2

Preliminaries on Systems Theory

In this chapter, basic concepts and analysis tools in systems theory are summarized. We begin with matrix algebra and matrix norms, the standard instruments for qualitatively and quantitatively analyzing linear time-invariant systems and their properties. While analysis of time-varying and non-linear systems often requires advanced tools and particular results, the simple mathematical concept of a contraction mapping can be used to describe such qualitative characteristics as stability and convergence. Accordingly, the contraction mapping theorem is introduced, and it, together with the Barbalat lemma and the comparison theorem, will facilitate the development of analysis tools for cooperative control.

Lyapunov direct method is the universal approach for analyzing general dynamical systems and their stability. Search for a successful Lyapunov function is the key, and relevant results on linear systems, non-linear systems, and switching systems are outlined. Standard results on controllability as well as control design methodologies are also reviewed.

2.1 Matrix Algebra

Let $\Re$ and $\mathcal{C}$ be the set of real and complex numbers, respectively. Then, $\Re^n$ represents the set of $n-$tuples for which all components belong to $\Re$, and $\Re^{n\times m}$ is the set of n-by-m matrices over $\Re$. Let $\Re_+ \stackrel{\triangle}{=} [0, +\infty)$ and $\Re_{+/0} \stackrel{\triangle}{=} (0, +\infty)$, and let $\Re^n_+$ denote the set of all n-tuples whose components belong to $\Re_+$.

Definition 2.1. *For any matrix $A \in \Re^{n\times n}$, its* eigenvalues λ_i *are the roots of the characteristic equation of $|\lambda I - A| = 0$, its* eigenvectors *are the non-zero vectors s_i satisfying the equation $As_i = \lambda_i s_i$. The set of all its eigenvalues $\lambda \in \mathcal{C}$ is called* spectrum *of A and is denoted by $\sigma(A)$, and its* spectral radius *is defined by $\rho(A) \stackrel{\triangle}{=} \max\{|\lambda| : \lambda \in \sigma(A)\}$.*

Set $\mathcal{X}$ is said to be a *linear vector space* if $x, y, z \in \mathcal{X}$ implies $x+y = y+x \in \mathcal{X}$, $x+(y+z) = (x+y)+z \in \mathcal{X}$, $-x \in \mathcal{X}$ with $x+(-x) = 0 \in \mathcal{X}$, $\alpha x \in \mathcal{X}$, and $\alpha(x+y) = \alpha x + \alpha y$, where $\alpha \in \Re$ is arbitrary. Clearly, $\Re^{n\times m}$ is a linear vector space. Operations of addition, subtraction, scalar multiplication, and matrix multiplication are standard in matrix algebra. The so-called *Kronecker product* is defined as $D \otimes E = [d_{ij}E]$. Among the properties of Kronecker product are: if matrix dimensions are compatible,

$$A \otimes (B+C) = A \otimes B + A \otimes C, \quad (A \otimes B)^T = (A^T \otimes B^T),$$

$$(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1}), \quad \text{and} \quad (A \otimes B)(C \otimes D) = ((AC) \otimes (BD)).$$

Definition 2.2. *A set of vectors $v_i \in \mathcal{X}$ for $i = 1, \cdots, l$ is said to be* linearly independent *if the equation*

$$\alpha_1 v_1 + \cdots + \alpha_l v_l = 0$$

has only the trivial solution $\alpha_1 = \cdots = \alpha_l = 0$. If there is a non-trivial solution α_i to the above equation, say $\alpha_k \neq 0$, the set of vector is said to be linear dependent, in which case one of the vectors can be expressed as a linear combination of the rest, i.e.,

$$v_k = \sum_{j=1, j\neq k}^{n} -\frac{\alpha_j}{\alpha_k} v_j.$$

Set $\mathcal{X} \subset \Re^n$ is said to be of rank p *if $\mathcal{X}$ is composed of exactly p linearly independent vectors.*

Matrix $S \in \Re^{n\times n}$ is invertible and its inverse S^{-1} exists as $SS^{-1} = S^{-1}S = I$ if and only if its rows (columns) are linearly independent. For matrix $A \in \Re^{n\times n}$, there are always exactly n eigenvalues, either complex or real, but there may not be n linearly independent eigenvectors. A square matrix whose number of linearly independent eigenvectors is less than its order is said to be *defective*. A sufficient condition under which there are n linearly independent eigenvectors is that there are n distinct eigenvalues. Matrix product $S^{-1}AS$ is called a *similarity transformation* on matrix A, and eigenvalues are invariant under such a transformation. Matrix A is said to be *diagonalizable* as $S^{-1}AS = \Lambda$ if and only if matrix A has n linearly independent eigenvectors s_i, where $S = \begin{bmatrix} s_1 \; s_2 \cdots s_n \end{bmatrix}$ and $\Lambda = \text{diag}\{\lambda_1, \cdots, \lambda_n\}$. Transformation matrix S is said to be a *unitary matrix* if $S^{-1} = S^T$. It is known that any symmetric matrix A can be diagonalized under a similarity transformation of unitary matrix S. A very special unitary matrix is the *permutation matrix* which is obtained by permuting the rows of an $n \times n$ identity matrix according to some new ordering of the numbers 1 to n. Therefore, for any permutation matrix P, every row and column contains precisely a single 1 with 0s everywhere else, its determinant is always ± 1, and $P^{-1} = P^T$ as a permutation is reversed by permutating according to the new (*i.e.* transposed) order.

Diagonalization fails only if (but not necessarily if) there are repeated eigenvalues. For an nth-order matrix whose eigenvalue λ_i repeats k times, the *null space* of matrix $(A - \lambda_i I)$ determines linearly independent eigenvectors s_i associated with λ_i. Specifically, the linearly independent eigenvectors of matrix $(A - \lambda_i I)$ span its null space, and their number is the dimension of the null space and can assume any integer value between 1 and k. Obviously, defectiveness of matrix A occurs when the dimension of the null space is less than k. If matrix $A \in \Re^{n\times n}$ is defective, its set of eigenvectors is incomplete in the sense that there is no similarity transformation (or an invertible eigenvector matrix) that diagonalizes A. Hence, the best one can do is to choose a similarity transformation matrix S such that $S^{-1}AS$ is as nearly diagonal as possible. A standardized matrix defined to be the closest to diagonal is the so-called *Jordan canonical form.* In (2.1) below, order n_i is determined by $(A - \lambda_i I)^{n_i} s_i' = 0$ but $(A - \lambda_i I)^{n_i - 1} s_i' \neq 0$. The corresponding transformation matrix S consists of the eigenvector(s) $s_i = (A - \lambda_i I)^{n_i-1} s_i'$ as well as generalized eigenvectors s_i' up to $(A - \lambda_i I)^{n_i-2} s_i'$.

Definition 2.3. *Matrix J is said to be in the* Jordan canonical form *if it is block diagonal where the diagonal blocks, the so-called Jordan blocks J_i, contain identical eigenvalues on the diagonal,* 1 *on the super-diagonal, and zero everywhere else. That is,*

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_l \end{bmatrix}, \quad \text{and} \quad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \Re^{n_i \times n_i}, \tag{2.1}$$

where λ_i may be identical to λ_j for some $i \neq j$ (that is, repeated eigenvalue may appear in different Jordan blocks, but eigenvalues are all the same in any given Jordan block), and $n_1 + \cdots + n_l = n$. The number of Jordan blocks with the same eigenvalue λ_i is called algebraical multiplicity *of λ_i. The order n_i of Jordan block J_i is called* geometrical multiplicity.

A generalization of distance or length in linear vector space $\mathcal{X}$, called *norm* and denoted by $\|\cdot\|$, can be defined as follows: for any $x, y \in \mathcal{X}$ and $\alpha \in \Re$, $\|x\| \geq 0$, $\|x\| = 0$ if and only if $x = 0$, $\|\alpha x\| = |\alpha| \cdot \|x\|$, and $\|x+y\| \leq \|x\| + \|y\|$. That is, a norm is a positive definite function (see Section 2.3.1) that satisfies the triangular inequality and is linear with respect to a real, positive constant multiplier. In $\Re^n$, the vector p-norm ($p \geq 1$) and ∞-norm are defined by

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}} \quad \text{and} \quad \|x\|_\infty = \max_{1\leq i \leq n} |x_i|,$$

respectively. The most commonly used are 1-norm ($p = 1$), Euclidean norm ($p = 2$) or 2-norm, and ∞-norm. All the vector norms are compatible with themselves as, for all $x, y \in \Re^n$,

$$\|x^T y\|_p \leq \|x\|_p \|y\|_p,$$

and they are also equivalent as, for any $p, q > 1$, there exist constants c_1 and c_2 such that

$$c_1 \|x\|_p \leq \|x\|_q \leq c_2 \|x\|_p.$$

The p-norm can also be defined in an infinite dimensional linear vector space. For an infinite sequence of scalars $\{x_i : \ i \in \aleph\} = \{x_1, x_2, \cdots\}$, p-norm $(p \geq 1)$ and ∞-norm are defined by

$$\|x\|_p = \left[\sum_{i=1}^{\infty} |x_i|^p \right]^{\frac{1}{p}} \qquad \text{and} \qquad \|x\|_\infty = \sup_i |x_i|,$$

respectively. A sequence $\{x_i : \ i \in \aleph\}$ is said to belong to l_p-space if $\|x\|_p < \infty$ and to l_∞-space if $\|x\|_\infty < \infty$.

In linear vector space $\mathcal{X} \subset \Re^n$, matrix $A \in \Re^{n \times n}$ in linear algebraic equation of $y = Ax$ represents a linear transformation, and its *induced matrix norms* are defined as

$$\|A\|_q \stackrel{\triangle}{=} \max_{\|x\|_q \neq 0} \frac{\|y\|_q}{\|x\|_q} = \max_{\|x\|_q = 1} \|y\|_q = \max_{\|x\|_q = 1} \|Ax\|_q,$$

where $q \geq 1$ is a positive real number including infinity. It follows that $\rho(A) \leq \|A\|$ for any induced matrix norm. According to the above definition, explicit solutions can be found for several induced matrix norms, in particular,

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|, \qquad \|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}, \qquad \|A\|_\infty = \|A^T\|_1,$$

where a_{ij} denotes the (i, j)th element of A, and $\lambda_{\max}(E)$ represents the operation of finding the maximum eigenvalue of E.

If a vector or a matrix is time-varying, the aforementioned vector norm or induced matrix norm can be applied pointwise at any instant of time, and hence the resulting norm value is time-varying. In this case, a functional norm defined below can be applied on top of the pointwise norm and over time: if $f : \Re^+ \to \Re$ is a Lebesgue measurable time function, then p-norm $(1 \leq p < \infty)$ and ∞-norm of $f(\cdot)$ are

$$\|f\|_p \stackrel{\triangle}{=} \left(\int_{t_0}^{\infty} |f(t)|^p dt \right)^{\frac{1}{p}}, \qquad \text{and} \qquad \|f\|_\infty \stackrel{\triangle}{=} \sup_{t_0 \leq t \leq \infty} |f(t)|,$$

respectively. Function $f(t)$ is said to belong to L_p*-space* if $\|f(t)\|_p < \infty$ and to L_∞*-space* if $\|f(t)\|_\infty < \infty$. Many results on stability analysis and in this chapter can be interpreted using the L_p-space or L_∞-space. For instance, any continuous and unbounded signal does not belong to either L_p-space or L_∞-space; and all uniformly bounded functions belong to L_∞-space. In fact, boundedness is always required to establish a stability result.

In a normed linear vector space, sequence $\{x_k : k \in \aleph\}$ is said to be a *Cauchy sequence* if, as $k, m \to \infty$, $\|x_k - x_m\| \to 0$. It is known and also easy to show that a convergent sequence must be a Cauchy sequence but not every Cauchy sequence is convergent. *Banach space* is a complete normed linear vector space in which every Cauchy sequence converges therein. Among the well known Banach spaces are $\Re^n$ with norm $\|\cdot\|_p$ with $1 \le p \le \infty$, and $C[a, b]$ (all continuous-time functions over interval $[a, b]$) with functional norm $\max_{t\in[a,b]} \|\cdot\|_p$.

2.2 Useful Theorems and Lemma

In this section, several mathematical theorems and lemma in systems and control theory are reviewed since they are simple yet very useful.

2.2.1 Contraction Mapping Theorem

The following theorem, the so-called *contraction mapping theorem*, is a special fixed-point theorem. It has the nice features that both existence and uniqueness of solution are ensured and that mapping T has its gain in norm less than 1 and hence is contractional. Its proof is also elementary as, by a successive application of mapping T, a convergent power series is obtained. Without $\lambda < 1$, Inequality 2.2 is referred to as the Lipschitz condition.

Theorem 2.4. *Let $\mathcal{S}$ be a closed sub-set of a Banach space $\mathcal{X}$ with norm $\|\cdot\|$ and let T be a mapping that maps $\mathcal{S}$ into $\mathcal{S}$. If there exists constant $0 \le \lambda < 1$ such that, for all $x, y \in \mathcal{S}$,*

$$\|T(x) - T(y)\| \le \lambda \|x - y\|, \tag{2.2}$$

then solution x^* to equation $x = T(x)$ exists and is a unique fixed-point in $\mathcal{S}$.

Proof: Consider sequence $\{x_{k+1} = T(x_k),\ \ k \in \aleph\}$. It follows that

$$\begin{aligned}
\|x_{k+l} - x_k\| &\le \sum_{j=1}^{l} \|x_{k+j} - x_{k+j-1}\| \\
&= \sum_{j=0}^{l-1} \|T(x_{k+j}) - T(x_{k+j-1})\| \\
&\le \sum_{j=0}^{l-1} \lambda \|x_{k+j} - x_{k+j-1}\| \\
&\le \sum_{j=0}^{l-1} \lambda^{k+j-1} \|x_2 - x_1\| \\
&\le \frac{\lambda^{k-1}}{1-\lambda} \|x_2 - x_1\|,
\end{aligned}$$

by which the sequence is a Cauchy sequence. By definition of $\mathcal{X}$, $x_k \to x^*$ for some x^* and hence x^* is a fixed-point since

$$\begin{aligned}\|T(x^*) - x^*\| &\leq \|T(x^*) - T(x_k)\| + \|x_{k+1} - x^*\| \\ &\leq \lambda\|x^* - x_k\| + \|x_{k+1} - x^*\| \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Uniqueness is obvious from the fact that, if $x = T(x)$ and $y = T(y)$,

$$\|x - y\| = \|T(x) - T(y)\| \leq \lambda\|x - y\|,$$

which holds only if $x = y$ since $\lambda < 1$. □

Computationally, fixed-point x^* is the limit of convergent sequence $\{x_{k+1} = T(x_k), \quad k \in \aleph\}$. Theorem 2.4 has important applications in control theory, starting with existence and uniqueness of a solution to differential equations [108, 192]. It will be shown in Section 2.3.2 that the Lyapunov direct method can be viewed as a special case of the contraction mapping theorem. Indeed, the concept of contraction mapping plays an important role in any convergence analysis.

