Chapter 2
Linear Periodic Systems

In this chapter, we set the basis for the study of linear periodic systems in discrete-time. We start with the impulse response description, thanks to which the notion of the periodic transfer operator is introduced. Then we move to the periodic input–output difference equation representation (PARMA model). This representation plays a major role in data analysis and periodic systems identification. The characterization in the frequency domain is carried out by means of the concept of exponentially modulated periodic signals. Finally, the state–space representation is discussed and related to the input–output descriptions. Historically, in the literature of the field, the state–space description has received more attention. However, herein we initially adopt an input–output viewpoint for reasons of generality, and postpone to later sections the state–space realm.

2.1 The Periodic Transfer Operator

In the description of linear dynamical systems, a most used model makes reference to the basic causal relationships supplying the output as a linear combination of past values of the input up to the current time instant. In discrete-time, by denoting with t the (integer) time variable, $u(t) \in \mathbb{R}^m$ the input and $y(t) \in \mathbb{R}^p$ the output, this amounts to writing:

$$\begin{aligned} y(t) &= M_0(t)u(t) + M_1(t)u(t-1) + M_2(t)u(t-2) + M_3(t)u(t-3) + \cdots \\ &= \sum_{j=0}^{\infty} M_j(t)u(t-j)\,. \end{aligned} \tag{2.1}$$

The matrix coefficients $M_i(t)$, $i = 0, 1, \cdots$, are known as *Markov coefficients* and completely capture the input–output behavior of the system. These Markov parameters are related to the impulsive response of the system.

Indeed, denote by $\delta(t)$ the impulsive function, i.e.,

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & otherwise\,. \end{cases} \tag{2.2}$$

When the input is a scalar variable ($m = 1$), from (2.1) it turns out that the response $y_{imp}^{(i)}(t)$ produced by $u(t) = \delta(t-i)$ is given by

$$y_{imp}^{(i)}(t) = M_{t-i}(t)\,.$$

To be precise, this is the output at time t when the input is an impulse applied at time $t = i$. In the general multi-input case, the j-th column of the Markov coefficients $M_{t-i}(t)$ represents the output response of the system to an input with all elements set to zero except the j-th entry $u_j(t)$ which is an impulse applied at time $t = i$.

Herein, we deal with *periodic systems*, namely systems for which there exists a positive integer T such that

$$M_i(t+T) = M_i(t)$$

for each t and for each $i \geq 0$. Obviously, this class of periodic systems includes the class of time-invariant ones, characterized by constant Markov coefficients.

A compact notation is obtained by introducing the concept of *periodic transfer operator*. To this purpose, introduce first the *unit delay operator* σ^{-1}. With reference to a generic discrete-time signal $v(t)$, this operator acts as follows:

$$\sigma^{-1} \cdot v(t) = v(t-1)\,.$$

Then, Expression (2.1) can be given the form

$$y(t) = G(\sigma,t) \cdot u(t)\,,$$

where the periodic transfer operator $G(\sigma,t)$ is defined as

$$G(\sigma,t) = M_0(t) + M_1(t)\sigma^{-1} + M_2(t)\sigma^{-2} + M_3(t)\sigma^{-3} + \cdots \tag{2.3}$$

Notice that for a given t in the interval $[0, T-1]$, this formula is just a power series in the variable σ^{-1}. Hence, convergence is ensured provided that the parameters $M_i(t)$ are bounded with respect to i.

The periodicity of the Markov parameters reflects into a specific structure of this operator. The first obvious consideration is that the operator $G(\sigma,t)$ is periodic:

$$G(\sigma,t+T) = G(\sigma,t)\,.$$

As for the dependence upon σ, it is easy to verify that

$$\sigma^{-k}G(\sigma,t) = G(\sigma,t-k)\sigma^{-k}\,.$$

This property will be referred to as *pseudo-commutative property* with respect to the delays. By setting the integer k as a multiple of the period T, say $k = iT$, this expression together with the previous one implies that the operators σ^{-iT} and $G(\sigma,t)$ do commute. Notice in passing that, by means of the pseudo-commutative property, one can write

$$G(\sigma,t) = M_0(t) + \sigma^{-1}M_1(t+1) + \sigma^{-2}M_2(t+2) + \sigma^{-3}M_3(t+3) + \cdots$$

in place of (2.3).

The periodicity of the Markov coefficients entails that the output response of the system at a generic time instant $t = kT + s$, with $s \in [0,\ T-1]$, can be written as a finite sum of the output responses of T time-invariant systems indexed in the integer s. As a matter of fact, consider again Eq. (2.1) and evaluate $y(t)$ in $t = kT + s$. It follows that

$$y(kT+s) = \sum_{i=0}^{T-1} \hat{y}_{i,s}(k)\,,$$

where

$$\hat{y}_{i,s}(k) = \sum_{j=0}^{\infty} M_{jT+i}(s)\hat{u}_{i,s}(k-j)\,,$$

and

$$\hat{u}_{i,s}(k) = u(kT+s-i)\,.$$

From these expressions, it is apparent that $\hat{y}_{i,s}(k)$ is the output of a time-invariant system having $M_i(s)$, $M_{i+T}(s)$, $M_{i+2T}(s)$, ..., as Markov parameters. Note the role of the different indexes appearing in these expressions: s is the chosen tag time index for the output variable, $s-i$ is the tag time index for the input variable, and k is the sampled time current variable. For each $i \in [0,T-1]$ and $t \in [0,T-1]$, one can define:

$$H_i(z,s) = \sum_{j=0}^{\infty} M_{jT+i}(s)z^{-j}\,. \tag{2.4}$$

This is the transfer function from $u_{i,s}(k)$ to $\hat{y}_{i,s}(k)$ (both signals seen as function of k). By using z as the 1-step-ahead shift operator in time k (namely, z is the T-*steps-ahead shift operator* in time t), and resorting to a mixed z/k notation, one can write

$$\begin{aligned} y(kT+s) &= H_0(z,s)u(kT+s) + H_1(z,s)u(kT+s-1) + \cdots \\ &\quad + H_{T-2}(z,s)u(kT+s-T+2) + H_{T-1}(z,s)u(kT+s-T+1) \end{aligned} \tag{2.5}$$

or, equivalently

$$\begin{aligned} y(kT+s) &= H_0(z,s)u(kT+s) + H_{T-1}(z,s)z^{-1}u(kT+s+1) + \cdots \\ &\quad + H_2(z,s)z^{-1}u(kT+s+T-2) + H_1(z,s)z^{-1}u(kT+s+T-1)\,. \end{aligned} \tag{2.6}$$

The function $H_i(z,s)$ will be referred to as the *sampled transfer function* at tag time s with input–output delay i.

