \end{aligned}$$
Hence if the owner wants to make a profit he must charge more than this so:
$$x > 20p$$

4. For the game to be fair now, we the cost of the game to equal the value at which the players rate the prize, so:

$$\begin{aligned} x &= 1.2 \cdot 10\frac{1}{50} \\ x &= \frac{6}{25} \\ &= 24p \end{aligned}$$

Hence if the owner should now charge $24p$.

5. Let's let p stand for the probability of a win. For the game to be fair we need our cost per game to equal our expected winnings, so:

$$\begin{aligned} 2 &= 100p \\ p &= \frac{1}{50} \end{aligned}$$

6. Let's let p stand for the probability of a win. For the game to be fair we need our cost per game to equal the value at which the players rate the prize, so:

$$\begin{aligned} 2 &= 0.8 \cdot 100p \\ p &= \frac{1}{40} \end{aligned}$$

Hence for people still to play the game should have a probability of winning $p > \frac{1}{40}$.

7. Let's let x stand for the total of the premiums that they pay in a year. For the company to expect to break even we need our x to equal the expected amount claimed, so:

$$\begin{aligned} x &= 1000\frac{1}{30} \\ x &= \frac{100}{3} \end{aligned}$$

For the monthly premium we need to divide this by 12, which gives $x = \frac{25}{9}$. Because the company should expect to make money (and pay peoples wages etc.) they should choose $x > £\frac{25}{9}$.

15.3 Decision Problems

1. We find the amount of profit in the following table

	Standard	Deluxe
Good	$1000 \times 20 - 1000 \times 5$	$1000 \times 35 - 1000 \times 8$
Bad	$1000 \times 20 - 400 \times 5$	$1000 \times 35 - 400 \times 8$

Which is the table:

	Standard	Deluxe
Good	15000	28000
Bad	3000	6000

It should be clear that regardless of whether the year is good or bad the deluxe mats make more profit. So the salesman should always buy deluxe mats.

2. We find the amount of profit in the following table

	500	1000
Corporate	$25 \times 500 - 5000$	$25 \times 950 - 9000$
No Corporate	$25 \times 200 - 5000$	$25 \times 200 - 9000$

Which is the table:

	500	1000
Corporate	1500	14750
No Corporate	0	-4000

This means that the points $(0, 0)$ and $(1, 1500)$ are on the line for the 500 box. So the equation is $y_1 = 1500x_1$. Also the points $(0, -4000)$ and $(1, 14750)$ are on the line for the 1000 box. So the equation is $y_2 = 18750x_1 - 4000$.

When $y_1 = y_2$:

$$\begin{aligned} 1500x_1 &= 18750x_1 - 4000 \\ 17250x_1 &= 4000 \\ x_1 &= \frac{16}{69} \end{aligned}$$

If the probability of getting the corporate is greater than $\frac{16}{69}$ then we choose the 1000 gifts. It is, hence choose 1000 gifts.

3. We find the amount of profit in the following table

	Large	Small
Popular	$5000 - 1000$	$2000 - 600$
Unpopular	$700 - 1000$	$700 - 600$

Which is the table:

	Large	Small
Popular	4000	1400
Unpopular	-300	100

This means that the points $(0, -300)$ and $(1, 4000)$ are on the line for the large unit. So the equation is $y_1 = 4300x_1 - 300$. Also the points $(0, 100)$ and $(1, 1400)$ are on the line for the small unit. So the equation is $y_2 = 1300x_1 + 100$.

When $y_1 = y_2$:

$$\begin{aligned} 4300x_1 - 300 &= 1300x_1 + 100 \\ 3000x_1 &= 400 \\ x_1 &= \frac{2}{15} \end{aligned}$$

If the probability of being popular is greater than $\frac{2}{15}$ then the shop should choose the large unit, otherwise choose the smaller one.

16
Geometry

Test Yourself

1. When we reflect about the line $x = y$ any point (x, y) becomes the point (y, x). The distance between two points (a_1, a_2) and (b_1, b_2) is equal to the distance between the points (a_2, a_1) and (b_2, b_1). So, it should be clear that this reflection preserves distance and is therefore an isometry.

2. This translation adds 3 to the x value and 1 to the y value, so a point (x, y) becomes the point $(x + 3, y + 1)$. The distance between two points (a_1, a_2) and (b_1, b_2) is equal to the distance between the points $(a_1 + 3, a_2 + 1)$ and $(b_1 + 3, b_2 + 1)$. So, it should be clear that this translation preserves distance and is therefore an isometry.

3. In the second triangle we can find the third angle by $\pi - \frac{3\pi}{5} - \frac{\pi}{5} = \frac{\pi}{5}$. In the first triangle we have a side of length 3, an angle of $\frac{\pi}{5}$ then a side of length 6. In the second triangle we have a side of length 3, an angle of $\frac{\pi}{5}$ then a side of length 6. So by SAS (side-angle-side) they are congruent.

4. We do not have enough information to find one of our criteria for congruence. But, we also don't have enough information to say that they aren't congruent. (In fact, it is possible to show that the triangles are not congruent, however this requires knowledge beyond the scope of this book).

5. We have 2 radii as sides of the smaller triangle hence it is an isosceles triangle. Therefore the two outer angles are the same so the inner angle is

$\pi - 2\frac{\pi}{8} = \frac{3\pi}{4}$. Then using the result from the previous question we have that angle x is a half of this so: $x = \frac{3\pi}{8}$.

6. There are a number of different ways to prove this, so as long as your proof is rigorous and unambiguous, that's fine. For the proof used in the chapter, see the subsection entitled "An Old Friend".

7. To find the other angle we do $\pi - \frac{7\pi}{12} - \frac{\pi}{6} = \frac{\pi}{4}$. Then we use the sine rule:

$$\begin{aligned}\frac{x}{\sin\frac{\pi}{4}} &= \frac{2}{\sin\frac{\pi}{6}}\\ x &= \frac{2\sin\frac{\pi}{4}}{\sin\frac{\pi}{6}}\\ &= \frac{2\frac{\sqrt{2}}{2}}{\frac{1}{2}}\\ &= 2\sqrt{2}\end{aligned}$$

8. Again, there are a number of different ways to prove this, so as long as your proof is rigorous and unambiguous, that's fine. For the proof used in the chapter see the subsection entitled "The Sine Rule".

9. By restricting the length of the lines to the distance between the equator and the north pole, our lines are each $\frac{1}{4}$ of the length of an equator. Then if we look at each point where the lines meet, from above we can see that the angle at each point must be $\frac{\pi}{2}$. This yields the angle sum of $\frac{3\pi}{2}$. This result is explained in more detail in the chapter, in the subsection entitled "Challenge Everything".

10. Two lines from the equator to the north pole that are coincident have $\frac{\pi}{2}$ at each side of the equator, and 0 at the north pole, summing in total to π. This is the smallest summation of the angles.

    Two lines from the equator to the north pole that are coincident have $\frac{\pi}{2}$ at each side of the equator, and *when traversing the reflex angle*, 2π at the north pole, summing in total to 3π. This is the largest summation of the angles.

    So if we call the sum of the angles x, then $\pi < x < 3\pi$.

16.1 Old Problems, New Tricks

1. In any rotation the points will simply move around the origin. This means that the distance between any two poins will stay the same, so it is an

isometry.

2. Let's take an example of the points $(0,0)$ and $(0,1)$. The distance between these 2 points is obviously 1. Then after a dilation of factor 2, we have the points $(0,0)$ and $(0,2)$, which has a distance of 2 between them. This means that distance isn't preserved, so it isn't an isometry.

3. The rotation will preserve the distance, but the dilation will not. So this is not an isometry.

4. Every dilation is not an isometry except for that of a factor 1. This preserves distance because all the points remain the same. Hence $z = 1$

5. a) These have 3 sides the same, so are congruent by the SSS congruence criteria.

   b) These 2 triangles have 2 sides of the same length, but the other is different. So, there is no way they can be congruent.

   c) We do not have enough information to find one of our criteria for congruence. But, we also don't have enough information to say that they aren't congruent.

   d) In the first triangle we have a side of length 12, an angle of $\frac{\pi}{4}$ then a side of length 6. In the second triangle we have a side of length 12, an angle of $\frac{\pi}{4}$ then a side of length 6. So by the SAS (side-angle-side) congruence criteria they are congruent.

   e) In the first triangle we have an angle of $\frac{\pi}{3}$, a side of length 7 then an angle of $\frac{\pi}{8}$. In the second triangle we have an angle of $\frac{\pi}{3}$, a side of length 7 then an angle of $\frac{\pi}{3}$. So by the ASA (angle-side-angle) congruence criteria they are congruent.

   f) The angles and sides are the same but do not come in the correct order. So we do not have enough information to find one of our criteria for congruence. But, we also don't have enough information to say that they aren't congruent.