2.2.2 Barbalat Lemma

The following lemma, known as the *Barbalat lemma* [141], is useful in convergence analysis.

Definition 2.5. *A function $w : \Re_+ \to \Re$ is said to be uniformly continuous if, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|t - s| < \delta$ implies $|w(t) - w(s)| < \epsilon$.*

Lemma 2.6. *Let $w : \Re_+ \to \Re$ be a uniformly continuous function. Then, if $\lim_{t\to\infty} \int_{t_0}^{t} w(\tau)d\tau$ exists and is finite, $\lim_{t\to\infty} w(t) = 0$.*

Proof: To prove by contradiction, assume that either $w(t)$ does not have a limit or $\lim_{t\to\infty} w(t) = c \neq 0$. In the case that $\lim_{t\to\infty} w(t) = c \neq 0$, there exists some finite constant α_0 such that $\lim_{t\to\infty} \int_{t_0}^{t} w(\tau)d\tau = \alpha_0 + ct \not\leq \infty$, and hence there is a contradiction. If $w(t)$ does not have a limit as $t \to \infty$, there exists an infinite time sub-sequence $\{t_i : \ i \in \aleph\}$ such that $\lim_{i\to\infty} t_i = +\infty$ and $|w(t_i)| \geq \epsilon_w > 0$. Since $w(t)$ is uniformly continuous, there exists interval $[t_i - \delta t_i, t_i + \delta t_i]$ within which $|w(t)| \geq 0.5|w(t_i)|$. Therefore, we have

$$\int_{t_0}^{t_i+\delta t_i} w(\tau)d\tau \geq \int_{t_0}^{t_i-\delta t_i} w(\tau)d\tau + w(t_i)\delta t_i, \quad \text{if } w(t_i) > 0,$$

and

$$\int_{t_0}^{t_i+\delta t_i} w(\tau)d\tau \leq \int_{t_0}^{t_i-\delta t_i} w(\tau)d\tau + w(t_i)\delta t_i, \quad \text{if } w(t_i) < 0.$$

Since $\{t_i\}$ is an infinite sequence and both $|w(t_i)|$ and δt_i are uniformly positive, we know by taking the limit of $t_i \to \infty$ on both sides of the above inequalities that $\int_{t_0}^{t} w(\tau)d\tau$ cannot have a finite limit, which contradicts the stated condition. □

The following example illustrates the necessity of $w(t)$ being uniformly continuous for Lemma 2.6, and it also shows that imposing $w(t) \geq 0$ does not add anything. In addition, note that $\lim_{t\to\infty} w(t) = 0$ (*e.g.*, $w(t) = 1/(1+t)$) may not imply $\lim_{t\to\infty} \int_{t_0}^{t} w(\tau)d\tau < \infty$ either.

Example 2.7. Consider scalar time function: for $n \in \aleph$,

$$w(t) = \begin{cases} 2^{n+1}(t-n) & \text{if } t \in [n,\ n+2^{-n-1}] \\ 2^{n+1}(n+2^{-n}-t) & \text{if } t \in [n+2^{-n-1},\ n+2^{-n}] \\ 0 & \text{everywhere else} \end{cases},$$

which is a triangle-wave sequence of constant height. It follows that $w(t)$ is continuous and that

$$\lim_{t\to\infty} \int_0^t w(\tau)d\tau = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 1 < \infty.$$

However, $w(t)$ does not have a limit as $t \to \infty$ as it approaches the sequence of discrete-time impulse functions in the limit. ◇

If $w(t)$ is differentiable, $w(t)$ is uniformly continuous if $\dot{w}(t)$ is uniformly bounded. This together with Lemma 2.6 has been widely used adaptive control [232], non-linear analysis [108], and robustness analysis [192], and it will be used in Chapter 6 to analyze non-linear cooperative systems.

2.2.3 Comparison Theorem

The following theorem, known as the *comparison theorem*, can be used to find explicitly either a lower bound or an upper bound on a non-linear differential equation which itself may not have an analytical solution.

Theorem 2.8. *Consider the scalar differential equation*

$$\dot{r} = \beta(r,t), \quad r(t_0) = r_0 \tag{2.3}$$

where $\beta(r,t)$ is continuous in t and locally Lipschitz in r for all $t \geq t_0$ and $r \in \Omega \subset \Re$. Let $[t_0, T)$ be the maximal interval of existence of the solution $r(t)$ such that $r(t) \in \Omega$, where T could be infinity. Suppose that, for $t \in [t_0, T)$,

$$\dot{v} \leq \beta(v,t), \quad v(t_0) \leq r_0, \quad v(t) \in \Omega. \tag{2.4}$$

Then, $v(t) \leq r(t)$ for all $t \in [t_0, T)$.

Proof: To prove by contradiction, assume that $t_1 \geq t_0$ and $\delta t_1 > 0$ exist such that

$$v(t_1) = r(t_1), \quad \text{and} \quad r(t) < v(t) \;\; \forall t \in (t_1, t_1 + \delta t_1]. \tag{2.5}$$

It follows that, for $0 < h < \delta t_1$,

$$\frac{r(t+h) - r(t_1)}{h} < \frac{v(t+h) - v(t_1)}{h},$$

which in turn implies

$$\dot{r}(t_1) < \dot{v}(t_1).$$

Combined with (2.3) and (2.4), the above inequality together with the equality in (2.5) leads to the contradiction

$$\beta(t_1, r(t_1)) < \beta(t_1, v(t_1)).$$

Thus, either t_1 or $\delta t_1 > 0$ does not exist, and the proof is completed. □

Extension of Theorem 2.8 to the multi-variable case is non-trivial and requires the so-called quasi-monotone property defined below. Should such a property hold, Theorem 2.10 can be applied. Nonetheless, the condition is usually too restrictive to be satisfied, as evident from the fact that, if $F(x,t) = Ax$ and is mixed monotone, all the entries in sub-matrices A_{11} and A_{22} must be non-negative while entries of A_{12} and A_{22} are all non-positive, where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{and} \quad A_{11} \in \Re^{k \times k}.$$

Definition 2.9. *Function $F(x,t) : \Re_+ \times \Re^n \to \Re^n$ is said to possess a* mixed quasi-monotone property *with respect to some fixed integer $k \in \{0, 1, \cdots, n\}$ if the following conditions hold:*

(a) For all $i \in \{1, \cdots, k\}$, function $F_i(x,t)$ is non-decreasing in x_j for $j = 1, \cdots, k$ and $j \neq i$, and it is non-increasing in x_l where $k < l \leq n$.
(b) For all $i \in \{k+1, \cdots, n\}$, function $F_i(x,t)$ is non-increasing in x_j for $j = 1, \cdots, k$, and it is non-decreasing x_l where $k < l \leq n$ and $l \neq i$.

In the special cases of $k = 0$ and $k = n$, function $F(x,t)$ is said to have quasi-monotone non-decreasing *and* quasi-monotone non-increasing *properties, respectively. Furthermore, function $F(x,t)$ is said to possess* mixed monotone *(or monotone non-decreasing or monotone non-increasing) property if $j \neq i$ and $l \neq i$ are not demanded in (a) and (b).*

Theorem 2.10. *Consider the following two sets of differential inequalities: for $v, w \in \Re^n$ and for some $k \in \{0, 1, \cdots, n\}$*

$$\begin{cases} \dot{v}_i \leq F_i(v,t), & i \in \{1,\cdots,k\} \\ \dot{v}_j > F_j(v,t), & j \in \{k+1,\cdots,n\} \end{cases} \quad \begin{cases} \dot{w}_i > F_i(w,t), & i \in \{1,\cdots,k\} \\ \dot{w}_j \leq F_j(w,t), & j \in \{k+1,\cdots,n\} \end{cases}. \tag{2.6}$$

or

$$\begin{cases} \dot{v}_i < F_i(v,t), & i \in \{1,\cdots,k\} \\ \dot{v}_j \geq F_j(v,t), & j \in \{k+1,\cdots,n\} \end{cases} \quad \begin{cases} \dot{w}_i \geq F_i(w,t), & i \in \{1,\cdots,k\} \\ \dot{w}_j < F_j(w,t), & j \in \{k+1,\cdots,n\} \end{cases}. \tag{2.7}$$

Suppose that $F(x,t)$ is continuous in t and locally Lipschitz in x for all $t \geq t_0$ and $x \in \Omega \subset \Re^n$, that initial conditions satisfy the inequalities of

$$\begin{cases} v_i(t_0) < w_i(t_0), & i \in \{1,\cdots,k\} \\ v_j(t_0) > w_j(t_0), & j \in \{k+1,\cdots,n\} \end{cases}, \tag{2.8}$$

that $F(x,t)$ has the mixed quasi-monotone property with respect to k, and that $[t_0, T)$ be the maximal interval of existence of solutions $v(t)$ and $w(t)$ such that $v(t), w(t) \in \Omega$, where T could be infinity. Then, over the interval of $t \in [t_0, T)$, inequalities $v_i(t) < w_i(t)$ and $v_j(t) > w_j(t)$ hold for all $i \in \{1,\cdots,k\}$ and for all $j \in \{k+1,\cdots,n\}$.

Proof: Define

$$\begin{cases} z_i(t) = v_i(t) - w_i(t), & i \in \{1,\cdots,k\} \\ z_j(t) = w_j(t) - v_j(t), & j \in \{k+1,\cdots,n\} \end{cases}.$$

It follows from (2.8) that $z_i(t_0) < 0$ for all i. To prove by contradiction, assume that $t_1 \geq t_0$ be the first time instant such that, for some l and for some $\delta t_1 > 0$,

$$z_l(t_1) = 0, \quad \text{and} \quad z_l(t) > 0 \;\; \forall t \in (t_1, t_1 + \delta t_1].$$

Hence, we know that, for all $\alpha \in \{1, \cdots, n\}$,

$$z_\alpha(t_1) \leq 0, \tag{2.9}$$

and that, for $0 < h < \delta t_1$,

$$\frac{z_l(t+h) - z_l(t_1)}{h} = \frac{z_l(t+h)}{h} > 0,$$

which in turn implies

$$\dot{z}_l(t_1) > 0.$$

Combined with (2.6) or (2.7), the above inequality becomes

$$\begin{cases} 0 < \dot{z}_l(t_1) < F_l(v(t_1),t) - F_l(w(t_1),t), & \text{if } l \leq k \\ 0 < \dot{z}_l(t_1) < F_l(w(t_1),t) - F_l(v(t_1),t), & \text{if } l > k \end{cases},$$

and, since $F(x,t)$ possesses the mixed quasi-monotone property with respect to k, the above set of inequality contradicts with (2.9) or simply

$$\begin{cases} v_i(t_1) \le w_i(t_1), & i \in \{1, \cdots, k\} \\ v_j(t_1) \ge w_j(t_1), & j \in \{k+1, \cdots, n\} \end{cases}.$$

Hence, t_1 and $\delta t_1 > 0$ cannot co-exist, and the proof is completed. □

In Section 2.3.2, Theorem 2.8 is used to facilitate stability analysis of nonlinear systems in terms of a Lyapunov function and to determine an explicit upper bound on state trajectory. In such applications as stability of large-scale interconnected systems, dynamic coupling among different sub-systems may have certain properties that render the vector inequalities of (2.6) and (2.7) in terms of a vector of Lyapunov functions v_k and, if so, Theorem 2.10 can be applied [118, 119, 156, 268]. Later in Section 6.2.1, another comparison theorem is developed for cooperative systems, and it is different from Theorem 2.10 because it does not require the mixed quasi-monotone property.

2.3 Lyapunov Stability Analysis

Dynamical systems can be described in general by the following vector differential equation:

$$\dot{x} = F'(x, u, t), \qquad y = H(x, t), \tag{2.10}$$

where $x \in \Re^n$ is the state, $y \in \Re^l$ is the output, and $u \in \Re^m$ is the input. Functions $F'(\cdot)$ and $H(\cdot)$ provide detailed dynamics of the system. System 2.10 is said to be *affine* if its differential equation can be written as

$$\dot{x} = f(x) + g(x)u. \tag{2.11}$$

The control problem is to choose a feedback law $u = u(x,t)$ as the input such that the controlled system has desired the properties including stability, performance and robustness. Upon choosing the control, the closed-loop system corresponding to (2.10) becomes

$$\dot{x} = F'(x, u(x,t), t) \stackrel{\triangle}{=} F(x,t), \qquad y = H(x,t). \tag{2.12}$$

The analysis problem is to study qualitative properties of System 2.12 for any given set of initial condition $x(t_0)$. Obviously, control design must be an integrated part of stability analysis, and quite often control $u = u(x,t)$ is chosen through stability analysis. Analysis becomes somewhat easier if functions $F(x,t)$ and $H(x,t)$ in (2.12) are independent of time. In this case, the system in the form of

$$\dot{x} = f(x), \quad y = h(x), \tag{2.13}$$

is called *autonomous*. In comparison, a system in the form of (2.12) is said to be *non-autonomous*.

As the first step of analysis, stationary points of the system, called *equilibrium points*, should be found by solving the algebraic equation: for all $t \geq t_0$,

$$F(0, t) = 0.$$

A system may have a unique equilibrium point (*e.g.*, all linear systems of $\dot{x} = Ax$ with invertible matrix A), or finite equilibrium points (*e.g.*, certain non-linear systems), or an infinite number of equilibrium points (*e.g.*, cooperative systems to be introduced later). In standard stability analysis, $x = 0$ is assumed since, if not, a constant translational transformation can be applied so that any specific equilibrium point of interest is moved to be the origin in the transformed state space. As will be shown in Chapter 5, a cooperative system typically has an infinite set of equilibrium points, its analysis needs to be done with respect to the whole set and hence should be handled differently. Should the system have periodic or chaotic solutions, stability analysis could also be done with respect to the solutions [2, 101]. In this section, the analysis is done only with respect to the equilibrium point of $x = 0$.

