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## Preface

This book is intended as an introduction to all the finite simple groups. During the monumental struggle to classify the finite simple groups (and indeed since), a huge amount of information about these groups has been accumulated. Conveying this information to the next generation of students and researchers, not to mention those who might wish to apply this knowledge, has become a major challenge.

With the publication of the two volumes by Aschbacher and Smith [12, 13] in 2004 we can reasonably regard the proof of the Classification Theorem for Finite Simple Groups (usually abbreviated CFSG) as complete. Thus it is timely to attempt an overview of all the (non-abelian) finite simple groups in one volume. For expository purposes it is convenient to divide them into four basic types, namely the alternating, classical, exceptional and sporadic groups.

The study of alternating groups soon develops into the theory of permutation groups, which is well served by the classic text of Wielandt [170] and more modern treatments such as the comprehensive introduction by Dixon and Mortimer [53] and more specialised texts such as that of Cameron [19]. The study of classical groups via vector spaces, matrices and forms encompasses such highlights as Dickson's classic book [48] of 1901, Dieudonné's [52] of 1955, and more modern treatments such as those of Taylor [162] and Grove [72]. The complete collection of groups of Lie type (comprising the classical and exceptional groups) is beautifully exposed in Carter's book [21] using the simple complex Lie algebras as a starting point. And sporadic attempts have been made to bring the structure of the sporadic groups to a wider audience—perhaps the most successful book-length introduction being that of Griess [69]. But no attempt has been made before to bring within a single cover an introductory overview of all the finite simple groups (unless one counts the 'Atlas of finite groups' [28], which might reasonably be considered to be an overview, but is certainly not introductory).

The remit I have given myself, to cover all of the finite simple groups, gives both advantages and disadvantages over books with more restricted subject

matter. On the one hand it allows me to point out connections, for example between exceptional behaviour of generic groups and the existence of sporadic groups. On the other hand it prevents me from proving everything in as much detail as the reader may desire. Thus the reader who wishes to understand everything in this book will have to do a lot of homework in filling in gaps, and following up references to more complete treatments. Some of the exercises are specifically designed to fill in gaps in the proofs, and to develop certain topics beyond the scope of the text.

One unconventional feature of this book is that Lie algebras are scarcely mentioned. The reasons for this are twofold. Firstly, it hardly seems possible to improve on Carter's exposition [21] (although this book is now out of print and secondhand copies change hands at astronomical prices). And secondly, the alternative approach to the exceptional groups of Lie type via octonions deserves to be better known: although real and complex octonions have been extensively studied by physicists, their finite analogues have been sadly neglected by mathematicians (with a few notable exceptions). Moreover, this approach yields easier access to certain key features, such as the orders of the groups, and the generic covering groups.

On the other hand, not all of the exceptional groups of Lie type have had effective constructions outside Lie theory. In the case of the family of large Ree groups I provide such a construction for the first time, and give an analogous description of the small Ree groups. The importance of the octonions in these descriptions led me also to a new octonionic description of the Leech lattice and Conway's group, and to an ambition, not yet realised, to see the octonions at the centre of the construction of all the exceptional groups of Lie type, and many of the sporadic groups, including of course the Monster.

Complete uniformity of treatment of all the finite simple groups is not possible, but my ideal (not always achieved) has been to begin by describing the appropriate geometric/algebraic/combinatorial structure, in enough detail to calculate the order of its automorphism group, and to prove simplicity of a clearly defined subquotient of this group. Then the underlying geometry/algebra/combinatorics is further developed in order to describe the subgroup structure in as much detail as space allows. Other salient features of the groups are then described in no particular order.

