Chapter 2
Mathematical Models

2.1 Nonlinear Differential Equations in Distributions

The present section develops a consistent modeling methodology which gives rise to mathematical models of nonlinear dynamical systems with nonlinear impulse responses. Modeling accommodates the details of the realization of the impulse input, and admits the impulse response to depend upon the manner in which the impulse is implemented. The set of all possible impulse responses is found from a certain auxiliary system with integrable inputs. Necessary and sufficient conditions are additionally obtained for the impulse response to be unique and independent of the impulse realization. The proposed modeling methodology is subsequently used to derive filtering equations over sampled-data measurements and to synthesize impulsive controllers.

2.1.1 Preliminaries

A linear continuous functional, mapping the space D_0 of continuous functions with compact support into the real line $\mathbf{R}^1$, is referred to as a *zero-order distribution*. Thus, the distributions are defined indirectly, by specifying their effect on the test functions. Recall that the support of a function $\varphi(t)$ specified point-wise is the closure of the set $\{t : \varphi(t) \neq 0\}$.

As usual, D_0^* denote the dual space of zero-order distributions. The dual product $< u, \varphi >$ of a distribution $u(t) \in D_0^*$ and a test function $\varphi(t) \in D_0$ is denoted by $\int_{-\infty}^{\infty} u(t)\phi(t)dt$. Alternatively, D_0^* may be viewed as the space of all (Borel) measures $d\mu(t)$ with locally bounded variations. The dual product is then explicitly defined by the Stieltjes integral $< d\mu, \varphi >= \int_{-\infty}^{\infty} \phi(t)d\mu(t)$.

A sequence of distributions $u_k(t) \in D_0^*,\ k = 1,2,\ldots$ converges to $u(t) \in D_0^*$ in weak* topology if

$$\lim_{k\to\infty}\int_{-\infty}^{\infty} u_k(t)\varphi(t)dt = \int_{-\infty}^{\infty} u(t)\varphi(t)dt$$

for any test function $\varphi(t) \in D_0$.

2.1.2 Instantaneous Impulse Response in a Nonlinear Setting

In what follows, we deal with affine systems, the dynamics of which are described by a nonlinear differential equation of the form

$$\dot{x}(t) = f(x,t) + b(x,t)u,\ x(0) = x_0 \tag{2.1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $x_0 \in \mathbf{R}^n$ is the initial state, $u(t) \in \mathbf{R}^m$ is the input, and $t \geq 0$ is the time variable. Without loss of generality, we have assumed that the system in question is initialized at $t = 0$; otherwise, a time substitution $s = t - t_0$ with an appropriate constant t_0 should be applied.

It is well known that the smoothness of f and b guarantees locally the existence of a unique Caratheodory trajectory, driven by an integrable input $u(t)$. For the convenience of the reader the following definitions are reviewed.

Definition 2.1. An absolutely continuous function $x(t)$, defined on some interval $[0,\tau)$, is said to be a Caratheodory solution of (2.1) iff it is initialized in accordance with (2.1), and it satisfies (2.1) for almost all $t \in [0,\tau)$.

Clearly, unbounded inputs can make trajectories of System 2.1 be arbitrary close to discontinuous ones. In order to admit discontinuous behavior of the system one should extend the description. If $b(x,t)$ is a state-invariant function $b(t)$, it appears that one can rigorously introduce discontinuous solutions into the equation by admitting the input to be a measure-type function (e.g., a δ-pulse). In general, this way is hampered, however, by the irregularity of a product of the impulsive input $u(t)$ and the discontinuous (in t) function $b(x(t),t)$. In order to avoid this we generalize the meaning of the differential equation as follows.

Definition 2.2. A sequence $\{u_k(t)\}$ of integrable inputs, the L_1-norms of which are uniformly bounded, is a generalized system input, if the solutions $x_k(t),\ k = 1,2,\ldots$ of (2.1), corresponding to the inputs $u(t) = u_k(t)$, converge to a left-continuous function $x(t)$ for all continuity points $t \geq 0$ of $x(t)$. The function $x(t)$ is referred to as a generalized solution of (2.1).

Relating the generalized control input $\{u_k(t)\}$ in the above definition to a peaking sequence of L_1 inputs, which weakly* converges to a δ-pulse, one can define the impulse response of a nonlinear system as the corresponding generalized solution of (2.1). Generally speaking, different approximations of the δ-pulse result in different generalized solutions. Thus, the impulse response depends upon the implementation of the impulse. The symbol $x_{\{u_k\}}(t)$ is used to denote the generalized solution of (2.1) corresponding to a generalized input $\{u_k(t)\}$.

The result stated below describes the set

$$X(\gamma,t_0)=\{x_{\{u_k\}}(t_0+):*-\lim_{k\to\infty}u_k(t)=\gamma\delta(t-t_0)\}$$

of the instantaneous impulse responses $x_{\{u_k\}}(t_0+)=\lim_{t\downarrow t_0}x_{\{u_k\}}(t)$ of (2.1) to all possible realizations $\{u_k(t)\}$ of the impulsive input $\gamma\delta(t-t_0)$, $\gamma\in\mathbf{R}^m$, $t_0\geq 0$. We shall prove that $X(\gamma,t_0)$ can be specified by means of the reachability set

$$R(y_0,\gamma,t_0)=\{\eta_w(y_0,1,t_0):\int_0^1 w(t)dt=\gamma\}$$

of the trajectories $\eta_w(y_0,t,t_0)$ of the auxiliary dynamical system

$$\dot\eta=b(\eta,t_0)w(t),\eta(0)=y_0 \tag{2.2}$$

with integrable inputs $w(t)$ of fixed integral power $\int_0^1 w(t)dt=\gamma$. Denoting the solution of the unforced system

$$\dot y=f(y,t),\ y(0)=x_0 \tag{2.3}$$

as $y(t)$, we arrive at the following [163].

Theorem 2.1. *Let functions $f(x,t)$, $b(x,t)\in C^1$ satisfy the linear growth condition in x. Then $X(\gamma,t_0)=R(y(t_0),\gamma,t_0)$. Moreover, if $y_1\in R(y(t_0),\gamma,t_0)$ and the auxiliary system (2.2) is driven from the initial state $y(t_0)$ to the terminal state y_1 for $t=1$ by an admissible input $w(t)$, i.e., $\eta_w(y(t_0),1,t_0)=y_1$, then the instantaneous impulse response $x_{\{u_k\}}(t_0+)=y_1$ of the affine system (2.1) is particularly forced by the generalized input $\{u_k(t)\}$ where*

$$u_k(t)=\begin{cases}kw(k(t-t_0)) & \textit{if } t\in[t_0,t_0+1/k]\\ 0 & \textit{otherwise,}\end{cases}$$
$$k=1,2,\ldots. \tag{2.4}$$

Proof. First, we shall demonstrate that functions (2.4) do converge to the impulsive input $\gamma\delta(t-t_0)$ in the weak* topology. Indeed,

$$\int_{-\infty}^{\infty}\varphi(t)u_k(t)dt=\int_{t_0}^{t_0+1/k}\varphi(t)kw(k(t-t_0))dt=$$
$$\int_0^1\varphi(t_0+\frac{t}{k})w(t)dt\to\varphi(t_0)\int_0^1 w(t)dt=\varphi(t_0)\gamma$$

as $k\to\infty$ for all $\varphi\in D_0$.

Now, in order to prove that $\{u_k(t)\}$ is a generalized input and $x_{\{u_k\}}(t_0+)=\eta_w(y(t_0),1,t_0)$, we introduce functions

$$\eta_k(s) = \begin{cases} x_k(t_0+s) & \text{if } s \le 0 \\ x_k(t_0+s/k) & \text{if } s \in [0,1] \\ x_k(t_0+1/k+s-1) & \text{if } s \ge 1 \end{cases} \tag{2.5}$$

where $x_k(t),\ k=1,2,\dots$ are the trajectories of (2.1) forced by the input actions (2.4). Then $\eta_k(s),\ k=1,2,\dots$ satisfy

$$\begin{aligned} \dot{\eta}_k(s) &= f(\eta_k,t_0+s),\ s \le 0 \\ \dot{\eta}_k(s) &= \frac{1}{k}f(\eta_k,t_0+\frac{s}{k})+ \\ b(\eta_k,t_0&+\frac{s}{k})w(s),\ 0 \le s \le 1 \\ \dot{\eta}_k(s) &= f(\eta_k,t_0+1/k+s-1),\ s \ge 1 \end{aligned} \tag{2.6}$$

and due to the continuous dependence of the solution on the small parameter $1/k$, the functions $\eta_k(s)$ converge point-wise to the solution of

$$\dot{\eta}(s) = \begin{cases} f(\eta,t_0+s) & \text{if } s \le 0 \\ b(\eta,t_0)w(s) & \text{if } s \in [0,1] \\ f(\eta,t_0+s-1) & \text{if } s \ge 1, \end{cases} \tag{2.7}$$

initialized with $\eta(-t_0) = x_0$, as $k \to \infty$. By virtue of (2.5) it follows that

$$x_{\{u_k\}}(t) = \begin{cases} \eta(t-t_0) & \text{if } t \le t_0 \\ \eta(t-t_0+1) & \text{if } t > t_0 \end{cases}$$

and $x_{\{u_k\}}(t+) = \eta_w(y(t_0),1,t_0)$. Hence, if $y_1 \in R(y(t_0),\gamma,t_0)$ and the auxiliary system (2.2) is driven from the initial state $y(t_0)$ to the terminal state y_1 for $t=1$ by an admissible input $w(t)$, then the instantaneous impulse response $x_{\{u_k\}}(t_0+) = y_1$ of (2.1) is particularly forced by the generalized input (2.4). Thus, the inclusion $R(y(t_0),\gamma,t_0) \subset X(\gamma,t_0)$ is shown.