Qualitative properties of System 2.12 can be measured using the following *Lyapunov stability* concepts [141]. The various stability concepts provide different characteristics on how close the solution of System 2.12 will remain around or approach the origin if the solution starts in some neighborhood of the origin.

Definition 2.11. *If $x = 0$ is the equilibrium point for System 2.12, then $x = 0$ is*

(a) Lyapunov stable *(or stable) if, for every pair of $\epsilon > 0$ and $t_0 > 0$, there exists a constant $\delta = \delta(\epsilon, t_0)$ such that $\|x(t_0)\| \leq \delta$ implies $\|x(t)\| \leq \epsilon$ for $t \geq t_0$.*
(b) Unstable *if it is not Lyapunov stable.*
(c) Uniformly stable *if it is Lyapunov stable and if $\delta(\epsilon, t_0) = \delta(\epsilon)$.*
(d) Asymptotically stable *if it is Lyapunov stable and if there exists a constant $\delta'(t_0)$ such that $\|x(t_0)\| \leq \delta'$ implies $\lim_{t\to\infty} \|x(t)\| = 0$.*
(e) Uniformly asymptotically stable *if it is uniformly stable and if $\delta'(t_0) = \delta'$.*
(f) Exponentially stable *if there are constants $\delta, \alpha, \beta > 0$ such that $\|x(t_0)\| \leq \delta$ implies*

$$\|x(t)\| \leq \alpha \|x(t_0)\| e^{-\beta(t-t_0)}.$$

Lyapunov stability states that, if the initial condition of the state is arbitrarily close to the origin, the state can remain within any small hyperball centered at the origin. The relevant quantity δ represents an estimate on the radius of *stability regions*. If δ is infinite, the stability results are *global*; otherwise, the stability results are *local*. Asymptotical stability requires additionally that the solution converges to the origin, and quantity δ' is an estimate on the radius of *convergence region*. Complementary to the above Lyapunov stability concepts are the concepts of *uniform boundedness*, *uniform ultimate* *boundedness*, *asymptotic convergence*, and *input-to-state* stability (ISS) [235].

Definition 2.12. *System 2.12 is said to be*

(a) Uniformly bounded *if, for any $c' > 0$ and for any initial condition $\|x(t_0)\| \leq c'$, there exists a constant c such that $\|x(t)\| \leq c$ for $t \geq t_0$.*
(b) Uniformly ultimately bounded *with respect to $\epsilon > 0$ if it is uniformly bounded and if there exists a finite time period τ such that $\|x(t)\| \leq \epsilon$ for $t \geq t_0 + \tau$.*
(c) Asymptotically convergent *if it is uniformly bounded and if $x(t) \to 0$ as $t \to \infty$.*

System 2.10 is input-to-state stable *if, under $u = 0$, it is uniformly asymptotically stable at $x = 0$ and if, under any bounded input u, the state is also uniformly bounded.*

For a control system, uniform boundedness is the minimum requirement. It is obvious that Lyapunov stability implies uniform boundedness, that asymptotic stability implies asymptotic convergence, and that asymptotic convergence implies uniform ultimate boundedness. But, their reverse statements are not true in general. Concepts (a), (b), and (c) in Definition 2.12 are often used in robustness analysis [192] because they do not require $x = 0$ be an equilibrium point.

Several of the above concepts are illustrated by the first-order systems in the following example.

Example 2.13. (1) The first-order, time-varying, linear system of $\dot{x} = a(t)x$ has the following solution:

$$x(t) = x(t_0)e^{\int_{t_0}^{t} a(\tau)d\tau}.$$

Using this solution, one can verify the following stability results:

(1a) System $\dot{x} = -x/(1+t)^2$ is stable but not asymptotically stable.
(1b) System $\dot{x} = (6t \sin t - 2t)x$ is stable but not uniformly stable.
(1c) System $\dot{x} = -x/(1+t)$ is uniformly stable and asymptotically stable, but not uniformly asymptotically stable.

(2) System $\dot{x} = -x^3$ has the solution of $x(t) = 1/\sqrt{2t + 1/x^2(0)}$. Hence, the system is asymptotically stable but not exponentially stable.
(3) System $\dot{x} = -x + \mathrm{sign}(x)e^{-t}$ has the solution $x(t) = e^{-(t-t_0)}x(t_0) + \mathrm{sign}(x(t_0))(e^{-t_0} - e^{-t})$, where $\mathrm{sign}(\cdot)$ is the standard sign function with $\mathrm{sign}(0) = 0$. Thus, the system is asymptotically convergent, but it is not stable.
(4) System $\dot{x} = -x + xu$ is exponentially stable with $u = 0$, but it is not input-to-state stable. ◇

In the above example, stability is determined by analytically solving differential equations. For time-invariant linear systems, a closed-form solution is found and used in Section 2.4 to determine stability conditions. For time-varying linear systems or for non-linear systems, there is in general no closed-form solution. In these cases, stability should be studied using the Lyapunov direct method.

2.3.1 Lyapunov Direct Method

The Lyapunov direct method can be utilized to conclude various stability results without the explicit knowledge of system trajectories. It is based on the simple mathematical fact that, if a scalar function is both bounded from below and decreasing, the function has a limit as time t approaches infinity. For stability analysis of System 2.12, we introduce the following definition.

Definition 2.14. *A time function* $\gamma(s)$ *is said to be* strictly monotone increasing *(or* strictly monotone decreasing*) if* $\gamma(s_1) < \gamma(s_2)$ *(or* $\gamma(s_1) > \gamma(s_2)$*) for any* $s_1 < s_2$.

Definition 2.15. *A scalar function* $V(x,t)$ *is said to be*

(a) Positive definite *(p.d.) if* $V(0,t) = 0$ *and if* $V(x,t) \geq \gamma_1(\|x\|)$ *for some scalar, strictly monotone increasing function* $\gamma_1(\cdot)$ *with* $\gamma_1(0) = 0$. *A positive definite function* $V(x,t)$ *is said to be* radially unbounded *if its associated lower bounding function* $\gamma_1(\cdot)$ *has the property that* $\gamma_1(r) \to \infty$ *as* $r \to +\infty$.
(b) Positive semi-definite *(p.s.d.) if* $V(x,t) \geq 0$ *for all* t *and* x.
(c) Negative definite or negative semi-definite *(n.d. or n.s.d.) if* $-V(x,t)$ *is positive definite or positive semi-definite, respectively.*
(d) Decrescent *if* $V(x,t) \leq \gamma_2(\|x\|)$ *for some scalar, strictly monotone increasing function* $\gamma_2(\cdot)$ *with* $\gamma_2(0) = 0$.

Clearly, $V(x,t)$ being p.d. ensures that scalar function $V(x,t)$ is bounded from below. To study stability of System 2.12, its time derivative along any of system trajectories can be evaluated by

$$\dot{V}(x,t) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right)^T \dot{x} = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right)^T F(x,t).$$

If $\dot{V}$ is n.d. (or n.s.d.), $V(x,t)$ keeps decreasing (or is non-increasing), asymptotic stability (or Lyapunov stability) can be concluded as stated in the following theorem, and $V(x,t)$ is referred to as a *Lyapunov function.*

Theorem 2.16. *System 2.12 is*

(a) Locally Lyapunov stable if, in a neighborhood around the origin, $V(x,t)$ *is p.d. and* $\dot{V}(x,t)$ *is n.s.d.*

(b) Locally uniformly stable as $\|x(t)\| \leq \gamma_1^{-1} \circ \gamma_2(\|x(t_0)\|)$ *if, for* $x \in \{x \in \Re^n : \|x\| < \eta\}$ *with* $\eta \geq \gamma_2(\|x(t_0)\|)$, $V(x,t)$ *is p.d. and decrescent and* $\dot{V}(x,t)$ *is negative semi-definite.*
(c) Uniformly bounded and uniformly ultimately bounded with respect to ϵ *if* $V(x,t)$ *is p.d. and decrescent and if, for* $x \in \{x \in \Re^n : \|x\| \geq \gamma_1^{-1} \circ \gamma_2(\epsilon)\}$, $\dot{V}(x,t)$ *is negative semi-definite.*
(d) Locally uniformly asymptotically stable in the region of $\{x \in \Re^n : \|x\| < \gamma_2^{-1}(\gamma_1(\eta))\}$ *if, for* $\|x\| < \eta$, $V(x,t)$ *is p.d. and decrescent and if* $\dot{V}(x,t)$ *is n.d.*
(e) Globally uniformly asymptotically stable if $V(x,t)$ *is p.d., radially unbounded and decrescent and if* $\dot{V}(x,t)$ *is n.d. everywhere.*
(f) Exponentially stable if $\gamma_1\|x\|^2 \leq V(x,t) \leq \gamma_2\|x\|^2$ *and* $\dot{V}(x,t) \leq -\gamma_3\|x\|^2$ *for positive constants* γ_i, $i = 1, 2, 3$.
(g) Unstable if, in every small neighborhood around the origin, $\dot{V}$ *is n.d. and* $V(x,t)$ *assumes a strictly negative value for some* x *therein.*

System 2.10 is ISS if $V(x,t)$ *is p.d., radially unbounded and decrescent and if* $\dot{V}(x,t) \leq -\gamma_3(\|x\|) + \gamma_3^{\beta}(\|x\|)\gamma_4(\|u\|)$, *where* $0 \leq \beta < 1$ *is a constant, and* $\gamma_i(\cdot)$ *with* $i = 3, 4$ *are scalar strictly monotone increasing functions with* $\gamma_i(0) = 0$.

A useful observation is that, if $\dot{V}(x,t) \leq 0$ for $\|x\| \leq \eta$ with $\eta \geq \gamma_2(\|x(t_0)\|)$ and if $\|x(t_0)\| < \gamma_2^{-1}(\gamma_1(\eta))$, $\dot{V}(x(t_0), t_0) \leq 0$ and, for any sufficiently small $\delta t > 0$, $V(x(t), t) \leq V(x(t_0), t_0) \leq \gamma_2(\|x(t_0)\|)$ and hence $\dot{V}(x, t + \delta t) \leq 0$. By applying the observation and repeating the argument inductively, all the statements in Theorem 2.16 except for (vi) can be proven. Proof of (vi) of Theorem 2.16 will be pursued in the next subsection. Counterexamples are provided in [108] to show that the condition of radial unboundedness is needed to conclude global stability and that the decrescent condition is required for both asymptotic stability and uniform stability.

The key of applying Lyapunov direct method is to find Lyapunov function $V(x,t)$. In searching of Lyapunov function, we should use the backward process of first finding $\dot{V}$ symbolically in terms of $\partial V/\partial x$ and then selecting $\partial V/\partial x$ so that V is found and $\dot{V}$ has one of the properties listed in Theorem 2.16. The process is illustrated by the following example.

Example 2.17. Consider the second-order system:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - x_2.$$

Let $V(x)$ be the Lyapunov function to be found. It follows that

$$\dot{V} = \frac{\partial V}{\partial x_1} x_2 + \frac{\partial V}{\partial x_2}(-x_1^3 - x_2).$$

In order to conclude Lyapunov stability, we need to show that $\dot{V}$ is at least n.s.d. To this end, we need to eliminate such sign-indefinite terms as the cross product terms of x_1 and x_2. One such choice is that

$$\frac{\partial V}{\partial x_1}x_2 = \frac{\partial V}{\partial x_2}x_1^3, \tag{2.14}$$

which holds if $V = x_1^4 + 2x_2^2$ is chosen. Consequently, we know that

$$\dot{V} = -4x_2^2$$

is n.s.d. and hence the system is globally (uniformly) Lyapunov stable.

Next, let us determine whether Lyapunov function V exists to make $\dot{V}$ n.d. It follows from the expression for $\dot{V}$ that $\dot{V}$ may become n.d. only if $\partial V/\partial x_2$ contains such a term as x_1. Accordingly, we introduce an additional term of $2ax_1$ into the previous expression for $\partial V/\partial x_2$, that is,

$$\frac{\partial V}{\partial x_2} = 2ax_1 + 4x_2$$

for some constant $a > 0$. Hence, it follows that

$$V(x) = 2ax_1x_2 + 2x_2^2 + h(x_1),$$

where function $h(x_1) > 0$ is desired. To ensure (2.14), $h(x_1)$ must contain x_1^4. On the other hand, $V(x)$ should be positive definite, and this can be made possible by including bx_1^2 in $h(x_1)$ for some constant $b > 0$. In summary, the simplest form of $V(x)$ should be

$$V(x) = 2ax_1x_2 + 2x_2^2 + x_1^4 + bx_1^2,$$

which is positive definite if $a < 2\sqrt{2b}$. Consequently, we have

$$\begin{aligned}\dot{V} &= (2ax_2 + 4x_1^3 + 2bx_1)x_2 + (2ax_1 + 4x_2)(-x_1^3 - x_2)\\ &= -2ax_1^4 - (4-2a)x_2^2 + (2b-2a)x_1x_2,\end{aligned}$$

which is n.d. under many choices of a and b (*e.g.*, $a = b = 1$). Since V is p.d. and $\dot{V}$ is n.d., the system is shown to be asymptotically stable.

While the system is stable, many choices of $V(x)$ (such as $V(x) = x_1^2 + x_2^2$) would yield a sign-indefinite expression of $\dot{V}$. ◇

The above example shows that the conditions in Theorem 2.16 are sufficient, Lyapunov function is not unique and, unless the backward process yields a Lyapunov function, stability analysis is inconclusive. Nonetheless, existence of a Lyapunov function is guaranteed for all stable systems. This result is stated as the following theorem, referred to *Lyapunov converse theorem*, and its proof based on qualitative properties of system trajectory can be found in [108].

Theorem 2.18. *Consider System 2.12. Then,*

(a) If the system is stable, there is a p.d. function $V(x,t)$ whose time derivative along the system trajectory is n.s.d.