The above "uncombing" procedure in the time-domain has a natural counterpart in the domain of transfer functions. Indeed, the transfer operator $G(\sigma,t)$ can be written as

$$G(\sigma,t) = \sum_{k=0}^{\infty} M_k(t)\sigma^{-k} = \sum_{i=0}^{T-1} \left[\sum_{j=0}^{\infty} M_{jT+i}(t)\sigma^{-jT} \right] \sigma^{-i}.$$

By taking into account the Expression (2.4), it finally follows

$$G(\sigma,t) = \sum_{i=0}^{T-1} H_i(\sigma^T,t)\sigma^{-i}. \tag{2.7}$$

This formula enables one to compute the periodic transfer operator from the sampled transfer functions, as depicted in Fig. 2.1. Expression (2.7) can be, so to say, "inverted". To this purpose, we introduce for the first time a symbol ϕ which will be often used in this book:

$$\phi = exp(\frac{2\pi j}{T}) = cos(\frac{2\pi}{T}) + jsin(\frac{2\pi}{T}). \tag{2.8}$$

Hence 1, ϕ, ϕ^2, $\cdots$, ϕ^{T-1} are the T-roots of the unit. Note that

$$\frac{1}{T}\sum_{k=0}^{T-1} \phi^{sk} = \begin{cases} 1 & s = 0, \pm T, \pm 2T, \cdots \\ 0 & otherwise. \end{cases} \tag{2.9}$$

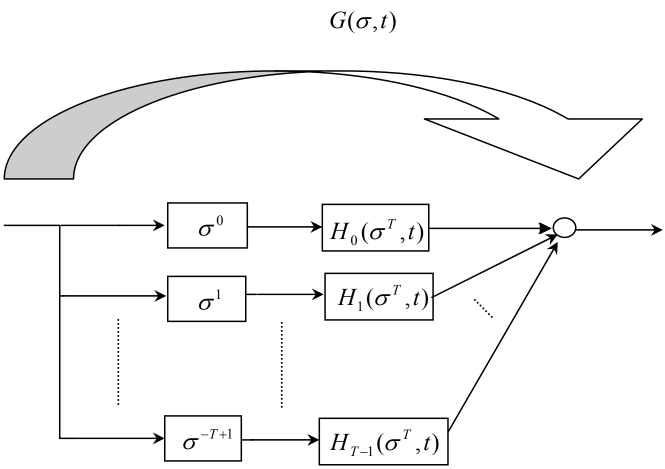

Fig. 2.1 The periodic transfer operator as a composition of (time-invariant) sampled transfer functions

Now, consider (2.7) evaluated in $\sigma\phi^k$. By multiplying both sides by ϕ^{kr} and summing w.r.t k, for k ranging from 0 to $T-1$, it easily follows from (2.9) that

$$H_r(\sigma^T,t) = \frac{\sigma^r}{T}\sum_{k=0}^{T-1} G(\sigma\phi^k, t-r)\phi^{kr}. \tag{2.10}$$

The input–output periodic model is said to be *rational* if all transfer functions (2.4) are indeed rational, i.e., they are transfer functions of finite–dimensional (time-invariant) systems. In view of (2.7), the periodic transfer operator $G(\sigma,t)$ is rational in σ for each t. In this case also the periodic system can be given a state–space finite–dimensional realization, as will be discussed in Chap. 7.

If the original periodic system is indeed time-invariant with transfer function $G(\sigma)$, the transfer function operator $G(\sigma,t)$ does not depend on t and $G(\sigma,t)$ actually reduces to $G(\sigma)$, i.e., $G(\sigma,t) = G(\sigma), \forall t$.

2.2 PARMA Model

A most useful input–output representation of a rational periodic system is given by the so-called *Periodic Auto-Regressive and Moving Average* (PARMA) model. This corresponds to writing:

$$\begin{aligned} y(t) = {} & a_1(t)y(t-1) + a_2(t)y(t-2) + \cdots + a_r(t)y(t-r) \ [\mathit{AR\ part}] \\ & + b_0(t)u(t) + b_1(t)u(t-1) + \cdots + b_s(t)u(t-s) \quad [\mathit{MA\ part}], \end{aligned} \tag{2.11}$$

where the coefficients $a_i(t)$ and $b_j(t)$ are T-periodic matrices of suitable dimensions and r and s are two positive integers. A generalization of this description is obtained by letting r and s be periodically time varying as well. However, as it is obvious, one can always rewrite the model with constant r and s, and for the sake of simplicity we will focus on the above description. Denoting by I the identity operator and letting

$$\begin{aligned} d_c(\sigma^{-1},t) &= I - a_1(t)\sigma^{-1} - a_2(t)\sigma^{-2} - \cdots - a_r(t)\sigma^{-r} \\ n_c(\sigma^{-1},t) &= b_0(t) + b_1(t)\sigma^{-1} + \cdots + b_s(t)\sigma^{-s} \end{aligned}$$

the Eq. (2.11) can be given the compact (operator) form

$$d_c(\sigma^{-1},t)\cdot y(t) = n_c(\sigma^{-1},t)\cdot u(t).$$

Define now the rational matrix $d_c^{-1}(\sigma^{-1},t)$ such that

$$d_c^{-1}(\sigma^{-1},t)d_c(\sigma^{-1},t) = d_c(\sigma^{-1},t)d_c^{-1}(\sigma^{-1},t) = I.$$

Then the periodic transfer operator of the PARMA system can be written as:

$$G(\sigma,t) = d_c^{-1}(\sigma^{-1},t)n_c(\sigma^{-1},t)\,.$$

The fractional representation of a PARMA model can be equivalently written in terms of polynomial matrices in the forward operator σ as well. Indeed, letting

$$d(\sigma,t) = \sigma^{max\{r,s\}}d_c(\sigma^{-1},t), \quad n(\sigma,t) = \sigma^{max\{r,s\}}n_c(\sigma^{-1},t)\,,$$

it follows that

$$G(\sigma,t) = d^{-1}(\sigma,t)n(\sigma,t)\,. \tag{2.12}$$

Notice that the ring of polynomials with periodic coefficients is not commutative, so that multiplication must be handled with care. Particularly important is the case in which the input and output variables are scalar, i.e., the system is SISO (Single Input Single Output). Then, the two polynomial matrices are scalar polynomials.

In the input–output representations seen so far, the polynomial have periodic coefficients. It is important to observe that, at the price of increasing the polynomials order, it is possible to work out an equivalent input–output representation with the "numerator" or the "denominator" having time-invariant coefficients. This fact is illustrated in the following simple example. The general underlying theory will be duly developed in Chap. 5.