16.2 Proof

1. Using the fact that we proved in the chapter that, from a chord, the angle to the centre is twice the angle to the edge we have that $x = \frac{2\pi}{7}$.

2. Because the traingle containing the angles a, d and c has two radii as its sides, it is isosceles. Hence the angles a and d are equal.

3. The angles a, b and c are the 3 angles in a triangle so $a+b+c=\pi$. Hence $a+b=\pi-c$. The angles c and d make a straight line so $c+d=\pi$, and then $d=\pi-c$. So we have that $a+b=d$, so the answer is d.

4. We have that:

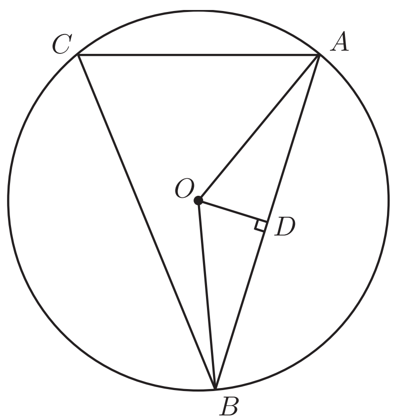
$$\begin{aligned} \sin\theta &= \frac{opposite}{hypotenuse} \\ \sin C\widehat{O}D &= \frac{\frac{b}{2}}{OC} \\ OC\sin C\widehat{O}D &= \frac{b}{2} \end{aligned}$$

5. From the previous question we have that,

$$OC\sin C\widehat{O}D = \frac{b}{2}$$

We have that $OC=R$ and that $C\widehat{O}D = C\widehat{B}A$ (because the angle at the centre is twice the angle at the edge). So $R\sin B = \frac{b}{2}$, and hence $\frac{b}{\sin B} = 2R$.

6. We draw the diagram:



$$OA\sin A\widehat{O}D = \frac{c}{2}$$

We have that $OA=R$ and thet $A\widehat{O}D = C$ So $R\sin C = \frac{c}{2}$, and hence $\frac{c}{\sin C} = 2R$.

16.3 3D Geometry

1. As the length of the sides change the angle between them will also change, so it isn't possible to say.
2. Two lines from the equator to the north pole that are coincident have $\frac{\pi}{2}$ at each side of the equator, and 0 at the north pole, summing in total to π.
3. Two lines from the equator to the north pole that are coincident have $\frac{\pi}{2}$ at each side of the equator, and *when traversing the reflex angle*, 2π at the north pole, summing in total to 3π.
4. If we divide the polygon into lots of triangles, then we can find the angles using the corresponding angles in the triangles.
5. It must be a straight line through the centre of the sphere, otherwise if we rotate around a different line it will take is off the sphere. This cannot happen as we're supposed to be doing geometry on it!
6. It must be a plane (flat surface) that passes through the centre of the sphere, otherwise it'll take us of the sphere again.
7. Any great circle is where the sphere joins a plane that passes through the center of the sphere. So, the diameter of a great cirle is always equal to the diameter of the sphere. In this case it is 6000km.
8. We find the volume of a sphere by $\frac{4}{3}\pi r^3$ which is, in our case, equal to $\frac{4}{3}\pi\mathrm{m}^3$. This means that the radius of our sphere, r, is 1. The circumfrance of a great circle around this sphere is therefore $2\pi r = 2\pi\mathrm{m}$. But to go from the top to the bottom, we only need a half of this. So the answer is $\pi\mathrm{m}$.

17
Hyperbolic Trigonometry

Test Yourself

1.

$$\begin{aligned}
\coth x &= \frac{1}{\tanh x} \\
&= \frac{e^x + e^{-x}}{e^x - e^{-x}} \\
&= \frac{e^{2x} + 1}{e^{2x} - 1} \\
\text{So, } \coth 3 &= \frac{e^6 + 1}{e^6 - 1}
\end{aligned}$$

2.

$$\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\frac{e^x + e^{-x}}{2} &= 2 \\
e^x + e^{-x} &= 4 \\
e^x - 4 + e^{-x} &= 0 \\
e^{2x} - 4e^x + 1 &= 0
\end{aligned}$$

Then we use the quadratic formula:

$$\begin{aligned} e^x &= \frac{4 \pm \sqrt{16-4}}{2} \\ &= \frac{4 \pm 2\sqrt{3}}{2} \\ &= 2 \pm \sqrt{3} \end{aligned}$$

So we have:

$$x = \ln(2 \pm \sqrt{3})$$

3.

$$\begin{aligned} \tanh x &= \frac{e^{2x}-1}{e^{2x}+1} \\ \frac{e^{2x}-1}{e^{2x}+1} &= \frac{1}{3} \\ 3e^{2x} - 3 &= e^{2x} + 1 \\ 2e^{2x} &= 4 \\ e^{2x} &= 2 \\ \text{So, } 2x &= \ln 2 \\ x &= \frac{1}{2}\ln 2 \\ &= \ln\sqrt{2} \end{aligned}$$

4. We will use the chain rule:

$$\begin{aligned} \frac{d}{dx}(2\sinh(4x)) &= 2\cosh(4x)\cdot 4 \\ &= 8\cosh(4x) \end{aligned}$$

5. We will use the chain rule, twice:

$$\begin{aligned} \frac{d}{dx}(\ln(\sinh(3x))) &= \frac{1}{\sinh(3x)}\cosh(3x)\cdot 3 \\ &= 3\coth(3x) \end{aligned}$$

6. We will use the product rule: Let's let $u = e^{2x}$ so $\frac{du}{dx} = 2e^{2x}$, and $v = \sinh x$ so $\frac{dv}{dx} = \cosh x$. Therefore we have:

$$\frac{d}{dx}(e^{2x}\sinh x) = 2e^{2x}\sinh x + e^{2x}\cosh x$$

7. We will use the chain rule:

$$\frac{d}{dx}(e^{\tanh x}) = e^{\tanh x} \cdot \text{sech}^2 x$$

8. Let's try $\tanh^3 x$:

$$\frac{d}{dx}(\tanh^3 x) = 3\tanh^2 x \cdot \text{sech}^2 x$$

So, we require a third of this for our solution. Hence:

$$\int \tanh^2 x \cdot \text{sech}^2 x \ dx = \frac{1}{3}\tanh^3 x + c$$

9.

$$\begin{aligned}\int 2\cosh(3x) \ dx &= 2\sinh(3x) \cdot \frac{1}{3} + c \\ &= \frac{2}{3}\sinh(3x) + c\end{aligned}$$

10. We have that $\cosh(2x) = \cosh^2 x + \sinh^2 x$ and $\cosh^2 x - \sinh^2 x = 1$, so $\cosh^2 x = 1 + \sinh^2 x$. Combining these 2 expressions:

$$\cosh(2x) = 1 + 2\sinh^2 x$$

Rearranging, we have:

$$\sinh^2 x = \frac{1}{2}(\cosh(2x) - 1)$$

So,

$$\begin{aligned}\int \sinh^2(3x) \ dx &= \frac{1}{2}\int (\cosh(6x) - 1) \ dx \\ &= \frac{1}{2}\left(\frac{1}{6}\sinh(6x) - x\right) + c \\ &= \frac{1}{12}\sinh(6x) - \frac{x}{2} + c\end{aligned}$$

17.1 Your New Best Friends

1.
$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\sinh 5 = \frac{e^5 - e^{-5}}{2}$$

2.
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\cosh \frac{3}{2} = \frac{e^{\frac{3}{2}} + e^{-\frac{3}{2}}}{2}$$

3.
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\tanh 2 = \frac{e^2 - e^{-2}}{e^2 + e^{-2}}$$

