2

State Feedback Control

If a system is controllable all eigenvalues can be placed at arbitrary locations by static state feedback. Therefore, state feedback control can be used to stabilize the closed-loop system and to achieve further design specifications. Hence, the parameterization of state feedback controllers is an important aspect of control theory.

In a system of the order n with one input u (*i.e.*, a single-input system), the desired eigenvalues of the closed-loop system specify all n elements of the $1 \times n$ state feedback gain k^T. If the system has $p > 1$ inputs (*i.e.*, in a multiple-input system), the desired eigenvalues again specify n elements of the $p \times n$ state feedback gain K. Therefore, after the assignment of the eigenvalues there remain $(p-1)n$ degrees of freedom parameterizing various properties of the closed-loop system as, *e.g.*, the zeros in the elements of its transfer matrix.

In the frequency domain, the state feedback is parameterized by a polynomial matrix $\tilde{D}(s)$. In the single-input case, all free coefficients in $\tilde{D}(s)$ are specified by the desired eigenvalues of the closed-loop system, whereas in the multiple-input case additional $(p-1)n$ degrees of freedom also exist.

If the system described by its transfer behaviour has n linearly independent measurable outputs, a pole-placing output feedback controller can be computed from the parameterizing polynomial matrix $\tilde{D}(s)$ without recourse to any state-space description.

The main results related to a time-domain representation of state feedback control are summarized briefly in Section 2.1. A more comprehensive treatment of the subject can be found in many text books on the control design of linear systems as, *e.g.*, [36]. Section 2.2 describes the frequency-domain design of state feedback control. It is based on the right coprime MFD of the system, and it is parameterized by the coefficients of the denominator matrix $\tilde{D}(s)$. It is shown that this polynomial matrix contains the same number of free parameters as the state feedback gain K. A connecting relation is also derived that, given a time-domain parameterization of state feedback, enables the designer to compute an equivalent frequency-domain parameterization and *vice versa*.

2.1 State Feedback in the Time Domain

Considered is the control of linear, time-invariant systems with the state equations

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

$$y(t) = Cx(t), \tag{2.2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, and $y \in \mathbb{R}^m$ with $m \geq p$ is the measurement. It is assumed that the inputs and outputs are linearly independent, *i.e.*, rank $B = p$ and rank $C = m$.

The controlled output $y_c \in \mathbb{R}^p$ is assumed to be measurable. Therefore, a $p \times m$ selection matrix

$$\varXi = \begin{bmatrix} e_{i_1}^T \\ e_{i_2}^T \\ \vdots \\ e_{i_p}^T \end{bmatrix}, \; i_\nu \in \{1, 2, \ldots, m\}, \; \nu = 1, 2, \ldots, p \tag{2.3}$$

exists with e_{i_ν} denoting the i_νth unit vector, so that

$$y_c(t) = \varXi y(t) = \varXi Cx(t) = C_c x(t), \tag{2.4}$$

i.e., one has $y_c^1 = y^{i_1}$, ..., $y_c^p = y^{i_p}$, where y^j denotes the jth element of the vector y.

Remark 2.1. In view of designing reduced-order observers the $m \times 1$ vector y of the measurements will be subdivided into an $(m - \kappa) \times 1$ vector y_1 and a $\kappa \times 1$ vector y_2 in many parts of this book. Therefore, the unusual notion y^j is adopted for the jth component of y.

On the one hand, only the controllable and observable part of a system can be influenced by output feedback (as, for example, observer-based) control. On the other hand, the coprime MFDs of the system exactly describe its controllable and observable part. Therefore, the time-domain representation (C, A, B) of the model of the system used for controller design is always assumed to be such that the pair (A, B) is controllable and the pair (C, A) is observable (see also Remark 1.1).