Proof of the converse inclusion follows the same method of time substitution. Let $\{u_k(t)\}$ be a generalized input such that the convergence

$$* - \lim_{k\to\infty} u_k(t) = \gamma\delta(t-t_0) \tag{2.8}$$

takes place in weak* topology and let $x_{\{u_k\}}(t)$ be the generalized solution of (2.1) corresponding to the generalized input, i.e.,

$$x_{\{u_k\}}(t) = \lim_{k\to\infty} x_k(t)\ \mathit{for\ all}\ t \ge t_0 \tag{2.9}$$

where $x_k(t)$ is the trajectory of (2.1) driven by the input $u_k(t)$. By Helly's theorem (see, e.g., [111]), the weak* convergence (2.8) means that

$$\lim_{k\to\infty}\int_0^t u_k(\tau)d\tau = \begin{cases} 0 \text{ for all } t < t_0 \\ \gamma \text{ for all } t > t_0 \end{cases} \tag{2.10}$$

and

$$\int_0^t \|u_k(\tau)\| d\tau \le M(t) \tag{2.11}$$

where $M(t) < \infty$ for all $t \ge 0$. Now define $\alpha_k(t),\ \beta_k(t),\ \nu_k,\ \mu_k(t),\ k = 1,2,\ldots$ by

$$\alpha_k(t) = t + \int_0^t \|u_k(s)\| ds,\ \beta_k(t) = \alpha_k^{-1}(t)$$
$$\nu_k = \alpha_k(t_0 + 1/k) - \alpha_k(t_0 - 1/k),$$
$$\mu_k(t) = \alpha_k(t_0 - 1/k) + \nu_k t$$

and introduce functions

$$\zeta_k(s) = \begin{cases} x_k(t_0 - 1/k + s) & \text{if } s \le 0, \\ x_k(\beta_k(\mu_k(s))) & \text{if } s \in [0,1] \\ x_k(t_0 + 1/k + s - 1) & \text{if } s \ge 1. \end{cases} \tag{2.12}$$

Then $\zeta_k(s),\ k = 1,2,\ldots$ satisfy the differential equations

$$\dot{\zeta}_k(s) = f(\zeta_k, t_0 - 1/k + s) +$$
$$b(\zeta_k, t_0 - 1/k + s) u_k(t_0 - 1/k + s),\ s \le 0, \tag{2.13}$$
$$\dot{\zeta}_k(s) = \frac{\nu_k f(\zeta_k, \beta_k(\mu_k(s)))}{1 + \|u_k(\beta_k(\mu_k(s)))\|} +$$
$$+ \frac{\nu_k b(\zeta_k, \beta_k(\mu_k(s))) u_k(\beta_k(\mu_k(s)))}{1 + \|u_k(\beta_k(\mu_k(s)))\|},\ 0 \le s \le 1, \tag{2.14}$$
$$\dot{\zeta}_k(s) = f(\zeta_k, t_0 + 1/k + s - 1) +$$
$$b(\zeta_k, t_0 + 1/k + s - 1) u_k(t_0 + 1/k + s - 1),\ s \ge 1. \tag{2.15}$$

Due to (2.8), (2.9), it follows that for all $s \le 0$ functions $\zeta_k(s)$ converge to the solution of the equation

$$\dot{\zeta}(s) = f(\zeta, t_0 + s),\ \zeta(-t_0) = x_0$$

as $k \to \infty$, whereas for $s \in [0,1]$ the point-wise convergence of $\zeta_k(s)$ is guaranteed by their uniform boundedness and equicontinuity. Indeed, due to our assumptions regarding the functions f, b and by virtue of inequalities

$$\nu_k = 2/k + \int_{t_0 - 1/k}^{t_0 + 1/k} \|u_k(\tau)\| d\tau \le 2 + M(t_0 + 1) < \infty,$$
$$\frac{1}{1 + \|u_k(\beta_k(\mu_k(s)))\|} \le 1,\ \frac{\|u_k(\beta_k(\mu_k(s)))\|}{1 + \|u_k(\beta_k(\mu_k(s)))\|} \le 1,$$

solutions of (2.14) are uniformly bounded, have uniformly bounded time derivatives, and converge at $s = 0$ as $k \to \infty$. Then, Arzela's theorem (see, e.g., [116]) ensures their point-wise convergence to a function that, according to Filippov's lemma [70], satisfies the equation

$$\dot{\zeta}(s) = f(\zeta, t_0)w_0(s) + b(\zeta, t_0)w(s).$$

Since (2.10) implies that

$$\lim_{k\to\infty} \frac{\nu_k}{1+\|u_k(\beta_k(\mu_k(s)))\|} = 0,$$
$$\lim_{k\to\infty} \int_0^1 \frac{\nu_k u_k(\beta_k(\mu_k(s)))}{1+\|u_k(\beta_k(\mu_k(s)))\|} ds =$$
$$\lim_{k\to\infty} \int_{t_0-1/k}^{t_0+1/k} u_k(t)dt = \gamma,$$

then $w_0(s) \equiv 0$ and $\int_0^1 w(s)ds = \gamma$. Therefore, in order to calculate the instantaneous impulse response $x_{\{u_k\}}(t_0+) = \zeta(1)$, we arrive at the same equation (2.2) subject to the same initial condition $y_0 = y(t_0)$ and the same integral condition $\int_0^1 w(s)ds = \gamma$. Clearly, $X(\gamma, t_0) \subset R(y(t_0), \gamma, t_0)$ and Theorem 2.1 is completely proved.

Thus, under the assumptions of Theorem 2.1, the set of all generalized solutions can be viewed as weak* closure of the set of the Caratheodory solutions corresponding to the integrable inputs. Due to the peaking phenomenon [219], the impulse response of the affine system (2.1) with no growth condition on the right-hand side can escape to infinity in infinitesimal time. The following example illustrates this property.

Example 2.1. Let (2.1) be specified to

$$\dot{x} = x^2 u(t),\ x(0) = 1 \tag{2.16}$$

where $f = 0$ and $b = x^2$, and let its generalized input $u(t) = \{u_k(t)\}$ be given by

$$u_k(t) = \begin{cases} kt \text{ if } t \in [0, 0+1/k] \\ 0 \text{ otherwise}, \end{cases}$$
$$k = 1, 2, \ldots. \tag{2.17}$$

It is then straightforwardly verified that the weak* convergence

$$* - \lim_{k\to\infty} u_k(t) = \delta(t) \tag{2.18}$$

holds and the Caratheodory solution $x_k(t)$ of (2.16) with input (2.17), being substituted into (2.16) for $u(t)$, is locally determined by $x_k(t) = (1-kt)^{-1}$ for $t \in [0, \frac{1}{k})$ so that

$$x_{\{u_k\}}(t+) = \lim_{k\to\infty} x_k(t+) = \infty.$$

Thus, the generalized solution $x_{\{u_k\}}(t)$ of (2.16) with the δ-wise input (2.17) escapes to infinity in infinitesimal time.

It is of interest to note that the Frobenius condition

$$\Sigma_{k=1}^{n} \frac{\partial b_{li}(x,t)}{\partial \xi_k} b_{kj}(x,t) = \Sigma_{k=1}^{n} \frac{\partial b_{lj}(x,t)}{\partial \xi_k} b_{ki}(x,t), \tag{2.19}$$

imposed on the matrix function $b(x,t) = \{b_{ki}(x,t)\}$ for all $l = 1,\dots,n,\ i,j = 1,\dots,m$, and $x \in \mathbf{R}^n,\ t \geq 0$, ensures the uniqueness of the impulse response [163].

Theorem 2.2. *Let the assumptions of Theorem 2.1 be satisfied and let $\{u_k(t)\}$ be a generalized input of (2.1) such that*

$$* - \lim_{k\to\infty} u_k(t) = \gamma\delta(t-t_0),\ \gamma \in \mathbf{R}^m,\ t_0 \geq 0 \tag{2.20}$$

where $\gamma \in \mathbf{R}^m$ and $t_0 \geq 0$ are arbitrary. Then, a generalized solution $x(t) = x_{\{u_k\}}(t)$ of (2.1) does not depend upon a choice of the approximating sequence $\{u_k(t)\}$ if and only if the Frobenius condition (2.19) holds for all $l = 1,\dots,n,\ i,j = 1,\dots,m$, and $x \in \mathbf{R}^n,\ t \geq 0$.

Proof. Let the Frobenius condition (2.19) hold for all $l = 1,\dots,n,\ i,j = 1,\dots,m$, and $x \in \mathbf{R}^n,\ t \geq 0$. Then, the corresponding Pfaffian equation

$$d\xi/dv = b(\xi,t_0),\ \xi \in \mathbf{R}^n,\ v \in \mathbf{R}^m,\ t_0 \geq 0 \tag{2.21}$$

is well known [42] to possess a solution for arbitrary initial conditions $\xi(0) = z \in \mathbf{R}^n$ and $t_0 \geq 0$. Just in case, the Pfaffian equation (2.21) integrates to the function $\xi(z,v,t_0)$ regardless of a path $v(s)$ between the initial point $v(0) = y_0$ and the terminal point $v(1) = \gamma$. Hence, the reachability set $R(y_0,\gamma,t_0)$ of (2.21), written in the parametric form (2.2) with $w(t) = \dot{v}(t)$, consists of the unique point $\xi(y_0,\gamma,t_0)$ and, according to Theorem 2.1, the instantaneous impulse response is uniquely defined as

$$x(t_0+) = \xi(y(t_0),\gamma,t_0) \tag{2.22}$$

where $y(t)$ satisfies (2.3).

To complete the proof it remains to demonstrate that the uniqueness of the impulse response guarantees the validity of the Frobenius condition.

For this purpose, we specify an initial condition $x_0 \in \mathbf{R}^n$ in (2.3) to ensure that the corresponding solution $y(t)$ of this equation takes an a priori given value $z \in \mathbf{R}^n$ at $t = t_0$. By virtue of the assumptions, imposed on the function $f(x,t)$, there always exists such an initial condition.

Then the generalized solution $x_{\{u_k\}}(t)$ of (2.1) takes the same value z at $t = t_0$ before it makes the jump enforced by the generalized input (2.20), and by Theorem 2.1, $x_{\{u_k\}}(t_0+) = \eta_\gamma(1)$ where $\eta_\gamma(\tau)$ is a solution of

$$\dot{\eta}_\gamma = b(\eta_\gamma,t_0)\gamma, \eta_\gamma(0) = z. \tag{2.23}$$

Based on $\eta_\gamma(\tau)$, thus specified, let us now introduce the function

$$\xi(z,v,t_0) = \eta_v(1) = z + \int_0^1 b(\eta_v(\tau),t_0)v d\tau \tag{2.24}$$

that proves to meet the condition

$$\frac{\partial \xi(z,v,t_0)}{\partial v}|_{v=0} = b(z,t_0), \tag{2.25}$$

computed at $v=0$. Indeed, by taking into account the trivial relation

$$\xi(z,0,t_0) = z, \tag{2.26}$$

one derives that

$$\begin{aligned} &\lim_{\|v\|\to 0} \frac{\|\xi(z,v,t_0) - \xi(z,0,t_0) - b(z,t_0)v\|}{\|v\|} \\ &= \lim_{\|v\|\to 0} \frac{\|\int_0^1 [b(\eta_v(\tau),t_0) - b(z,t_0)]v d\tau\|}{\|v\|} \\ &\leq \lim_{\|v\|\to 0} \|\int_0^1 [b(\eta_v(\tau),t_0) - b(z,t_0)]d\tau\| = 0, \end{aligned} \tag{2.27}$$

because $\lim_{v\to 0} \eta_v(\tau) = z$ for all $\tau \in [0,1]$.

Furthermore, the function $\xi(z,v,t_0)$, governed by (2.24), satisfies (2.21). To establish this, let us first note that

$$\xi(z,\gamma_1+\gamma,t_0) = \xi(\xi(z,\gamma_1,t_0),\gamma,t_0). \tag{2.28}$$

In order to prove (2.28) it suffices to consider the generalized inputs

$$\begin{aligned} u_k(t) &= \begin{cases} k(\gamma_1+\gamma) & \text{if } t \in [t_0, t_0+1/k] \\ 0 & \text{otherwise} \end{cases}, \\ u_k'(t) &= \begin{cases} k\gamma_1 & \text{if } t \in [t_0, t_0+1/k] \\ k\gamma & \text{if } t \in [t_0+1/k, t_0+2/k] \\ 0 & \text{otherwise} \end{cases}, \\ & k = 1,2,\ldots. \end{aligned} \tag{2.29}$$

Similar to (2.4), both inputs in (2.29) are straightforwardly verified to converge to $(\gamma_1+\gamma)\delta(t-t_0)$ in the weak* topology, so that they produce the same generalized solution $x_{\{u_k\}}(t) = x_{\{u_k'\}}(t)$ of (2.1). It follows that

$$\xi(z,\gamma_1+\gamma,t_0) = x_{\{u_k\}}(t_0+) = x_{\{u_k'\}}(t_0+) = \xi(\xi(z,\gamma_1,t_0),\gamma,t_0),$$

thereby yielding (2.28).