Example 2.1 *Consider the univariate PARMA model*

$$y(t) = a(t)y(t-2) + b_0(t)u(t) + b_1(t)u(t-1) + b_2(t)u(t-2)\,,$$

where all parameters are periodic of period $T = 2$. The corresponding periodic transfer operator is

$$G(\sigma,t) = (\sigma^2 - a(t))^{-1}(b_0(t)\sigma^2 + b_1(t)\sigma + b_2(t))\,.$$

The sampled transfer functions can easily be computed from (2.10) as

$$H_0(\sigma^2,t) = \frac{b_0(t)\sigma^2 + b_2(t)}{\sigma^2 - a(t)}, \quad H_1(\sigma^2,t) = \frac{b_1(t)\sigma^2}{\sigma^2 - a(t+1)}\,.$$

Interestingly enough, the original periodic transfer operator can be equivalently rewritten with a denominator given by a polynomial in σ^2 with constant parameters. *Indeed, write the PARMA model in the operator form (2.12) with*

$$d(\sigma,t) = \sigma^2 - a(t), \quad n(\sigma,t) = b_0(t)\sigma^2 + b_1(t)\sigma + b_2(t)\,.$$

Pre-multiplying both polynomials by

$$d_*(\sigma,t) = \sigma^2 - a(t+1)\,,$$

the transfer function $G(\sigma,t)$ becomes

$$G(\sigma,t) = (d_*(\sigma,t)d(\sigma,t))^{-1}d_*(\sigma,t)n(\sigma,t)\,,$$

and the new denominator is

$$d_*(\sigma,t)d(\sigma,t) = \sigma^4 - \bar{\alpha}_1\sigma^2 - \bar{\alpha}_2$$

with

$$\bar{\alpha}_1 = (a(t)+a(t+1)), \quad \bar{\alpha}_2 = -a(t)a(t+1)\,.$$

This is a time-invariant polynomial in σ^2. Notice that the numerator is given by

$$d_*(\sigma,t)n(\sigma,t) = \beta_0(t)\sigma^4 + \beta_1(t)\sigma^3 + \cdots + \beta_4(t)\,,$$

where

$$\begin{aligned}\beta_0(t) &= b_0(t), \quad \beta_1(t) = b_1(t), \quad \beta_2(t) = b_2(t) - a(t+1)b_0(t)\\ \beta_3(t) &= -a(t+1)b_1(t), \quad \beta_4(t) = -a(t+1)b_2(t)\,.\end{aligned}$$

Thus, the numerator still has periodic coefficients. With this new transfer function, the corresponding PARMA representation becomes

$$y(t) = \bar{\alpha}_1 y(t-2) + \bar{\alpha}_2 y(t-4) + \beta_0(t)u(t) + \beta_1(t)u(t-1) + \cdots \beta_4(t)u(t-4)\,.$$

The distinctive feature of this new PARMA representation is that the autoregressive part of the model is a time-invariant polynomial in σ^2. One can therefore conjecture that the underlying dynamics of the system is dominated by the roots of the polynomial

$$z^2 - \bar{\alpha}_1 z - \bar{\alpha}_2\,.$$

As a matter of fact, the roots of this polynomial determine the input–output stability of the system, as will be explained in a forthcoming chapter.
Conversely, it is also possible to give $G(\sigma,t)$ an expression in which the numerator is a polynomial with constant coefficients. *To this end, it suffices to pre-multiply $n(\sigma,t)$ and $d(\sigma,t)$ by*

$$n_*(\sigma,t) = b_0(t+1)\sigma^2 - b_1(t)\sigma + b_2(t+1)\,,$$

so yielding

$$n_*(\sigma,t)n(\sigma,t) = \bar{\beta}_0\sigma^4 + \bar{\beta}_1\sigma^2 + \bar{\beta}_2\,,$$

where

$$\begin{aligned}\bar{\beta}_0 &= b_0(t+1)b_0(t)\\ \bar{\beta}_1 &= (b_0(t+1) - b_1(t+1))b_1(t) + b_0(t)b_2(t+1)\\ \bar{\beta}_2 &= b_2(t)b_2(t+1)\,.\end{aligned}$$

In this second form, letting

$$\begin{aligned}\alpha_0(t) &= b_0(t+1), \quad \alpha_1(t) = b_1(t), \quad \alpha_2(t) = -b_2(t+1) + b_0(t+1)a(t) \\ \alpha_3(t) &= -b_1(t)a(t+1), \quad \alpha_4(t) = b_2(t+1)a(t),\end{aligned}$$

the denominator is

$$n_*(\sigma,t)d(\sigma,t) = \alpha_0(t)\sigma^4 - \alpha_1(t)\sigma^3 - \cdots - \alpha_4(t).$$

This is a polynomial with periodic coefficients. The associated PARMA representation is

$$\alpha_0(t)y(t) = \alpha_1(t)y(t-1) + \cdots \alpha_4(t)y(t-4) + \bar{\beta}_0 u(t) + \bar{\beta}_1 u(t-2) + \bar{\beta}_2 u(t-4).$$

Now, it is the moving-average part of the model to be time-invariant in the power of σ^2. One can therefore conjecture that the underlying inverse dynamics is dominated by the roots of the polynomial

$$\bar{\beta}_0 z^2 + \bar{\beta}_1 z + \bar{\beta}_2.$$

Later on we will see that the roots of this polynomial are the transmission zeros of the periodic system. ■

Among other things, in the example it is shown that the AR part or the MA part of a PARMA model can be given an "invariantized" form. This holds in general. Precisely, consider any T-periodic polynomial $p(\sigma,t)$ whose coefficients are matrices of dimension $n \times m$. If $n \geq m$ $[n \leq m]$, there exists an associated matrix periodic polynomial $p_*(\sigma,t)$ with coefficients of dimension $n \times n$ $[m \times m]$ such that the product $p_*(\sigma,t)p(\sigma,t)$ $[p(\sigma,t)p_*(\sigma,t)]$ is indeed a polynomial in σ^T with *constant coefficients*. This can be shown by preliminarily introducing an auxiliary matrix $P(\sigma^T,t)$, whose entries are polynomials in σ^T with periodic coefficients. Precisely, if $p(\sigma,t)$ is $n \times m$, the associated $P(\sigma^T,t)$ is the $nT \times mT$ matrix defined by the expression

$$\begin{bmatrix} I_n \\ \sigma I_n \\ \vdots \\ \sigma^{T-1} I_n \end{bmatrix} p(\sigma,t) = P(\sigma^T,t) \begin{bmatrix} I_m \\ \sigma I_m \\ \vdots \\ \sigma^{T-1} I_m \end{bmatrix}. \tag{2.13}$$