The stabilizing state feedback has the form

$$u(t) = -Kx(t) + Mr(t), \tag{2.5}$$

where $r \in \mathbb{R}^p$ is the reference input. In (2.5) the $p \times n$ feedback gain K must be chosen such that the eigenvalues $\tilde{s}_\nu$, $\nu = 1, 2, \ldots, n$, of the controlled system, *i.e.*, the zeros of the characteristic polynomial $\det(sI - A + BK)$,

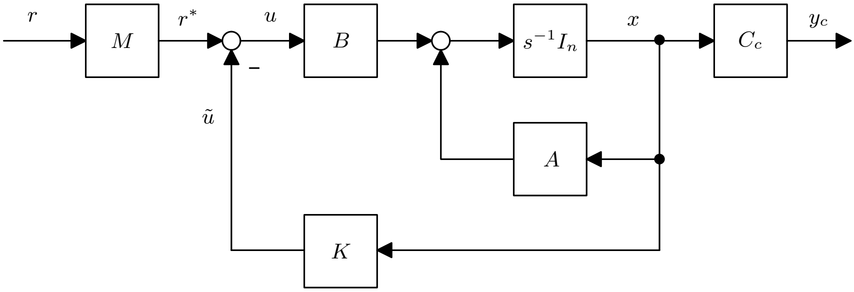

Figure 2.1. State feedback control in the time domain

are located in the complex open left-half plane. The remaining $(p-1)n$ free parameters in K also influence the properties of the closed-loop system. They can, for example, be exploited to assure a reference transfer behaviour that is decoupled (see also Chapter 6).

By applying the control (2.5) to (2.1) and (2.2) the closed-loop system is described by the state equations

$$\dot{x}(t) = (A - BK)x(t) + BMr(t), \tag{2.6}$$

$$y_c(t) = C_c x(t). \tag{2.7}$$

A block diagram representation of the closed-loop system (2.6) and (2.7) is shown in Figure 2.1.

A reference transfer behaviour

$$y_c(s) = C_c(sI - A + BK)^{-1}BMr(s) = G_r(s)r(s) \tag{2.8}$$

with vanishing steady-state error $y_c(\infty) - r(\infty)$ for stationary constant reference signals (*i.e.*, $r(\infty) = const$) requires $G_r(0) = I$. For all stabilizing gains K, *i.e.*, the inverse of $A - BK$ exists, this is assured by the constant $p \times p$ matrix

$$M = \left[C_c(-A + BK)^{-1}B\right]^{-1} \tag{2.9}$$

(see (2.5)). The matrix in square brackets in (2.9) has full rank iff the system (C_c, A, B) has no invariant zero at $s = 0$. This can be shown by elementary operations applied to Rosenbrock's system matrix (see [58]) defined for the system (2.6) and (2.7). In the following, it will always be assumed that no invariant zero of the system (C_c, A, B) is located at $s = 0$.

2.2 Parameterization of the State Feedback in the Frequency Domain

The system (2.1) and (2.2) is now assumed to be described by its transfer behaviour $y(s) = G(s)u(s)$, where the $m \times p$ transfer matrix $G(s)$ is related to the time-domain quantities by

$$G(s) = C(sI - A)^{-1}B. \tag{2.10}$$

The transfer matrix $G(s)$ of the system is represented with the aid of an $m \times p$ numerator matrix $N(s)$ and a $p \times p$ denominator matrix $D(s)$ by a right coprime MFD

$$G(s) = N(s)D^{-1}(s). \tag{2.11}$$

The controlled output $y_c = \Xi y$ (see also (2.4)) is characterized by its transfer behaviour $y_c(s) = G_c(s)u(s)$ and the transfer matrix $G_c(s)$ is represented by the right MFD

$$G_c(s) = N_c(s)D^{-1}(s), \tag{2.12}$$

where $N_c(s)$ is a $p \times p$ polynomial matrix. This polynomial matrix has the form $N_c(s) = \Xi N(s)$ (see also (2.4)).

Remark 2.2. In general, the MFD (2.12) need not be coprime, *i.e.*, the system can be such that the order of $y_c(s) = G_c(s)u(s)$ is smaller than the order of $y(s) = G(s)u(s)$. A right coprime MFD of $G_c(s)$ then has the form $G_c(s) = N_c^*(s)D^{*-1}(s)$ with $\deg(\det D^*(s)) < \deg(\det D(s))$. Throughout this book, the MFD of $G_c(s)$ is assumed to have the form (2.12) with $D(s)$ as defined in (2.11) and $N_c(s) = \Xi N(s)$.

The MFDs (2.11) and (2.12) are assumed to be such that $D(s)$ is column reduced. The poles of $G(s)$ are the zeros of $\det D(s)$ and the invariant zeros of the system (2.1)–(2.2) are the zeros of $\det N_c(s)$. If the MFD (2.12) is coprime the zeros of $\det N_c(s)$ are the transmission zeros of $G_c(s)$.

