

## Chapter 2

# Approaches to Robust $\mathcal{H}_\infty$ Controller Synthesis of Nonlinear Discrete-time-delay Systems via Takagi-Sugeno Fuzzy Models

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**Abstract** This chapter investigates the problem of robust  $\mathcal{H}_\infty$  piecewise state-feedback control for a class of nonlinear discrete-time-delay systems via Takagi-Sugeno (T-S) fuzzy models. The state delay is assumed to be time-varying and of an interval-like type with the lower and upper bounds. The parameter uncertainties are assumed to have a structured linear-fractional form. Based on two novel piecewise Lyapunov-Krasovskii functionals and some matrix inequality convexifying techniques, both delay-independent and delay-dependent controller design approaches are developed in terms of a set of linear matrix inequalities (LMIs). Numerical examples are also provided to illustrate the effectiveness and less conservatism of the proposed methods.

## 2.1 Introduction

Fuzzy logic control is a simple and effective approach to the control of many complex nonlinear systems [1–7]. During the past decades, a great number of industrial applications of fuzzy logic control have been reported in the open literature [1–4]. Among various model-based fuzzy control approaches, in particular, the method

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based on Takagi-Sugeno (T-S) model is well suited to model-based nonlinear control [5–7]. A T-S fuzzy model consists of a set of local linear models in different premise variable space regions which are blended through fuzzy membership functions.

During the past few years, significant research efforts have been devoted to the stability analysis and controller design of T-S fuzzy systems. The reader can refer to the survey papers [8, 9] and the references cited therein for the most recent advances on this topic. The appeal of T-S fuzzy models is that the stability analysis and controller synthesis of the overall fuzzy systems can be carried out in the Lyapunov function based framework. It has been shown in [5–11] that the stability and stabilization of the T-S fuzzy systems can be determined by searching for a common positive definite Lyapunov matrix. However, there are many fuzzy systems that do not admit a common Lyapunov function, whereas they may still be asymptotically stable [12, 14]. Recently, there have appeared some results on stability analysis and controller synthesis of fuzzy systems based on piecewise/fuzzy Lyapunov functions instead of a common Lyapunov function [12–20]. It has been shown that piecewise/fuzzy Lyapunov functions are much richer classes of Lyapunov function candidates than a common Lyapunov function candidate, and thus are able to deal with a larger class of fuzzy dynamic systems. The analysis and design results based on piecewise/fuzzy Lyapunov functions are generally less conservative than those based on a common Lyapunov function.

On the other hand, it is well-known that time-delays are frequently encountered in various complex nonlinear systems, such as chemical systems, mechanical systems, and communication networks. It has been well recognized that the presence of time-delays may result in instability, chaotic mode, and/or poor performance of control systems [21–27]. Control of dynamic systems with time-delays is a research subject of great practical and theoretical significance, which has received considerable attention in the past decades. Over the past few years, increasing attention has been drawn to the study of stability analysis and controller design for T-S fuzzy systems with time-delays [28–43]. In the context of discrete-time T-S fuzzy delay systems, some sufficient conditions for the solvability of stability analysis, state-feedback/output-feedback stabilization/ $\mathcal{H}_\infty$  control problems were obtained in [28, 30, 31] in terms of LMIs. It is known that since the delay-dependent conditions include the information on the size of delays, they are usually less conservative than the delay-independent ones, especially when the size of delays is small. To reduce the design conservatism, recently, based on a piecewise Lyapunov-Krasovskii functional, the authors in [32] studied the problems of delay-dependent stability analysis and controller synthesis for a class of discrete-time T-S fuzzy systems with time-delays and it was shown that the obtained delay-dependent results are less conservative than the existing delay-independent ones. However, it is noted that the delay-dependent approaches presented in [32] suffer from a couple of drawbacks. Firstly, the delay-dependent criteria was realized by utilizing a system model transformation technique incorporating *Moon's bounding inequality* [21] to estimate the inner product of the involved crossing terms, which would lead to significant conservatism. Secondly, the results given in [32] only considered the case of

time-invariant delays. Those results are unfortunately not applicable to the case of time-varying delays. In addition, the system parameter uncertainties were not taken into account in [32]. These motivate the present research.