It is worthy to point out that the periodic coefficients of matrix $P(\sigma,t)$ obey the following recursive formula

$$P(\sigma^T,t+1) = \Delta_n'(\sigma^{-T})P(\sigma^T,t)\Delta_m(\sigma^T), \tag{2.14}$$

where the symbol $\Delta_k(z)$ denotes the matrix of dimension $kT \times kT$ given by

$$\Delta_k(z) = \begin{bmatrix} 0 & z^{-1}I_k \\ I_{k(T-1)} & 0 \end{bmatrix}.$$

This recursion is easily verified starting from Eq. (2.13). Matrix $P(\sigma,t)$ is called *lifted polynomial* at time t associated with the given periodic polynomial $p(\sigma,t)$. In the square case ($n=m$), the lifted polynomial has the distinctive feature that its determinant has constant coefficients

$$det[P(\sigma^T,t)] = \bar{p}(\sigma^T)\,,$$

namely $\bar{p}(\sigma^T)$ is a polynomial in σ^T with constant coefficients. This fact derives from (2.14) by noticing that $det[\Delta_n(\sigma^T)] = \sigma^{-nT}$.

The representation of a periodic system in terms of a couple of lifted polynomials will be extensively studied in Chap. 5. Here we limit ourselves to exploiting the above representation in order to understand the rationale underlying the polynomial manipulations in Example 2.1. Precisely, the problem there is to pass from a PARMA scalar model to a new PARMA model in which the AR or MA part is time-invariant. In general, the procedure of "invariantization" of a given polynomial $p(\sigma,t)$ amounts to finding a polynomial $p_*(\sigma,t)$ such that $p_*(\sigma,t)p(\sigma,t)$ is a time-invariant polynomial in σ^T. This problem can be solved by pre-multiplying (2.13) by the row vector $\left[det[P(\sigma^T,t)]\ 0\ \cdots\ 0\right](P(\sigma^T,t))^{-1}$ which is indeed polynomial in σ^T. In this way, one obtains

$$\left[det[P(\sigma^T,t)]\ 0\ \cdots\ 0\right](P(\sigma^T,t))^{-1}\begin{bmatrix}1\\ \sigma\\ \vdots\\ \sigma^{T-1}\end{bmatrix}p(\sigma,t) = det[P(\sigma^T,t)]\,.$$

As already observed $det[P(\sigma^T,t)]$ has constant coefficients. Therefore the polynomial $p_*(\sigma,t)$ is simply given by

$$p_*(\sigma,t) = \left[det[P(\sigma^T,t)]\ 0\ \cdots\ 0\right](P(\sigma^T,t))^{-1}\begin{bmatrix}1\\ \sigma\\ \vdots\\ \sigma^{T-1}\end{bmatrix},$$

and it turns out that

$$p_*(\sigma,t)p(\sigma,t) = det[P(\sigma^T,t)] = \bar{p}(\sigma^T)\,.$$

2.3 The Transfer Operator in Frequency Domain

Periodic systems can be studied in the frequency domain by making reference to the concept of *exponentially modulated periodic* (EMP) signal.

A signal $v(\cdot)$ is said to be EMP if there exists a (complex) number $\lambda \neq 0$ such that

$$v(t+kT) = v(t)\lambda^{kT}, \; t \in [\tau, \tau+T-1].$$

Obviously, if one imposes $\lambda = 1$ (or equal to any T-th root of the unit), a T-periodic signal is eventually recovered.

Notice that if $v(\cdot)$ is EMP relative to a (non-null) complex number $\lambda \neq 0$, then $\bar{v}(t) = v(t)\lambda^{-t}$ is T-periodic. Conversely, if $\bar{v}(\cdot)$ is T-periodic, then, for each non-null λ, $v(t) = \bar{v}(t)\lambda^{t}$ is EMP. Therefore any EMP signal can be written as

$$v(t) = \sum_{k=0}^{T-1} \bar{v}_k \phi^{kt} \lambda^{t}, \tag{2.15}$$

where v_k are the coefficients of the Fourier expansion of the periodic signal $v(t)\lambda^{-t}$. Expression (2.15) will be referred to as the *EMP Fourier expansion*.

Suppose now we feed the T-periodic System (2.1) with a λ-EMP input $u(t)$, namely $u(t+kT) = u(t)\lambda^{kT}$, $t \in [0, T-1]$ and assume that all the sampled transfer functions $H_i(z,t)$ are well defined in $z = \lambda^T$ for each t. Notice that $H_i(z,t)$ represents a time-invariant system operating over the sampled time-variable k. If such a system is subject to an exponential signal μ^k, then in an exponential regime, the output is simply obtained by evaluating the transfer function in $z = \mu$. In particular, this applies when $\mu = \lambda^T$. Then, considering that $z = \sigma^T$ is the one period ahead shift operator (one shift ahead in k), it follows that

$$H_i(z,t+r)u(kT+t+r) = H_i(\lambda^T,t)u(t+r)\lambda^{kT}, \quad \forall r \in [0, T-1].$$

Consequently, from (2.6), for each $t = 0, 1, \cdots, T-1$:

$$\begin{aligned}
y(t+kT) &= \sum_{i=0}^{T-1} H_i(z,t)u(t+kT-i) \\
&= H_0(z,t)u(t+kT) + \left[\sum_{i=1}^{T-1} H_i(z,t)z^{-1}u(t+kT+T-i)\right] \\
&= H_0(\lambda^T,t)\lambda^{kT}u(t) + \left[\sum_{i=1}^{T-1} H_i(\lambda^T,t)\lambda^{kT-T}u(t+T-i)\right] \\
&= \left[H_0(\lambda^T,t) + \left[\sum_{i=1}^{T-1} H_i(\lambda^T,t)\lambda^{-T}\sigma^{T-i}\right]u(t)\right]\lambda^{kT}.
\end{aligned}$$

In an EMP regime, it suffices to set $y(t+kT) = y(t)\lambda^{kT}$. Hence in the EMP regime, the values of the input and output over one period are related as follows

$$y(t) = G_\lambda(\sigma,t)\cdot u(t), \quad t = 0, 1, \cdots, T-1 \tag{2.16}$$

with

$$G_\lambda(\sigma,t) = H_0(\lambda^T,t) + \sum_{i=1}^{T-1} H_i(\lambda^T,t)\lambda^{-T}\sigma^{T-i}. \tag{2.17}$$

It is worthy to point out the relationship between the periodic transfer function $G(\sigma,t)$ as characterized in (2.7) and the EMP transfer operator $G_\lambda(\sigma,t)$ in (2.17). A direct inspection leads to

$$G(\sigma,t) = G_\lambda(\sigma,t)|_{\lambda=\sigma}. \tag{2.18}$$

In conclusion, it is possible to pass from the periodic transfer operator to the EMP transfer operator and vice versa by means of (2.10), (2.17) and (2.18), respectively.