In the discussion of state feedback control it is always assumed that the state x of the system is directly measurable. Therefore, the right coprime MFD in the transfer behaviour

$$x(s) = N_x(s)D^{-1}(s)u(s) = (sI - A)^{-1}Bu(s) \tag{2.13}$$

is also considered with $D(s)$ as defined in (2.11) and $N_x(s)$ being an $n \times p$ polynomial matrix. Obviously, the equalities

$$N(s) = CN_x(s) \;\text{ and }\; N_c(s) = C_cN_x(s) \tag{2.14}$$

then also hold.

Using the MFDs (2.12) and (2.13), the transfer behaviour of the system in Figure 2.1 can be equally represented by the block diagram in Figure 2.2 with a feedback

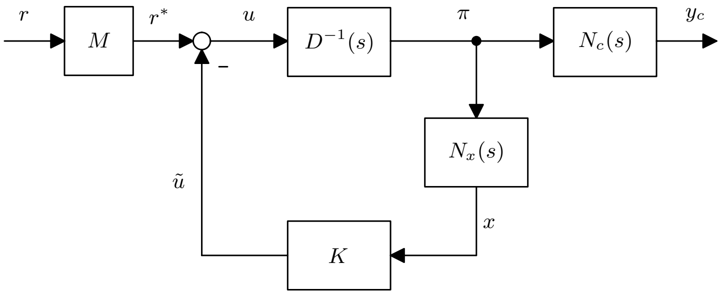

Figure 2.2. State feedback control in the frequency domain

$$\tilde{u}(s) = KN_x(s)\pi(s) \tag{2.15}$$

of the partial state π (see (1.29)).

In the closed-loop system of Figure 2.1 the transfer behaviour between the modified reference input r^* and the controlled output y_c is

$$y_c(s) = C_c(sI - A + BK)^{-1}Br^*(s). \tag{2.16}$$

In the closed-loop system of Figure 2.2 this transfer behaviour has the form

$$y_c(s) = N_c(s)\left[D(s) + KN_x(s)\right]^{-1} r^*(s). \tag{2.17}$$

Thus, the $p \times p$ denominator matrix

$$\tilde{D}(s) = D(s) + KN_x(s) \tag{2.18}$$

in the MFD (2.17) characterizes the dynamics of the closed-loop system in Figure 2.2.

By using the substitution (2.18) the Equations (2.16) and (2.17) can be written as

$$y_c(s) = \frac{C_c\mathrm{adj}\{sI - A + BK\}B}{\det(sI - A + BK)}\, r^*(s) = \frac{N_c(s)\mathrm{adj}\{\tilde{D}(s)\}}{\det \tilde{D}(s)}\, r^*(s), \tag{2.19}$$

where adj$\{\cdot\}$ denotes the adjoint of a matrix. This shows that

$$\det \tilde{D}(s) = \det(sI - A + BK). \tag{2.20}$$

By inserting (2.18) and $r^* = Mr$ in (2.17) one obtains the reference transfer behaviour

$$y_c(s) = G_r(s)r(s) = N_c(s)\tilde{D}^{-1}(s)Mr(s) \tag{2.21}$$

of the state feedback control in the frequency domain. This also shows that the invariant zeros of the system (*i.e.*, the zeros of $\det N_c(s)$) are not influenced by state feedback.

A vanishing steady-state error $y_c(\infty) - r(\infty)$ for stationary constant reference signals (*i.e.*, $r(\infty) = const$) is assured by $G_r(0) = I$ which yields

$$M = \tilde{D}(0)N_c^{-1}(0), \tag{2.22}$$

in view of (2.21). The matrix M in (2.22) only exists if $\det N_c(s)$ has no root at $s = 0$ and this is the frequency-domain equivalent to the corresponding results in Section 2.1.

Given a frequency-domain parameterization of the state feedback control by $\tilde{D}(s)$, the Relation (2.18) can be used to obtain an equivalent feedback gain K or *vice versa*. When multiplying (2.18) from the right by $D^{-1}(s)$ and using (2.13) one obtains the well-known connecting relation

$$\tilde{D}(s)D^{-1}(s) = I + K(sI - A)^{-1}B \tag{2.23}$$

between the time- and the frequency-domain representations of state feedback control (see, *e.g.*, [2, 36]).