(b) If the system is (globally) uniformly asymptotically stable, there is a p.d. decrescent (and radially unbounded) function $V(x,t)$ such that $\dot{V}$ is n.d.
(c) If the system is exponentially stable, there are Lyapunov function $V(x,t)$ and positive constants γ_i such that $\gamma_1\|x\|^2 \leq V(x,t) \leq \gamma_2\|x\|^2$, $\dot{V}(x,t) \leq -\gamma_3\|x\|^2$, and $\|\partial V/\partial x\| \leq \gamma_4\|x\|$.

It is the converse theorem that guarantees existence of the Lyapunov functions and makes Lyapunov direct method a universal approach for non-linear systems. Nonetheless, the theorem provides a promise rather than a recipe for finding a Lyapunov function. As a result, Lyapunov function has to be found for specific classes of non-linear systems. In the subsequent subsections, system properties are explored to search for Lyapunov functions.

2.3.2 Explanations and Enhancements

A Lyapunov function can be sought by exploiting either physical, mathematical, or structural properties of system dynamics. In what follows, physical and mathematical features of the Lyapunov direct method are explained, while structural properties of dynamics will be explored in Section 2.6.1 to construct the Lyapunov function. In Section 2.6.4, the Lyapunov function is related to a performance function in an optimal control design.

Interpretation as an Energy Function

In essence, stability describes whether and how the system trajectory moves toward its equilibrium point, and hence the motion can be analyzed using a physical measure such as an energy function or its generalization. Intuitively, if the total energy of a system keeps dissipating over time, the system will eventually lose all of its initial energy and consequently settle down to an equilibrium point. Indeed, Lyapunov function is a measure of generalized energy, and the Lyapunov direct method formalizes the dissipative argument. As an illustration, consider an one-dimensional rigid-body motion for which kinetic and potential energies are assumed to be

$$\mathcal{K} = \frac{1}{2}m(x)\dot{x}^2, \qquad \mathcal{P} = \mathcal{P}(x),$$

respectively, where $m(x)$ is a positive number or function. It follows from (1.16) that the dynamic equation of motion is

$$m(x)\ddot{x} + \frac{1}{2}\frac{dm(x)}{dx}\dot{x}^2 + \frac{d\mathcal{P}(x)}{dx} = \tau. \tag{2.15}$$

Obviously, the *energy function* of the system is

$$\mathcal{E} = \mathcal{K} + \mathcal{P} = \frac{1}{2}m(x)\dot{x}^2 + \mathcal{P}(x). \tag{2.16}$$

It follows that the time derivative of $\mathcal{E}$ along trajectories of System 2.15 is

$$\dot{\mathcal{E}} = \dot{x}\tau.$$

If the net external force is zero, $\dot{\mathcal{E}} = 0$, the total energy of systems is conservative, and hence System 2.15 with $\tau = 0$ is called a *conservative system*. On the other hand, if $\tau = -k\dot{x}$ for some $k > 0$, the external input is a dynamic friction force, and System 2.15 is dissipative as

$$\dot{\mathcal{E}} = -k\dot{x}^2,$$

which is n.s.d. Depending upon the property of potential energy $\mathcal{P}(x)$, equilibrium point(s) can be found, and asymptotic stability could be concluded by following Example 2.17 and finding an appropriate Lyapunov function. In other words, an energy function such as the one in (2.16) can be used as the starting point to search for the Lyapunov function.

Interpretation as a Contraction Mapping

Suppose that, for a given system $\dot{x} = F(x,t)$, Lyapunov function $V(x)$ is found such that $V(x)$ is positive definite and

$$\dot{V}(x) = \left[\frac{\partial V(x)}{\partial x}\right]^T F(x,t)$$

is negative definite. It follows that, for any infinite time sequence $\{t_k : k \in \aleph\}$, $V(x(t)) \leq V(x(t_i))$ for all $t \in [t_i, t_{i+1})$ and

$$V(x(t_{i+1})) = V(x(t_i)) + \int_{t_i}^{t_{i+1}} \dot{V}(x)d\tau \leq -\lambda V(x(t_i)),$$

for some $\lambda \in [0,1)$. It follows from Theorem 2.4 that Lyapunov function $V(x(t_i))$ itself is a contraction mapping from which asymptotic stability can be concluded.

Enhancement by Comparison Theorem

The comparison theorem, Theorem 2.8, can be used to facilitate a generalized Lyapunov argument if the Lyapunov function and its time derivative render a solvable inequality. The following lemma is such a result in which $\dot{V}$ is not negative definite in the neighborhood around the origin.

Lemma 2.19. *Let V be a (generalized) Lyapunov function for System 2.12 such that*

$$\gamma_1(\|x(t)\|) \leq V(x,t) \leq \gamma_2(\|x\|),$$

and

$$\dot{V}(x,t) \leq -\lambda\gamma_2(\|x\|) + \epsilon\varphi(t),$$

where $\gamma_i(\cdot)$ are strictly monotone increasing functions with $\gamma_i(0) = 0$, $\lambda, \epsilon > 0$ are constants, and $0 \leq \varphi(t) \leq 1$. Then, the system is

(a) Uniformly ultimately bounded with respect to $\gamma_1^{-1}(\epsilon/\lambda)$.
(b) Asymptotically convergent if $\varphi(t)$ converges to zero.
(c) Exponentially convergent if $\varphi(t) = e^{-\beta t}$ for some $\beta > 0$.

Proof: It follows that $\dot{V}(x,t) \leq -\lambda V(x,t) + \epsilon\varphi(t)$ and hence, by Theorem 2.8, $V(x,t) \leq w(t)$, where

$$\dot{w} = -\lambda w + \epsilon\varphi(t), \quad w(t_0) = V(x(t_0), t_0).$$

Solving the scalar differential equation yields

$$V(x,t) \leq e^{-\lambda(t-t_0)}V(x(t_0),t_0) + \epsilon \int_{t_0}^{t} e^{-\lambda(t-s)}\varphi(s)ds,$$

from which the statements become obvious. □

If $\epsilon = 0$ in the statement, Lemma 2.19 reduces to some of the stability results in Theorem 2.16.

Enhancement by Barbalat Lemma

As a useful tool in Lyapunov stability analysis, the Barbalat lemma, Lemma 2.6, can be used to conclude convergence for the cases that $\dot{V}$ is merely n.s.d. with respect to state x and may also contain an L_1-space time function. The following lemma illustrates such a result.

Lemma 2.20. *Consider System 2.12 in which function $F(x,t)$ is locally uniformly bounded with respect to x and uniformly bounded with respect to t. Let V be its Lyapunov function such that*

$$\gamma_1(\|x(t)\|) \leq V(x,t) \leq \gamma_2(\|x\|),$$

and

$$\dot{V}(x,t) \leq -\gamma_3(\|z(t)\|) + \varphi(t),$$

where $\gamma_i(\cdot)$ are strictly monotone increasing functions with $\gamma_i(0) = 0$, $\gamma_3(\cdot)$ is locally Lipschitz, $z(t)$ is a sub-vector of $x(t)$, and $\varphi(t)$ belongs to L_1-space. Then, the system is uniformly bounded and the sub-state $z(t)$ is asymptotically convergent.

Proof: It follows from the expression of $\dot{V}$ that

$$\begin{aligned} V(x(t),t) + \int_{t_0}^{t} \gamma_3(\|z(\tau)\|)d\tau &\leq V(x_0,t_0) + \int_{t_0}^{t} \varphi(s)ds \\ &\leq V(x_0,t_0) + \int_{t_0}^{\infty} \varphi(s)ds \\ &< \infty. \end{aligned}$$

The above inequality implies that $V(x(t), t)$ and hence $x(t)$ are uniformly bounded. Recalling properties of $F(x, t)$, we know from (2.12) that $\dot{x}$ is uniformly bounded and thus $x(t)$ as well as $z(t)$ is uniformly continuous. On the other hand, the above inequality also shows that $\gamma_3(\|z(\tau)\|)$ belongs to L_1-space. Thus, by Lemma 2.6, $z(t)$ is asymptotically convergent. □

For the autonomous system in (2.13), Lemma 2.20 reduces to the famous LaSalle's *invariant set theorem*, given below. Set Ω is said to be *invariant* for a dynamic system if, by starting within Ω, its trajectory remains there for all future time.

Theorem 2.21. *Suppose that $V(x)$ is p.s.d. for $\|x\| < \eta$ and, along any trajectory of System 2.13, $\dot{V}$ is n.s.d. Then, state $x(t)$ converges either to a periodic trajectory or an equilibrium point in set $\Omega \triangleq \{x \in \Re^n : \dot{V}(x) = 0, \|x\| < \eta\}$. System 2.13 is asymptotically stable if set Ω contains no periodic trajectory but only equilibrium point $x \equiv 0$. Moreover, asymptotic stability becomes global if $\eta = \infty$ and $V(x)$ is radially unbounded.*

It must be emphasized that Theorem 2.21 only holds for autonomous systems. It holds even without $V(x)$ being positive semi-definite as long as, for any constant $l > 0$, the set defined by $V(x) < l$ is closed and bounded.

Directional Derivative

Consider the autonomous system in (2.13). The corresponding Lyapunov function is time-invariant as $V = V(x)$, and its time derivative along trajectories of System 2.13 is

$$\dot{V} = \left(\frac{\partial V}{\partial x}\right)^T f(x), \quad \text{or} \quad \dot{V} = (\bigtriangledown_x V)^T f(x),$$

which is the dot product of system dynamics and gradient of Lyapunov function. Hence, as the projection of the gradient along the direction of motion, $\dot{V}$ is called directional derivative.

To simplify the notations in the subsequent discussions, the so-called Lie derivative is introduced. *Lie derivative* of scalar function $\xi(x)$ with respect to vector function $f(x)$, denoted by $L_f\xi$, is a scalar function defined by

$$L_f\xi = (\bigtriangledown_x^T \xi) f.$$

High-order Lie derivatives can be defined recursively as, for $i = 1, \cdots$,

$$L_f^0\xi = \xi, \quad L_f^i\xi = L_f(L_f^{i-1}\xi) = [\bigtriangledown_x^T(L_f^{i-1}\xi)]f, \quad \text{and} \quad L_g L_f\xi = [\bigtriangledown_x^T(L_f\xi)]g,$$

where $g(x)$ is another vector field of the same dimension. It is obvious that, if $y_j = h_j(x)$ is a scalar output of System 2.13, the ith-order time derivative of this output is simply $y_j^{(i)} = L_f^i h_j$ and that, if $V(x)$ is the Lyapunov function for System 2.13, $\dot{V} = L_f V$.

2.3.3 Control Lyapunov Function

Consider the special case that dynamics of Control System 2.10 do not explicitly depend on time, that is,

$$\dot{x} = \mathcal{F}'(x, u), \tag{2.17}$$

where $\mathcal{F}'(0, 0) = 0$. For System 2.17, existence of both a stabilizing control and its corresponding Lyapunov function is captured by the following concept of control Lyapunov function.

Definition 2.22. *A smooth and positive definite function $V(x)$ is said to be a* control Lyapunov function *for System 2.17 if, for any $x \neq 0$,*

$$\inf_{u \in \Re^m} L_{\mathcal{F}'(x,u)} V(x) < 0. \tag{2.18}$$

Control Lyapunov function $V(\cdot)$ is said to satisfy the small control property *if, for every $\epsilon > 0$, there exists $\delta > 0$ such that Inequality 2.18 holds for any x with $0 < \|x\| < \delta$ and for some $u(x)$ with $\|u(x)\| < \epsilon$.*

Clearly, in light of (2.18), the time derivative of the control Lyapunov function along the system trajectory can always be made negative definite by properly choosing a feedback control $u(x)$, which is sufficient for concluding at least local asymptotic stability. The converse Lyapunov theorem, Theorem 2.18, also ensures the existence of a control Lyapunov function if System 2.17 is asymptotically stabilized. The following lemma due to [8, 165] provides a sufficient condition for constructing control and concluding stability.

Lemma 2.23. *Suppose that System 2.17 has a control Lyapunov function $V(x)$. If the mapping $u \to L_{\mathcal{F}'(x,u)} V(x)$ is convex for all $x \neq 0$, then the system is globally asymptotically stable under a feedback control $u(x)$ which is continuous for all $x \neq 0$. In addition, if $V(x)$ satisfies the small control property, the control $u(x)$ is continuous everywhere.*

Given a control Lyapunov function, construction of a stabilizing control for System 2.17 is generally non-trivial. In addition, the convex condition required by the above lemma may not be valid in general. For the affine nonlinear control system in (2.11), existence of control Lyapunov function $V(x)$ is equivalent to the requirement that $L_{g(x)} V(x) = 0$ implies $L_{f(x)} V(x) < 0$. Should the control Lyapunov function be known for Affine System 2.11, a universal feedback controller is available in terms of the following Sontag formula [234]:

$$u(x) = \begin{cases} -\frac{L_f V + \sqrt{(L_f V)^2 + \|L_g V\|^2 \alpha(x)}}{\|L_g V\|^2} (L_g V)^T & \text{if } L_g V \neq 0 \\ 0 & \text{if } L_g V = 0 \end{cases}, \tag{2.19}$$

where $\alpha(x) \geq 0$ is a scalar function. It is straightforward to show that $u(x)$ is stabilizing as inequality $L_f V + [L_g V]u(x) < 0$ holds everywhere, that $u(x)$ is continuous except at those points satisfying $L_g V = 0$, and that $u(x)$ becomes continuous everywhere under the choice of $\alpha(x) = \xi(\|L_g V\|)$, where $\xi(\cdot)$ is a scalar function satisfying $\xi(0) = 0$ and $\xi(a) > 0$ for $a > 0$.

The problem of finding a control Lyapunov function for System 2.17 can be studied by imposing certain conditions on function $F(x, u)$, either analytical properties or special structures. In the first approach, one typically assumes that a positive definite function $V_0(x)$ (as a weak version of Lyapunov function) is already known to yield $L_{f(x)} V_0(x) \le 0$ and that, for controllability (which will be discussed in Section 2.5), vector fields $\{f,\ ad_f g_k,\ ad_f^2 g_k,\ \cdots\}$ satisfy certain rank conditions, where $f(x) = \mathcal{F}'(x, 0)$ and $g_k(x) = \partial \mathcal{F}'(x, u)/\partial u_k|_{u=0}$. Based on the knowledge of $V_0(x)$ as well as f and g_k, a control Lyapunov function $V(x)$ can be constructed [152]. In the second approach, a special structural property of the system is utilized to explicitly search for control Lyapunov function, which will be carried out in Section 2.6.