If the system is actually time-invariant, the EMP external signals take the form $u(t) = \bar{u}\lambda^t$ and $y(t) = \bar{y}\lambda^t$. Therefore, one obtains $G_\lambda(\sigma,t) = G(\lambda)$, which is indeed the gain between the input and output "amplitudes" $\bar{u}$-$\bar{y}$.

Remark 2.1 *Notice that $G_\lambda(\sigma,t)$ is polynomial in σ. It is an anti-causal operator, in that it maps $u(t)$, $u(t+1)$, $\cdots$, $u(t+T-1)$ to $y(t)$, $t \in [0,T-1]$. Instead, one can consider a causal operator $H_\lambda(\sigma,t)$ defined as*

$$H_\lambda(\sigma,t) = G_\lambda(\sigma,t)\lambda^T\sigma^{-T},$$

which maps $u(t-T)$, $u(t-T+1)$, $\cdots$, $u(t-1)$ to $y(t)$. If one makes reference to operator $H_\lambda(\sigma,t)$, then the relation

$$G(\sigma,t) = H_\lambda(\sigma,t)|_{\lambda=\sigma}.$$

follows. ∎

The EMP transfer operator plays in the periodic realm the role that the transfer function plays in time-invariant systems. Indeed, consider, for each $t \in [\tau,\tau+T-1]$, the formal discrete-time series $\{u(t+kT)\}$ and $\{y(t+kT)\}$, obtained by uniformly sampling (with tag t) the input and output signals, and define the bilateral z-transform

$$U(z,t) = \sum_{k=-\infty}^{\infty} u(t+kT)z^{-k}$$
$$Y(z,t) = \sum_{k=-\infty}^{\infty} y(t+kT)z^{-k}.$$

Notice that $U(z^T,t)$ and $Y(Z^T,t)$ are EMP signals in t, since $Y(z^T,t+kT) = Y(z^T,t)z^{kT}$, $t \in [\tau,\tau+T-1]$ and analogously for $U(z^T,t)$. Hence, Eq. (2.16) can be used yielding

$$Y(z^T,t) = G_z(\sigma,t)\cdot U(z^T,t).$$

The above equation shows that $G_z(\sigma,t)$ can be interpreted as "transfer function at t" since it maps the input z-transform $U(z^T,t)$ into the z-transform of the forced output $Y(z^T,t)$. Of course, when the system is time-invariant ($T = 1$) one can set

$z = \sigma$. Therefore, if $G(z)$ is the transfer function of the time-invariant system, then $G_z(\sigma,t) = G(z)$.

Example 2.2 *Consider a periodic system of period $T = 2$, characterized by the PARMA model*

$$y(t) = a(t)y(t-1) + b(t)u(t-1)\,.$$

The associated periodic transfer operator is

$$G(\sigma,t) = (1 - a(t)\sigma^{-1})^{-1}b(t)\sigma^{-1} = (\sigma - a(t+1))^{-1}b(t+1)\,.$$

The sampled transfer functions can be derived by making reference to (2.10), thus obtaining:

$$H_0(\sigma^2,t) = 0.5\left((\sigma - a(t+1))^{-1}b(t+1) - (\sigma + a(t+1))^{-1}b(t+1)\right)\,.$$

The denominators of the two terms in this sum neither coincide nor commute. To work out a common denominator, we preliminarily pre-multiply the first term by $(\sigma + a(t))(\sigma + a(t))^{-1}$ and the second term by $(\sigma - a(t))(\sigma - a(t))^{-1}$. In this way, one obtains:

$$H_0(\sigma^2,t) = \frac{a(t)b(t+1)}{\sigma^2 - a(t)a(t+1)}, \quad t = 0,1\,.$$

Analogously, it is possible to determine $H_1(\sigma^2,t)$ as

$$H_1(\sigma^2,t) = \frac{b(t)\sigma^2}{\sigma^2 - a(t)a(t+1)}, \quad t = 0,1\,.$$

Notice that, as expected, all the four sampled transfer functions are constant coefficient rational functions of σ^2, a fact which was not explicit in the previous expression. These expressions for $H_i(\sigma^2,t)$ can alternatively be obtained by considering the equivalent form of the periodic transfer operator

$$G(\sigma,t) = \frac{a(t)b(t+1) + b(t)\sigma}{\sigma^2 - a(t)a(t+1)}\,.$$

which can be derived from the previously given $G(\sigma,t)$ by pre-multiplying it by $(\sigma + a(t))(\sigma + a(t))^{-1}$.
Finally, the EMP transfer operator can be obtained from Eq. (2.17) in view of the previous expression of $H_i(\sigma,t)$:

$$G_\lambda(\sigma,t) = \frac{a(t)b(t+1) + b(t)\sigma}{\lambda^2 - a(t)a(t+1)}\,.$$

The causal equivalent form of this operator is

$$H_\lambda(\sigma,0) = \frac{a(t)b(t+1)\sigma^{-2} + b(t)\sigma^{-1}}{1 - a(t)a(t+1)\lambda^{-2}}\,.$$

2.4 State–Space Description

A celebrated alternative to the input–output representation is provided by the state–space description. In the finite–dimensional case, this amounts to consider the model:

$$x(t+1) = A(t)x(t) + B(t)u(t) \tag{2.19a}$$

$$y(t) = C(t)x(t) + D(t)u(t)\,, \tag{2.19b}$$

where $u(t) \in R^m$, $x(t) \in R^n$, $y(t) \in R^p$ are the input, state and output vectors, respectively. Matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are real matrices, of appropriate dimensions, which depend periodically on t:

$$A(t+T) = A(t),\ B(t+T) = B(t),\ C(t+T) = C(t),\ D(t+T) = D(t)\,.$$

A most useful concept is that of state-transition matrix; this matrix denoted by $\Phi_A(t,\tau)$ with $t \geq \tau$ is defined as:

$$\Phi_A(t,\tau) = \begin{cases} A(t-1)A(t-2)\cdots A(\tau), & t > \tau \\ I, & t = \tau\,. \end{cases}$$

The solution of System (2.19a) is given by

$$x(t) = \Phi_A(t,\tau)x(\tau) + \sum_{j=\tau}^{t-1} \Phi_A(t, j+1)B(j)u(j)\,. \tag{2.20}$$

In this expression we distinguish the free state motion $x_L(t)$ and the forced state motion $x_F(t)$

$$x_L(t) = \Phi_A(t,\tau)x(\tau), \quad x_F(t) = \sum_{j=\tau}^{t-1} \Phi_A(t, j+1)B(j)u(j)\,.$$

In this way, one can see the state evolution $x(\cdot)$ as the superposition of the free and forced motions $x(t) = x_L(t) + x_F(t)$.

