

# Preface

The contents of this monograph fall within the general area of nonlinear functional analysis and applications. We focus on an important topic within this area: *geometric properties of Banach spaces and nonlinear iterations*, a topic of intensive research efforts, especially within the past 30 years, or so.

In this theory, some geometric properties of Banach spaces play a crucial role. In the first part of the monograph, we expose these geometric properties most of which are well known. As is well known, among all infinite dimensional Banach spaces, Hilbert spaces have the *niciest* geometric properties. The availability of the inner product, the fact that the *proximity map or nearest point map* of a real Hilbert space  $H$  onto a closed convex subset  $K$  of  $H$  is Lipschitzian with constant 1, and the following two identities

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2, \quad (*)$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \quad (**)$$

which hold for all  $x, y \in H$ , are some of the geometric properties that characterize inner product spaces and also make certain problems posed in Hilbert spaces more manageable than those in general Banach spaces. However, as has been rightly observed by M. Hazewinkel, "... many, and probably most, mathematical objects and models do not naturally live in Hilbert spaces". Consequently, to extend some of the Hilbert space techniques to more general Banach spaces, analogues of the identities (\*) and (\*\*) have to be developed. For this development, the duality map which has become a most important tool in nonlinear functional analysis plays a central role. In 1976, Bynum [61] obtained the following analogue of (\*) for  $l_p$  spaces,  $1 < p < \infty$ :

$$\|x + y\|^2 \leq (p - 1)\|x\|^2 + \|y\|^2 + 2\langle x, j(y) \rangle, \quad 2 \leq p < \infty,$$

$$(p - 1)\|x + y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\langle x, j(y) \rangle, \quad 1 < p \leq 2.$$

Analogues of (\*\*) were also obtained by Bynum. In 1979, Reich [408] obtained an analogue of (\*) in uniformly smooth Banach spaces. Other analogues of (\*) and (\*\*) obtained in 1991 and later can be found, for example, in Xu [509] and in Xu and Roach [525].

In Chapters 1 and 2, basic well-known facts on geometric properties of Banach spaces which are used in the monograph are presented. The materials here (and much more) can be found in any of the excellent books on this topic (e.g., Diestel [206]; Lindenstrauss and Tzafriri [312]). The *duality map* which is central in our work is presented in Chapter 3. Here, we have also computed explicitly the duality maps in some concrete Banach spaces. In Chapters 4 and 5, we sketch the proofs of the analogues of the identities (\*) and (\*\*) obtained in 1991 and later. In the last section of Chapter 5, we present characterizations of real uniformly smooth Banach spaces and Banach spaces with uniformly Gâteaux differentiable norms by means of continuity properties of the normalized duality maps. Applications of the geometric properties of Banach spaces presented in Chapters 1 to 5 to iterative algorithms for solutions of nonlinear equations, an intensive and extensive area of research work (Berinde [28] contains 1575 entries in the reference list on this topic) begin in Chapter 6. To motivate some of the reasons for our choices of the classes of nonlinear operators studied in this monograph, we begin with the following.

Let  $K$  be a nonempty subset of a real normed linear space  $E$  and let  $T : K \rightarrow K$  be a map. A point  $x \in K$  is said to be a *fixed point* of  $T$  if  $Tx = x$ . Now, consider the differential equation  $\frac{du}{dt} + Au(t) = 0$  which describes an evolution system where  $A$  is an *accretive map* from a Banach space  $E$  to itself. In Hilbert spaces, accretive operators are called *monotone*. At equilibrium state,  $\frac{du}{dt} = 0$ , and so a solution of  $Au = 0$  describes the equilibrium or stable state of the system. This is very desirable in many applications in, for example, ecology, economics, physics, to name a few. Consequently, considerable research efforts have been devoted to methods of solving the equation  $Au = 0$  when  $A$  is accretive. Since generally  $A$  is nonlinear, there is no closed form solution of this equation. The standard technique is to introduce an operator  $T$  defined by  $T := I - A$  where  $I$  is the identity map on  $E$ . Such a  $T$  is called a *pseudo-contraction* (or is called *pseudo-contractive*). It is then clear that any zero of  $A$  is a fixed point of  $T$ . As a result of this, the study of fixed point theory for pseudo-contractive maps has attracted the interest of numerous scientists and has become a flourishing area of research, especially within the past 30 years or so, for numerous mathematicians. A very important subclass of the class of pseudo-contractive mappings is that of *nonexpansive* mappings, where  $T : K \rightarrow K$  is called nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$  holds for arbitrary  $x, y \in K$ .