2.3.4 Lyapunov Analysis of Switching Systems

Among non-autonomous systems of form (2.12), there are *switching systems* whose dynamics experience instantaneous changes at certain time instants and are described by

$$\dot{x} = F(x, s(t)), \quad s(t) \in \mathcal{I}, \tag{2.20}$$

where $s(\cdot)$ is the switching function (or selection function), and $\mathcal{I}$ is the corresponding value set (or index set). In the discussion of this subsection, it is assumed that set $\mathcal{I}$ be a finite sub-set of $\aleph$ and that, uniformly with respect to all $s(t) \in \mathcal{I}$, function $F(x, s(t))$ be bounded for each x and also be locally Lipschitz in x.

In principle, analysis of Switching System 2.20 is not much different from that of Non-autonomous System 2.12. Specifically, Lyapunov direct method can readily be applied to System 2.20, and the successful Lyapunov function is usually time dependent as $V(x, s(t))$. The dependence of V on discontinuous function $s(t)$ introduces the technical difficulty that time derivative $\dot{V}$ contains singular points. Consequently, non-smooth analysis such as semi-continuity, set valued map, generalized solution, and differential inclusions [11, 65, 110, 219, 220] need be used in stability analysis.

One way to avoid the complication of non-smooth analysis is to find a *common Lyapunov function*, that is, Lyapunov function $V(x)$ that is positive definite and whose time derivative $L_{F(x,s(t))} V(x)$ is negative definite no matter what *fixed* value $s(t)$ assumes in set $\mathcal{I}$. It is obvious that, if a (radially-unbounded) common Lyapunov function exists, System 2.20 is (globally) asymptotically stable and so are the family of autonomous systems

$$\dot{x}_s = F(x_s, s) \stackrel{\triangle}{=} \mathcal{F}_s(x_s), \quad s \in \mathcal{I}. \tag{2.21}$$

However, stability of all the autonomous systems in (2.21) is not necessary or sufficient for concluding stability of Switching System 2.20. It will be shown in Section 2.4.3 that a system switching between two stable linear autonomous systems may be unstable. Similarly, a system switching between two unstable linear autonomous systems may be stable. In other words, an asymptotically stable switching system in the form of (2.20) may not have a common Lyapunov function, nor do its induced family of autonomous systems in (2.21) have to be asymptotically stable. On the other hand, if the switching is arbitrary in the sense that switching function $s(t)$ is allowed to assume any value in index set $\mathcal{I}$ at any time, stability of Switching System 2.20 implies stability of all the autonomous systems in (2.21). Indeed, under the assumption of arbitrary switching, the corresponding Lyapunov converse theorem is available [49, 142, 155] to ensure the existence of a common Lyapunov function for an asymptotically stable switching system of (2.20) and for all the autonomous systems in (2.21). In short, stability analysis under arbitrary switching can and should be done in terms of a common Lyapunov function.

In most cases, switching occurs according to a sequence of time instants that are either fixed or unknown *a priori* and, if any, the unknown switching time instants are not arbitrary because they are determined by uncertain exogenous dynamics. In these cases, System 2.20 is piecewise-autonomous as, for a finite family of autonomous functions $\{\mathcal{F}_s(x) : \ s \in \mathcal{I}\}$, and for some sequence of time instances $\{t_i : \ i \in \aleph\}$,

$$\dot{x} = F(x, s(t_i)) \stackrel{\triangle}{=} \mathcal{F}_{s(t_i)}(x), \quad \forall t \in [t_i, t_{i+1}). \tag{2.22}$$

System 2.22 may not have a common Lyapunov function, nor may its induced family of autonomous systems in (2.21). The following theorem provides the condition under which stability can be concluded using a family of Lyapunov functions defined for Autonomous Systems 2.21 and invoked over consecutive time intervals.

Theorem 2.24. *Consider the piecewise-autonomous system in (2.22). Suppose that the corresponding autonomous systems in (2.21) are all globally asymptotically stable and hence have radially unbounded positive definite Lyapunov functions $V_s(x)$ and that, along any trajectory of System 2.22, the following inequality holds: for every pair of switching times (t_i, t_j) satisfying $t_i < t_j$, $s(t_i) = s(t_j) \in \mathcal{I}$, and $s(t_k) \neq s(t_j)$ for any t_k of $t_i < t_k < t_j$,*

$$V_{s(t_j)}(x(t_j)) - V_{s(t_i)}(x(t_i)) \leq -W_{s(t_i)}(x(t_i)), \tag{2.23}$$

where $W_s(x)$ with $s \in \mathcal{I}$ are a family of positive definite continuous functions. Then, Switched System 2.22 is globally asymptotically stable.

Proof: We first show global Lyapunov stability. Suppose $\|x(t_0)\| < \delta$ for some $\delta > 0$. Since $V_s(x)$ is radially unbounded and positive definite, there exist scalar strictly monotone increasing functions $\gamma_{s1}(\cdot)$ and $\gamma_{s2}(\cdot)$ such that

$\gamma_{s1}(\|x\|) \leq V_s(x) \leq \gamma_{s2}(\|x\|)$. Therefore, it follows from (2.23) that, for any t_j,

$$V_{s(t_j)}(x(t_j)) \leq \max_{s \in \mathcal{I}} \gamma_{s2}(\delta),$$

which yields

$$\|x(t_j)\| \leq \max_{\sigma \in \mathcal{I}} \gamma_{\sigma 1}^{-1} \circ \max_{s \in \mathcal{I}} \gamma_{s2}(\delta) \triangleq \epsilon.$$

The above inequality together with asymptotic stability of autonomous systems in (2.21) imply that System 2.22 is globally Lyapunov stable.

Asymptotic stability is obvious if time sequence $\{t_i : \ i \in \aleph\}$ is finite. If the sequence is infinite, let $\{t_{\sigma_j} : \ \sigma_j \in \aleph, \ j \in \aleph\}$ denote its infinite sub-sequence containing all the entries of $s(t_i) = s(t_{\sigma_j}) = \sigma \in \mathcal{I}$. It follows from (2.23) that Lyapunov sequence $V_\sigma(x(t_{\sigma_j}))$ is monotone decreasing and hence has a limit c and that

$$0 = c - c = \lim_{j \to \infty} [V_\sigma(x(t_{\sigma_j})) - V_\sigma(x(t_{\sigma_{j+1}})] \leq - \lim_{j \to \infty} W_\sigma(x(t_{\sigma_j})) \leq 0.$$

Since $W_s(\cdot)$ is positive definite and $\sigma \in \mathcal{I}$ is arbitrary, $x(t_i)$ converges to zero. The proof is completed by recalling asymptotic stability of autonomous systems in (2.21). □

In stability analysis of switching systems using multiple Lyapunov functions over time, Theorem 2.24 is representative among the available results [28, 133, 185]. Extensions such as that in [91] can be made so that some Lyapunov functions are allowed to increase during their active time intervals as long as these increases are bounded by positive definite functions properly incorporated into Inequality 2.23. Similarly, some of the systems in (2.21) may not have to be asymptotically stable provided that an inequality in the form of (2.23) holds over time. As will be illustrated in Section 2.4.3 for linear switching systems, Inequality 2.23 implies monotone decreasing over sequences of consecutive intervals, while System 2.22 can have transient increases such as overshoots during some of the intervals. To satisfy Inequality 2.23, either the solution to (2.22) or a quantitative bound on the solution should be found, which is the main difficulty in an application of Theorem 2.24. In general, Inequality 2.23 can always be ensured if the so-called *dwell time*, the length of the intervals for System 2.22 to stay as one of the systems in (2.21), is long if the corresponding system in (2.21) is asymptotically stable and sufficiently short if otherwise. Further discussions on dwell time can be found in Section 2.4.3 and in [133].

It is worth noting that, while multiple Lyapunov functions are used in Theorem 2.24 to establish stability, only one of the Lyapunov functions is actively used at every instant of time to capture the instantaneous system behavior and their cumulative effect determines the stability outcome. This is different from Comparison Theorem 2.10 by which a vector of Lyapunov functions is used simultaneously to determine qualitative behavior of a system.

2.4 Stability Analysis of Linear Systems

State space model of a linear dynamical system is

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u, \tag{2.24}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the input, and $y \in \Re^p$ is the output. Matrices A, B, C, D are called system matrix, input matrix, output matrix, and direct coupling matrix, respectively. System 2.24 is called time-invariant if these matrices are all constant. In what follows, analysis tools of linear systems are reviewed.

2.4.1 Eigenvalue Analysis of Linear Time-invariant Systems

Eigenvalue analysis is both fundamental and the easiest to understanding stability of the linear time-invariant system:

$$\dot{x} = Ax + Bu. \tag{2.25}$$

Defining *matrix exponential function* e^{At} as

$$e^{At} = \sum_{j=0}^{\infty} \frac{1}{j!} A^j t^j, \tag{2.26}$$

we know that e^{At} satisfies the property of

$$\frac{d}{dt} e^{At} = A e^{At} = e^{At} A,$$

that the solution to $\dot{x} = Ax$ is $x(t) = e^{A(t-t_0)}x(t_0)$ and hence $e^{A(t-t_0)}$ is called *state transition matrix*, and that the solution to (2.25) is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{2.27}$$

The application of a similarity transformation, reviewed in Section 2.1, is one of the time-domain approaches for solving state space equations and revealing stability properties. Consider System 2.25 and assume that, under similarity transformation $z = S^{-1}x$, $\dot{z} = S^{-1}ASz = Jz$ where J is the Jordan canonical form in (2.1). It follows from the structure of J and the Taylor series expansion in (2.26) that

$$e^{At} = S \begin{bmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_l t} \end{bmatrix} S^{-1},$$

in which

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{n_i-1}}{(n_i-1)!}e^{\lambda_i t} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & te^{\lambda_i t} \\ 0 & 0 & \cdots & e^{\lambda_i t} \end{bmatrix}. \tag{2.28}$$

Therefore, the following necessary and sufficient conditions [102, 176] can be concluded from the solutions of (2.27) and (2.28):

(a) System 2.25 with $u = 0$ is Lyapunov stable if and only if its eigenvalues are not in the right open half plane and those on the imaginary axis are of geometrical multiplicity one.
(b) System 2.25 with $u = 0$ is asymptotically stable if and only if matrix A is Hurwitz (*i.e.*, its eigenvalues are all in the left open half plane).
(c) System 2.25 is input-to-state stable if and only if it is asymptotically stable.
(d) System 2.25 with $u = 0$ is exponentially stable if and only if it is asymptotically stable.

2.4.2 Stability of Linear Time-varying Systems

Consider first the continuous-time linear time-varying system

$$\dot{x} = A(t)x, \quad x \in \Re^n. \tag{2.29}$$

Its solution can be expressed as

$$x(t) = \Phi(t, t_0)x(t_0), \tag{2.30}$$

where state transition matrix $\Phi(\cdot)$ has the properties that $\Phi(t_0, t_0) = I$, $\Phi(t, s) = \Phi^{-1}(s, t)$, and $\partial\Phi(t, t_0)/\partial t = A(t)\Phi(t, t_0)$. It follows that the above system is asymptotically stable if and only if

$$\lim_{t\to\infty} \Phi(t, t_0) = 0.$$

However, an analytical solution of $\Phi(t, t_0)$ is generally difficult to find except for simple systems. The following simple example shows that eigenvalue analysis does not generally reveal stability of linear time-varying systems.

Example 2.25. Consider System 2.29 with

$$A(t) = \begin{bmatrix} -1 + \sqrt{2}\cos^2 t & 1 - \sqrt{2}\sin t \cos t \\ -1 - \sqrt{2}\sin t \cos t & -1 + \sqrt{2}\sin^2 t \end{bmatrix}.$$

Matrix $A(t)$ is continuous and uniformly bounded. By direct computation, one can verify that the corresponding state transition matrix is

$$\Phi(t,0) = \begin{bmatrix} e^{(\sqrt{2}-1)t} & e^{-t}\sin t \\ -e^{(\sqrt{2}-1)t}\sin t & e^{-t}\cos t \end{bmatrix}.$$

Hence, the system is unstable. Nonetheless, eigenvalues of matrix $A(t)$ are at $[-(2-\sqrt{2}) \pm \sqrt{2}j]/2$, both of which are time-invariant and in the left open half plan. ◇

Piecewise-constant systems are a special class of time-varying systems. Consider a continuous-time switching system which is in the form of (2.29) and whose system matrix $A(t)$ switches between two constant matrices as, for some $\tau > 0$ and for $k \in \aleph$,

$$A(t) = \begin{cases} A_1 \text{ if } t \in [2k\tau, 2k\tau + \tau) \\ A_2 \text{ if } t \in [2k\tau + \tau, 2k\tau + 2\tau) \end{cases}. \tag{2.31}$$

Since the system is piecewise-constant, the state transition matrix is known to be

$$\Phi(t,0) = \begin{cases} e^{A_1(t-2k\tau)}H^k & \text{for } [2k\tau, 2k\tau+\tau) \\ e^{A_2(t-2k\tau-\tau)}e^{A_1\tau}H^k & \text{for } [2k\tau+\tau, 2k\tau+2\tau) \end{cases}, \quad H = e^{A_2\tau}e^{A_1\tau},$$

whose convergence depends upon the property of matrix exponential H^k. Indeed, for System 2.29 satisfying (2.31), we can define the so-called average system as

$$\dot{x}_a = A_a x_a, \quad x \in \Re^n. \tag{2.32}$$

where $x_a(2k\tau) = x(2k\tau)$ for all $k \in \aleph$, $e^{2A_a\tau} = H$, and matrix A_a can be expressed in terms of Baker-Campbell-Hausdorff formula [213] as

$$A_a = A_1 + A_2 + \frac{1}{2}ad_{A_1}(A_2) + \frac{1}{12}\{ad_{A_1}(ad_{A_1}A_2) + ad_{A_2}(ad_{A_1}A_2)\} \\ + \cdots, \tag{2.33}$$

and $ad_A B = AB - BA$ is the so-called Lie bracket (see Section 2.5 for more details). Note that A_a in (2.33) is an infinite convergent sequence in which the generic expression of ith term is not available. If $ad_A B = 0$, the system given by (2.31) is said to have commuting matrices A_i and, by invoking (2.32) and (2.33), it has the following solution:

$$x(t) = e^{A_1(\tau+\tau+\cdots)}e^{A_2(\tau+\tau+\cdots)}x(0),$$

which is exponentially stable if both A_1 and A_2 are Hurwitz. By induction, a linear system whose system matrix switches among a finite set of commuting Hurwitz matrices is exponentially stable. As a relaxation, the set of matrices are said to have *nilpotent Lie algebra* if all Lie brackets of sufficiently high-order are zero, and it is shown in [84] that a linear system whose system matrix switches among a finite set of Hurwitz matrices of nilpotent Lie algebra is also exponentially stable.