Remark 2.2 *The free motion expression can be derived in an operator way. For, write the state–space equation for the free motion originating in* $x(\tau-1) = 0$ *as*

$$x(t+1) = A(t)x(t) + x(\tau)\delta(t+1-\tau)\,,$$

where $\delta(\cdot)$ *is the Kronecker function (2.2). By resorting to the* σ *operator, one obtains*

$$x(t) = (\sigma I - A(t))^{-1}\,\sigma x(\tau)\delta(t-\tau)$$

This operator expression can be elaborated as follows:

$$
\begin{aligned}
x(t) &= (\sigma I - A(t))^{-1}\,\sigma x(\tau)\delta(t-\tau) \\
&= \left(I - \sigma^{-1}A(t)\right)^{-1} x(\tau)\delta(t-\tau) \\
&= \sum_{k=0}^{\infty} \Phi_A(t,t-k)\sigma^{-k}x(\tau)\delta(t-\tau) \\
&= \sum_{k=0}^{\infty} \Phi_A(t,t-k)x(\tau)\delta(t-\tau-k) \\
&= \Phi_A(t,\tau)x(\tau)
\end{aligned}
$$

so that the classical free motion formula is recovered. ■

As for the output, it is straightforward to see that

$$
y(t) = C(t)\Phi_A(t,\tau)x(\tau) + \sum_{j=\tau}^{t-1} C(t)\Phi_A(t,j+1)B(j)u(j) + D(t)u(t)\,. \tag{2.21}
$$

Again one can write $y(t) = y_L(t) + y_F(t)$ where

$$
y_L(t) = C(t)\Phi_A(t,\tau)x(\tau), \quad y_F(t) = \sum_{j=\tau}^{t-1} C(t)\Phi_A(t,j+1)B(j)u(j) + D(t)u(t)
$$

are the free and forced output motion, respectively. Letting

$$
\begin{aligned}
M_0(t) &= D(t), \quad M_1(t) = C(t)B(t-1), \quad M_2(t) = C(t)A(t-1)B(t-2), \\
M_3(t) &= C(t)A(t-1)A(t-2)B(t-3), \cdots,
\end{aligned}
$$

the forced output motion can be written as

$$
y_F(t) = \sum_{j=0}^{t-\tau} M_j(t)u(t-j)\,.
$$

In this formula, if one let $\tau \to -\infty$, then Expression (2.1) is eventually recovered. As already said, the parameters $M_i(t)$ are known as *Markov coefficients*.

A particularly significant role is played by the so-called *monodromy matrix*, which is defined as the transition matrix over a period $[\tau,\ \tau+T-1]$:

$$
\Psi_A(\tau) = \Phi_A(\tau+T,\tau)\,.
$$

The eigenvalues of this matrix are named *characteristic multipliers*. The monodromy matrix and the characteristic multipliers are fundamental tools in the analysis of periodic systems, noticeably for the stability analysis, and will be extensively studied in the next chapter.

Thanks to periodicity, the Markov parameters can be written as

$$M_0(t) = D(t), \quad M_{jT+i}(t) = C(t)\Psi_A(t)^j \Phi_A(t,t-i+1)B(t-i), \tag{2.22}$$

where $j = 0,1,2,\cdots$ and $i \in [1,T]$. Moreover, a simple computation shows that the sampled transfer functions $H_i(z,t)$ defined in (2.4) are given by:

$$H_0(z,t) = D(t) + C(t)(zI - \Psi_A(t))^{-1}\Phi_A(t,t-T+1)B(t) \tag{2.23a}$$
$$H_i(z,t) = zC(t)(zI - \Psi_A(t))^{-1}\Phi_A(t,t-i+1)B(t-i), \quad i \in [1,T-1]. \tag{2.23b}$$

Hence, all $H_i(z,t)$ are rational functions of z, and, according to the definition in Sect. 2.1, the system is rational. From Expressions (2.7), (2.23), the periodic transfer operator $G(\sigma,t)$ can be given the compact form

$$G(\sigma,t) = D(t) + C(t)(\sigma^T I - \Psi_A(t))^{-1}\mathscr{B}(\sigma,t) \tag{2.24a}$$
$$\mathscr{B}(\sigma,t) = \sum_{j=0}^{T-1} \Phi_A(t,t+j-T+1)B(t+j)\sigma^j. \tag{2.24b}$$

This expression is reminiscent of the classical formula of a transfer function of a time-invariant system, the only difference being that the input matrix is replaced by a polynomial matrix of σ-powers up to $T-1$.

Finally, consider the periodic System (2.19) fed by the EMP input function

$$u(t+kT) = u(t)\lambda^{kT}, \; t \in [\tau, \tau+T-1],$$

and assume that λ^T does not coincide with any characteristic multiplier of $A(\cdot)$. Then the initial state

$$x_\lambda(\tau) = (\lambda^T I - \Psi_A(\tau))^{-1} \sum_{i=\tau}^{\tau+T-1} \Phi_A(\tau+T, i+1)B(i)u(i) \tag{2.25}$$

is such that both the state and the corresponding output are still EMP signals, i.e.,

$$x(t+kT) = x(t)\lambda^{kT}, \quad t \in [\tau, \tau+T-1]$$
$$y(t+kT) = y(t)\lambda^{kT}, \quad t \in [\tau, \tau+T-1].$$

Note that, for any $t \in [\tau, \tau+T-1]$, the EMP output $y(t)$ can be written as

$$y(t) = C(t)\Phi_A(t,\tau)x_\lambda(\tau) + C(t)\sum_{i=\tau}^{t-1} \Phi_A(t,i+1)B(i)u(i) + D(t)u(t).$$

Since $x_\lambda(\tau)$ is given by (2.25) and the following property holds true

$$\Phi_A(t,\tau)(\lambda^T I - \Psi_A(\tau))^{-1} = (\lambda^T I - \Psi_A(t))^{-1}\Phi_A(t,\tau),$$

a cumbersome computation shows that $y(\cdot)$ can be equivalently rewritten as

$$y(t) = C(t)(\lambda^T I - \Psi_A(t))^{-1} \sum_{i=t}^{t+T-1} \Phi_A(t+T, i+1)B(i)u(i) + D(t)u(t) .$$

This expression can be given a compact operator form as

$$y(t) = G_\lambda(\sigma,t)u(t) ,$$

where

$$G_\lambda(\sigma,t) = D(t) + C(t)(\lambda^T I - \Psi_A(t))^{-1}\mathscr{B}(\sigma,t) , \tag{2.26}$$

and $\mathscr{B}(\sigma,t)$ was defined in (2.24b). This is the state–space version of the EMP transfer operator introduced in the previous section.