To this end, relations (2.25), (2.26), and (2.28), coupled together, lead to

$$\begin{aligned} &\lim_{\|\Delta v\|\to 0} \frac{\|\xi(z,v+\Delta v,t_0) - \xi(z,v,t_0) - b(z,t_0)\Delta v\|}{\|\Delta v\|} \\ &= \lim_{\|\Delta v\|\to 0} \frac{\|\xi(\xi(z,v,t_0),\Delta v,t_0) - \xi(\xi(z,v,t_0),0,t_0) - b(z,t_0)\Delta v\|}{\|\Delta v\|} = 0. \end{aligned} \tag{2.30}$$

Hence, the function $\xi(z,v,t_0)$, governed by (2.24), solves the Pfaffian equation (2.21) under arbitrary initial conditions $\xi(0) = z \in \mathbf{R}^n$ and $t_0 \geq 0$. It is well known [42] that this ensures the Frobenius condition (2.19) to hold for all $l = 1,\ldots,n,\ i,j = 1,\ldots,m$ and $x \in \mathbf{R}^n,\ t \geq 0$. The proof is completed.

2.1.3 Vibroimpact Solutions

As shown, modeling the nonlinear impulsive system (2.1) should, generally speaking, accommodate details of the impulse implementation, and the set of all possible instantaneous impulse responses may be found from the auxiliary system (2.2) with integrable inputs. In turn, imposing the Frobenius condition on the underlying system allows one to replace peak functions in modeling such a system by δ-pulses. The following definition is thus in order.

Definition 2.3. System 2.1 is said to be vibrocorrect if given an initial condition x_0 and a generalized input (2.20), the system possesses a unique generalized solution $x(t)$, regardless of a choice of the δ-approximating sequence $\{u_k(t)\}$ in (2.20). The generalized solution $x(t)$ is then referred to as a vibroimpact solution of (2.1) under the impulsive input $u(t) = \gamma\delta(t-t_0)$.

By Theorem 2.2, the instantaneous impulse response (2.22) of the vibrocorrect system (2.1) is uniquely determined from the solution of the Pfaffian system (2.21). Particularly, in the case where $b(x,t) = b(t)$ is a state-independent function, and hence the mild solution of (2.1) is well-defined, the solution $\xi(z,v,s)$ of the Pfaffian system is given by $\xi(z,v,s) = z + b(s)v$ and the instantaneous impulse response $x(0+) = y(t_0) + b(0)\gamma$ is the same as if the conventional mild solution would be under consideration.

We are now in a position to address the problem of reconstructing a nonlinear vibrocorrect system (2.1), (2.20), based on its discrete-continuous representation

$$\dot{x}(t) = f(x,t),\ x(0) = x_0 \tag{2.31}$$

$$\Delta x(t_0) = \theta(x(t_0-),\gamma,t_0), \tag{2.32}$$

similar to that of (1.3), which we discussed in the introduction.

Theorem 2.3. *Let functions $f(x,t),\ b(x,t) \in C^1$ satisfy the linear growth condition in x and let (2.1) be driven by the generalized input (2.20). Suppose that, given an arbitrary initial condition $x_0 \in \mathbf{R}^n$, and an arbitrary impulse magnitude $\gamma \in \mathbf{R}^m$, (2.1), (2.20) and the discrete-continuous system (2.31), (2.32) possess the same solutions. Then (2.1) is vibrocorrect if and only if the following relations hold:*

$$\frac{\partial\theta(z,\gamma,t_0)}{\partial\gamma}\Big|_{\gamma=0} = b(z,t_0), \tag{2.33}$$

$$\theta(z,\gamma_1+\gamma,t_0) = \theta(z,\gamma_1,t_0) + \theta(z+\theta(z,\gamma_1,t_0),\gamma,t_0). \tag{2.34}$$

Proof. Let us assume that (2.1) is vibrocorrect. Then relations (2.22) and (2.32), coupled together, yield

$$\xi(z,v,t_0) = z + \theta(z,v,t_0). \tag{2.35}$$

Taking (2.35) into account, relation (2.33) results from (2.25), whereas (2.34) is guaranteed by (2.28).

Conversely, once relations (2.33), (2.34) are satisfied, the function $\xi(z,v,t_0)$, governed by (2.35), meets Conditions 2.25, 2.28. Then, following the line of reasoning, used in the proof of Theorem 2.2, this function is shown to solve the Pfaffian equation (2.21) under arbitrary initial conditions $\xi(0) = z \in \mathbf{R}^n$ and $t_0 \geq 0$. Since, due to [42], this ensures the validity of the Frobenius condition (2.19) for all $l = 1,\ldots,n,\ i,j = 1,\ldots,m$ and $x \in \mathbf{R}^n,\ t \geq 0$ [42], the vibrocorrectness of (2.1) is straightforwardly verified by applying Theorem 2.2. The proof of Theorem 2.3 is completed.

Clearly, Theorem 2.3 allows one to reconstruct an accessible input, resulting in the prescribed restitution rule (2.32). Indeed, provided that (2.34) is satisfied, applying the impulsive input $b(x,t)\gamma\delta(t-t_0)$ with the feedback gain $b(x,t)$, determined by (2.33), ensures the desired state restitution (2.32) for (2.1), thus defined. A simple example, given below, illustrates the capabilities of the distributions theory in a nonlinear setting.

Example 2.2. Let (2.1), (2.20) be specified to

$$\dot{x} = x\gamma\delta(t-t_0), x(0) = x_0 \in \mathbf{R}^1 \tag{2.36}$$

where $f = 0$ and $b = x$. Then, the Pfaffian equation (2.21) is simplified to the ordinary differential equation

$$d\xi/dv = \xi,\ \xi, v \in \mathbf{R}^1, \tag{2.37}$$

which integrates to $\xi(v) = \xi(0)e^v$. By taking into account Theorems 2.1 and 2.2, (2.36) is vibrocorrect and its instantaneous impulse response is uniquely defined by

$$x(t_0+) = x_0 e^\gamma \tag{2.38}$$

whereas the discrete-continuous representation (2.31), (2.32) of the system is given by

$$\dot{x}(t) = 0,\ x(0) = x_0 \tag{2.39}$$

$$\Delta x(t_0) = x_0(e^\gamma - 1). \tag{2.40}$$

In turn, applying Theorem 2.3 to the discrete-continuous system (2.39), (2.40) yields $\frac{\partial\{x(e^\gamma-1)\}}{\partial\gamma}|_{\gamma=0} = x$, thus converting the solutions of (2.39), (2.40) into the vibroimpact solutions of (2.36).

2.2 Differential Equations with a Piece-wise Continuous Right-hand Side

Consider a non-autonomous differential equation

$$\dot{x} = \varphi(x,t), \tag{2.41}$$

with the state vector $x = (x_1, \ldots, x_n)^T$, with the time variable $t \in \mathbf{R}$, and with a piece-wise continuous right-hand side $\varphi = (\varphi_1, \ldots, \varphi_n)^T$.

Recall that the function $\varphi : \mathbf{R}^{n+1} \to \mathbf{R}^n$ is piece-wise continuous iff $\mathbf{R}^{n+1}$ is partitioned into a finite number of domains $G_j \subset \mathbf{R}^{n+1}$, $j = 1, \ldots, N$, with disjoint interiors and boundaries ∂G_j of measure zero such that each of the restrictions $\varphi_j = \varphi|_{G_j}$, $j = 1, \ldots, N$ of φ to these domains is continuous within G_j where it has a finite limit $\varphi^j(x,t)$ as the argument $(x^*, t^*) \in G_j$ approaches a boundary point $(x,t) \in \partial G_j$.

The solutions of (2.41) are defined in the conventional Caratheodory sense whenever they are within the domains G_j $j = 1, \ldots, N$. In order to describe possible generalized solutions of (2.41) on the boundaries ∂G_j, $j = 1, \ldots, N$, which are further referred to as *sliding modes*, the following regularization technique is normally utilized. In a vicinity of the boundaries, the original equation is replaced by a related equation, the Caratheodory solutions of which are well-posed. The sliding modes are then defined by making the characteristics of the new equation approach those of the original one. The rigorous introduction of the solution concept is as follows.

Definition 2.4. An absolutely continuous function $x^\delta(t)$, defined on some interval I, is said to be an approximate δ-solution of (2.41) if it is a Caratheodory solution of

$$\dot{x}^\delta = \phi^\delta(x^\delta, t)$$

with some $\phi^\delta(x,t)$ such that

$$\|\phi^\delta(x,t) - \phi(x,t)\| \le \delta$$

for almost all $(x,t) \in \mathbf{R}^n \times I$, satisfying $sup_{(\xi,\tau)\in\cup_{j=1}^N \partial G_j}[\|x-\xi\| + \|t-\tau\|] \ge \delta$.

Definition 2.5. An absolutely continuous function $x(t)$, defined on some interval I, is said to be a generalized solution of (2.41) if there exists a family of approximate δ-solutions $x^\delta(t)$ of the equation such that

$$\lim_{\delta\to 0} \|x^\delta(t) - x(t)\| = 0 \; uniformly \; in \; t \in I.$$

Being defined as a generalized solution, a sliding mode of (2.41) appears to depend on a particular approximation of the right-hand side of the equation, but it does not depend on the precise specification of the right-hand side on the discontinuity boundaries ∂G_j, $j = 1, \ldots, N$.

2.2.1 Filippov Solutions

A particular regularization of (2.41) with the discontinuous right-hand side occurs if the switch between the structures $\varphi_j = \varphi|_{G_j},\ j = 1,\dots,N$ has a hysteresis. As mentioned in Sect. 1.3, such a regularization results in the meaning of the differential equation (2.41) to be defined in the sense of A. F. Filippov [71].

Definition 2.6. Given the differential equation (2.41) let us introduce for each point $(x,t) \in \mathbf{R}^n \times \mathbf{R}$ the smallest convex closed set $\Phi(x,t)$ which contains all the limit points of $\varphi(x^*,t)$ as $x^* \to x, t = const$, and $(x^*,t) \in \mathbf{R}^{n+1} \setminus (\cup_{j=1}^{N} \partial G_j)$. An absolutely continuous function $x(\cdot)$, defined on an interval I, is said to be a Filippov solution of (2.41) if it satisfies the differential inclusion

$$\dot{x} \in \Phi(x,t) \tag{2.42}$$

almost everywhere on I. The set $\Phi(x,t)$ is further referred to as the Filippov set.