Apart from being an obvious generalization of the contraction mappings, nonexpansive maps are important, as has been observed by Bruck [59], mainly for the following two reasons:

- Nonexpansive maps are intimately connected with the monotonicity methods developed since the early 1960's and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.
- Nonexpansive mappings appear in applications as the transition operators for initial value problems of differential inclusions of the form  $0 \in \frac{du}{dt} + T(t)u$ , where the operators  $\{T(t)\}$  are, in general, set-valued and are *accretive* or *dissipative* and *minimally continuous*.

If  $K$  is a closed nonempty subset of a Banach space and  $T : K \rightarrow K$  is nonexpansive, it is known that  $T$  may not have a fixed point (unlike the case if  $T$  is a strict contraction), and even when it has, the sequence  $\{x_n\}$  defined by  $x_{n+1} = Tx_n, n \geq 1$  (the so-called *Picard sequence*) may fail to converge to such a fixed point. This can be seen by considering an anti-clockwise rotation of the unit disc of  $\mathbb{R}^2$  about the origin through an angle of say,  $\frac{\pi}{4}$ . This map is nonexpansive with the origin as the unique fixed point, but the Picard sequence fails to converge with any starting point  $x_0 \neq 0$ . Krasnosel'skii [291], however, showed that in this example, if the Picard iteration formula is replaced by the following formula,

$$x_0 \in K, \quad x_{n+1} = \frac{1}{2}(x_n + Tx_n), n \geq 0, \quad (0.1)$$

then the iterative sequence converges to the unique fixed point. In general, if  $E$  is a normed linear space and  $T$  is a nonexpansive mapping, the following generalization of (0.1) which has proved successful in the approximation of a fixed point of  $T$  (when it exists) was given by Schaefer [431]:

$$x_0 \in K, \quad x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n \geq 0, \quad \lambda \in (0, 1). \quad (0.2)$$

However, the most general iterative formula for approximation of fixed points of nonexpansive mappings, which is called the *Mann iteration formula* (in the light of Mann [319]), is the following:

$$x_0 \in K, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \quad (0.3)$$

where  $\{\alpha_n\}$  is a sequence in the interval  $(0, 1)$  satisfying the following conditions: (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . The recursion formula (0.2) is consequently called the *Krasnoselskii-Mann (KM) formula* for finding fixed points of *nonexpansive (ne)* mappings. The following quotation indicates part of the interest in iterative approximation of fixed points of *nonexpansive mappings*.

- “Many well-known algorithms in signal processing and image reconstruction are iterative in nature ... A wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the KM iteration procedure, for particular choices of the ne operator...” (Charles Byrne, [63]).

For the past 30 years or so, the study of the Krasnoselskii-Mann iterative procedures for the approximation of fixed points of nonexpansive mappings and fixed points of some of their generalizations, and approximation of zeros of accretive-type operators have been flourishing areas of research for many mathematicians. Numerous applications of analogues of (\*) and (\*\*) to nonlinear iterations involving various classes of nonlinear operators have since then been topics of intensive research. Today, substantial definitive results have been proved, some of the methods have reached their boundaries while others are still subjects of intensive research activity. However, it is apparent that the theory has now reached a level of maturity appropriate for an examination of its central themes.

The aim of this monograph is to present an in-depth and up-to-date coverage of the main ideas, concepts and most important results on iterative algorithms for approximation of fixed points of nonlinear nonexpansive mappings and some of their important generalizations; iterative approximation of zeros of accretive-type operators; iterative approximation of solutions of variational inequality problems involving these operators; iterative algorithms for solutions of Hammerstein integral equations; and iterative approximation of common fixed points (and common zeros) of families of these mappings. Furthermore, some important open questions related to these selected topics are included.

We assume familiarity with basic concepts of analysis and topology. The monograph is addressed to graduate students of mathematics, computer science, statistics, informatics, engineering, to mathematicians interested in learning about the subject, and to numerous specialists in the area.

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