In the following discussion of the properties of $\tilde{D}(s)$ the notion of the polynomial part of a rational matrix is required. Recall that given a rational matrix $G(s)$ and its corresponding limit value $G_\infty = \lim_{s\to\infty} G(s)$, this matrix is called *strictly proper* if $G_\infty = 0$, *proper* if G_∞ is finite and not vanishing, and *improper* if at least one element of G_∞ is not finite.

Definition 2.1 (Polynomial part and strictly proper part of a rational matrix). *Any given transfer matrix $G(s)$ can be represented as*

$$G(s) = \Pi\{G(s)\} + SP\{G(s)\}, \tag{2.24}$$

where the polynomial part $\Pi\{\cdot\}$ *is a polynomial matrix and the* strictly proper part $SP\{\cdot\}$ *is a strictly proper rational matrix.*

Therefore, in a proper rational matrix $G(s)$ the polynomial part $\Pi\{G(s)\}$ coincides with the direct feedthrough G_∞ from the input u to the output y (see also (1.22) and (1.75)).

Example 2.1. Decomposition of a rational matrix into its polynomial and strictly proper parts

For the rational matrix

$$G(s) = \frac{1}{s+1}\begin{bmatrix} s+3 & 2s+4 \\ 3 & s^2+3s+2 \end{bmatrix} \tag{2.25}$$

the polynomial part is given by

$$\Pi\{G(s)\} = \begin{bmatrix} 1 & 2 \\ 0 & s+2 \end{bmatrix}, \tag{2.26}$$

and the strictly proper part by

$$SP\{G(s)\} = \frac{1}{s+1}\begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}. \tag{2.27}$$

In the time domain the state feedback is parameterized by the $p \times n$ feedback gain K, which contains pn free parameters. The following theorem shows that the $p \times p$ polynomial matrix $\tilde{D}(s)$ plays the corresponding role in the frequency domain.

Theorem 2.1 (Parameterizing polynomial matrix of state feedback). *The $p \times p$ polynomial matrix $\tilde{D}(s)$ characterizing the dynamics of the state feedback loop in the frequency domain has the properties*

$$\delta_{ci}[\tilde{D}(s)] = \delta_{ci}[D(s)], \quad i = 1, 2, \ldots, p, \tag{2.28}$$

and

$$\mathit{\Gamma}_c[\tilde{D}(s)] = \mathit{\Gamma}_c[D(s)], \tag{2.29}$$

and it has exactly the same number of free parameters as the state feedback gain K, namely pn.

Proof. Since $K(sI - A)^{-1}B$ is strictly proper the right-hand side of (2.23) indicates that the polynomial part $\mathit{\Pi}\{\tilde{D}(s)D^{-1}(s)\}$ is the identity matrix. This implies that $\tilde{D}(s)D^{-1}(s)$ is proper and as $D(s)$ is column reduced (*i.e.*, $\det \mathit{\Gamma}_c[D(s)] \neq 0$) this further implies that $\delta_{ci}[\tilde{D}(s)] \leq \delta_{ci}[D(s)]$, $i = 1, 2, \ldots, p$ (see Section 1.1). Applying (1.22) one obtains

$$\mathit{\Pi}\{\tilde{D}(s)D^{-1}(s)\} = \mathit{\Gamma}_{\delta_c[D(s)]}[\tilde{D}(s)]\mathit{\Gamma}_c^{-1}[D(s)] = I. \tag{2.30}$$

Postmultiplying this by $\mathit{\Gamma}_c[D(s)]$ leads to

$$\mathit{\Gamma}_{\delta_c[D(s)]}[\tilde{D}(s)] = \mathit{\Gamma}_c[D(s)]. \tag{2.31}$$

Since $D(s)$ is column reduced also $\det \mathit{\Gamma}_{\delta_c[D(s)]}[\tilde{D}(s)] \neq 0$. This, on the other hand, shows that (2.28) is satisfied, because if $\delta_{ci}[\tilde{D}(s)] < \delta_{ci}[D(s)]$ for any i, the corresponding column of $\mathit{\Gamma}_{\delta_c[D(s)]}[\tilde{D}(s)]$ would vanish, contradicting $\det \mathit{\Gamma}_{\delta_c[D(s)]}[\tilde{D}(s)] \neq 0$. With (2.28) satisfied, (2.29) directly follows from (2.31).