4.
$$\tanh x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$
$$\tanh \sqrt{2} = \frac{e^{\sqrt{2}} + e^{-\sqrt{2}}}{e^{\sqrt{2}} - e^{-\sqrt{2}}}$$

5.
$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} \\ \frac{e^x - e^{-x}}{2} &= 3 \\ e^x - e^{-x} &= 6 \\ e^x - 6 - e^{-x} &= 0 \\ \text{Multiplying through by } e^x &: \\ e^{2x} - 6e^x - 1 &= 0 \end{aligned}$$

Then we use the quadratic formula:
$$\begin{aligned} e^x &= \frac{6 \pm \sqrt{36+4}}{2} \\ &= \frac{6 \pm 2\sqrt{10}}{2} \\ &= 3 \pm \sqrt{10} \end{aligned}$$

So we have:

$$x = \ln(3 \pm \sqrt{10})$$

But $3 - \sqrt{10}$ is negative, so our only answer is:

$$x = \ln(3 + \sqrt{10})$$

6.

$$\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\frac{e^x + e^{-x}}{2} &= 4 \\
e^x + e^{-x} &= 8 \\
e^x - 8 + e^{-x} &= 0 \\
e^{2x} - 8e^x + 1 &= 0
\end{aligned}$$

Then we use the quadratic formula:

$$\begin{aligned}
e^x &= \frac{8 \pm \sqrt{64 - 4}}{2} \\
&= \frac{8 \pm 2\sqrt{15}}{2} \\
&= 4 \pm \sqrt{15}
\end{aligned}$$

So we have:

$$x = \ln(4 \pm \sqrt{15})$$

But $4 - \sqrt{15}$ is still positive, so we have two possible values of x:

$$x = \ln(4 \pm \sqrt{15})$$

7.

$$\begin{aligned}
\tanh x &= \frac{e^{2x} - 1}{e^{2x} + 1} \\
\frac{e^{2x} - 1}{e^{2x} + 1} &= \frac{1}{2} \\
2e^{2x} - 2 &= e^{2x} + 1 \\
e^{2x} &= 3 \\
2x &= \ln 3 \\
x &= \frac{1}{2} \ln 3 \\
x &= \ln 3^{\frac{1}{2}} \\
x &= \ln \sqrt{3}
\end{aligned}$$

8.

$$\begin{aligned}
\coth x &= \frac{e^{2x}+1}{e^{2x}-1} \\
\frac{e^{2x}+1}{e^{2x}-1} &= 6 \\
e^{2x}+1 &= 6e^{2x}-6 \\
5e^{2x} &= 7 \\
e^{2x} &= \frac{7}{5} \\
2x &= \ln\frac{7}{5} \\
x &= \ln\sqrt{\frac{7}{5}}
\end{aligned}$$

17.2 Identities and Derivatives

1.

$$\begin{aligned}
RHS &= (\frac{e^x+e^{-x}}{2})(\frac{e^y+e^{-y}}{2}) + (\frac{e^x-e^{-x}}{2})(\frac{e^y-e^{-y}}{2}) \\
&= \frac{1}{4}(e^x e^y + e^x e^{-y} + e^y e^{-x} + e^{-x}e^{-y} + e^x e^y - e^x r^{-y} - e^y e^{-x} + e^{-x}e^{-y}) \\
&= \frac{1}{4}(e^{x+y} + e^{x-y} + e^{y-x} + e^{-(x+y)} + e^{x+y} - e^{x-y} - e^{y-x} + e^{-(x+y)}) \\
&= \frac{1}{2}\left(e^{x+y} + e^{-(x+y)}\right) \\
&= \cosh(x+y) \\
&= LHS
\end{aligned}$$

2.

$$\tanh(2x) = \frac{\sinh(2x)}{\cosh(2x)}$$
$$= \frac{2\sinh x \cosh x}{\cosh^2 x + \sinh^2 x}$$

Divide top and bottom by $\cosh^2 x$:

$$= \frac{2\frac{\sinh x \cosh x}{\cosh^2 x}}{\frac{\cosh^2 x}{\cosh^2 x} + \frac{\sinh^2 x}{\cosh^2 x}}$$
$$= \frac{2\frac{\sinh x}{\cosh x}}{1 + \frac{\sinh^2 x}{\cosh^2 x}}$$
$$= \frac{2\tanh x}{1 + \tanh^2 x}$$

3. a) We will use the chain rule:

$$\frac{d}{dx}(\sinh(3x)) = \cosh(3x) \cdot 3$$
$$= 3\cosh(3x)$$

b) We will use the quotient rule as:

$$2\text{sech}x = \frac{2}{\cosh x}$$

Let $u = 2$ and $v = \cosh x$. Then $\frac{du}{dx} = 0$ and $\frac{dv}{dx} = \sinh x$

$$\frac{d}{dx}(2\text{sech}x) = \frac{\cosh x \cdot 0 - 2\sinh x}{\cosh^2 x}$$
$$= -2\frac{\sinh x}{\cosh x}\frac{1}{\cosh x}$$
$$= -2\tanh x\text{sech}x$$

c) We will use the product rule. Let $u = e^x$ and $v = \cosh x$. Then $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = \sinh x$. So

$$\frac{d}{dx}(e^x \cosh x) = e^x \sinh x + e^x \cosh x$$

d) We will use the chain rule:

$$\frac{d}{dx}(\tanh(x^2)) = \text{sech}^2(x^2) \cdot 2x$$
$$= 2x\text{sech}^2(x^2)$$

e) We will use the quotient rule and the chain rule. We have:

$$x\text{cosech}(\ln x) = \frac{x}{\sinh(\ln x)}$$

For the quotent rule we will let $u = x$ and $v = \sinh(\ln x)$. Then $\frac{du}{dx} = 1$ and we require the chain rule to find $\frac{dv}{dx}$:

$$\begin{aligned}\frac{dv}{dx} &= \cosh(\ln x) \cdot \frac{1}{x} \\ &= \frac{\cosh(\ln x)}{x}\end{aligned}$$

So we have:

$$\begin{aligned}\frac{d}{dx}\left(\frac{x}{\sinh(\ln x)}\right) &= \frac{\sinh(\ln x) \cdot 1 - x\frac{\cosh(\ln x)}{x}}{\sinh^2(\ln x)} \\ &= \text{cosech}(\ln x) - \text{cosech}(\ln x)\coth(\ln x)\end{aligned}$$

f) We will use the chain rule:

$$\begin{aligned}\frac{d}{dx}(\ln(\cosh x)) &= \frac{1}{\cosh x} \cdot \sinh x \\ &= \tanh x\end{aligned}$$

g) We will use the chain rule:

$$\frac{d}{dx}(e^{\sinh x}) = e^{\sinh x}\cosh x$$

h) Let's look at what we have:

$$\coth x \sin x = \frac{\cosh x \sin x}{\sinh x}$$

Note that the term on the right is $\sin x$, *not* $\sinh x$. We will now use the quotient rule with $u_q = \cosh x \sin x$ and $v_q = \sinh x$. Then, to find $\frac{du_q}{dx}$ we will need the product rule with $u_p = \cosh x$ and $v_p = \sin x$. So $\frac{du_p}{dx} = \sinh x$ and $\frac{dv_p}{dx} = \cos x$. So we have that:

$$\frac{du_q}{dx} = \sin x \sinh x + \cosh x \cos x$$

And, we also have that $\frac{dv_q}{dx} = \cosh x$ Then the quotient rule says:

$$\begin{aligned}
\frac{d}{dx}(\coth x \sin x) &= \frac{\sinh x(\sin x \sinh x + \cosh x \cos x) - (\cosh x \sin x)\cosh x}{\sinh^2 x} \\
&= \frac{(\sin x \sinh^2 x + \sinh x \cosh x \cos x) - \cosh^2 x \sin x}{\sinh^2 x} \\
&= \sin x + \cos x \coth x - \sin x \coth^2 x \\
&= \cos x \coth x - \sin x(\coth^2 x - 1)
\end{aligned}$$