Similarly, consider a discrete-time linear time-varying system

$$x_{k+1} = A_k x_k, \quad x_k \in \Re^n. \tag{2.34}$$

It is obvious that the above system is asymptotically stable if and only if

$$\lim_{k\to\infty} A_k A_{k-1} \cdots A_2 A_1 \triangleq \lim_{k\to\infty} \prod_{j=1}^{k} A_j = 0. \tag{2.35}$$

In the case that A_k arbitrarily switches among a finite number matrices $\{D_1, \cdots, D_l\}$, The sequence convergence of (2.35) can be embedded into stability analysis of the switching system:

$$z_{k+1} = \overline{A}_k z_k, \tag{2.36}$$

where $\overline{A}_k$ switches between two constant matrices $\overline{D}_1$ and $\overline{D}_2$,

$$\overline{D}_1 = \mathrm{diag}\{D_1, \cdots, D_l\}, \quad \overline{D}_2 = T \otimes I,$$

and $T \in \Re^{l\times l}$ is any of the cyclic permutation matrices (see Sections 4.1.1 and 4.1.3 for more details). Only in the trivial case of $n = 1$ do we know that the standard time-invariant stability condition, $|D_i| < 1$, is both necessary and sufficient for stability of System 2.36. The following example shows that, even in the simple case of $n = 2$, the resulting stability condition on System 2.36 is not general or consistent enough to become satisfactory. That is, there is no simple stability test for switching systems in general.

Example 2.26. Consider the pair of matrices: for constants $\alpha, \beta > 0$,

$$D_1 = \sqrt{\beta}\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \quad D_2 = D_1^T.$$

It follows that, if the system matrix switches among D_1 and D_2, product $\overline{A}_k\overline{A}_{k-1}$ has four distinct choices: D_1^2, D_2^2, D_1D_2, and D_2D_1. Obviously, both D_1^2 and D_2^2 are Schur (*i.e.*, asymptotically stable) if and only if $\beta < 1$. On the other hand, matrix

$$D_1D_2 = \beta\begin{bmatrix} 1+\alpha^2 & \alpha \\ \alpha & 1 \end{bmatrix}$$

is Schur if and only if $\beta < 2/[2+\alpha^2+\sqrt{(2+\alpha^2)^2-4}]$, in which β has to approach zero as α becomes sufficiently large.

Such a matrix pair of $\{D_1, D_2\}$ may arise from System 2.29. For instance, consider the case of (2.31) with $c > 0$:

$$A_1 = \begin{bmatrix} -1 & c \\ 0 & -1 \end{bmatrix}, \quad A_2 = A_1^T, \quad \text{and} \quad e^{A_1\tau} = e^{-\tau}\begin{bmatrix} 1 & c\tau \\ 0 & 1 \end{bmatrix}.$$

Naturally, System 2.34 can exhibit similar properties. ◇

The above discussions demonstrate that eigenvalue analysis does not generally apply to time-varying systems and that stability depends explicitly upon changes of the dynamics. In the case that all the changes are known *a priori*, eigenvalue analysis could be applied using the equivalent time-invariant system of (2.32) (if found). In general, stability of time-varying systems should be analyzed using such tools as the Lyapunov direct method.

2.4.3 Lyapunov Analysis of Linear Systems

The Lyapunov direct method can be used to handle linear and non-linear systems in a unified and systematic manner. Even for linear time-invariant systems, it provides different perspectives and insights than linear tools such as impulse response and eigenvalues. For Linear System 2.29, the Lyapunov function can always be chosen to be a *quadratic function* of form

$$V(x,t) = x^T P(t) x,$$

where $P(t)$ is a symmetric matrix and is called *Lyapunov function matrix*. Its time derivative along trajectories of System 2.29 is also quadratic as

$$\dot{V}(x,t) = -x^T Q(t) x,$$

where matrices $P(t)$ and $Q(t)$ are related by the so-called differential Lyapunov equation

$$\dot{P}(t) = -A^T(t)P(t) - P(t)A(t) - Q(t). \tag{2.37}$$

In the case that the system is time-invariant, $Q(t)$ and hence $P(t)$ can be selected to be constant, and (2.37) becomes algebraic.

To conclude asymptotic stability, we need to determine whether the two quadratic functions $x^T P(t) x$ and $x^T Q(t) x$ are positive definite. To this end, the following Rayleigh-Ritz inequality should be used. For any symmetric matrix H:

$$\lambda_{min}(H)\|x\|^2 \leq x^T H x \leq \lambda_{max}(H)\|x\|^2,$$

where $\lambda_{min}(H)$ and $\lambda_{max}(H)$ are the minimum and maximum eigenvalues of H, respectively. Thus, stability of linear systems can be determined by checking whether matrices P and Q are positive definite (p.d.). Positive definiteness of a symmetric matrix can be checked using one of the following tests:

(a) Eigenvalue test: a symmetric matrix P is p.d. (or p.s.d.) if and only if all its eigenvalues are positive (or non-negative).
(b) Principal minor test: a symmetric matrix P is p.d. (or p.s.d.) if and only if all its leading principal minors are positive (or non-negative).
(c) Factorization test: a symmetric matrix P is p.s.d. (or p.d.) if and only if $P = WW^T$ for some (invertible) matrix W.

(d) Gershgorin test: a symmetric matrix P is p.d. or p.s.d. if, for all $i \in \{1, \cdots, n\}$,

$$p_{ii} > \sum_{j=1, j \neq i}^{n} |p_{ij}| \quad \text{or} \quad p_{ii} \geq \sum_{j=1, j \neq i}^{n} |p_{ij}|,$$

respectively.

For time-varying matrices $P(t)$ and $Q(t)$, the above tests can be applied but should be strengthened to be uniform with respect to t.

As discussed in Section 2.3.1, Lyapunov function $V(x, t)$ and its matrix $P(t)$ should be solved using the backward process. The following theorem provides such a result.

Lemma 2.27. *Consider System 2.29 with uniformly bounded matrix $A(t)$. Then, it is uniformly asymptotically stable and exponentially stable if and only if, for every uniformly bounded and p.d. matrix $Q(t)$, solution $P(t)$ to (2.37) is positive definite and uniformly bounded.*

Proof: Sufficiency follows directly from a Lyapunov argument with Lyapunov function $V(x, t) = x^T P(t) x$. To show necessity, choose

$$P(t) = \int_t^\infty \Phi^T(\tau, t) Q(\tau) \Phi(\tau, t) d\tau, \tag{2.38}$$

where Q be p.d. and uniformly bounded, and $\Phi(\cdot, \cdot)$ is the state transition matrix in (2.30). It follows from the factorization test that $P(t)$ is positive definite. Since matrix $A(t)$ is uniformly bounded, $\Phi^T(t, \tau)$ is exponentially convergent and uniformly bounded if and only if the system is exponentially stable. Thus, $P(t)$ defined above exists and is also uniformly bounded if and only if the system is exponentially stable. In addition, it follows that

$$\begin{aligned}\dot{P}(t) &= \int_t^\infty \frac{\partial \Phi^T(\tau, t)}{\partial t} Q(\tau) \Phi(\tau, t) d\tau + \int_t^\infty \Phi^T(\tau, t) Q(\tau) \frac{\partial \Phi(\tau, t)}{\partial t} d\tau - Q(t) \\ &= -A^T(t) P(t) - P(t) A(t) - Q(t),\end{aligned}$$

which is (2.37). This completes the proof. □

To solve for Lyapunov function $P(t)$ from Lyapunov Equation 2.37, $A(t)$ needs to be known. Since finding $P(t)$ is computationally similar to solving for state transition matrix $\Phi(\cdot, \cdot)$, finding a Lyapunov function may be quite difficult for some linear time-varying systems. For linear time-invariant systems, Lyapunov function matrix P is constant as the solution to either the algebraic Lyapunov equation or the integral given below:

$$A^T P + PA = -Q, \quad P = \int_0^\infty e^{A^T \tau} Q e^{A\tau} d\tau, \tag{2.39}$$

where Q is any positive definite matrix.

The Lyapunov direct method can be applied to analyze stability of linear piecewise-constant systems. If System 2.20 is linear and has arbitrary switching, a common Lyapunov function exists, but may not be quadratic as indicated by the counterexample in [49]. Nonetheless, it is shown in [158] that the common Lyapunov function can be chosen to be of the following homogeneous form of degree 2: for some vectors $c_i \in \Re^n$,

$$V(x) = \max_{1 \le i \le k} V_i(x), \quad V_i(x) = x^T c_i c_i^T x.$$

Using the notations in (2.22), a linear piecewise-constant system can be expressed as

$$\dot{x} = A_{s(t_i)} x, \quad s(t_i) \in \mathcal{I}. \tag{2.40}$$

Assuming that A_s be Hurwitz for all $s \in \mathcal{I}$, we have $V_s(x) = x^T P_s x$ where P_s is the solution to algebraic Lyapunov equation

$$A_s^T P_s + P_s A_s = -I.$$

Suppose that $t_i < t_{i+1} < t_j$ and that $s(t_i) = s(t_j) = p$ and $s(t_{i+1}) = q \neq p$. It follows that, for $t \in [t_i, t_{i+1})$,

$$\dot{V}_p = -x^T x \le -\sigma_p V_p,$$

and hence

$$V_p(x(t_{i+1})) \le e^{-\sigma_p \tau_{i+1}} V_p(x(t_i)),$$

where $\tau_{i+1} = t_{i+1} - t_i$ is the dwell time, and $\sigma_p = 1/\lambda_{\max}(P_p)$ is the time constant. Similarly, it follows that, over the interval $[t_{i+1}, t_{i+2})$

$$V_q(x(t_{i+2})) \le e^{-\sigma_q \tau_{i+2}} V_q(x(t_{i+1})).$$

Therefore, we have

$$\begin{aligned}
& V_p(x(t_{i+2})) - V_p(x(t_i)) \\
& \le \lambda_{max}(P_p) \|x(t_{i+2})\|^2 - V_p(x(t_i)) \\
& \le \frac{\lambda_{max}(P_p)}{\lambda_{min}(P_q)} V_q(x(t_{i+2})) - V_p(x(t_i)) \\
& \le \frac{\lambda_{max}(P_p)}{\lambda_{min}(P_q)} e^{-\sigma_q \tau_{i+2}} V_q(x(t_{i+1})) - V_p(x(t_i)) \\
& \le \frac{\lambda_{max}(P_p)}{\lambda_{min}(P_p)} \frac{\lambda_{max}(P_q)}{\lambda_{min}(P_q)} e^{-\sigma_q \tau_{i+2}} V_p(x(t_{i+1})) - V_p(x(t_i)) \\
& \le - \left[1 - \frac{\lambda_{max}(P_p)}{\lambda_{min}(P_p)} \frac{\lambda_{max}(P_q)}{\lambda_{min}(P_q)} e^{-\sigma_q \tau_{i+2}} e^{-\sigma_p \tau_{i+1}} \right] V_p(x(t_i)),
\end{aligned}$$

which is always negative definite if either time constants σ_s are sufficiently small or dwell times τ_k are sufficiently long or both, while the ratios of

$\lambda_{max}(P_s)/\lambda_{min}(P_s)$ remain bounded. By induction and by applying Theorem 2.24, we know that asymptotic stability is maintained for a linear system whose dynamics switch relatively slowly among a finite number of relatively fast dynamics of exponentially stable autonomous systems. This result illustrates robustness of asymptotic stability with respect to switching, but it is quite conservative (since the above inequality provides the worst estimate on transient overshoots of the switching system). Similar analysis can also be done for Discrete-time System 2.36.

In the special case that matrices A_s are all commuting Hurwitz matrices, it has been shown in Section 2.4.2 that System 2.40 is always exponentially stable for all values of time constants and dwell times. Furthermore, System 2.40 has a quadratic common Lyapunov function in this case. For instance, consider the case that $\mathcal{I} = \{1, 2\}$ and let P_1 and P_2 be the solutions to Lyapunov equations

$$A_1^T P_1 + P_1 A_1 = -I, \;\; A_2^T P_2 + P_2 A_2 = -P_1.$$

Then, P_2 is the common Lyapunov function matrix since $e^{A_1\tau_1}$ and $e^{A_2\tau_2}$ commute and, by (2.39),

$$P_2 = \int_0^\infty e^{A_2^T\tau_2} P_1 e^{A_2\tau_2} d\tau_2 = \int_0^\infty e^{A_1^T\tau_1} W_2 e^{A_1\tau_1} d\tau_1,$$

where

$$W_2 = \int_0^\infty e^{A_2^T\tau_2} e^{A_2\tau_2} d\tau_2$$

is positive definite. Given any finite index set $\mathcal{I}$, the above observation leads to a recursive process of finding the quadratic common Lyapunov function [173].

2.5 Controllability

Roughly speaking, a control system is locally controllable if a steering law for its control can be found to move the state to any specified point in a neighborhood of its initial condition. On the other hand, Affine System 2.11 can be rewritten as

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i = \sum_{i=0}^{m} g_i(x) u_i, \tag{2.41}$$

where $x \in \Re^n$, $u_0 = 1$, $g_0(x) \stackrel{\triangle}{=} f(x)$, and vector fields $g_i(x)$ are assumed to be analytic and linearly independent. Thus, controllability should be determined by investigating time evolution of the state trajectory of (2.41) under all possible control actions.