Remark 2.3 *As seen in Remark 2.1, the EMP transfer operator can be given a causal expression $H_\lambda(\sigma,t)$. In the state–space context, the causal expression is*

$$H_\lambda(\sigma,t) = D(t) + C(t)(I - \Psi_A(t)\lambda^{-T})^{-1}\sigma^{-T}\mathscr{B}(\sigma,t) . \tag{2.27}$$

Example 2.3 *Consider System (2.19) with $T = 2$, $D(0) = D(1) = 0$ and*

$$A(t) = \begin{cases} 2 & t = 0 \\ -5 & t = 1 \end{cases}, \; B(t) = \begin{cases} 1 & t = 0 \\ -2 & t = 1 \end{cases}, \; C(t) = \begin{cases} 0.5 & t = 0 \\ 3 & t = 1 \end{cases} .$$

From (2.23), it follows that

$$\begin{aligned} H_0(z,0) &= \frac{-2.5}{z+10}, \;\; H_1(z,0) = \frac{-z}{z+10} \\ H_0(z,1) &= \frac{-12}{z+10}, \;\; H_1(z,1) = \frac{3z}{z+10} . \end{aligned}$$

As for the polynomial $\mathscr{B}(\sigma,t)$, it can be computed from (2.24b):

$$\mathscr{B}(\sigma,t) = \begin{cases} -5-2\sigma & t = 0 \\ -4+\sigma & t = 1 , \end{cases}$$

so that the periodic transfer operator as in (2.24a) turns out to be given by:

$$G(\sigma,0) = \frac{-2.5-\sigma}{\sigma^2+10}, \quad G(\sigma,1) = \frac{-12+3\sigma}{\sigma^2+10} .$$

Finally, from (2.26) and (2.27), the following relations are obtained:

$$\begin{aligned} G_\lambda(\sigma,0) &= \frac{-2.5-\sigma}{\lambda^2+10}, \;\; G_\lambda(\sigma,1) = \frac{-12+3\sigma}{\lambda^2+10} \\ H_\lambda(\sigma,0) &= \frac{-2.5\sigma^{-2}-\sigma^{-1}}{1+10\lambda^{-2}}, \;\; H_\lambda(\sigma,1) = \frac{-12\sigma^{-2}+3\sigma^{-1}}{1+10\lambda^{-2}} . \end{aligned}$$

■

A different way to work out the transfer operator from the state–space description of the system is that of considering directly the action of the 1-step-ahead shift operator σ on the state, input and output signals. Indeed, consider the System (2.19) and notice that it is possible to regard the action of the initial state $x_0 = x(0)$ in the following way

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) + x_0\delta(t+1) \\ y(t) &= C(t)x(t) + D(t)u(t) \\ x(-1) &= 0\,, \end{aligned} \tag{2.28}$$

where $\delta(t)$ is the (discrete) impulse function defined in (2.2). Obviously in (2.28) is has tacitly assumed that $u(\tau) = 0$ for $\tau < 0$. By resorting to the operator σ it follows that

$$x(t) = (\sigma I - A(t))^{-1}\sigma x_0 \cdot \delta(t) + (\sigma I - A(t))^{-1}B(t) \cdot u(t)\,,$$

and

$$y(t) = C(t)(\sigma I - A(t))^{-1}\sigma x_0 \cdot \delta(t) + \left[C(t)(\sigma I - A(t))^{-1}B(t) + D(t)\right] \cdot u(t)\,.$$

Of course the transfer operator can be written as

$$G(\sigma,t) = C(t)(\sigma I - A(t))^{-1}B(t) + D(t)\,. \tag{2.29}$$

Equation (2.29) is formally identical to the usual formula of the transfer function for time-invariant systems. Obviously, when computing the inverse of $\sigma I - A(t)$, the skew commutative role of the operator σ must be taken into account, as illustrated in the subsequent example.

Example 2.4 *Consider the matrix*

$$A(t) = \begin{bmatrix} 0 & 1 \\ 1 & \alpha(t) \end{bmatrix}$$

where $\alpha(t)$ is a 2-periodic scalar function. The inverse of the operator matrix $\sigma I - A(t)$ is given by:

$$\begin{aligned} (\sigma I - A(t))^{-1} &= \begin{bmatrix} (\sigma - \alpha(t)) & 1 \\ 1 & \sigma \end{bmatrix} \begin{bmatrix} (\sigma^2 - \alpha(t+1)\sigma - 1)^{-1} & 0 \\ 0 & (\sigma^2 - \alpha(t)\sigma - 1)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (\sigma^2 - \alpha(t)\sigma - 1)^{-1} & 0 \\ 0 & (\sigma^2 - \alpha(t+1)\sigma - 1)^{-1} \end{bmatrix} \begin{bmatrix} (\sigma - \alpha(t)) & 1 \\ 1 & \sigma \end{bmatrix} \end{aligned}$$

The easy check of the above expression is left to the reader. ■

Compare the Expression (2.29) with the EMP transfer function $G_\lambda(\sigma,t)$ as given by Eq. (2.26). Since Relation (2.18) holds true, such comparison suggests that

$$(\sigma^T I - \Psi_A(t))^{-1}\mathscr{B}(\sigma,t) = (\sigma I - A(t))^{-1}B(t)\,.$$

The validity of such identity can be proven as follows:

$$\begin{aligned}
(\sigma I - A(t))^{-1}B(t) &= (I-\sigma^{-1}A(t))^{-1}\sigma^{-1}B(t) \\
&= \sum_{k=0}^{\infty} \Phi_A(t,t-k)\sigma^{-k-1}B(t) \\
&= \sum_{p=0}^{\infty}\sum_{i=0}^{T-1} \Phi_A(t,t-i-pT)B(t-i-1)\sigma^{-i-1-pT} \\
&= \sum_{p=0}^{\infty} \Psi_A(t)\sigma^{-pT} \sum_{i=0}^{T-1} \Phi_A(t,t-i))B(t-i-1)\sigma^{-i-1} \\
&= (I-\sigma^{-T}\Psi_A(t))^{-1} \sum_{i=0}^{T-1} \Phi_A(t,t-i)B(t-i-1)\sigma^{-i-1} \\
&= (\sigma^T I - \Psi_A(t))^{-1} \sum_{j=0}^{T-1} \Phi_A(t,t+j-T+1)B(t+j)\sigma^{j} \\
&= (\sigma^T I - \Psi_A(t))^{-1}\mathscr{B}(\sigma,t)\,.
\end{aligned}$$