Since $D(s)$ is column reduced the order of the system is

$$n = \sum_{i=1}^{p} \delta_{ci}[D(s)] = \sum_{i=1}^{p} \delta_{ci}[\tilde{D}(s)] \tag{2.32}$$

in view of (2.28). Because of (2.29) the number n_{fi} of free parameters in each column of $\tilde{D}(s)$ is $n_{fi} = p\delta_{ci}[\tilde{D}(s)]$ and consequently, the total number n_f of free parameters in $\tilde{D}(s)$ is

$$n_f = \sum_{i=1}^{p} p\, \delta_{ci}[\tilde{D}(s)] = pn \tag{2.33}$$

in view of (2.32). □

Remark 2.3. Representing $\tilde{D}(s)$, which has the properties (2.28) and (2.29), in the form

$$\tilde{D}(s) = \Gamma_c[D(s)]\mathrm{diag}\left(s^{\delta_{ci}[D(s)]}\right) + \tilde{D}_c S(s), \tag{2.34}$$

with

$$S(s) = \mathrm{diag}\left([s^{\delta_{c1}[D(s)]-1} \dots s\ 1]^T, \dots, [s^{\delta_{cp}[D(s)]-1} \dots s\ 1]^T\right) \tag{2.35}$$

it becomes obvious that the degrees of freedom in $\tilde{D}(s)$ are contained in the constant matrix $\tilde{D}_c$ of freely assignable coefficients in (2.34). It has the same dimensions as K, *i.e.*, it is a $p \times n$ matrix since $\sum_{i=1}^{p} \delta_{ci}[D(s)] = n$ (see (2.32)).

Therefore, the linear state feedback control can either be parameterized in the time domain by the $p \times n$ constant matrix K or in the frequency domain by the $p \times p$ polynomial matrix $\tilde{D}(s)$. In both cases, the same number pn of free parameters exists.

Given a set of desired eigenvalues of the closed-loop system, the parameterization via $\tilde{D}(s)$ or K is uniquely defined in the case of single-input systems. For multiple-input systems, there exist $(p-1)n$ additional degrees of freedom in $\tilde{D}(s)$ and K, so that the parameterization of a state feedback control yielding desired eigenvalues of the closed-loop system is more complicated in this case.

Remark 2.4. In the frequency domain there exists a simple way to solve the eigenvalue-assignment problem by using the parameterizing matrix

$$\tilde{D}(s) = \Gamma_c[D(s)] \begin{bmatrix} \tilde{d}_1(s) & & \\ & \ddots & \\ & & \tilde{d}_p(s) \end{bmatrix}, \tag{2.36}$$

with

$$\deg \tilde{d}_i(s) = \delta_{ci}[D(s)], \quad i = 1, 2, \dots, p, \tag{2.37}$$

so that one obtains

$$\det \tilde{D}(s) = \tilde{d}_1(s) \cdot \dots \cdot \tilde{d}_p(s). \tag{2.38}$$

More degrees of freedom can possibly be used when choosing the free coefficients in $\tilde{D}(s)$ in an appropriate way, so that (2.38) is satisfied for given polynomials $\tilde{d}_i(s)$ (see, *e.g.*, Example 2.2). If all degrees of freedom need to be used, the parametric approach presented in Chapter 5 can be used.

Equation (2.18) is a special form of the Diophantine equation

$$P(s)N_x(s) + Q(s)D(s) = \tilde{D}(s) \tag{2.39}$$

with polynomial matrices $P(s)$ and $Q(s)$. A solution of this Diophantine equation is given by the pair of constant matrices

$$P(s) = P = K \text{ and } Q(s) = Q = I, \tag{2.40}$$

which becomes obvious by comparison of (2.39) and (2.18).

Remark 2.5. If the pair $(N_x(s), D(s))$ is right coprime (*i.e.*, if the pair (A, B) is controllable) the Diophantine equation (2.39) can be solved for arbitrary right-hand sides $\tilde{D}(s)$ having the properties (2.28) and (2.29), *i.e.*, the dynamics of the closed-loop system can be assigned arbitrarily. If a controllability defect occurs, the matrices $N_x(s)$ and $D(s)$ have a greatest common right divisor $R(s)$, which is no longer a unimodular matrix. Then, the Diophantine equation (2.39) takes the form

$$P(s)N_x^*(s)R(s) + Q(s)D^*(s)R(s) = \tilde{D}(s), \tag{2.41}$$

and this makes it obvious that a solution only exists if the polynomial matrix $\tilde{D}(s)$ also has the form $\tilde{D}(s) = \tilde{D}^*(s)R(s)$. This immediately shows which part of the system dynamics cannot be changed by state feedback control.