Recall that $\cosh^2 x - \sinh^2 x = 1$ so dividing by $\sinh^2 x$ we have $\coth^2 x - 1 = \text{cosech}^2 x$. So the answer is:

$$\cos x \coth x - \sin x \text{cosech}^2 x$$

i) We have:

$$e^{\cosh x} \cdot \text{cosech} x = \frac{e^{\cosh x}}{\sinh x}$$

We will us the quotient rule with $u = e^{\cosh x}$ and $v = \sinh x$. Then we reiquire the chain rule to find that $\frac{du}{dx} = \sinh x e^{\cosh x}$, and also $\frac{dv}{dx} = \cosh x$. So we have:

$$\begin{aligned}
\frac{d}{dx}(e^{\cosh x}) &= \frac{\sinh x(\sinh x e^{\cosh x}) - e^{\cosh x}\cosh x}{\sinh^2 x} \\
&= \frac{\sinh^2 x e^{\cosh x} - \cosh x e^{\cosh x}}{\sinh^2 x} \\
&= e^{\cosh x} - \text{cosech} x \coth x e^{\cosh x}
\end{aligned}$$

17.3 Integration

1.

$$\begin{aligned}
\int \cosh(2x)\ dx &= \sinh(2x) \cdot \frac{1}{2} + c \\
&= \frac{1}{2}\sinh(2x) + c
\end{aligned}$$

2.

$$\begin{aligned}
\int 2\sinh(4x)\ dx &= 2\cosh(4x) \cdot \frac{1}{4} + c \\
&= \frac{1}{2}\cosh(4x) + c
\end{aligned}$$

3. Use the exponential form of $\cosh x$:

$$\begin{aligned} 2e^x \cosh x &= 2e^x \frac{e^x + e^{-x}}{2} \\ &= e^2 x + 1 \end{aligned}$$

So we have that:

$$\begin{aligned} \int 2e^x \cosh x \ dx &= \int e^2 x + 1 dx \\ &= \frac{1}{2} e^{2x} + x + c \end{aligned}$$

4. Using the exponential form of $\tanh x$ we have:

$$\begin{aligned} e^{-x}(e^{2x} + 1) \tanh x &= e^{-x}(e^{2x} + 1) \frac{e^2 x - 1}{e^{2x} + 1} \\ &= e^{-x}(e^{2x} - 1) \\ &= e^x - e^{-x} \end{aligned}$$

So we have:

$$\begin{aligned} \int e^{-x}(e^{2x} + 1) \tanh x \ dx &= \int e^x - e^{-x} \ dx \\ &= e^x + e^{-x} + c \end{aligned}$$

5. Let's try $\tanh(3x)$:

$$\begin{aligned} \frac{d}{dx}(\tanh(3x)) &= \mathrm{sech}^2(3x) \cdot 3 \\ &= 3\mathrm{sech}^2(3x) \end{aligned}$$

But we require 3 times this so our answer is:

$$\int 9\mathrm{sech}^2(3x) \ dx = 3 \tanh(3x) + c$$

6. We have that $\cosh(2x) = \cosh^2 x + \sinh^2 x$ and $\cosh^2 x - \sinh^2 x = 1$, so $\sinh^2 x = \cosh^2 x - 1$. Combining these 2 expressions:

$$\cosh(2x) = 2 \cosh^2 x - 1$$

Rearranging, we have:

$$2 \cosh^2 x = \cosh(2x) + 1$$

So,

$$\begin{aligned}\int 2\cosh^2 x \; dx &= \int (\cosh(2x)+1) \; dx \\ &= \frac{1}{2}\sinh 2x + x + c\end{aligned}$$

7. Let's try $\sinh^3 x$:

$$\frac{d}{dx}(\sinh^3 x) = 3\sinh^2 x \cdot \cosh x$$

We only require a third of this so:

$$\int \sinh^2 x \cosh x \; dx = \frac{1}{3}\sinh^3 x + c$$

8. We have:

$$\begin{aligned}\tanh(3x)\mathrm{sech}(3x) &= \frac{\sinh(3x)}{\cosh^2(3x)} \\ &= \sinh(3x)\cosh^{-2}(3x)\end{aligned}$$

So, let's try $\cosh^{-1}(3x)$

$$\begin{aligned}\frac{d}{dx}(\cosh^{-1}(3x)) &= -3\cosh^{-2}(3x)\cdot\sinh(3x) \\ &= -3\frac{\sinh(3x)}{\cosh^2(3x)} \\ &= -3\tanh(3x)\mathrm{sech}^2(3x)\end{aligned}$$

We require minus a third of this, so:

$$\int \tanh(3x)\mathrm{sech}^2(3x) \; dx = -\frac{1}{3}\mathrm{sech}(3x) + c$$

9. Let's try $\tanh^4 x$:

$$\frac{d}{dx}(\tanh^4 x) = 4\tanh^3 x \cdot \mathrm{sech}^2 x$$

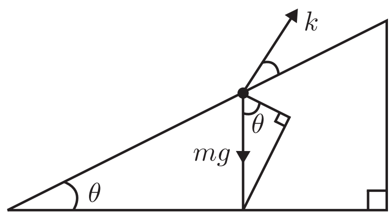
Which is what we require, so:

$$\int 4\tanh^3 x\mathrm{sech}^2 x \; dx = \tanh^4 x + c$$

18
Motion and Curvature

Test Yourself

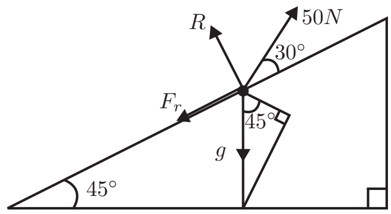
1. We have the following situation:



Resolving the forces parallel to the plane, we have:

$$\begin{aligned} k\cos\theta &= mg\sin\theta \\ k &= mg\frac{\cos\theta}{\sin\theta} \\ &= mg\tan\theta \end{aligned}$$

2. We have the following situation:



Resolving the forces perpendicular to the plane, we have:

$$R + 50\sin 30 = 1g\cos 45$$
$$R = \frac{\sqrt{2}}{2}g - 25$$

Then we can use:

$$\begin{aligned} F_r &= \mu R \\ &= 0.2(\frac{\sqrt{2}}{2}g - 25) \\ &= 0.1\sqrt{2}g - 5 \end{aligned}$$

Next, we resolve the forces parallel to the slope (using $F = ma$):

$$50\cos 30 - g\sin 45 - F_r = a$$
$$25\sqrt{3} - \frac{\sqrt{2}}{2}g - (0.1\sqrt{2}g - 5) = a$$
$$a = \left(25\sqrt{3} + 5 - \frac{3\sqrt{2}}{5}g\right)\text{ms}^{-2}$$

3. We are told that $r = 3$ and $\omega = 10$. So we'll use:

$$\begin{aligned} a &= r\omega^2 \\ &= 3 \cdot 10^2 \\ &= 3 \cdot 100 \\ &= 300\text{ms}^{-2} \end{aligned}$$

4. We are told that $r = 2$ and $v = 8$. So we'll use:

$$\begin{aligned} a &= \frac{v^2}{r} \\ &= \frac{8^2}{2} \\ &= \frac{64}{2} \\ &= 32\text{ms}^{-2} \end{aligned}$$

5. We are told that $m = 4$, $r = 2$ and $\omega = 1$. If we combine $a = r\omega^2$ with $F = ma$, we get:

$$\begin{aligned} F &= mr\omega^2 \\ &= 4 \cdot 2 \cdot 1^2 \\ &= 4 \cdot 2 \\ &= 8\text{N} \end{aligned}$$

6. We are told that $10 = 2$, $r = 2$ and $v = 3$. If we combine $a = \frac{v^2}{r}$ with $F = ma$, we get:

$$\begin{aligned} F &= \frac{mv^2}{r} \\ &= \frac{10 \cdot 3^2}{2} \\ &= \frac{10 \cdot 9}{2} \\ &= \frac{90}{2} \\ &= 45\text{N} \end{aligned}$$

7. If we let $x = t$ then $y = 3t + 2$ and t can be any real number. So, our parameterisation is:

$$\{(t, 3t + 2) | t \in \mathbb{R}\}$$

8. If we let $x = t$ then $y = \sin t$ and t can be any real number. So, our parameterisation is:

$$\{(t, \sin t) | t \in \mathbb{R}\}$$

9. So,$\mathbf{r} = (t, 7t + 1)$ and then we have that:

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= (1, 7) \\ \text{So, } \left\| \frac{d\mathbf{r}}{dt} \right\| &= \sqrt{1^2 + 7^2} \\ &= \sqrt{1 + 49} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \end{aligned}$$