Equation 2.41 with a piece-wise continuous right-hand side is well known [71] to possess a solution for arbitrary initial conditions $x(t_0) = x^0 \in \mathbf{R}^n,\ t_0 \in \mathbf{R}$. This solution is locally defined on some time interval $[t_0,t_1)$; however, generally speaking, it is nonunique.

At any continuity point $(x,t) \in \cup_{i=1}^{N} G_i$ of the function φ the Filippov set $\Phi(x,t)$ consists of the only point $\varphi(x,t)$, and the Filippov solution satisfies (2.41) in the conventional sense. Given $(x,t) \in \cap_{k=1}^{L} \partial G_{j_k}$ from the intersection of the boundaries ∂G_{j_k} of several domains $G_{j_k},\ k = 1,\dots,L$, the Filippov set $\Phi(x,t)$ is either a segment, or a convex polygon, or a convex polyhedron with vertices

$$\varphi_{j_k}(x,t) = \lim_{(\xi,t)\in G_{j_k},\ \xi\to x} \varphi(\xi,t),\ k = 1,\dots,L.$$

Apparently, the points $\varphi_{j_k}(x,t),\ k = 1,\dots,L$ belong to the Filippov set $\Phi(x,t)$, but not all these points are the vertices, forming this set.

Let us assume now that the function $\varphi(x,t)$ undergoes discontinuities on a smooth surface S only, and let this surface be governed by the equation $s(x) = 0$. Then the discontinuity set S separates the x space into domains $G^- = \{x \in \mathbf{R}^n : s(x) < 0\}$ and $G^+ = \{x \in \mathbf{R}^n : s(x) > 0\}$. Given t, the Filippov set $\Phi(x,t)$ would be a linear segment joining the endpoints of the vectors

$$\varphi^-(x,t) = \lim_{(\xi,t)\in G^-,\ \xi\to x} \varphi(\xi,t),\ \varphi^+(x,t) = \lim_{(\xi,t)\in G^+,\ \xi\to x} \varphi(\xi,t).$$

Hereinafter, these vectors are assumed to have the same initial point x.

If for $t \in [t_0,t_1]$, the vectors $\varphi^-(x,t)$ and $\varphi^+(x,t)$ point toward the same region, the Filippov segment $\Phi(x,t)$ would be located on one side of the plane T, tangential to the discontinuity surface S so that the Filippov solutions of (2.41) would cross S for these t. Such a situation appears if either $grad^T s(x)\ \varphi^-(x,t) < 0$ or $grad^T s(x)\ \varphi^+(x,t) > 0$ where $grad^T s = (\frac{\partial s}{\partial x_1},\dots,\frac{\partial s}{\partial x_n})$.

On the contrary, if

$$grad^T s(x)\ \varphi^-(x,t) > 0,\ grad^T s(x)\ \varphi^+(x,t) < 0 \tag{2.43}$$

for all $t \in [t_0, t_1]$, the vectors $\varphi^-(x,t)$ and $\varphi^+(x,t)$ are directed to opposite directions, and the Filippov segment $\Phi(x,t)$ intersects the tangential plane T. Then a sliding mode occurs on the discontinuity surface S for these t. According to Definition 2.6, this mode is governed by

$$\dot{x} = \varphi^0(x,t) \tag{2.44}$$

where the intersection of the Filippov segment $\Phi(x,t)$ and the plane T, tangential to S, determines the endpoint of the vector $\varphi^0(x,t)$. Analytically, this vector is expressed in the form

$$\varphi^0(x,t) = \mu(x,t)\varphi^+(x,t) + [1-\mu(x,t)]\varphi^-(x,t),\ \mu(x,t) \in [0,1] \tag{2.45}$$

where

$$\mu(x,t) = \frac{grad^T s(x)\ \varphi^-(x,t)}{grad^T s(x)\ [\varphi^-(x,t) - \varphi^+(x,t)]} \tag{2.46}$$

is found from the condition

$$grad^T s(x)\ \{\mu(x,t)\varphi^+(x,t) + [1-\mu(x,t)]\varphi^-(x,t)\} = 0 \tag{2.47}$$

that the velocity vector (2.45) is in the plane T, tangential to S.

Recall that the same expression (2.45), specified with (2.46), has appeared in Example 1.4 of the introduction to describe the average velocity (1.23) of the VSS (1.11) subject to the hysteresis switching (1.22) between its two structures.

It is clear that the velocity vector (2.45) is uniquely determined by (2.46) iff the segment with the ends $\varphi^-(x,t)$ and $\varphi^+(x,t)$ intersects the tangential plane T but it does not lie in it. If this segment however is entirely in the tangential plane T, i.e.,

$$grad^T s(x)\ [\varphi^-(x,t) - \varphi^+(x,t)] = 0,$$

the Filippov solution is not uniquely defined.

The same line of finding the Filippov set $\Phi(x,t)$ applies to the (x,t) space with$(n+1)$-dimensional vectors $(\varphi^-(x,t),1)$ and $(\varphi^+(x,t),1)$ in the case where the discontinuity surface S is governed by the time-dependent equation $s(x,t) = 0$. Setting $s_t = \frac{\partial s}{\partial t}$, Condition 2.43 of a sliding mode to exist is then modified to

$$s_t(x,t) +\ grad^T s(x,t)\ \varphi^-(x,t) > 0,\ s_t(x,t) + grad^T s(x,t)\ \varphi^+(x,t) < 0, \tag{2.48}$$

whereas the velocity vector (2.45) is determined by

$$\mu(x,t) = \frac{s_t(x,t) + grad^T s(x,t)\ \varphi^-(x,t)}{grad^T s(x,t)\ [\varphi^-(x,t) - \varphi^+(x,t)]}, \tag{2.49}$$

being found from the condition

$$s_t(x,t) + grad^T s(x,t)\ \{\mu(x,t)\varphi^+(x,t) + [1-\mu(x,t)]\varphi^-(x,t)\} = 0, \tag{2.50}$$

which ensures that $\frac{ds(x(t),t)}{dt} = 0$ for the sliding mode $x(t)$, evolving along the discontinuity surface S.

2.2.2 Equivalent Control Method

An alternative approach of defining the velocity vector on a discontinuity set is based on the equivalent control method, developed by V.I. Utkin [227] for controlled systems of the form

$$\dot{x} = f(x, u(x,t), t) \tag{2.51}$$

with a continuous (in all arguments) right-hand side $f = (f_1, \ldots, f_n)^T$ where the state vector $x = (x_1, \ldots, x_n)^T$ and the time variable $t \in \mathbf{R}$ are the same as before, whereas the controlled input $u = (u_1, \ldots, u_m)^T$ is a piece-wise continuous function of the state and time variables. The components $u_i(x,t),\ i = 1, \ldots, m$ of the control signal $u(x,t)$ are assumed to undergo discontinuities on possibly intersecting smooth surfaces $S_i = \{(x,t) \in \mathbf{R}^{n+1} : s_i(x,t) = 0\}$, and to vary within the state/time dependent segments $U_i(x,t) = [u_i^-(x,t), u_i^+(x,t)]$ where

$$u_i^-(x,t) = \lim_{(\xi,t)\in S_i^-,\ \xi\to x} u_i(\xi,t),\ S_i^- = \{(x,t) \in \mathbf{R}^{n+1} : s_i(x,t) < 0\}$$

and

$$u_i^+(x,t) = \lim_{(\xi,t)\in S^+,\ \xi\to x} u_i(\xi,t),\ S_i^+ = \{(x,t) \in \mathbf{R}^{n+1} : s_i(x,t) > 0\}.$$

Apparently, at the continuity points (x,t) of $u_i(x,t)$, the corresponding set $U_i(x,t)$ consists of a unique point.

According to the equivalent control method, the potential sliding modes of (2.51) at an intersection of some sets $S_{j_k},\ k = 1, \ldots, r$ are governed by

$$\dot{x} = f(x, u^{eq}(x,t), t) \tag{2.52}$$

where the components

$$u_i^{eq}(x,t) \in [u_i^+(x,t), u_i^+(x,t)],\ i = 1, \ldots, m \tag{2.53}$$

of the equivalent control input $u^{eq}(x,t)$ are such that the velocity vector f in (2.52) is tangent to the sets $S_{j_k},\ k = 1, \ldots, r$, i.e., for $(x,t) \in \cap_{k=1}^r S_{j_k}$, the equivalent control vector with components (2.53) is to satisfy

$$\frac{\partial s_{j_k}}{\partial t}(x,t) + \ grad^T s_{j_k}(x,t)\ f(x, u^{eq}(x,t), t) = 0,\ k = 1, \ldots, r. \tag{2.54}$$

Definition 2.7. An absolutely continuous function $x(\cdot)$, defined on an interval I, is said to be an Utkin solution of (2.51) if it satisfies (2.51) beyond the surfaces S_i, $i = 1,\dots,m$ and it satisfies equations of the form (2.52) on these surfaces and their intersections.

The physical meaning behind Utkin solutions is as follows [228]. While approximating the discontinuous input signal u on the time interval I by its continuous counterpart u^δ, providing system oscillations within a δ-vicinity of the discontinuity set $S = \cup_{i=1}^m S_i$, the equivalent control value u^{eq} proves to be approached by the output of a lowpass filter

$$\tau \dot{z}^\delta + z^\delta = u^\delta, \tag{2.55}$$

i.e., $\lim z(t) = u^{eq}(t)$ uniformly in $t \in I$ as $\tau \downarrow 0$ and $\delta/\tau \downarrow 0$. In other words, the approximate δ-solution of (2.51), driven by the slow component $u = z^\delta$ of the control signal u^δ, passing through filter (2.55), approaches the sliding mode $x(t)$, computed according to the equivalent control method:

$$\lim_{\tau\downarrow 0, \frac{\delta}{\tau}\downarrow 0} x^\delta(t) = x(t) \text{ uniformly in } t \in I. \tag{2.56}$$

Thus, Utkin solutions constitute a special class of generalized solutions under a particular regularization of (2.51) with the output z of filter (2.55), being substituted into (2.51) for the control input u.

Generally speaking, Utkin solutions do not coincide with corresponding Filippov solutions. The following example drawn from [227] illustrates this feature.

Example 2.3. Consider the discontinuous system

$$\dot{x}_1 = 0.3x_2 + x_1 u,\ \dot{x}_2 = -0.7x_1 + 4x_1 u^3 \tag{2.57}$$

with the input function u, governed by

$$u = \begin{cases} 1 & \text{if } (x_1 + x_2)x_1 < 0 \\ -1 & \text{if } (x_1 + x_2)x_1 > 0 \end{cases}. \tag{2.58}$$

By inspection, the sliding mode existence condition (2.43), being specified for (2.57), (2.58), appears to hold on the discontinuity line $x_1 + x_2 = 0$, whereas it does not hold on the discontinuity line $x_1 = 0$. Thus, sliding modes of the system in question occur along the line $x_1 + x_2 = 0$.