To simplify the notation, we introduce the so-called *Lie bracket*: for any pair of $f(x), g(x) \in \Re^n$, $[f, g]$ or $ad_f g$ (where ad stands for "adjoint") is a vector function of the two vector fields and is defined by

$$[f, g] = (\nabla_x^T g) f - (\nabla_x^T f) g.$$

High-order Lie brackets can be defined recursively as, for $i = 1, \cdots$,

$$ad_f^0 g = g, \qquad \text{and} \qquad ad_f^i g = [f, ad_f^{i-1} g].$$

An important property associated with the Lie derivative and Lie bracket is the so-called *Jacobi identity*: for any $f(x), g(x) \in \Re^n$ and $\xi(x) \in \Re$,

$$L_{ad_f g}\xi = L_f L_g \xi - L_g L_f \xi, \tag{2.42}$$

and it can easily be verified by definition. The following are the concepts associated with Lie bracket.

Definition 2.28. *A set of linearly independent vector fields* $\{\xi_l(x) : l = 1, \cdots, k\}$ *is said to be* involutive *or* closed under Lie bracket *if, for all* $i, j \in \{1, \cdots, k\}$*, the Lie bracket* $[\xi_i, \xi_j]$ *can be expressed as a linear combination of* ξ_1 *up to* ξ_k.

Definition 2.29. *Given smooth linearly independent vector fields* $\{\xi_l(x) : l = 1, \cdots, k\}$*, the tangent-space* distribution *is defined as*

$$\Delta(x) = span\{\xi_1(x), \xi_2(x), \cdots, \xi_k(x)\}.$$

$\Delta(x)$ *is* regular *if the dimension of* $\Delta(x)$ *does not vary with* x. $\Delta(x)$ *is involutive if* $[f, g] \in \Delta$ *for all* $f, g \in \Delta$. Involutive closure $\overline{\Delta}$ *of* $\Delta(x)$*, also called* Lie algebra*, is the smallest distribution that contains* Δ *and is closed under Lie bracket.*

While Lie brackets of $g_i(x)$ do not have an explicit physical meaning, the span of their values consists of all the incremental movements achievable for System 2.41 under modulated inputs. For instance, consider the following two-input system:

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2, \quad x(0) = x_0.$$

Should piecewise-constant inputs be used, there are only two linearly independent values for the input vector at any fixed instant of time, and they can be used to construct any steering control input over time. For instance, the following is a piecewise-constant steering control input:

$$\begin{bmatrix} u_1(t) \; u_2(t) \end{bmatrix}^T = \begin{cases} \begin{bmatrix} 1 \; 0 \end{bmatrix}^T & t \in (0, \varepsilon] \\ \begin{bmatrix} 0 \; 1 \end{bmatrix}^T & t \in (\varepsilon, 2\varepsilon] \\ \begin{bmatrix} -1 \; 0 \end{bmatrix}^T & t \in (2\varepsilon, 3\varepsilon] \\ \begin{bmatrix} 0 \; -1 \end{bmatrix}^T & t \in (3\varepsilon, 4\varepsilon] \end{cases},$$

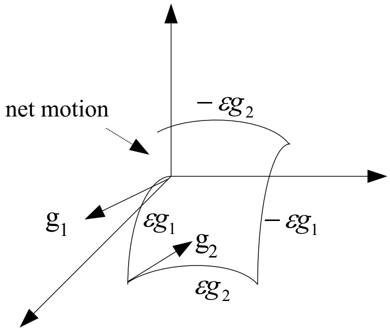

Fig. 2.1. Incremental motion in terms of Lie bracket

where $\varepsilon > 0$ is some sufficiently small constant. One can calculate the second-order Taylor series expansions of $x(i\varepsilon)$ recursively for $i = 1, 2, 3, 4$ and show that

$$[g_1, g_2](x_0) = \lim_{h \to 0} \frac{x(4\varepsilon) - x_0}{\varepsilon^2}.$$

As shown by Fig. 2.1, Lie bracket $[g_1, g_2](x_0)$ is the net motion generated under the steering control. In general, controllability of Affine System 2.11 can be determined in general by a rank condition on its Lie algebra, as stated by the following theorem [40, 114].

Definition 2.30. *A system is* controllable *if, for any two points $x_0, x_f \in \Re^n$, there exist a finite time T and a control $u(x, t)$ such that the system solution satisfies $x(t_0) = x_0$ and $x(t_0 + T) = x_f$. The system is* small-time locally controllable *at x_1 if $u(x, t)$ can be found such that $x(t_0 + \delta t) = x_1$ for sufficiently small $\delta t > 0$ and that $x(t)$ with $x(t_0) = x_0$ stays near x_1 at all times.*

Theorem 2.31. *Affine System 2.41 is small-time locally controllable at x if the involutive closure of*

$$\Delta(x) = \{f(x),\ g_1(x),\ \cdots,\ g_m(x)\}$$

is of dimension n, that is, the rank of the controllability Lie algebra is n.

If an affine system is not controllable, controllability decomposition can be applied to separate controllable dynamics from uncontrollable dynamics. For non-affine systems, conditions could be obtained using local linearization with respect to control u. More details on non-linear controllability can be found in texts [12, 96, 175, 223]. For Linear System 2.25, the controllability Lie algebra reduces to

$$\{g, ad_f g, \cdots, ad_f^{n-1} g\} = \{B, AB, \cdots, A^{n-1}B\},$$

which is the linear controllability matrix $\mathcal{C} = \left[B \; AB \; A^2B \cdots A^{n-1}B \right]$. And, System 2.25 is said to be in the controllable canonical form if the matrix pair (A, B) is given by

$$A_c \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_c \triangleq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{2.43}$$

Consider the linear time-varying system:

$$\dot{x} = A(t)x + B(t)u, , \tag{2.44}$$

where matrices $A(t)$ and $B(t)$ are uniformly bounded and of proper dimension. Their controllability can directly be studied using its solution

$$\begin{aligned} x(t) &= \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau \\ &= \Phi(t, t_0) \left[x(t_0) + \int_{t_0}^{t} \Phi(t_0, \tau)B(\tau)u(\tau)d\tau \right]. \end{aligned}$$

It is straightforward to verify that, if the matrix

$$W_c(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_0, \tau)B(\tau)B^T(\tau)\Phi^T(t_0, \tau)d\tau \tag{2.45}$$

is invertible, the state can be moved from any x_0 to any x_f under control

$$u(t) = -B^T(t)\Phi^T(t_0, t)W_c^{-1}(t_0, t_f)[x_0 - \Phi(t_0, t_f)x_f].$$

Matrix $W_c(t_0, t_f)$ is the so-called *controllability Gramian*, and it is easy to show [39] that $W_c(t_0, t_f)$ is invertible and hence System 2.44 is controllable if and only if the n rows of n-by-m matrix function $\Phi(t_0, t)B(t)$ are linear independent over time interval $[t_0, t_f]$ for some $t_f > t_0$. To develop a controllability condition without requiring the solution of state transition matrix $\Phi(t_0, t)$, it follows that

$$\frac{\partial^k}{\partial t^k} \Phi(t_0, t)B(t) = \Phi(t_0, t)E_k(t), \quad k = 0, 1, \cdots, (n-1),$$

where $E_0(t) = B(t)$ and $E_{k+1}(t) = -A(t)E_k(t) + \dot{E}_k(t)$ for $k = 0, 1, \cdots, (n-1)$. Hence, controllability of System 2.44 is equivalent to the Kalman rank condition of

$$\text{rank} \left[E_0(t) \; E_1(t) \cdots E_{n-1}(t) \right] = n,$$

Comparing (2.45) and (2.38), we know that, under the choice of positive semi-definite matrix $Q(t) = B(t)B^T(t)$, Lyapunov Equation 2.37 still yields positive definite Lyapunov function $P(t)$ provided that System 2.44 is controllable and that its uncontrolled dynamics in (2.29) are asymptotically stable. To design a Lyapunov-based control and to ensure global and uniformly asymptotic stability, it is necessary that the system solution is uniformly bounded and that Lyapunov function matrix $P(t)$ exists and is also uniformly bounded, which leads to the following definition and theorem [103]. Later in Section 2.6.3, Control 2.46 below is shown to be optimal as well.

Definition 2.32. *System 2.44 is said to be* uniformly completely controllable *if the following two inequalities hold for all t:*

$$0 < \alpha_{c1}(\delta)I \leq W_c(t, t+\delta) \leq \alpha_{c2}(\delta)I, \quad \|\Phi(t, t+\delta)\| \leq \alpha_{c3}(\delta),$$

where $W_c(t_0, t_f)$ is defined by (2.45), $\delta > 0$ is a fixed constant, and $\alpha_{ci}(\cdot)$ are fixed positively-valued functions.

Theorem 2.33. *Consider System 2.44 under control*

$$u = -R^{-1}(t)B^T(t)P(t)x, \tag{2.46}$$

where matrices $Q(t)$ and $R(t)$ are chosen to be positive definite and uniformly bounded, and matrix $P(t)$ is the solution to the differential Riccati equation

$$\dot{P} + [PA + A^T P - PBR^{-1}B^T P + Q] = 0 \tag{2.47}$$

with terminal condition of $P(\infty)$ being positive definite. Then, if System 2.44 is uniformly completely controllable, Lyapunov function $V(x,t) = x^T P(t)x$ is positive definite and decrescent, and Control 2.46 is asymptotically stabilizing.

2.6 Non-linear Design Approaches

In this section, three popular design methods are outlined for non-linear systems; they are backstepping design, feedback linearization, and optimal control.

2.6.1 Recursive Design

In this section, we focus upon the following class of *feedback systems*:

$$\begin{cases} \dot{x}_1 = f_1(x_1, t) + x_2, \\ \dot{x}_2 = f_2(x_1, x_2, t) + x_3, \\ \vdots \\ \dot{x}_n = f_n(x_1, \cdots, x_n, t) + u, \end{cases} \tag{2.48}$$

where $x_i \in \Re^l$, x_1 is the output, and u is the input. Letting x_n up to x_1 be the outputs of a chain of pure integrators, we can graphically connect the inputs to the integrators by using dynamic equations in (2.48). In the resulting block diagram, there is a feedforward chain of integrators, and all other connections are feedback. In particular, the ith integrator has x_{i+1} and x_i as its input and output, respectively. Should $f_i(x_1, \cdots, x_i, t) = f_i(x_i, t)$, System 2.48 would become a cascaded chain of first-order non-linear sub-systems. As such, System 2.48 is also referred to as a *cascaded system*. Several classes of physical systems, especially electromechanical systems [130, 195, 242], have by nature this structural property on their dynamics, and their controllability is guaranteed.

The aforementioned structural property of cascaded dynamics provides a natural and intuitive way for finding both Lyapunov function and a stabilizing control. The systematic procedure, called *backstepping* or *backward recursive design* [72, 116, 192], is a step-by-step design process in which the first-order sub-systems are handled one-by-one and backwards from the output x_1 back to the input u. Specifically, let us consider the case of $n = 2$ and begin with the first sub-system in (2.48), that is,

$$\dot{x}_1 = f_1(x_1, t) + x_2.$$

If x_2 were a control variable free to be selected, a choice of x_2 would easily be found to stabilize the first sub-system. Because x_2 is a state variable rather than a control, we rewrite the first sub-system as

$$\dot{x}_1 = f_1(x_1, t) + x_2^d(x_1, t) + [x_2 - x_2^d(x_1, t)],$$

and choose the fictitious control $x_2^d(x_1, t)$ (such as the most obvious choice of $x_2^d(x_1, t) = -f_1(x_1, t) - x_1$) to stabilize asymptotically fictitious system $\dot{x}_1 = f_1(x_1, t) + x_2^d(x_1, t)$ by ensuring the inequality

$$2x_1^T[f_1(x_1, t) + x_2^d(x_1, t)] \le -\|x_1\|^2.$$

Thus, choosing Lyapunov sub-function $V_1(x_1) = \|x_1\|^2$, we have that, along the trajectory of (2.48),

$$\dot{V}_1 \le -2\|x_1\|^2 + 2x_1^T z_2,$$

where $z_2 = x_2 - x_2^d(x_1, t)$ is a transformed state variable of x_2. As shown in Section 2.3.2, the above inequality of $\dot{V}_1$ implies that, if z_2 is asymptotically convergent, so is x_1. To ensure asymptotic convergence of z_2, we know from its definition that

$$\dot{z}_2 = u - \frac{\partial x_2^d}{\partial t} - \left(\frac{\partial x_2^d}{\partial x_1}\right)^T [f_1(x_1, t) + x_2].$$

As before, the above dynamic equation of z_2 is of first-order and hence control u can be easily found (for instance, the choice rendering $\dot{z}_2 = -z_2$) such that, with $V_2(x_1, x_2) = \|z_2\|^2$,

$$\dot{V}_2 \leq -2\|z_2\|^2.$$

Therefore, we now have found Lyapunov function $V = V_1 + \alpha_2 V_2$ whose time derivative is

$$\dot{V} = -2\|x_1\|^2 + 2x_1^T z_2 - 2\alpha_2\|z_2\|^2,$$

which is negative definite for any choice of $\alpha_2 > 1/4$. Hence, under the control u selected, both x_1 and z_2 and consequently both x_1 and x_2 are globally asymptotically stable. It is straightforward to see that, by induction, a stabilizing control and the corresponding Lyapunov function can be found recursively for System 2.48 of any finite-order n.

2.6.2 Feedback Linearization

The feedback linearization approach provides the conditions under which a pair of state and control transformations exist such that a non-linear system is mapped (either locally or globally) into the linear controllable canonical form. Specifically, the objective of feedback linearization is to map System 2.11 into the form

$$\begin{aligned} \dot{z} &= A_c z + B_c v, \\ \dot{w} &= \phi(z, w), \end{aligned} \tag{2.49}$$

where the pair $\{A_c, B_c\}$ is that in (2.43), $z(x) \in \Re^r$ is the state of the feedback linearized sub-system, r is the so-called relative degree of the system, w is the state of so-called internal dynamics, $[z^T \;\; w^T] = [z^T(x) \;\; w^T(x)]$ is the state transformation, and $v = v(u, x)$ is the control transformation. Then, standard linear control results can be applied through the transformations to the original non-linear system provided that the internal dynamics are minimum phase (*i.e.*, the zero dynamics of $\dot{w} = \phi(0, w)$ are asymptotically stable).