This book may be read in sequence as a story of all the finite simple groups, or it may be read piecemeal by a reader who wants an introduction to a particular group or family of groups. The latter reader must however be prepared to chase up references to earlier parts of the book if necessary, and/or make use of the index. Chapters 4 and 5 are largely (but not entirely) independent of each other, but both rely heavily on Chapters 2 and 3. The sections of Chapter 4 are arranged in what I believe to be the most appropriate order pedagogically, rather than logically or historically, but could be read in a different order. For example, one could begin with Section 4.3 on  $G_2(q)$  and proceed via triality (Section 4.7) to  $F_4(q)$  (Section 4.8) and  $E_6(q)$  (Section 4.10), postponing the twisted groups until later. The ordering of sections

in Chapter 5 is traditional, but a more avant-garde approach might begin with  $J_1$  (Section 5.9.1), and follow this with the exotic incarnation of (the double cover of)  $J_2$  as a quaternionic reflection group (in the first few parts of Section 5.6), and/or the octonionic Leech lattice (Section 5.6.12). But one cannot go far in the study of the sporadic groups without a thorough understanding of  $M_{24}$ .

I was introduced to the weird and wonderful world of finite simple groups by a course of lectures on the sporadic simple groups given by John Conway in Cambridge in the academic year 1978–9. During that course, and the following three years when he was my Ph.D. supervisor, he taught me most of what subsequently appeared in the ‘Atlas of finite groups’ [28], and a large part of what now appears in this book. I am of course extremely indebted to him for this thorough initiation.

Especial thanks go also to my former colleague at the University of Birmingham, Chris Parker, who, early in 2003, fuelled by a couple of pints of beer, persuaded me there was a need for a book of this kind, and volunteered to write half of it; who persuaded the Head of School to let us teach a two-semester course on finite simple groups in 2003–4; who developed the original idea into a detailed project plan; and who then quietly left me to get on and write the book. It is not entirely his fault that the book which you now have in your hands bears only a superficial resemblance to that original plan: I excised the chapters which he was going to write, on Lie algebras and algebraic groups, and shovelled far more into the other chapters than we ever anticipated. At the same time the planned 150 pages grew to nearly 300. Indeed, the more I wrote, the more I became aware of how much I had left out. I would need at least another 300 pages to do justice to the material, but one has to stop somewhere. I apologise to those readers who find that I stopped just at the point where they started to get interested.

Several colleagues have read substantial parts of various drafts of this book, and made many valuable comments. I particularly thank John Bray, whose keen nose for errors and assiduousness in sorting out some of the finer points has improved the accuracy and reliability of the text enormously; John Bradley, whose refusal to accept woolly arguments helped me tighten up the exposition in many places; and Peter Cameron whose comments on some early draft chapters have led to significant improvements, and whose encouragement has helped to keep me working on this book.

I owe a great deal also to my students for their careful reading of various versions of parts of the text, and their uncovering of countless errors, some minor, some serious. I used to tell them that if they had not found any errors, it was because they had not read it properly. I hope that this is now less true than it used to be. I thank in particular Jonathan Ward, Johanna Rämö, Simon Nickerson, Nicholas Krempel, and Richard Barraclough, and apologise to any whose names I have inadvertently omitted.

It is a truism that errors remain, and the fault (if fault there be), human nature. By convention, the responsibility is mine, but in fact that is unrealistic.

As Gauss himself said, “In science and mathematics we do not appeal to authority, but rather *you are responsible for what you believe*.” Nevertheless, I shall endeavour to maintain a web-site of corrections that have been brought to my attention, and will be grateful for notification of any further errors that you may find.

Thanks go also to Karen Borthwick at Springer-Verlag London for her gentle but persistent pressure, and to the anonymous referees for their enthusiasm for this project and their many helpful suggestions. I am grateful to Queen Mary, University of London, for their initially relatively light demands on me when I moved there in September 2004, which left me time to indulge in the pleasures of writing. It is entirely my own fault that I did not finish the book before those demands increased to the point where only a sabbatical would suffice to bring this project to a conclusion. I am therefore grateful to Jianbei An and the University of Auckland, and John Cannon and the University of Sydney, for providing me with time, space, and financial support during the last six months which enabled me, among other things, to sign off this book.

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