The equivalent control value $u^{eq} = 0.5$ is then obtained from (2.54), which is now specified to $(-1 + u^{eq} + 4(u^{eq})^3 + u^{eq} - 1 = 0$. Substituting the equivalent control value into (2.57) for u yields the instable equation $\dot{x}_1 = 0.2x_1$, governing Utkin solutions on the discontinuity line $x_1 + x_2 = 0$.

In turn, Filippov solutions are governed by another equation $\dot{x}_1 = -0.1x_1$, which is obtained by specifying the Filippov velocity (2.45), (2.46) for the present system and which is asymptotically stable.

It is worth noticing that both Filippov sliding modes and Utkin sliding modes are described by reduced-order equations because they are confined to the one-dimensional state subspace $x_1 + x_2 = 0$.

The equivalent control method, applied to an affine system (2.51), which admits representation in the form

$$\dot{x} = \eta(x,t) + b(x,t)u \tag{2.59}$$

yields

$$u^{eq}(x) = -(\frac{\partial s}{\partial x}b)^{-1}(\frac{\partial s}{\partial t} + \frac{\partial s}{\partial x}\eta) \tag{2.60}$$

provided that the matrix function $\frac{\partial s(x,t)}{\partial x}b(x,t)$ is nonsingular for all $(x,t) \in \mathbf{R}^{n+1}$. By substituting the equivalent control value (2.60) into the affine system (2.59) for the input function u, the equation

$$\dot{x} = \eta - b(\frac{\partial s}{\partial x}b)^{-1}(\frac{\partial s}{\partial t} + \frac{\partial s}{\partial x}\eta) \tag{2.61}$$

is derived to describe potential Utkin solutions of the affine system (2.59) on the discontinuity manifold $s(x,t) = 0$.

As shown in [227, Sect. 3 of Chapt. II], the sliding mode equation (2.61) appears to hold for arbitrary Filippov solutions, and moreover, for arbitrary generalized solutions that can occur in the affine systems of the form (2.59). Our next goal is to extend this result to differential equations in a Hilbert space.

2.2.3 Sliding Modes in a Hilbert Space

Following [162], consider a differential equation

$$\dot{x} = Ax + f(x,t) + bu(x,t),\ t > 0,\ x(0) = x^0 \in \mathscr{D}(A) \tag{2.62}$$

where the state variable $x(t)$ and the input signal $u(x,t)$ are abstract functions with values in Hilbert spaces H and U, respectively, the infinitesimal operator A with domain $\mathscr{D}(A)$ generates a strongly continuous semigroup $T_A(t)$ on H, the operator function $f(x,t)$ with values in H is of class C^1 in all arguments, and b is a linear bounded operator, acting from U to H.

For later use, recall the following (see, e.g., [55] for relevant background materials on Hilbert space-valued dynamical systems). A family $\{T(t)\}_{t\geq 0}$ of linear bounded operators $T(t),\ t \geq 0$ forms a strongly continuous semigroup on a Hilbert space H if the identity $T(t+\tau) = T(t)T(\tau)$ is satisfied for all $t, \tau \geq 0$, and the functions $T(t)x$ are continuous with respect to $t \geq 0$ for all $x \in H$. The induced operator norm $\|T(t)\|$ of the semigroup satisfies the inequality $\|T(t)\| \leq \omega e^{\beta t}, t \geq 0$ with some growth bound β and some $\omega > 0$.

The domain of an operator A, generating a strongly continuous semigroup, forms the Hilbert space $\mathscr{D}(A)$ with the graph inner product $\langle\cdot,\cdot\rangle_{\mathscr{D}(A)}$ defined by means of the inner product $\langle\cdot,\cdot\rangle_H$ of the underlying Hilbert space H:

$$\langle x,y\rangle_{\mathscr{D}(A)} = \langle x,y\rangle_H + \langle Ax,Ay\rangle_H,\ x,y \in \mathscr{D}(A).$$

If β is a growth bound of the semigroup, then given $\lambda > \beta$, there holds $(A-\lambda I)^{-1}H = \mathscr{D}(A)$ where I is the identity operator, and the norm of $x \in \mathscr{D}(A)$ given by $\|(A-\lambda I)x\|_H$ is equivalent to the graph norm $\|x\|_{\mathscr{D}(A)}$ of $\mathscr{D}(A)$. In particular, $\|x\|_{\mathscr{D}(A)} = \|Ax\|_H$ if A possesses a growth bound $\beta < 0$. It should be noted that $\mathscr{D}(A) \hookrightarrow H$, i.e., $\mathscr{D}(A) \subset H$, $\mathscr{D}(A)$ is dense in H and the inequality $\|x\|_H \leq \omega_0\|x\|_{\mathscr{D}(A)}$ holds for all $x \in \mathscr{D}(A)$ and some constant $\omega_0 > 0$.

If the input function u meets the same smoothness conditions as that imposed on the system nonlinearity f, the above equation locally has a unique strong solution $x(t)$ which is defined as follows.

Definition 2.8. A continuous function $x(t)$, defined on $[0,T)$, is a strong solution of the initial value-problem (2.62) with a continuously differentiable input $u(x,t)$ iff $\lim_{t\downarrow 0}\|x(t)-x^0\|_H = 0$, and $x(t)$ is continuously differentiable and satisfies the equation for $t \in (0,T)$.

The precise meaning of the solutions of (2.62), for inputs which are only piecewise continuously differentiable, is defined as a limiting result obtained through the regularization procedure, similar to that proposed for finite-dimensional systems.

Let the input $u(x,t)$ be continuously differentiable beyond a linear manifold

$$cx = 0 \tag{2.63}$$

with $c \in \mathscr{L}(H,S)$ being a linear bounded operator from H to some Hilbert space S, and let $u(x,t)$ undergo discontinuities on this manifold. Then the strong solutions of (2.62) are only considered whenever they are beyond the discontinuity manifold (2.63), whereas in a vicinity of this manifold, the original system is replaced by a related system, which takes into account all possible imperfections in the new input function $u^\delta(x,t)$ (e.g., delay, hysteresis, saturation, etc.) and for which there exists a strong solution. A generalized solution of (2.62) is then obtained by making the characteristics of the new system approach those of the original one. As in the finite-dimensional case, a motion along the discontinuity manifold is referred to as a sliding mode.

To rigorously introduce a sliding mode in the infinite-dimensional system, let us complement the subspace

$$H_1 = ker\ c = \{x_1 \in H :\ cx_1 = 0\} \subseteq H$$

by the subspace $H_2 \subseteq H$ such that

$$H = H_1 \oplus H_2.$$

Clearly, the discontinuity manifold (2.63), written through the new coordinates $x_1(t) = P_1 x(t) \in H_1$ and $x_2(t) = P_2 x(t) \in H_2$, takes the form $x_2 = 0$. Hereinafter, P_i is the projector on the subspace H_i, $A_i = A|_{H_i}$ is the operator restriction on H_i, $i = 1,2$.

Definition 2.9. An absolutely continuous function $x^\delta(t)$, defined on some interval $[0,\tau)$, is said to be an approximate δ-solution of (2.62) if it is a strong solution of

$$\dot{x}^\delta = Ax^\delta + f(x^\delta, t) + bu^\delta(x^\delta, t) \tag{2.64}$$

with some $u^\delta(x,t)$ such that

$$\|u^\delta(x,t) - u(x,t)\| \leq \delta \tag{2.65}$$

for all $t \geq 0$ and for all $x = (x_1, x_2) \in H = H_1 \oplus H_2$ subject to $\|x_2\|_{\mathscr{D}(A_2)} \geq \delta$.

Definition 2.10. An absolutely continuous function $x(t)$, defined on some interval $[0,\tau)$, is said to be a generalized solution of (2.62) if there exists a family of approximate δ-solutions $x^\delta(t)$ of the system such that

$$\lim_{\delta \to 0} \|x^\delta(t) - x(t)\|_{\mathscr{D}(A)} = 0 \; uniformly \; in \; t \in [0,\tau).$$

Although beyond the discontinuity manifold, our investigation is confined to strong solutions of the initial-value problem, an extension to the case where such a solution is defined in a mild sense as a solution to a corresponding integral equation, is possible.

By definition, the sliding motion, which is in general non-unique, does not depend on the precise specification of the discontinuous input on the discontinuity manifold. Moreover, an equivalent control value $u^{eq}(x,t)$, maintaining the system motion on this manifold, is imposed by the original system itself.

To describe sliding modes in the infinite-dimensional system let us rewrite (2.62) in terms of variables $x_1(t) \in H_1$ and $x_2(t) \in H_2$:

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + f_1(x_1, x_2, t) + b_1 u(x_1, x_2, t), \; x_1(0) = x_1^0, \tag{2.66}$$

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + f_2(x_1, x_2, t) + b_2 u(x_1, x_2, t), \; x_2(0) = x_2^0 \tag{2.67}$$

where $A_{ij} = P_i A_j$, $i,j = 1,2$ are the operators from H_j to H_i, and $f_i = P_i f$, $b_i = P_i b$. If the operator b_2 is non-singular and the inverse operator b_2^{-1} is bounded, there exists a unique solution of the algebraic equation $A_{21}x_1 + f_2(x_1, 0, t) + b_2 u(x,t) = 0$ with respect to u. This solution

$$u^{eq}(x_1) = -b_2^{-1}[A_{21}x_1 + f_2(x_1, 0, t)] \tag{2.68}$$

is accepted as the equivalent control value because (2.68) is the only input ensuring that $\dot{x}_2 = 0$ on the discontinuity manifold $x_2 = 0$, thereby maintaining (2.62) with appropriate initial conditions on the manifold $x_2 = 0$. Setting $\tilde{A} = A_{11} - b_1 b_2^{-1} A_{21}$ and $f_0(x_1, x_2, t) = f_1(x_1, x_2, t) - b_1 b_2^{-1} f_2(x_1, x_2, t)$, the sliding mode equation

$$\dot{x} = \tilde{A}x_1 + f_0(x_1, 0, t), \tag{2.69}$$

governing the system motion on the discontinuity manifold $cx = 0$, is then obtained by substituting the equivalent control value (2.68) into (2.67) for u.

Under the assumptions

1. The linear operator b is bounded and its projection b_2 on the subspace H_2 is continuously invertible, i.e., the operator b_2^{-1} from H_2 to U is bounded, too;
2. The infinitesimal operators A and $\tilde{A} = A_{11} - b_1 b_2^{-1} A_{21}$ generate strongly continuous semigroups $T_A(t), t \geq 0$ and $T_{\tilde{A}}(t), t \geq 0$ on the Hilbert spaces H and H_1, respectively;
3. The system nonlinearity $f(x,t)$ is everywhere continuously differentiable in (x,t) and it satisfies the linear growth condition in x;
4. The input function $u(x,t)$ is everywhere continuously differentiable but on the discontinuity manifold $x_2 = 0$;
5. The operator $G_0 = \tilde{A} b_1 b_2^{-1}$ from H_2 to H_1 is governed by A_{12} in the sense that $\mathscr{D}(G_0) \subseteq \mathscr{D}(A_{12})$ and $\|G_0 y\| \leq K \|A_{12} y\|$ for all $y \in \mathscr{D}(G_0)$ and some $K > 0$;

the result, given below, extends the equivalent control method to infinite-dimensional systems.