If n linearly independent measured outputs y exist, an arbitrary assignment of all eigenvalues of the closed-loop system is also possible by an output feedback $u = -Py$ with the constant feedback gain P. This output feedback can be directly computed in the frequency domain without recourse to a time-domain representation of the system by replacing the polynomial matrix $N_x(s)$ by $N(s)$ in the Diophantine equation (2.39).

Example 2.2. Parameterization of a state feedback control in the frequency domain

Considered is a system with two inputs u and three measured outputs y. The right coprime MFD of its transfer matrix $G(s)$ has the form

$$G(s) = N(s)D^{-1}(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s^2+3s+3 & -1 \\ -1 & s+3 \end{bmatrix}^{-1}. \tag{2.42}$$

The characteristic polynomial of the system is $\det D(s) = (s+2)^3$, *i.e.*, it has the order $n = 3$ and its three poles are located at $s = -2$. As no constant left annihilator $N^{\perp}$ of $N(s)$ exists the $m = 3$ outputs are linearly independent, which means that no output can be represented as a linear combination of the other two.

It is assumed that the controlled outputs y_c coincide with the first two measurements. Therefore, the selection matrix in $y_c = \Xi y$ (see (2.4)) is

$$\Xi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{2.43}$$

giving the corresponding numerator matrix

$$N_c(s) = \Xi N(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}. \tag{2.44}$$

Since $\det N_c(s) = 1$ the transfer matrix $G_c(s) = N_c(s)D^{-1}(s)$ has no zero.

The control is intended to place all three eigenvalues of the system at $\tilde{s}_i = -5$, $i = 1, 2, 3$, and since n linearly independent measurements exist, it can be realized by an output feedback

$$u(s) = -Py(s) + Mr(s). \tag{2.45}$$

The column degrees of $D(s)$ are $\delta_{c1}[D(s)] = 2$, $\delta_{c2}[D(s)] = 1$ and the highest column-degree-coefficient matrix is $\Gamma_c[D(s)] = I$. Therefore, the parameterizing matrix $\tilde{D}(s)$ has the general form

$$\tilde{D}(s) = \begin{bmatrix} s^2 + \alpha s + \beta & \gamma \\ \delta s + \varepsilon & s + \varphi \end{bmatrix}, \tag{2.46}$$

i.e., there are $pn = 6$ free parameters α, β, γ, δ, ε, and φ.

A choice of $\alpha = 10$, $\beta = 25$, $\gamma = 0$, $\delta = 2$, $\varepsilon = 10$, and $\varphi = 5$ assures the desired characteristic polynomial $\det \tilde{D}(s) = (s + 5)^3$ and leads to

$$\tilde{D}(s) = \begin{bmatrix} s^2 + 10s + 25 & 0 \\ 2s + 10 & s + 5 \end{bmatrix}. \tag{2.47}$$

If vanishing steady-state errors for constant reference inputs are required this is assured by the constant matrix

$$M = \tilde{D}(0)N_c^{-1}(0) = \begin{bmatrix} 25 & 0 \\ 10 & 5 \end{bmatrix}. \tag{2.48}$$

In order to obtain the feedback gain P in (2.45) the Diophantine equation (2.39) with $N_x(s) = N(s)$, *i.e.*,

$$P(s)N(s) + Q(s)D(s) = \tilde{D}(s) \tag{2.49}$$

has to be solved. By inserting the above matrices $N(s)$, $D(s)$ and $\tilde{D}(s)$ one obtains $Q(s) = Q = I$ and

$$P(s) = P = \begin{bmatrix} 22 & 7 & -6 \\ 11 & 2 & 0 \end{bmatrix}. \tag{2.50}$$

To assign the desired eigenvalues of the closed-loop system it is obviously much easier to look for a polynomial matrix $\tilde{D}(s)$ in the frequency domain than to determine K from $\det(sI - A + BK) = \prod_{i=1}^{n}(s - \tilde{s}_i)$ in a time-domain approach.

http://www.springer.com/978-1-84882-536-9

Design of Observer-based Compensators
From the Time to the Frequency Domain
Hippe, P.; Deutscher, J.
2009, XIII, 285 p., Hardcover
ISBN: 978-1-84882-536-9