So the length of the curve is found by:

$$\begin{aligned} \int_0^3 5\sqrt{2}\ dt &= [5\sqrt{2}t]_0^3 \\ &= 15\sqrt{2} \end{aligned}$$

10. We can parameterise this curve by:

$$\{(t, \cosh t) | t \in [0, \ln 10]\}$$

So, $\mathbf{r} = (t, \cosh t)$ and then we have that:

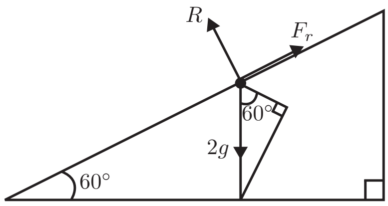
$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= (1, \sinh t) \\ \text{So, } \left\| \frac{d\mathbf{r}}{dt} \right\| &= \sqrt{1^2 + (\sinh t)^2} \\ &= \sqrt{1 + \sinh^2 t} \\ &= \sqrt{\cosh^2 t} \text{ (See hyperbolic trigonometry)} \\ &= \cosh t \end{aligned}$$

So the length of the curve is found by:

$$\begin{aligned} \int_0^{\ln 10} \cosh t\ dt &= [\sinh t]_0^{\ln 10} \\ &= \sinh \ln 10 \\ &= \frac{99}{20} \end{aligned}$$

18.1 Loose Ends or New Beginnings

1. We have the following situation:



Resolving the forces perpendicular to the plane, we have:

$$\begin{aligned} R &= 2g\cos 60 \\ &= 2g\frac{1}{2} \\ &= g \end{aligned}$$

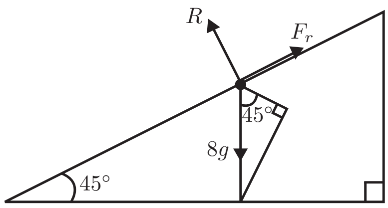
Then we can use:

$$\begin{aligned} F_r &= \mu R \\ &= 0.4g \end{aligned}$$

Next, we resolve the forces parallel to the slope (using $F = ma$):

$$\begin{aligned} 2g\sin 60 - F_r &= 2a \\ \sqrt{3}g - 0.4g &= 2a \\ a &= \frac{\sqrt{3}}{2}g - \frac{1}{5}g \end{aligned}$$

2. We have the following situation:



Resolving the forces perpendicular to the plane, we have:

$$\begin{aligned} R &= 8g\cos 45 \\ &= 8g\frac{\sqrt{2}}{2} \\ &= 4\sqrt{2}g \end{aligned}$$

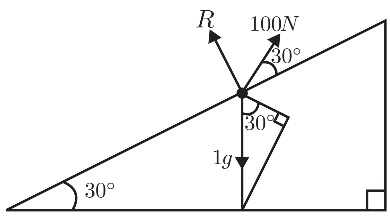
Then we can use:

$$\begin{aligned} F_r &= \mu R \\ &= 0.5 \cdot 4\sqrt{2}g \\ &= 2\sqrt{2}g \end{aligned}$$

Next, we resolve the forces parallel to the slope (using $F = ma$):

$$\begin{aligned} 8g\sin 45 - F_r &= 8a \\ 4\sqrt{2}g - 2\sqrt{2}g &= 8a \\ a &= \frac{\sqrt{2}}{4}g \\ a &= \frac{g}{2\sqrt{2}} \end{aligned}$$

3. We have the following situation:

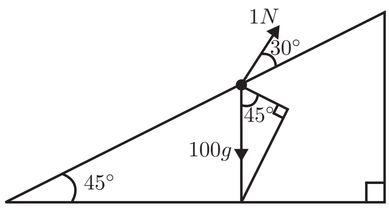


Resolving the forces parallel to the plane in the direction up the slope, we have:

$$\begin{aligned} 100\cos 30 - 1g\sin 30 &= 100\frac{\sqrt{3}}{2} - g\frac{1}{2} \\ &= 50\sqrt{3} - \frac{1}{2}g \\ &\approx 81.7\text{N} \end{aligned}$$

This is positive. Hence the resultant force acts *up* the slope, so the friction will act downwards.

4. We have the following situation:

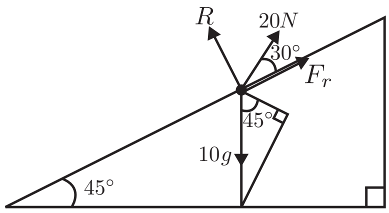


Resolving the forces parallel to the plane in the direction down the slope, we have:

$$\begin{aligned} 100g \sin 45 - 1 \cdot \cos 30 &= 100g\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \\ &= 50g\sqrt{2} - \frac{\sqrt{3}}{2} \\ &\approx 692\text{N} \end{aligned}$$

This is positive. Hence the resultant force acts *down* the slope, so the friction act upwards.

5. We have the following situation:



Resolving the forces perpendicular to the plane, we have:

$$\begin{aligned} R + 20 \sin 30 &= 10g \cos 45 \\ R &= 10g\frac{\sqrt{2}}{2} - 20 \cdot \frac{1}{2} \\ &= 5\sqrt{2}g - 10 \end{aligned}$$

Then we can use:

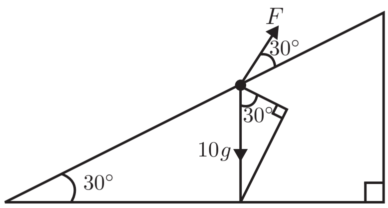
$$\begin{aligned} F_r &= \mu R \\ &= 0.5(5\sqrt{2}g - 10) \\ &= \frac{5\sqrt{2}}{2}g - 5 \end{aligned}$$

Next, we resolve the forces parallel to the slope (using $F = ma$):

$$\begin{aligned}10g\sin 45 - 20\cos 30 - F_r &= 10a\\ 10g\frac{\sqrt{2}}{2} - 30\cdot\frac{\sqrt{3}}{2} - \left(\frac{5\sqrt{2}}{2}g - 5\right) &= 10a\end{aligned}$$

$$\begin{aligned}a &= g\frac{\sqrt{2}}{2} - \frac{2\sqrt{3}}{2} - \frac{\sqrt{2}}{4}g + \frac{1}{2}\\ &= \left(\frac{\sqrt{2}}{4}g + \frac{1}{2} - \sqrt{3}\right)\text{ms}^{-2} \quad \text{down the slope.}\end{aligned}$$

6. We have the following situation:



We can assume that the forces are equal because the particle is motionless. Resolving the forces perpendicular to the plane, we have:

$$\begin{aligned}F\cos 30 &= 10g\sin 30\\ F &= 10g\frac{\sin 30}{\cos 30}\\ &= \frac{10}{\sqrt{3}}g\end{aligned}$$

18.2 Circular Motion

1. In circular motion acceleration is always towards the centre of the circle. This means that the linear velocity, v, is constant as acceleration is perpendicular to the direction of motion.

2. We are told that $r = 5$ and $\omega = \pi$. So we'll use:

$$\begin{aligned}v &= \omega r\\ &= 5\pi\text{ms}^{-1}\end{aligned}$$

3. We are told that $r = 8$ and $v = 4$. So we'll use:

$$\begin{aligned} a &= \frac{v^2}{r} \\ &= \frac{4^2}{8} \\ &= \frac{16}{8} \\ &= 2\text{ms}^{-2} \end{aligned}$$

4. We are told that $r = 5$ and $\omega = 3$. So we'll use:

$$\begin{aligned} a &= r\omega^2 \\ &= 5 \cdot 3^2 \\ &= 5 \cdot 9 \\ &= 45\text{ms}^{-2} \end{aligned}$$

5. We are told that $m = 2$, $r = 2$ and $v = 6$. If we combine $a = \frac{v^2}{r}$ with $F = ma$, we get:

$$\begin{aligned} F &= \frac{mv^2}{r} \\ &= \frac{2 \cdot 6^2}{2} \\ &= 36\text{N} \end{aligned}$$

6. We are told that $m = 5$, $r = 1$ and $\omega = 2$. If we combine $a = r\omega^2$ with $F = ma$, we get:

$$\begin{aligned} F &= mr\omega^2 \\ &= 5 \cdot 1 \cdot 2^2 \\ &= 5 \cdot 4 \\ &= 20\text{N} \end{aligned}$$