By (2.49), System 2.11 with $m = 1$ is feedback linearizable if function $h(x) \in \Re$ exists such that

$$z_1 = h(x); \;\; \dot{z}_i = z_{i+1}, \; i = 1, \cdots, r-1; \;\; \dot{z}_r = v; \;\; \dot{w} = \phi(z, w) \tag{2.50}$$

for some vector functions $w(x)$ and $\phi(\cdot)$ and for transformed control $v(u, x)$. It follows from dynamics of System 2.11 that $\dot{z}_i = L_f z_i + (L_g z_i)u$. Therefore, equations of (2.50) are equivalent to

$$\begin{cases} L_g w_j = 0, \; j = 1, \cdots, n-r; \\ L_g L_f^{i-1} h = 0, \; i = 1, \cdots, r-1; \\ L_g L_f^r h \neq 0, \end{cases} \tag{2.51}$$

while the control transformation and linearized state variables are defined by

$$v = L_f z_r + (L_g z_r)u; \quad z_1 = h(x), \;\; z_{i+1} = L_f z_i \;\; i = 1, \cdots, r-1. \tag{2.52}$$

By Jacobi Identity 2.42, $L_g h = L_g L_f h = 0$ if and only if $L_g h = L_{ad_f g} h = 0$. By induction, we can rewrite the partial differential equations in (2.51) as

$$\begin{cases} L_g w_j = 0, \ j = 1, \cdots, n-r; \\ L_{ad^i_f g} h = 0, \ i = 1, \cdots, r-1; \\ L_g L^r_f h \neq 0. \end{cases} \tag{2.53}$$

In (2.53), there are $(n-1)$ partial differential equations that are all homogeneous and of first-order. Solutions to these equations can be found under rank-based conditions, and they are provided by the following theorem often referred to as the Frobenius theorem [96]. For control design and stability analysis, the transformation from x to $[z^T, \ w^T]^T$ needs to be diffeomorphic (*i.e.*, have a unique inverse), which is also ensured by Theorem 2.34 since, by implicit function theorem [87], the transformation is diffeomorphic if its Jacobian matrix is invertible and since the Jacobian matrix consists of $\bigtriangledown q_j(x)$, the gradients of the solutions.

Theorem 2.34. *Consider $k(n-k)$ first-order homogeneous partial differential equations:*

$$L_{\xi_i} q_j = 0, \qquad i = 1, \cdots, k; \quad j = 1, \cdots, n-k, \tag{2.54}$$

where $\{\xi_1(x), \xi_2(x), \cdots, \xi_k(x)\}$ is a set of linearly independent vectors in $\Re^n$, and $q_j(x)$ are the functions to be determined. Then, the solutions $q_j(x)$ to (2.54) exist if and only if distribution $\{\xi_1(x), \xi_2(x), \cdots, \xi_k(x)\}$ is involutive. Moreover, under the involutivity condition, the gradients of the solutions, $\bigtriangledown q_j(x)$, are linearly independent.

Applying Theorem 2.34 to (2.53), we know that System 2.11 is feedback linearizable with relative degree $r = n$ (*i.e.*, full state feedback linearizable) if the set of vector fields $\{g, ad_f g, \cdots, ad^{n-2}_f g\}$ is involutive and if matrix $[g \ ad_f g \ \cdots \ ad^{n-1}_f g]$ is of rank n. Upon verifying the conditions, the first state variable $z_1 = h(x)$ can be found such that $L_{ad^i_f g} h = 0$ for $i = 1, \cdots, n-1$ but $L_g L^n_f h \neq 0$, the rest of state variables and the control mapping are given by (2.52). The following theorem deals with the case of $m \geq 1$ and $r = n$, and a similar result can be applied for the general case of $r < n$ [96].

Theorem 2.35. *System 2.11 is feedback linearizable (i.e., can be mapped into the linear controllable canonical form) under a diffeomorphic state transformation $z = T(x)$ and a control mapping $u = \alpha(x) + \beta(x)v$ if and only if the nested distributions defined by $\mathcal{D}_0 = span \ \left\{ g_1 \cdots g_m \right\}$ and $\mathcal{D}_i = \mathcal{D}_{i-1} + span \ \left\{ ad^i_f g_1 \cdots ad^i_f g_m \right\}$ with $i = 1, \cdots, n-1$ have the properties that $\mathcal{D}_l$ are all involutive and of constant rank for $0 \leq l \leq n-2$ and that rank $\mathcal{D}_{n-1} = n$.*

As shown in Section 2.5, matrix $[g \ ad_f g \ \cdots \ ad^{n-1}_f g]$ having rank n is ensured by non-linear controllability. The involutivity condition on set

$\{g \ ad_f g \ \cdots \ ad_f^{n-2} g\}$ ensures the existence and diffeomorphism of state and control transformations but does not have a clear intuitive explanation. Note that the involutivity condition is always met for linear systems (since the set consists of constant vectors only).

2.6.3 Optimal Control

For simplicity, consider Affine System 2.11 and its optimal control problem over the infinite horizon. That is, our goal is to find control u such that the following performance index

$$J(x(t_0)) = \int_{t_0}^{\infty} L(x(t), u(t))dt \tag{2.55}$$

is minimized, subject to Dynamic Equation 2.11 and terminal condition of $x(\infty) = 0$. There are two approaches to solve the optimal control problem: the Euler-Lagrange method based on Pontryagin minimum principle, and the principle of optimality in dynamic programming.

By using a Lagrange multiplier, the Euler-Lagrange method converts an optimization problem with equality constraints into one without any constraint. That is, System Equation 2.11 is adjoined into performance index J as

$$\mathcal{J}(t_0) = \int_{t_0}^{\infty} \{L(x,u) + \lambda^T[f(x) + g(x)u - \dot{x}]\}dt = \int_{t_0}^{\infty} [H(x,u,\lambda,t) - \lambda^T \dot{x}]dt,$$

where $\lambda \in \Re^n$ is the Lagrange multiplier, and

$$H(x,u,\lambda,t) = L(x,u) + \lambda^T[f(x) + g(x)u] \tag{2.56}$$

is the Hamiltonian. Based on calculus of variations [187], $\mathcal{J}$ is optimized locally by u if $\delta^1 \mathcal{J} = 0$, where $\delta^1 \mathcal{J}$ is the first-order variation of $\mathcal{J}$ due to variation δu in u and its resulting state variation δx in x. Integrating in part the term containing $\dot{x}$ in $\mathcal{J}$ and then finding the expression of its variation, one can show that $\delta^1 \mathcal{J} = 0$ holds under the following equations:

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \left(\frac{\partial f}{\partial x}\right)^T \lambda - \sum_{i=1}^{n}\sum_{j=1}^{n_u} \lambda_i \frac{\partial g_{ij}}{\partial x} u_j, \quad \lambda(\infty) = 0, \tag{2.57}$$

$$0 = \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + g^T \lambda, \quad \text{or} \quad H(x,\lambda,u) = \min_u H(x,\lambda,u). \tag{2.58}$$

If instantaneous cost $L(\cdot)$ is a p.d. function, optimal control can be solved from (2.58). Equation 2.58 is referred to as the *Pontryagin minimum principle*. Equation 2.57 is the so-called costate equation; it should be solved simultaneously with State Equation 2.11. Equations 2.11 and 2.57 represent a two-point boundary-value problem, and they are necessary for optimality but generally not sufficient. To ensure that the value of J is a local minimum, its second-order variation $\delta^2 \mathcal{J}$ must be positive, that is, the following Hessian matrix of Hamiltonian H should be positive definite:

$$\mathcal{H}_H \triangleq \begin{bmatrix} \dfrac{\partial^2 H}{\partial x^2} & \dfrac{\partial^2 H}{\partial x \partial u} \\ \dfrac{\partial^2 H}{\partial u \partial x} & \dfrac{\partial^2 H}{\partial u^2} \end{bmatrix}.$$

For the linear time-varying system in (2.44), it is straightforward to verify that $\lambda = P(t)x$ and $\mathcal{J}(t_0) = V(x(t_0), t_0)$ and that Control 2.46 is optimal.

Alternatively, an optimal control can be derived by generalizing Performance Index 2.55 into

$$J^*(x(t)) = \inf_u \int_t^\infty L(x(\tau), u(\tau)) d\tau,$$

which can be solved using dynamic programming. The *principle of optimality* states that, if u^* is the optimal control under Performance Index 2.55, control u^* is also optimal with respect to the above measure for the same system but with initial condition $x(t) = x^*(t)$. Applying the principle of optimality to $J^*(x(t + \delta t))$ and invoking Taylor series expansion yield the following partial differential equation:

$$\min_u H(x, u, \lambda)\Big|_{\lambda = \bigtriangledown_x J^*(x)} = - \bigtriangledown_t J^*(x) = 0. \tag{2.59}$$

which is so-called Hamilton-Jacobi-Bellman (HJB) equation [10, 16, 33, 97].

For Affine System 2.11 and under the choice of $L(x, u) = \alpha(x) + \|u\|^2$, the HJB equation reduces to

$$L_f J^* - \frac{1}{4}\|L_g J^*\|^2 + \alpha(x) = 0, \tag{2.60}$$

where $\alpha(x) \geq 0$ is any scalar function. If the above partial differential equation has a continuously differentiable and positive definite solution $J^*(x)$, the optimal control is

$$u^*(x) = -\frac{1}{2}(L_g J^*)^T,$$

under which the closed-loop system is asymptotically stable. However, solving HJB Equation 2.60 in general is quite difficult even for Affine System 2.11.

2.6.4 Inverse Optimality and Lyapunov Function

Solving HJB Equation 2.60 is equivalent to finding control Lyapunov function $V(x)$. If J^* is known, the Lyapunov function can be set as $V(x) = J^*(x)$, and the optimal control is stabilizing since

$$L_f J^* + (L_g J^*) u^* = -\frac{1}{4} \|L_g J^*\|^2 - \alpha(x) \leq 0.$$

On the other hand, given a control Lyapunov function $V(x)$, a class of stabilizing controls for System 2.11 can be found, and performance under any of these controls can be evaluated. For instance, consider Control 2.19 in Section 2.3.3. Although optimal value function $J^*(x)$ is generally different from $V(x)$, we can assume that, for some scalar function $k(x)$, $\bigtriangledown_x J^* = 2k(x) \bigtriangledown_x V$. That is, in light of stability outcome, J^* has the same level curves as those of V. Substituting the relationship into (2.60) yields

$$2k(x) L_f V - k^2(x) \|L_g V\|^2 + \alpha(x) = 0,$$

which is a quadratic equation in $k(x)$. Choosing the solution with positive square root, we know that

$$k(x) = \frac{L_f V + \sqrt{(L_f V)^2 + \|L_g V\|^2 \alpha(x)}}{\|L_g V\|^2}$$

and that optimal control $u^*(x)$ reduces to Control 2.19. That is, for any choice of $\alpha(x)$, Control 2.19 is optimal with respect to J^*, which is called inverse optimality [72, 234].

In the special case of linear time-invariant systems, the relationship between the Lyapunov-based control design and inverse optimality is more straightforward. Should System 2.25 be controllable, a stabilizing control of general form $u = -Kx$ exists such that $A - BK$ is Hurwitz. For any pair of p.d. matrices $Q \in \Re^{n \times n}$ and $R \in \Re^{m \times m}$, matrix sum $(Q + K^T RK)$ is p.d. and hence, by Lemma 2.27, solution P to Lyapunov equation

$$P(A - BK) + (A - BK)^T P = -(Q + K^T RK)$$

is p.d. Thus, it follows from System 2.25 under control $u = -Kx$ that

$$x^T(t) P x(t) = -\int_t^\infty \frac{dx^T P x}{d\tau} d\tau = \int_t^\infty (x^T Q x + u^T R u) d\tau.$$

Thus, every stabilizing control is inversely optimal with respect to a quadratic performance index, and the optimal performance value over $[t, \infty)$ is the corresponding quadratic Lyapunov function.

2.7 Notes and Summary

The Lyapunov direct method [141] is a universal approach for both stability analysis and control synthesis of general systems, and finding a Lyapunov function is the key [108, 264]. As shown in Sections 2.3.2 and 2.6, the Lyapunov function [141] or the pair of control Lyapunov function [235, 236] and the corresponding controller can be searched for by analytically exploiting system properties, that is, a control Lyapunov function [8] can be constructed for systems of special forms [261], for systems satisfying the Jurdejevic-Quinn conditions [61, 152], for feedback linearizable systems [94, 96, 99, 175], and for all the systems to which recursive approaches are applicable [116, 192]. Numerically, a Lyapunov function can also be found by searching for its dual of density function [210] or the corresponding sum-of-square numerical representation in terms of a polynomial basis [190]. Alternative approaches are that a numerical solution to the HJB equation yields the Lyapunov function as an optimal value function [113] and that set-oriented partition of the state space renders discretization and approximation to which graph theoretical algorithms are applicable [81]. For linear control systems, the problem reduces to a set of linear matrix inequalities (LMI) which are convex and can be solved using semi-definite programming tools [27], while linear stochastic systems can be handled using Perron-Frobenius operator [121].

Under controllability and through the search of control Lyapunov function, a stabilizing control can be designed for dynamic systems. In Chapter 3, several vehicle-level controls are designed for the class of non-holonomic systems that include various vehicles as special cases. To achieve cooperative behaviors, a team of vehicles needs to be controlled through a shared sensing/communication network. In Chapter 4, a mathematical representation of the network is introduced, and a matrix-theoretical approach is developed by extending the results on piecewise-constant linear systems in Section 2.4.2. In Chapter 5, cooperative stability is defined, and the matrix-theoretical approach is used to study cooperative controllability over a network, to design a linear cooperative control, and to search for the corresponding control Lyapunov function. Due to the changes in the network, a family of control Lyapunov functions would exist over consecutive time intervals in a way similar to those in Theorem 2.24. Since the networked changes are uncertain, the control Lyapunov functions cannot be determined and their changes over time cannot be assessed. Nonetheless, all the control Lyapunov functions are always quadratic and have the same square components, and these components can be used individually and together as a vector of Lyapunov function components to study stability. This observation leads to the Lyapunov function component-based methodology which is developed in Chapter 6 to extend both Theorems 2.10 and 2.24 and to analyze and synthesize non-linear cooperative systems.

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