In this chapter, we revisit the problem of robust  $\mathcal{H}_\infty$  piecewise state-feedback control for a class of nonlinear discrete-time-delay systems via T-S fuzzy models. The state delay is assumed to be time-varying and of an interval-like type with the lower and upper bounds. The parameter uncertainties are assumed to have a structured linear-fractional form. Based on two novel piecewise Lyapunov-Krasovskii functionals combined with some matrix inequality convexifying techniques, both delay-independent and delay-dependent controller design approaches have been developed. It is shown that the controller gains can be obtained by solving a set of LMIs. Two simulation examples are also provided to illustrate the effectiveness and less conservatism of the proposed methods.

The rest of this chapter is structured as follows. Section 2.2 is devoted to the model description and problem formulation. The main results for robust  $\mathcal{H}_\infty$  piecewise controller designs are given in Section 2.3. Two simulation examples are presented in Section 2.4 to demonstrate the applicability of the proposed approaches. In Section 2.5, some conclusions are presented.

The notations used throughout this chapter are standard.  $Z^+$  denotes the set of nonnegative integer number.  $\mathfrak{R}^n$  denotes the  $n$ -dimensional Euclidean space.  $\mathfrak{R}^{n \times m}$  denotes the set of  $n \times m$  real matrices. A real symmetric matrix  $P > 0 (\geq 0)$  denotes  $P$  being positive definite (or being positive semi-definite). For a matrix  $A \in \mathfrak{R}^{n \times n}$ ,  $A^{-1}$  and  $A^T$  are the inverse and transpose of the matrix  $A$ , respectively, and  $A^{-T}$  denotes  $(A^{-1})^T$ .  $\text{Sym}\{A\}$  is the shorthand notation for  $A + A^T$ .  $\mathbf{I}_n$  and  $\mathbf{0}_{n \times m}$  are used to denote the  $n \times n$  identity matrix and  $n \times m$  zero matrix, respectively. The subscripts  $n$  and  $n \times m$  are omitted when the size is not relevant or can be determined from the context.  $\text{diag}\{\dots\}$  denotes a block-diagonal matrix. The notation  $\star$  in a symmetric matrix always denotes the symmetric block in the matrix.  $l_2[0, \infty)$  refers to the space of square summable infinite vector sequences with the Euclidean norm  $\|\bullet\|_2$ . Matrices, if not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

## 2.2 Model Description and Robust $\mathcal{H}_\infty$ Piecewise Control Problem

The T-S fuzzy dynamic model is described by fuzzy IF-THEN rules, each of which represents the local linear input-output relationship of nonlinear systems. Similar to [2–7], a discrete-time T-S fuzzy dynamic model with time-delay and parametric uncertainties can be described as

**Plant Rule  $R^l$ : IF  $\zeta_1(t)$  is  $F_1^l$  and  $\zeta_2(t)$  is  $F_2^l$  and  $\dots$  and  $\zeta_g(t)$  is  $F_g^l$ , THEN**

$$\begin{cases} x(t+1) = A_l(t)x(t) + A_{dl}(t)x(t-\tau(t)) + B_{1l}(t)u(t) + D_{1l}(t)w(t) \\ z(t) = C_l(t)x(t) + C_{dl}(t)x(t-\tau(t)) + B_{2l}(t)u(t) + D_{2l}(t)w(t) \\ x(t) = \phi(t), -\tau_2 \leq t \leq 0, l \in \mathcal{L} := \{1, 2, \dots, r\} \end{cases} \quad (2.1)$$