18.3 Curves

1. The line y=x is a straight line that passes through the point $(0, 0)$. For every 1 that x increases by 1, so we can use the direction vector $(1, 1)$. So we can write

$$(x, y) = (0, 0) + t(1, 1)$$

We can also write this

$$\{(t,t)|t \in \mathbb{R}\}$$

2. We once again our curve goes through the point $(0,0)$. And as $y = x^2$, we want the y value to increase by x^2 as x increases. If we let $x = t$ then y must always be t^2, so we have:

$$\{(t,t^2)|t \in \mathbb{R}\}$$

3. Recall the parameterisation of a circle of radius r and centre (m,n) is

$$\{(m + r\cos\theta, n + r\sin\theta)|\theta \in [0,2\pi)\}$$

So we have the parameterisation:

$$\{(3 + 2\cos\theta, 3 + 2\sin\theta)|\theta \in [0,2\pi)\}$$

4. The centre of this circle is at $(2,0)$ and has radius 2. So, if we had a full circle our parameterisation would be:

$$\{(2 + 2\cos\theta, 2\sin\theta)|\theta \in [0,2\pi)\}$$

We can see that when $\theta = 0$ we're at the point $(4,0)$, when $\theta = \frac{\pi}{2}$ we're at the point $(2,2)$ and when $\theta = \pi$ we're at the point $(0,0)$. So, we must use the parameterisation.

$$\{(2\cos\theta, 2 + 2\sin\theta)|\theta \in [0,\pi]\}$$

5. Recall that we find the length of a curve by:

$$\int_a^b \left|\left|\frac{d\mathbf{r}}{dt}\right|\right| dt$$

When differentiating vector functions, we simply differentiate each component seperately:

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= (5,12) \\ \text{So, } \left|\left|\frac{d\mathbf{r}}{dt}\right|\right| &= \sqrt{5^2 + 12^2} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

So the length of the curve is found by:

$$\begin{aligned} \int_0^4 13\ dt &= [13t]_0^4 \\ &= 52 \end{aligned}$$

6.

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= (16t, 12t)\\ \text{So, } \left|\left|\frac{d\mathbf{r}}{dt}\right|\right| &= \sqrt{(16t)^2 + (12t)^2}\\ &= \sqrt{256t^2 + 144t^2}\\ &= \sqrt{400t^2}\\ &= 20t\end{aligned}$$

So the length of the curve is found by:

$$\begin{aligned}\int_1^3 20t\ dt &= [20t^2]_1^3\\ &= 90 - 10\\ &= 80\end{aligned}$$

7. We can parameterise this curve by:

$$\{(t, t^3)|t \in [0, 5]\}$$

So, $\mathbf{r} = (t, t^3)$ and then we have that:

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= (1, 3t^2)\\ \text{So, } \left|\left|\frac{d\mathbf{r}}{dt}\right|\right| &= \sqrt{1^2 + (3t^2)^2}\\ &= \sqrt{1 + 9t^4}\end{aligned}$$

So the length of the curve is found by:

$$\int_0^5 \sqrt{1 + 9t^4}\ dt$$

8. We can parameterise this curve by:

$$\{(t, 2t^2)|t \in [0, 18]\}$$

So, $\mathbf{r} = (t, 2t^2)$ and then we have that:

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= (1, 4t)\\ \text{So, } \left|\left|\frac{d\mathbf{r}}{dt}\right|\right| &= \sqrt{1^2 + (4t)^2}\\ &= \sqrt{1 + 16t^2}\end{aligned}$$

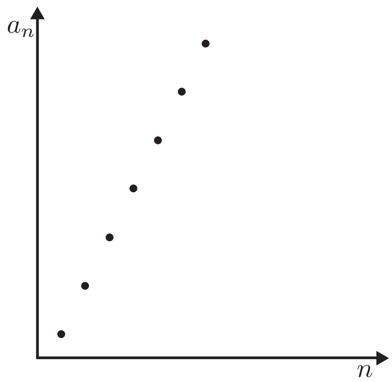
So the length of the curve is found by:

$$\int_0^3 \sqrt{1 + 16t^2}\ dt$$

19
Sequences

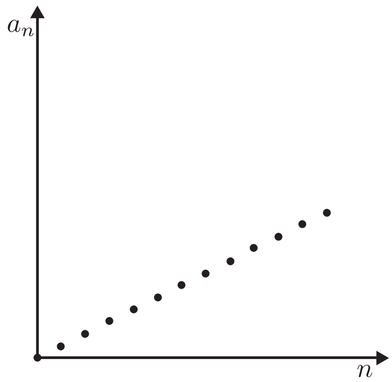
Test Yourself

1. We simply plot the points against the number, that they occur in:



Remember: we don't join the dots as this is a sequence!

2. We simply plot the points against the number, that they occur in:



Remember: we don't join the dots as this is a sequence!

3. Let's take a look at the values: $a_n = 2, 7, 8, 13, 14...$ Every term is strictly greater than the previous term, so we say that this is a strictly increasing sequence, hence monotonic.

4. It should be clear that if we pick any number then, although there are points where we will be below this number, there will be a point at which we go above this number and *never* return below this number. So, this sequence *does* tend to infinity.

5. Let's write some terms down: $a_n = -1, 2, -3, 4, -5, ...$ The definition of tending to infinity has to work for *any* number that we choose. So, if we find a number that it doesn't work for, then the sequence doesn't tend to infinity. So, let's pick the number 0. Although the sequence does go above it, it will also go below 0, as every other term is negative.

6. Let's write some terms down: $a_n = 1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, ...$ All of the terms in this sequence are positive, so the absolute value is simply just the same sequence. The definition of tending to 0 has to work for *any* number that we choose. So, if we find a number that it doesn't work for, then the sequence doesn't tend to 0. So, let's pick the number 1. Although the sequence does hit 1, it then goes above 1 and will stay above it.

7. Let's write some terms down: $a_n = -\frac{1}{5}, \frac{1}{10}, -\frac{1}{15}, \frac{1}{20}, -\frac{1}{25}...$ We then take the absolute value of our sequence to get $|a_n| = \frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \frac{1}{25}....$ The definition of tending to 0 has to work for *any* number that we choose. This is the sequence $a_n = \frac{1}{5n}$. So, for any number, say x, that we pick we require $a_n < x$, which means $\frac{1}{5n} < x$. Therefore, if we pick any n such that $n > \frac{1}{5x}$ then every number after this is less than x. So, our sequence does tend to 0.

8. Let's write some terms down: $a_n = \frac{3}{10}, \frac{3}{20}, \frac{3}{30}, \frac{3}{40}, \frac{3}{}...$ This doesn't look like it will tend to infinity, but could tend to 0:

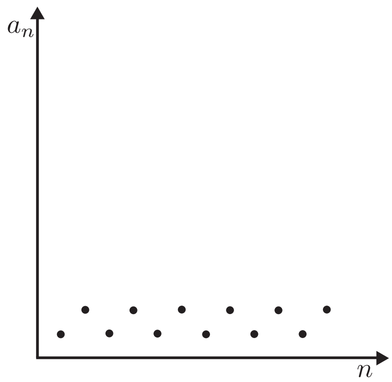
   All of the terms in this sequence are positive, so the absolute value is simply just the same sequence. The definition of tending to 0 has to work for *any* number that we choose. This is the sequence $a_n = \frac{3}{10n}$. So, for any number, say x, that we pick we require $a_n < x$, which means $\frac{3}{10n} < x$. Therefore, if we pick any n such that $n > \frac{3}{10x}$ then every number after this is less than x. So, our sequence does tend to 0.

9. Let's write some terms down: $a_n = -1, 4, -27, 256, -3125...$ Every other term is negative, so this sequence can't tend to infinity. Also, $|a_n| = a_n = 1, 4, 27, 256, 3125...$ which obviously doesn't tend to 0.

10. Regardless of what number we pick, say x, we can choose an n such that $n > x^2$. Then for all numbers larger $\sqrt{n} > x$, so this sequence does tend to infinity.