Assumption 1, inherited from the finite-dimensional case, ensures the uniqueness of the sliding mode equation. Assumptions 2–4 are made for technical reasons. Coupled to Assumption 1, they ensure the existence and uniqueness of local strong solutions of (2.62) beyond the discontinuity manifold $x_2 = 0$. As a matter of fact, the operator function $f_0(x_1, x_2, t) = f_1(x_1, x_2, t) - b_1 b_2^{-1} f_2(x_1, x_2, t)$ meets Assumption 4, too, so that the existence and uniqueness of local strong solutions of the sliding mode equation (2.69) are also guaranteed. Assumption 5 is intrinsic for infinite-dimensional systems in the sense that if it would fail to hold other generalized solutions, not governed by (2.69), could appear.

Theorem 2.4. *Consider the dynamic system (2.62) under Assumptions 1–5. Let the system, being initialized in the discontinuity manifold $x_2 = 0$ with $x^0 = (x_1^0, 0) \in H$, start evolving in this manifold on some time interval $[0, \tau)$. Then the initial value problem (2.62) possesses a unique generalized solution $x(t)$, and on the time interval $[0, \tau)$ this solution is governed by the sliding mode equation (2.69) under the initial condition $x_1(0) = x_1^0$.*

Proof. By definition 2.10, on the time interval $[0, \tau)$ the generalized solution $x(t)$ of the discontinuous system (2.62) is approximated by δ-solutions $x^\delta(t)$, corresponding to inputs $u^\delta(x,t)$ and evolving within the δ-vicinity

$$\|x_2\|_{\mathscr{D}(A_2)} \leq \delta \tag{2.70}$$

of the discontinuity manifold $x_2 = 0$. It follows that

$$u^\delta(x,t) = -b_2^{-1}[A_{21}x_1^\delta + A_{22}x_2^\delta + f_2(x_1^\delta, x_2^\delta, t) - \dot{x}_2^\delta].$$

By substituting the first component $x_1^\delta(t)$ of the approximate δ-solution $x^\delta(t)$ of the discontinuous system (2.62) is governed by

$$\dot{x}_1^\delta = \tilde{A}x_1^\delta + f_0(x_1^\delta, x_2^\delta, t) + b_1 b_2^{-1}\dot{x}_2^\delta + [A_{12} - b_1 b_2^{-1} A_{22}]x_2^\delta,\ x_1^\delta(0) = x_1^0. \quad (2.71)$$

Let on the time interval $[0,\tau)$ the notation $z^\delta(t) = x_1^\delta(t) - x_1(t)$ stands for the deviation of the strong solution $x_1(t)$ of the sliding mode equation (2.69) from the approximate solution $x_1^\delta(t)$. Then subtracting (2.69) from (2.71) yields the deviation equation

$$\dot{z}^\delta = \tilde{A}z^\delta + [f_0(x_1^\delta, x_2^\delta, t) - f_0(x_1, 0, t)] + b_1 b_2^{-1}\dot{x}_2^\delta + [A_{12} - b_1 b_2^{-1} A_{22}]x_2^\delta,$$

whose solutions are representable in the integral form

$$\begin{aligned} z^\delta(t) = T_{\tilde{A}}(t)z^\delta(0) + \int_0^t T_{\tilde{A}}(t-s)\{[f_0(x_1^\delta, x_2^\delta, s) - f_0(x_1, 0, s)] + \\ b_1 b_2^{-1}\dot{x}_2^\delta(s) + [A_{12} - b_1 b_2^{-1} A_{22}]x_2^\delta(s)\}ds. \end{aligned} \quad (2.72)$$

Taking into account that $z^\delta(0) = 0$, employing integration by parts, and making use of the well-known property

$$\dot{T}_{\tilde{A}}(t) = \tilde{A}T_{\tilde{A}}(t) = T_{\tilde{A}}(t)\tilde{A}$$

of the infinitesimal operator $\tilde{A}$, one can rewrite (2.72) as follows:

$$\begin{aligned} z^\delta(t) = T_{\tilde{A}}(0)b_1 b_2^{-1}x_2^\delta(t) - T_{\tilde{A}}(t)b_1 b_2^{-1}x_2^\delta(0) + \int_0^t T_{\tilde{A}}(t-s)\{f_0(x_1^\delta, x_2^\delta, s) - \\ f_0(x_1, 0, s) - [\tilde{A}b_1 b_2^{-1} - A_{12} + b_1 b_2^{-1} A_{22}]x_2^\delta(s)\}ds. \end{aligned}$$

For the purpose of estimating the norm of the deviation $z^\delta(t)$ we utilize Assumption 2 to note that the inequality $\|T_{\tilde{A}}(t)\| \le \omega e^{\tilde{\beta}t}$ holds for all $t \ge 0$, for some growth bound $\tilde{\beta}$, and for some constant $\omega > 0$. Without loss of generality, we assume that $\tilde{\beta} > 0$. Apart from this, we apply Assumptions 1–3, coupled together, to ensure that both the strong solution $x_1(t)$ of (2.69) and the strong solution $x_1^\delta(t)$ of (2.71) cannot escape to infinity on any finite time interval while the strong solution $x_2^\delta(t)$ of (2.67) under $u = u^\delta$ remains bounded in $\mathscr{D}(A_2)$. With this in mind, we conclude that on the time interval $[0,\tau)$ the deviation of the function $f_0(x_1^\delta(t), x_2^\delta(t), t)$ from $f_0(x_1(t), 0, t)$ can be estimated

$$\|f_0(x_1^\delta(t), x_2^\delta(t), t) - f_0(x_1(t), 0, t)\| \le L(\|x_1^\delta(t) - x_1(t)\| + \delta)$$

with some Lipschitz constant $L > 0$, the existence of which is guaranteed by Assumption 3.

Summarizing, Assumptions 1–5 admit straightforward estimation of the state deviation $z^\delta(t)$ on the time interval $[0,\tau)$ when the approximate solution $x^\delta = (x_1^\delta, x_2^\delta)$ is assumed to evolve within the δ-vicinity (2.70) of the discontinuity manifold

$x_2 = 0$:

$$\|z^\delta(t)\| \le \omega\|b_1 b_2^{-1}\|[\|x_2^\delta(0)\| + \|x_2^\delta(t)\|] +$$
$$\int_0^t \|T_{\tilde{A}}(t-s)\|[L(\|x_1^\delta(s) - x_1(s)\| + \delta) + \|\tilde{A}b_1 b_2^{-1} x_2^\delta(s)\| + \|A_{12}x_2^\delta(s)\| +$$
$$\|b_1 b_2^{-1} A_{22} x_2^\delta(s)]ds \le \omega(1 + \|b_1 b_2^{-1}\|)\delta + \omega \int_0^t e^{\tilde{\beta}(t-s)}[L(\|z_1^\delta(s)\| + \delta) +$$
$$(K+1)\|A_{12}x_2^\delta(\tau)\| + \|b_1 b_2^{-1}\|\|A_{22}x_2^\delta(\tau)\|]ds \le \omega(1 + \|b_1 b_2^{-1}\|)\delta +$$
$$\omega \int_0^t e^{\tilde{\beta}(t-s)}[L\|z_1^\delta(s)\| + L\delta + (K + 1 + \|b_1 b_2^{-1}\|)\|x_2^\delta(s)\|_{\mathscr{D}(A_2)} ds \le$$
$$\omega(1 + \|b_1 b_2^{-1}\|)\delta + \omega(L + K + 1 + \|b_1 b_2^{-1}\|)\delta \int_0^t e^{\tilde{\beta}(t-s)} ds +$$
$$\omega L \int_0^t e^{\tilde{\beta}(t-s)}\|z_1^\delta(s)\|ds \le L_0\delta + L_1 \int_0^t \|z_1^\delta(s)\|ds$$

where $L_0 = \omega[1 + \|b_1 b_2^{-1}\| + (L + K + 1 + \|b_1 b_2^{-1}\|)\tau e^{\tilde{\beta}\tau}]$ and $L_1 = \omega L e^{\tilde{\beta}\tau}$. By applying the Bellman–Gronwell lemma, it follows that the norm of the deviation $z^\delta(t)$ tends to zero uniformly in $t \in [0,\tau)$ as $\delta \to 0$. In other words, once a generalized solution $x(t) = (x_1(t), x_2(t))$ of the discontinuous system (2.62) starts evolving in the discontinuity manifold $x_2 = 0$, its first component $x_1(t)$ is governed by the sliding mode equation (2.69), strong solutions of which are uniquely defined. Theorem 2.4 is thus proved.

2.3 Modeling of Electromechanical Nonlinear Phenomena

The dynamic model of an electromechanical system can be derived in a systematic way with a *Lagrange* formulation (see, e.g., [11]). The *Lagrangian*

$$\mathscr{L} = \mathscr{T} - \mathscr{U} \tag{2.73}$$

of such a system is defined in terms of the total kinetic energy $\mathscr{T}$ and potential energy $\mathscr{U}$ as a function of the vector $q = (q_1, \ldots, q_n)^T$ of generalized coordinates $q_1, \ldots, q_n$ which effectively describe the state of the system. The Lagrange's state equation is then expressed by

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} - \frac{\partial \mathscr{L}}{\partial q} = F(q, \dot{q}) \tag{2.74}$$

where the component $F_i,\ i = 1, \ldots, n$ of the vector $F = (F_1, \ldots, F_n)^T$ is the generalized force associated with the generalized coordinate q_i.

The contributions to the generalized forces are given by the nonconservative forces, such as the joint actuator torques and the joint friction torques. The Lagrange's equation (2.74) establishes the relation between the generalized force F, applied to the system, and the generalized position q, the generalized velocity $v = \dot{q}$, and the generalized acceleration $a = \ddot{q}$.

2.3.1 Friction Models

Friction is a natural phenomenon, representing the tangential reaction force between two surfaces in contact. Since these reaction forces depend on many factors such as contact geometry and surface materials, and the displacement and relative velocities of contacting bodies, and the presence of lubrication, among others, it is hardly possible to deduce a general friction model from physical first principles. Instead, phenomenological models, capturing essential friction features, are normally brought into play. Some friction models of interest are reviewed below (see surveys [10, 158] for details and for other existing friction models).

2.3.1.1 Static Models

The classical friction models are described by static maps between velocity and friction forces. The main idea of such a model is that friction opposes motion and its magnitude is independent of velocity v and contact area.

The Coulomb friction model

$$F(v) = F_C \text{sign } v \tag{2.75}$$

is an ideal relay model, multi-valued for zero velocity:

$$\text{sign } v = \begin{cases} 1, & \text{if } v > 0 \\ [-1,1], & \text{if } v = 0 \,. \\ -1, & \text{if } v < 0 \end{cases} \tag{2.76}$$

Since it does not specify the friction force $F(v)$ for zero velocity, the static force $F(0)$ is admitted to counteract external forces below the Coulomb friction level F_C. Thus, stiction, describing the Coulomb friction force at rest, can take on any value in the segment $[-F_C, F_C]$, thereby yielding the meaning of the corresponding state equation (2.74) in the sense of Filippov.