Example 2.5 *Consider again the* 2*-periodic* 1*-dimensional system defined in Example 2.3. The periodic transfer operator, as defined in (2.29), is*

$$G(\sigma,t) = C(t)(\sigma - A(t))^{-1}B(t)\,.$$

By noticing that

$$C(t)(\sigma - A(t))^{-1} = (\sigma - A(t)C(t+1)C(t)^{-1})^{-1}C(t+1)\,,$$

it follows

$$G(\sigma,t) = (\sigma - A(t)C(t+1)C(t)^{-1})^{-1}C(t+1)B(t)\,.$$

Multiplying by

$$(\sigma + C(t)C(t+1)^{-1}A(t+1))(\sigma + C(t)C(t+1)^{-1}A(t+1))^{-1}\,,$$

we have

$$\begin{aligned}
G(\sigma,t) &= (\sigma^2 - A(t)A(t+1))^{-1})(B(t+1)C(t)\sigma + C(t)A(t+1)B(t)) \\
&= \frac{B(t+1)C(t)\sigma + C(t)A(t+1)B(t)}{\sigma^2 - A(t)A(t+1)}\,,
\end{aligned}$$

which is easily proven to coincide with the one worked out in Example 2.3. ■

We end this section by pointing out the effect of a change of basis in the state–space.

Consider the new state coordinates

$$\tilde{x}(t) = S(t)x(t)$$

where $S(\cdot)$ is a $n \times n$ T-periodic matrix, invertible for each t. The new representation of the given system is

$$\begin{aligned} \tilde{x}(t+1) &= \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)u(t) \\ y(t) &= \tilde{C}(t)\tilde{x}(t) + \tilde{D}(t)u(t), \end{aligned}$$

with

$$\tilde{A}(t) = S(t+1)A(t)S(t)^{-1}, \ \tilde{B}(t) = S(t+1)B(t) \tag{2.30a}$$
$$\tilde{C}(t) = C(t)S(t)^{-1}, \ \tilde{D}(t) = D(t). \tag{2.30b}$$

In system theory, two systems defined by the quadruplets $(A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ and $(\tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot), \tilde{D}(\cdot)$ related as in (2.30) are said to be *algebraically equivalent*. It is easy to see that the transition matrices and the monodromy matrices associated with $A(\cdot)$ and $\tilde{A}(\cdot)$ are related to each other as follows

$$\Phi_{\tilde{A}}(t,i) = S(t)\Phi_A(t,i)S(i)^{-1} \tag{2.31a}$$
$$\Psi_{\tilde{A}}(t) = S(t)\Psi_A(t)S(t)^{-1}. \tag{2.31b}$$

Formula (2.31b) is a similarity transformation which, as is well-known, preserves eigenvalues. This means that the characteristic multipliers of a periodic system are independent of the state representation.
In the same vein, one can observe that the Markov parameters are invariant with respect to the change of basis, as an easy computation starting from Expressions (2.22) and (2.31a) reveals. As a consequence, the periodic transfer operator, the sampled transfer functions, and the EMP transfer operators are invariant as well. This conclusion is expected since all these quantities have to deal with the input–output behavior of the system, and, as such, must be insensitive to the choice of basis in state–space.

2.5 Adjoint Operator and Its State–Space Description

For a (discrete-time) time-invariant system with transfer function $W(z)$, the adjoint system is defined as the system with transfer function $W^{\sim}(z) = W(z^{-1})'$. Notice that the adjoint transfer function $W^{\sim}(z)$ may correspond to a system which is not causal. This is indeed a major difference between discrete-time and continuous-time systems. If (A, B, C, D) is a state–space realization of $W(z)$, then

$$W(z) = D + C(zI - A)^{-1}B, \quad W^{\sim}(z) = D' + B'(z^{-1}I - A')^{-1}C'.$$

Now, consider the periodic System (2.19) and the transfer operator

$$\begin{aligned} G(\sigma,t) &= M_0(t)+M_1(t)\sigma^{-1}+M_2(t)\sigma^{-2}+M_3(t)\sigma^{-3}+\cdots \\ &= D(t)+C(t)\left(\sigma I-A(t)\right)^{-1}B(t)\,. \end{aligned}$$

The adjoint operator is defined as

$$\begin{aligned} G^{\sim}(\sigma,t) &= M_0(t)'+\sigma M_1(t)'+\sigma^2 M_2(t)'+\sigma^3 M_3(t)+\cdots \\ &= M_0(t)'+M_1(t+1)'\sigma+M_2(t+2)'\sigma^2+M_3(t+3)\sigma^3+\cdots \\ &= D(t)'+B(t)'\left(\sigma^{-1}I-A(t)'\right)^{-1}C(t)'\,. \end{aligned}$$

It is easy to check that the state–space system corresponding to the adjoint operator is written in the form

$$A(t)'\lambda(t+1) = \lambda(t)-C(t)'v(t) \tag{2.32a}$$

$$\varsigma(t) = B(t)'\lambda(t+1)+D(t)'v(t)\,. \tag{2.32b}$$

This form is referred to in the literature as the *descriptor form*. In general, a periodic system in descriptor form is characterized by the modified state equation $E(t)x(t+1)=A(t)x(t)+B(t)u(t)$, where $E(\cdot)$ is T-periodic.

2.6 Bibliographical Notes

Most of the fundamental concepts presented in this chapter for periodic systems are rooted in the core of basic system theory; classical textbooks are [85, 203, 204, 255, 316]. For time-varying systems, the notion of transfer function operator goes back to the early 1950s, [315], and has been subsequently studied and elaborated further by a number of authors, see e.g., [205, 206]. For the fractional representation of PARMA models we refer to [132] and [47].

In the developments of the theory, further families of periodic models have been introduced in the literature. Among them, we mention the so-called *descriptor* periodic systems which, as said above, are characterized by the modified state equation $E(t)x(t+1)=A(t)x(t)+B(t)\,u(t)$, where $E(t)$ is T-periodic too. Another interesting approach is supplied by the behavioral modelization. We will not cover these aspects, and the interested reader can refer to the literature, e.g., [215, 277].

http://www.springer.com/978-1-84800-910-3

Periodic Systems
Filtering and Control
Bittanti, S.; Colaneri, P.
2009, XIX, 424 p., Hardcover
ISBN: 978-1-84800-910-3