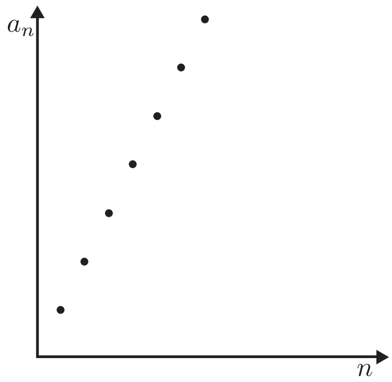
19.1 (Re)Starting Afresh

1. We simply plot the points against the number, that they occur in:



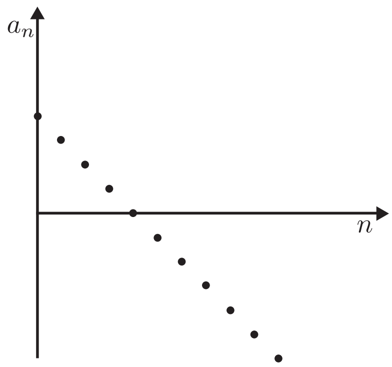
   Remember: we don't join the dots as this is a sequence!

2. We simply plot the points against the number, that they occur in:



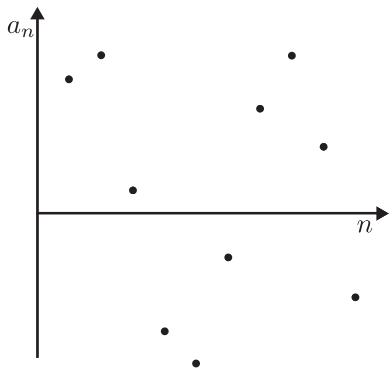
Remember: we don't join the dots as this is a sequence!

3. We simply plot the points against the number, that they occur in:



Remember: we don't join the dots as this is a sequence!

4. We simply plot the points against the number, that they occur in:



Remember: we don't join the dots as this is a sequence!

5. a) Let's write some terms down: $a_n = 1, 2, 3, 4, 5, ...$ Every term is strictly greater than the previous term (they are never equal). So this sequence is strictly increasing and therefore increasing. Hence, it is monotonic (see definition of monotonic in the chapter).

   b) Every term is either less than or equal to the previous term. So it is decreasing (not strictly decreasing as the terms are sometimes equal). Hence, it is monotonic (see definition of monotonic in the chapter).

   c) Let's write some terms down: $a_n = -1, 1, -1, 1, -1, ...$ This is neither increasing nor decreasing, so it is non-monotonic.

   d) The terms in this sequence will follow the shape of the sin curve. Let's write some terms down to $2d.p.$: $a_n = 0.84, 0.91, 0.14, , -0.76, -0.96, ...$ This is neither increasing nor decreasing, so it is non-monotonic.

19.2 To Infinity (not beyond)

1. Let's write some terms down: $a_n = 2, 4, 6, 8, 10, ...$ If we pick any n such that $n > 6$, then $2n$ will always be greater than 12.

   We cannot conclude anything from this fact alone: we need to test whether there's a point after which the sequence is *always* greater than this value, and then whether this still holds if we choose *any* number, not just 12.

2. Let's write some terms down: $a_n = -1, -2, -3, -4, -5, ...$ This is strictly decreasing and starts below 7, so will never go above 7.

   We can conclude for certain that $a_n = -n$ does not tend to ∞.

3. Let's write some terms down: $a_n = 1, 4, 9, 16, 25, ...$ So, if we pick any $n \geq 7$ then $n^2 > 42$.

   We cannot conclude anything from this fact alone: we need to test whether this still holds if we choose *any* number, not just 42.

4. Let's write some terms down: $a_n = -2, 4, -8, 16, -32, ...$ Although some individual terms are greater than 20, there is no point after which *every* term is, as there will be a negative number that occurs every other number. We can conclude for certain that $a_n = (-2)^n$ *does not* tend to ∞.

5. Let's write some terms down: $a_n = 2, 6, 12, 20, 30, ...$ It should be clear that if we pick any number then there will be a point at which we go above

this number and *never* return below this number. So, this sequence *does* tend to infinity.

6. Let's write some terms down: $a_n = 0, 3, 2, 5, 4, ...$ It should be clear that if we pick any number then, although there are points where we will be below this number, there will be a point at which we go above this number and *never* return below this number. So, this sequence *does* tend to infinity.

7. Let's write some terms down: $a_n = 99, 99\frac{1}{2}, 99\frac{2}{3}, 99\frac{3}{4}, 99\frac{4}{5}, ...$ If we pick our number as 100, then the sequence will obviousy never go above this. So the sequence can't tend to infinity.

8. Let's write some terms down: $a_n = 2, 2, 6, 4, 10, ...$ It should be clear that if we pick any number then, although there are points where we will be below this number, there will be a point at which we go above this number and *never* return below this number. So, this sequence *does* tend to infinity.

19.3 Nothing at All

1. Let's write some terms down: $a_n = 1, 2, 3, 4, 5, ...$ We do have terms less than 7 at the beginning of the sequence. But, we cannot conclude anything from this.

2. Let's write some terms down: $a_n = 3, 6, 9, 12, 15, ...$ We do not have any terms less than 3. We can conclude for certain that $a_n = 3n$ does not tend to 0.

3. Let's write some terms down: $a_n = 4, 3, 2, 1, 0, ...$ So there are terms at which the sequence is less than 3. We cannot conclude anything from this.

4. Let's write some terms down: $a_n = -3, 9, -27, 81, -243, ...$The absolute value of this is therefore:$|a_n| = 3, 9, 27, 81, 243, ...$This is below 5 at the beginning of the sequence, but it does go above 5 and stays there. Hence, we can conclude for certain that $a_n = (-3)^n$ does not tend to 0.

5. Let's write some terms down: $a_n = 3, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}...$ Taking the absolute value gives us the same sequence. The definition of tending to 0 has to work for *any* number that we choose. This is the sequence $a_n = \frac{3}{n}$. So, for any number, say x, that we pick we require $a_n < x$, which means $\frac{3}{n} < x$. Therefore, if we pick any n such that $n > \frac{3}{x}$ then every number after this is less than x. So, our sequence does tend to 0.

6. Let's write some terms down: $a_n = 9, 8, ..., 1, 0, -1, -2...$ We then take the absolute value of our sequence to get $|a_n| = 9, 8, ..., 1, 0, 1, 2....$ The

definition of tending to 0 has to work for *any* number that we choose. Pick the number 2. When the absolute value of the sequence hits 2 the second time, it never goes below it again. So this sequence does not tend to 0.

7. Let's write some terms down: $a_n = 1, 0, 3, 0, 5...$ Taking the absolute value gives us the same sequence. The definition of tending to 0 has to work for *any* number that we choose. Pick the number 2. When the sequence goes above 2, it does go below it again, but will always have a later number that is above it. So this sequence does not tend to 0.

8. Let's write some terms down: $a_n = 1, 0, \frac{1}{3}, 0, \frac{1}{5}...$ Taking the absolute value gives us the same sequence. The definition of tending to 0 has to work for *any* number that we choose. For any number, say x, that we pick we require $a_n < x$, which means $\frac{1}{n} < x$. Therefore, if we pick any n such that $n > \frac{1}{x}$ then every number after this is less than x. So, our sequence does tend to 0.

20
Series

Test Yourself

1. Let's write out the first n terms: $a_n = 2 + 4 + 6 + ... + 2n$. If we take a factor of 2 out of this we have that: $a_n = 2(1 + 2 + 3 + ... + n)$. Then we can use the fact that we know: $1 + 2 + 3 + ... + n = \frac{n}{2}(n + 1)$. Hence we have $a_n = n(n + 1) = n^2 + n$

2. Let's write down some terms $a_n = 3, -1, 5, -3, 7, -5....$ If we let S_n be the sum of the first n terms then we have that $S_1 = 3$, $S_2 = 2$, $S_3 = 7$, $S_4 = 4$, $S_5 = 11$, $S_6 = 6....$ From this it should be clear that if n is even then the sum of the first n terms is n. For odd n we can see that the sum of the first n terms is $2n + 1$. So we have:

$$S_n = \begin{cases} 2n+1, & n \text{ odd} \\ n, & n \text{ even} \end{cases}$$

   So the sum to n terms varies depending on whether n is odd or even. This means we are not able to find as sum to infinity because as n increases indefinitely the sum will switch between the 2 possibilities.