The viscous friction model

$$F(v) = F_v v \tag{2.77}$$

with a viscous friction coefficient $F_v > 0$ is used for the friction force caused by the viscosity of lubricants. By combining with the Coulomb friction, it is often modified to

$$F(v) = F_v v + F_C \text{sign } v. \tag{2.78}$$

In order to account for the observed destabilizing Stribeck phenomenon at very low velocities the latter model is augmented with the Stribeck friction model $\sigma_s e^{-(v/v_s)^2}$sign v where the constants $\sigma_s > 0$ and $v_s > 0$ stand for the Stribeck level and for the Stribeck velocity, respectively. The resulting model is then given by

$$F(v) = F_v v + [F_C + \sigma_s e^{-(\frac{v}{v_s})^2}]\text{sign } v \tag{2.79}$$

where the stiction force level $F_S = \sigma_s + F_C$ is admitted to be higher than the Coulomb level F_C.

2.3.1.2 Dynamic Models

To better match experimental data, the dynamic modeling of friction is typically involved. Proposed by Dahl [56], the following dynamic model

$$\dot{F}_D = \sigma_1 v - \sigma_1 |v| \frac{F_D}{F_C} \tag{2.80}$$

where $\sigma_1 > 0$, and $F_C > 0$ are the stiffness and the Coulomb friction level, respectively, appears to describe the spring-like behavior of the friction force F_D during stiction when the velocity v of the contacting body is infinitesimal. Formally setting $\sigma_1 = \infty$, the above dynamic model (2.80) specializes to the static Coulomb model (2.75). Since the Dahl model (2.80) is nonsmooth rather than discontinuous, it can be viewed as a regularization of the discontinuous Coulomb model (2.75) as $\sigma_1 \rightarrow \infty$.

While being essentially Coulomb friction with a lag in the change of the friction force when the motion direction is changed, the Dahl model (2.80) does not capture the Stribeck effect. In order to account for the Stribeck effect, the LuGre friction model from [41]

$$F_L = F_v v + \sigma_1 \eta + \sigma_2 \frac{d\eta}{dt} \tag{2.81}$$

can be utilized. In the LuGre model (2.81) the friction interface is thought of as a contact between bristles [41], $F_v > 0$ is a viscous friction coefficient, $\sigma_1 > 0$ is the stiffness, $\sigma_2 > 0$ is a damping coefficient, η is the average deflection of the bristles, whose dynamics are governed by

$$\frac{d\eta}{dt} = v - \frac{\sigma_1 |v|}{F_C + [F_S - F_C)e^{-(\frac{v}{v_S})^2}]}\eta, \tag{2.82}$$

where $F_C > 0$ is the Coulomb friction level, $F_S > 0$ is the level of the stiction force, $v_s > 0$ is the Stribeck velocity, and v is the actual velocity of the contacting body.

Thus, the complete LuGre model (2.81), (2.82) is characterized by six parameters $F_v, \sigma_1, \sigma_2, F_C, F_S, v_s$. It reduces to the Dahl model (2.80) if $F_v = 0, \sigma_2 = 0$, and $F_S = F_C$. In turn, for steady state motion when v is constant and $\dot{\eta} = 0$, the relation between the velocity and the LuGre friction force (2.81) is given by the classical model (2.79).

2.3.2 *The Multistable Backlash Model and Its Single-stability Approximation*

Backlash occurs in any mechanical system where a driving part (motor) is not directly connected with a driven part (load). An electrical actuator consists of a DC motor and a load coupled by a gear reduction part. The dynamics of the actuator are modeled as follows [144]:

$$J_o N^{-1}\ddot{q}_o + f_o N^{-1}\dot{q}_o = T + w_o$$
$$J_i\ddot{q}_i + f_i\dot{q}_i + T = \tau_m + w_i. \tag{2.83}$$

Hereinafter, J_o, f_o, $\ddot{q}_o$, $\dot{q}_o$, and $q_o(t)$ are, respectively, the inertia of the load and the reducer, the viscous output friction, the load acceleration, the load velocity, and the angular position of the load. The inertia of the motor, the viscous motor friction, the motor acceleration, the motor velocity, and the angular of the motor are denoted by J_i, f_i, $\ddot{q}_i$, $\dot{q}_i$, and $q_i(t)$, respectively. The input torque τ_m serves as a control action, and T stands for the transmitted torque. The external disturbances $w_i(t)$, $w_o(t)$ have been introduced into the driver equation (2.83) to account for destabilizing model discrepancies due to hard-to-model nonlinear phenomena such as friction and backlash.

The transmitted torque T through a backlash with an amplitude j is typically modeled by a dead-zone characteristic [156, p. 8]:

$$T(\Delta q) = \begin{cases} 0 & \text{if } |\Delta q| \le j \\ K\Delta q - Kj\, sign(\Delta q) & \text{otherwise} \end{cases} \tag{2.84}$$

where

$$\Delta q = q_i - Nq_o, \tag{2.85}$$

K is the stiffness, and N is the reducer ratio. Such a model is depicted in Fig. 2.1a. Provided the servomotor position $q_i(t)$ is the only available measurement on the system, the above model (2.83)–(2.85) appears to be non-minimum phase, because along with the origin, the unforced system possesses a multi-valued set of equilibria (q_i, q_o) with $q_i = 0$ and $q_o \in [-j, j]$.

To avoid dealing with a non-minimum phase system, the backlash model (2.84) is replaced with its monotonic approximation (see Fig. 2.1b)

$$T = K\Delta q + K\eta(\Delta q) \tag{2.86}$$

where

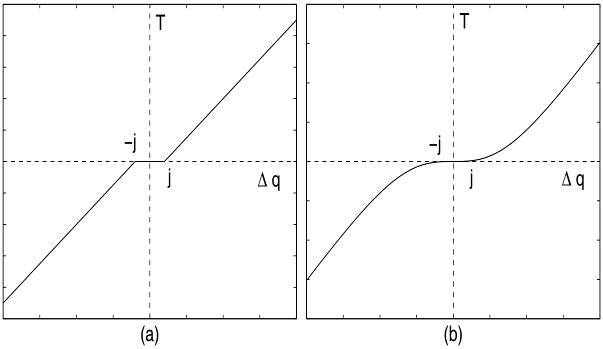

Fig. 2.1 a) The dead-zone model of backlash; b) The monotonic approximation of the dead-zone model

$$\eta = -2j\frac{1-e^{-\frac{\Delta q}{j}}}{1+e^{-\frac{\Delta q}{j}}}. \tag{2.87}$$

The present backlash approximation is inspired from [144]. Coupled to the drive system (2.83), the motor position of which is only available for measurements, it is shown to constitute an internally minimum phase approximation of the underlying servomotor, operating under uncertainties $w_i(t)$, $w_o(t)$. As a matter of fact, these uncertainties involve discrepancies between the physical backlash model (2.84) and its approximation (2.86)–(2.87).

For later use, let us introduce the state deviation vector $x = [x_1, x_2, x_3, x_4]^T$ from a desired load position q_o with

$$\begin{aligned} x_1 &= q_o - q_d \\ x_2 &= \dot{q}_o \\ x_3 &= q_i - Nq_d \\ x_4 &= \dot{q}_i \end{aligned}$$

where x_1 is the load position error, x_2 is the load velocity, x_3 is the motor position deviation from its nominal value, and x_4 is the motor velocity. The nominal motor position Nq_d has been pre-specified in such a way to guarantee that $\Delta q = \Delta x$ where

$$\Delta x = x_3 - Nx_1.$$

Then (2.83)–(2.87), represented in terms of the deviation vector x, takes the form:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= J_o^{-1}[KNx_3 - KN^2x_1 - f_o x_2 + KN\eta(\Delta x) + w_o] \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= J_i^{-1}[\tau_m + KNx_1 - Kx_3 - f_i x_4 - K\eta(\Delta x) + w_i].
\end{aligned} \tag{2.88}
$$

The internal zero dynamics

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= J_o^{-1}[-KN^2x_1 - f_o x_2 + NK\eta(-Nx_1)]
\end{aligned} \tag{2.89}
$$

of the nominal system (2.88) with respect to the output

$$y = x_3 \tag{2.90}$$

is formally obtained (see [104] for details) by setting $w_i = w_o = 0$ and specifying the control law that maintains the output identically zero. The following result, extracted from [5], guarantees that the error system (2.88), (2.90) is globally minimum phase.

Theorem 2.5. *Let the nonlinearity η be governed by (2.87). Then (2.89) is globally asymptotically stable.*

Proof. To begin with, let us consider a Lyapunov function of the form

$$
\begin{aligned}
V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{2}KN^2J_o^{-1}x_1^2 + \\
2j^2KJ_o^{-1}[2\ln\frac{e^{\frac{Nx_1}{j}}+1}{2} - \frac{Nx_1}{j}].
\end{aligned} \tag{2.91}
$$

Since

$$2\ln\frac{e^{\frac{Nx_1}{j}}+1}{2} \geq \frac{Nx_1}{j} \ \mathit{for\ all}\ x_1 \in \mathbf{R}$$

by inspection, the function V, governed by (2.91), appears to be positive definite.

Next, let us compute the time derivative of the Lyapunov function on the trajectories of (2.89):

$$
\begin{aligned}
\dot{V} = -J_o^{-1}(KN^2x_1 + f_o x_2 + 2jKN\frac{e^{\frac{Nx_1}{j}}-1}{e^{\frac{Nx_1}{j}}+1})x_2 \\
+KN^2J_o^{-1}x_1x_2 + 2j^2KJ_o^{-1} \\
\times[\frac{4Ne^{\frac{Nx_1}{j}}}{2j(e^{\frac{Nx_1}{j}}+1)}x_2 - \frac{Nx_2}{j}] = -J_o^{-1}f_o x_2^2 \leq 0.
\end{aligned} \tag{2.92}
$$

Let us now observe that the internal zero dynamics (2.89) of the system have no trivial solutions on the manifold $x_2 = 0$ where the time derivative of the Lyapunov function equals to zero. Indeed, if $x_2 = 0$ then due to (2.89)

$$\frac{Nx_1}{j} + 2\frac{1-e^{\frac{Nx_1}{j}}}{1+e^{\frac{Nx_1}{j}}} = 0, \tag{2.93}$$

thus concluding that $x_1 = 0$. To reproduce the latter conclusion, it suffices to represent (2.93) in terms of $z = \frac{Nx_1}{j}$ and note that the left-hand side of the inequality

$$z + 2\frac{1-e^z}{1+e^z} = 0 \tag{2.94}$$

thus obtained is a strictly increasing function of z because its derivative is positive definite by inspection:

$$1 - \frac{4e^z}{(1+e^z)^2} = \frac{(1-e^z)^2}{(1+e^z)^2} > 0 \; for \; all \; z \neq 0. \tag{2.95}$$

In order to complete the proof it remains to apply the LaSalle–Krasovskii invariance principle [110] to the system in question.

2.3.3 Limit Cycles and Nonlinear Asymptotic Harmonic Generators

Oscillation is among the important phenomena that occur in electromechanical systems. While linear time-invariant systems can only generate oscillations of amplitude dependent on the initial state, nonlinear systems can go into an oscillation of a fixed amplitude irrespective of the initial conditions. Such an oscillation is referred to as a *limit cycle* [110, 213].

The Van der Pol equation, whose general representation is given by the second-order scalar nonlinear differential equation

$$\ddot{x} + \varepsilon[(x-x_0)^2 - \rho^2]\dot{x} + \mu^2(x-x_0) = 0 \tag{2.96}$$

with positive parameters ε, ρ, μ, is a special case of the Lienard (circuit) equation (see, e.g., [110])

$$\ddot{v} + r(v)\dot{v} + g(v) = 0 \tag{2.97}$$

where the functions $r(v)$ and $g(v)$ are continuously differentiable. Equation 2.96, initially used by Van der Pol to study oscillations in vacuum tube circuits, presents a fundamental example in nonlinear oscillation theory. It possesses a stable limit cycle that attracts every other solutions except the unique equilibrium point $(x, \dot{x}) = (x_0, 0)$. The parameter ρ controls the amplitude of this limit cycle, the parameter

μ controls its frequency, the parameter ε controls the speed of the limit cycle transients, and the parameter x_0 is for the offset of x (see [238] for details).

Being proposed in [196], the Van der Pol modification

$$\ddot{x} + \varepsilon[(x^2 + \frac{\dot{x}^2}{\mu^2}) - \rho^2]\dot{x} + \mu^2 x = 0 \tag{2.98}$$

appears if (2.96) admits no offset of x, i.e., the parameter $x_0 = 0$ is used, and if the additional term $\frac{\varepsilon}{\mu^2}\dot{x}^3$ is involved. As opposed to the Van der Pol equation (2.96), the above modification has nothing to do with the Lienard equation (2.97). Meanwhile, it still possesses a stable limit cycle, being expressible in the explicit form

$$x^2 + \frac{\dot{x}^2}{\mu^2} = \rho^2 \tag{2.99}$$

(unlike that of the Van der Pol oscillator, exhibiting a nonsinusoidal periodic response in its limit cycle!).

The following result is in order [181].

Theorem 2.6. *Consider the modified Van der Pol equation (2.98) with positive parameters ε, μ, ρ. Then this equation has a stable limit cycle, given by (2.99), so that every other solution except the equilibrium point $x = \dot{x} = 0$ converges to the limit cycle (2.99) as $t \to \infty$.*

Proof. By inspection, the origin $x = \dot{x} = 0$ is a unique equilibrium point of (2.98). Hence, Poincare–Bendixson criterion [110, p. 61] is applicable to the modified Van der Pol equation (2.98). By applying Poincare–Bendixson criterion, the existence of a periodic orbit is concluded for this equation. Its analytical representation (2.99) comes from the expression of the time derivative of the positive definite function

$$V(x,\dot{x}) = \frac{1}{2}x^2 + \frac{1}{2\mu^2}\dot{x}^2, \tag{2.100}$$

computed along the trajectories of (2.98):

$$\dot{V}(x,\dot{x}) = x\dot{x} + \frac{1}{\mu^2}\dot{x}\,\ddot{x} = \frac{\varepsilon}{\mu^2}[\rho^2 - (x^2 + \frac{\dot{x}^2}{\mu^2})]\dot{x}^2. \tag{2.101}$$

It follows that

$$\dot{V}(x,\dot{x}) \begin{cases} > 0 & \text{if } (x^2 + \frac{\dot{x}^2}{\mu^2}) < \rho^2 \ \& \ \dot{x} \neq 0 \\ < 0 & \text{if } (x^2 + \frac{\dot{x}^2}{\mu^2}) > \rho^2 \ \& \ \dot{x} \neq 0 \\ = 0 & \text{if } [\rho^2 - (x^2 + \frac{\dot{x}^2}{\mu^2})]\dot{x} = 0 \end{cases} \tag{2.102}$$

on the trajectories of (2.98). By applying the LaSalle–Krasovskii invariance principle [110] to (2.102), one concludes that a periodic solution of (2.98) has to oscillate within the set $\{(x,\dot{x}) : \dot{V}(x,\dot{x}) = 0\}$ where

$$[\rho^2 - (x^2 + \frac{\dot{x}^2}{\mu^2})]\dot{x} = 0. \tag{2.103}$$

Since the origin is a unique equilibrium point of (2.98) all the trajectories of (2.98) cross the axis $\dot{x} = 0$ everywhere except the origin. Hence, the largest invariant manifold of set (2.103) coincides with ellipse (2.99) and it is straightforwardly verified that (2.99) is a limit cycle of the modified Van der Pol equation (2.98).

To complete the proof, it remains to note that due to (2.102), the norm $\|x(t)\| = \sqrt{V(x(t),\dot{x}(t))}$ of any trajectory of (2.98), initialized inside the limit cycle (2.99), must grow with time. Conversely, the norm of any trajectory of (2.98), initialized outside the limit cycle, must shrink with time. Thus, any trajectory of (2.98) except the equilibrium point $x = \dot{x} = 0$ is attracted by the limit cycle (2.99). The proof is completed.

Now, it becomes clear that in (2.98), the parameter ρ stands for the amplitude of the limit cycle whereas μ is for its frequency. Furthermore, by substituting the orbit equation (2.99) into (2.98) we conclude that the limit cycle of the modified Van der Pol equation (2.98) is remarkably generated by a standard linear harmonic oscillator

$$\ddot{x} + \mu^2 x = 0, \tag{2.104}$$

initialized on (2.99). Thus, we arrive at a *nonlinear asymptotic harmonic generator* (2.98) which naturally exhibits an ideal sinusoidal signal (2.104) in its limit cycle (2.99). The amplitude and frequency of this sinusoidal signal can be varied at will by tuning the parameters ρ and μ of the harmonic generator (2.98).

The modified Van der Pol oscillator (2.98), the phase portrait of which is shown in Fig. 2.2 for the parameter values $\varepsilon = 1000$, $\rho = 0.01$, $\mu = 1$, still belongs to a class of damped systems. In the region of negative damping, occurring within the limit cycle where the signals are small, the damping increases the energy level of the response (see the proof of Theorem 2.6). Conversely, outside the limit cycle, the damping becomes positive, thus decreasing the energy of the output signal. As a result, the motion approaches the limit cycle whose energy is determined by its amplitude ρ and frequency μ and therefore a desired level of the energy can be attained by assigning appropriate values of the oscillator parameters ρ and μ.

2.3.4 Vibroimpact Modeling

Following [164], we demonstrate how impact dynamics (collisions, percussions, etc.) of mechanical systems can be treated within the framework of nonlinear differential equations in distributions. The vertical launch of a spacecraft, governed by

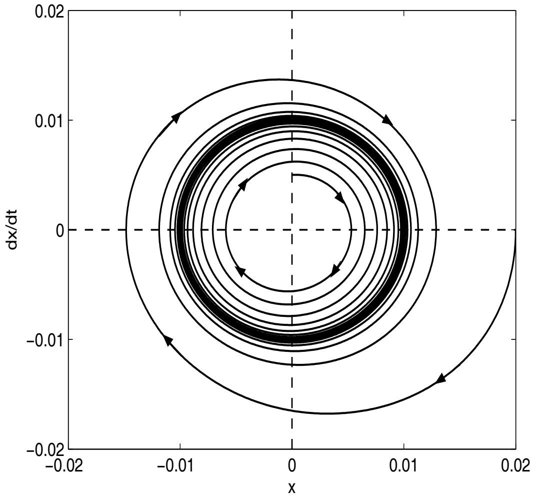

Fig. 2.2 Phase portrait of the modified Van der Pol oscillator

$$
\begin{aligned}
\dot{h} &= V,\ h(0) = 0 \\
\dot{V} &= \frac{p - mg - q(h,V)}{m},\ V(0) = 0 \\
\dot{m} &= -\frac{p}{c},\ m(0) = m_o,
\end{aligned} \tag{2.105}
$$

serves here as an illustrative example. In the above equations, $h(t), V(t)$, $m(t)$ are, respectively, the altitude, the velocity, and the mass of the rocket at the time instant t, p is the thrust, q is the air resistance, g is the gravity acceleration, c is the specific impulse, and m_0 is the initial mass.

System 2.105 implies a continuous change of the mass $m(t)$, whereas a discrete change $m(0+) - m_0 = \Delta m_0 < 0$, corresponding to an instantaneous fuel combustion in the spacecraft jet, is also possible and it is caused by the impulsive thrust

$$
p(t) = -c\Delta m_0 \delta(t). \tag{2.106}
$$

From the physical point of view, the impulsive thrust (2.106) indicates that the cumulative power of the jet is limited while attaining a very high value during a very short period of time. With this in mind, the meaning of (2.105) subject to (2.106) has to be treated in the generalized sense because of the irregularity of the product $c\Delta m_0 \delta(t) m^{-1}(t)$ of the impulsive thrust (2.106) and the discontinuous function $m^{-1}(t)$ that appears in the right-hand side of the second equation of (2.105). Since (2.105) is driven by the scalar impulsive action (2.106), the corresponding Frobenius condition is therefore satisfied, and by Theorem 2.2 the system has a unique vibroimpact solution. Apparently, the altitude h has no jump whereas the change

Δm_0 of the mass $m(t)$ at the initial time moment is straightforwardly computed by integrating the latter equation of (2.105) which has no irregular product. In turn, the instantaneous change $\Delta V_0 = V(0+)$ of the velocity is found by applying Theorem 2.3 through the relation

$$V(0+) = \xi_2(1), \tag{2.107}$$

where $\xi(v) = (\xi_1(v), \xi_2(v), \xi_3(v))^T$ solves the Cauchy problem

$$\begin{aligned} \frac{d\xi_1}{dv} &= 0,\ \xi_1(0) = 0 \\ \frac{d\xi_2}{dv} &= -\frac{c\Delta m_0}{\xi_3},\ \xi_2(0) = 0 \\ \frac{d\xi_3}{dv} &= \Delta m_0,\ \xi_1(0) = m_0. \end{aligned} \tag{2.108}$$

Since (2.108) integrates to $\xi_1(v) = 0,\ \xi_2(v) = -c\ln\frac{m_0+\Delta m_0 v}{m_0},\ \xi_3(v) = m_0 + \Delta m_0 v$, relation (2.107) results in the formula

$$V(0+) = -c\ln(1 + \frac{\Delta m_0}{m_0}), \tag{2.109}$$

for the instantaneous change of the velocity.

Thus, the spacecraft dynamics enforced by the impulsive jet is modeled by vibroimpact solutions of the nonlinear differential equation (2.105) with the impulsive input (2.106). Such a nonlinear distribution formalism allows one to qualitatively analyze these dynamics.

It is anticipated (but it is not the aim of the book to demonstrate) that the nonlinear distribution formalism relies on measure differential inclusions from [151] introduced for the adequate modeling of post-impact behavior of mechanical systems. Finding the post-impact velocity of such a system is, however, far from being trivial, as this represents a large combinatorial problem that can be formulated as a complementarity problem or as a normal cone inclusion (see [33, 81, 125, 150] for details).

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