3. Let's write down some terms $a_n = 4, -4, \frac{4}{3}, -\frac{4}{3}, \frac{4}{5}, -\frac{4}{5}....$ It should be clear that is n is even then the sum of the first n terms is 0 as the pairs of terms cancel each other out. If n is odd the sum of the first n terms is simply the nth term, which is $\frac{4}{n}$. As n gets very large (to infinity) $\frac{4}{n}$ gets very small

so it is approximately equal to 0. Hence, whether n is odd or even, the sum to infinity is 0. So the sum to infinity of this sequence is 0.

4. If the sum to infinity of a sequence is finite then it means that the sequence tends to 0. This *does not* mean that every sequence that tends to 0 has an infinite sum that is a finite number.

5. We can always find a finite number of terms that sum to at least 1, then of the remaining terms we can always find a finite number of terms that sum to at least 1, then of the remaining terms we can always find a finite number of terms that sum to at least 1... This looks like:

$$a_n = \underbrace{1}_{\geq 1}, \underbrace{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}}_{\geq 1}, \underbrace{\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}}_{\geq 1} \ldots$$

6. The first pie has $\frac{1}{3}$ filled in, the second has $\frac{1}{3} + \frac{1}{9} = \frac{4}{9}$ filled in, the third has $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{13}{27}$ filled in, and so on. Visually, it is clear that there will never be more than $\frac{1}{2}$ a pie, and so a finite value exists for the summation of infinitely many terms.

7. Let's look at some values: $a_n = 7, \frac{7}{2}, \frac{7}{3}, \frac{7}{4}, \frac{7}{5}, \frac{7}{6}...$ When n is $7, 14, 21, ...$ we have the values of a_n that are $1, \frac{1}{2}, \frac{1}{3}...$ Which are the values of the harmonic sequence. We are adding on extra values as well so the sum to infinity is infinite.

8. We can use the comparison test with $b_n = \frac{1}{2^n}$ and $K = 7$. Then $a_n < K \cdot b_n$ and the infinite sum of b_n is finite. So $\sum_{n=1}^{\infty} a_n$ is finite.

9. $\sum_{n=1}^{\infty} a_n$ is positive and if we let $b_n = \frac{1}{2^n}$ therefore $\sum_{n=1}^{\infty} b_n$ is finite. We have that $a_n < 0.2 \cdot b_n$. So, by the comparison test $\sum_{n=1}^{\infty} a_n$ is finite.

10. $\sum_{n=1}^{\infty} a_n$ is positive and if we let $b_n = \frac{1}{n}$ therefore $\sum_{n=1}^{\infty} b_n$ is infinite. We have that $a_n > 0.3b_n$. So, by the reverse comparison test $\sum_{n=1}^{\infty} a_n$ is infinite.

20.1 Various Series

1. Let's look at some terms and the difference between the terms:

$$a_n = 1 \underbrace{,}_{1} 4 \underbrace{,}_{3} 9 \underbrace{,}_{5} 16 \underbrace{,}_{7} 25, \underbrace{,}_{9} 36, ..., n^2$$

So the differences are the same on the third line. Hence we're looking at a rule that will have an n^3 term. If we let S_n be the sum of the first n terms then we have that:

$$S_n = 1, 5, 14, 30, 55, 91...$$

Other than finding this fact it is diffilut to find the rule fot n^2, so it is something you should probably memorise. It is:

$$S_n = \frac{n}{6}(n+1)(2n+1)$$

2. Lets look at some terms: $a_n = 1, 3, 5, 7, 9, ..., 2n-1$. So, if we let S_n be the sum of the first n terms then we have that: $S_n = 1, 4, 9, 16, 25, ...$ It should be clear that this is simply n^2.

3. Neither of these sequences tend to 0, so their infinite sums cannot be a finite number. Hence, the answer is no.

4. Let's write down some terms $a_n = 2, -2, 4, -4, 6, -6....$ If we let S_n be the sum of the first n terms then we have that $S_1 = 2$, $S_2 = 0$, $S_3 = 4$, $S_4 = 0$, $S_5 = 6$, $S_6 = 0....$ From this it should be clear that if n is even then the sum of the first n terms is 0. For odd n we can see that the sum of the first n terms is n. So we have:

$$S_n = \begin{cases} 0, & n \text{ odd} \\ n, & n \text{ even} \end{cases}$$

So the sum to n terms varies depending on whether n is odd or even. This means we are not able to find as sum to infinity because as n increases indefinitely the sum will switch between the 2 possibilities.

5. Let's write down some terms $a_n = 3, -3, 1, -1, \frac{3}{5}, -\frac{3}{5}....$ It should be clear that if n is even then the sum of the first n terms is 0 as the pairs of terms cancel each other out. If n is odd the sum of the first n terms is simply the nth term, which is $\frac{3}{n}$. As n gets very large (to infinity) $\frac{3}{n}$ gets very small so it is approximately equal to 0. Hence, whether n is odd or even, the sum to infinity is 0. So the sum to infinity of this sequence is 0.

6. Let's write down some terms $a_n = 1, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{5}, \frac{1}{5}, ...$ It should be clear that if n is odd then the sum of the first n terms is 1 as the pairs of terms cancel each other out and we are left with the first term. If n is een the sum of the first n terms is the first term plus the nth term, which is $1 - \frac{1}{n+1}$. As n gets very large (to infinity) $\frac{1}{n+1}$ gets very small so it is approximately equal to 0 and the sum to infinity is 1. Hence, whether n is odd or even, the sum to infinity is 1. So the sum to infinity of this sequence is 1.

7. It is *not* true that every sequence that tends to 0 has an infinite sum that is a finite number.
8. If the sum to infinity of a sequence is finite then it means that the sequence must tend to 0.

20.2 Harmonics and Infinites

1. The sequence $a_n = \frac{1}{n}$ is called "The Harmonic Sequence".
2. We can always find a finite number of terms that sum to at least 1, then of the remaining terms we can always find a finite number of terms that sum to at least 1, then of the remaining terms we can always find a finite number of terms that sum to at least 1... This looks like:

$$a_n = \underbrace{1}_{\geq 1}, \underbrace{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}}_{\geq 1}, \underbrace{\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}}_{\geq 1} \cdots$$

   Hence we cannot find a value for its infinite sum.
3. The sum to n of this is n plus the sum to n of the harmonic sequence. Hence, for the sum to infinity both of these terms will be very large, so the sum to infinity is not a finite number.
4. For these sequence we are adding on a "third", then "a third of a third", then "a third of a third of a third"... This will always be greater than 0 and less than 1, therefore the sum to infinity will be finite.
5. To start we add on $\frac{1}{3}$ of the pie. This is $\frac{2}{3}$ of half a pie. We therefore have $\frac{1}{3}$ of half a pie ($\frac{1}{6}$ of the pie). We add on $\frac{1}{9}$ which is $\frac{2}{3}$ of what is left to half the pie. Hence we are adding $\frac{2}{3}$ of what we need to make half the pie each time.
6. For these sequence we are adding on "$\frac{1}{10}$th", then "$\frac{1}{10}$th of $\frac{1}{10}$th", then "$\frac{1}{10}$th of $\frac{1}{10}$th of $\frac{1}{10}$th"... This will always be greater than 0 and less than 1, therefore the sum to infinity will be finite.

20.3 Comparision Testing

1. We require the conditions that : $a_n \leq K \cdot b_n$, for all n and that $\sum_{n=1}^{\infty} b_n$ is a finite number.

2. We must have that K is a positive, real number.
3. Recall that we can only use the comparison test on non-negative sequences. So the answer is no.
4. The sequence has all positive terms so the sum of the terms is always positive. Hence we can use the comparison test on this sequence.
5. $\sum_{n=1}^{\infty} a_n$ is positive and if we let $b_n = \frac{1}{2^n}$ therefore $\sum_{n=1}^{\infty} b_n$ is finite. We have that $a_n < 3 \cdot b_n$. So, by the comparison test $\sum_{n=1}^{\infty} a_n$ is finite.
6. $\sum_{n=1}^{\infty} a_n$ is positive and if we let $b_n = 1$ therefore $\sum_{n=1}^{\infty} b_n$ is infinite. We have that $a_n > b_n$. So, by the reverse comparison test $\sum_{n=1}^{\infty} a_n$ is infinite.

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