where  $R^l$  denotes the  $l$ th fuzzy inference rule;  $r$  is the number of inference rules;  $F_q^l (q = 1, 2, \dots, g)$  are fuzzy sets;  $x(t) \in \mathfrak{R}^{n_x}$  is the system state;  $u(t) \in \mathfrak{R}^{n_u}$  is the control input;  $z(t) \in \mathfrak{R}^{n_z}$  is the regulated output;  $w(t) \in \mathfrak{R}^{n_w}$  is the disturbance input which is assumed to belong to  $l_2[0, \infty)$ ;  $\zeta(t) := [\zeta_1(t), \zeta_2(t), \dots, \zeta_g(t)]$  are some measurable variables of the system, for example, the state variables;  $\tau(t)$  is a positive integer function representing the time-varying state delay of the system (2.1) and satisfying the following assumption

$$\tau_1 \leq \tau(t) \leq \tau_2 \quad (2.2)$$

with  $\tau_1$  and  $\tau_2$  being two constant positive integers representing the minimum and maximum time-delay, respectively. In this case,  $\tau(t)$  is called an *interval-like* or *range-like* time-varying delay [24, 25, 37, 43]. It is noted that this kind of time-delay describes the real situation in many practical engineering systems. For example, in the field of networked control systems, the network transmission induced delays (either from the sensor to the controller or from the controller to the plant) can be assumed to satisfy (2.2) without loss of generality [26, 27].  $\phi(t)$  is a real-valued initial condition sequence on  $[-\tau_2, 0]$ .  $A_l(t)$ ,  $A_{dl}(t)$ ,  $B_{1l}(t)$ ,  $B_{2l}(t)$ ,  $C_l(t)$ ,  $C_{dl}(t)$ ,  $D_{1l}(t)$  and  $D_{2l}(t)$ ,  $l \in \mathcal{L}$  are appropriately dimensioned system matrices with time-varying parametric uncertainties, which are assumed to be of the form

$$\begin{bmatrix} A_l(t) & A_{dl}(t) & B_{1l}(t) & D_{1l}(t) \\ C_l(t) & C_{dl}(t) & B_{2l}(t) & D_{2l}(t) \end{bmatrix} = \begin{bmatrix} A_l & A_{dl} & B_{1l} & D_{1l} \\ C_l & C_{dl} & B_{2l} & D_{2l} \end{bmatrix} + \begin{bmatrix} W_{1l} \\ W_{2l} \end{bmatrix} \Delta(t) \begin{bmatrix} E_{1l} & E_{2l} & E_{3l} & E_{4l} \end{bmatrix} \quad (2.3)$$

$$\Delta(t) = \Lambda(t) [\mathbf{I}_{s_2} - J\Lambda(t)]^{-1} \quad (2.4)$$

$$0 < \mathbf{I}_{s_2} - JJ^T \quad (2.5)$$

where  $A_l$ ,  $A_{dl}$ ,  $B_{1l}$ ,  $B_{2l}$ ,  $C_l$ ,  $C_{dl}$ ,  $D_{1l}$ ,  $D_{2l}$ ,  $W_{1l}$ ,  $W_{2l}$ ,  $E_{1l}$ ,  $E_{2l}$ ,  $E_{3l}$ ,  $E_{4l}$  and  $J$  are known real constant matrices of appropriate dimensions.  $\Lambda(t) : Z^+ \rightarrow \mathfrak{R}^{s_1 \times s_2}$  is an unknown real-valued time-varying matrix function with Lebesgue measurable elements satisfying

$$\Lambda^T(t)\Lambda(t) \leq \mathbf{I}_{s_2}. \quad (2.6)$$

The parameter uncertainties are said to be admissible if (2.3)-(2.6) hold. It is noted that interval bounded parameters can also be utilized to describe uncertain systems. For the discrete-time case, interval model control and applications can be found in [44] and the references therein.

*Remark 2.1.* The parametric uncertainties are assumed to have a structured linear-fractional form. It is noted that this kind of parameter uncertainties has been fairly investigated in the robust control theory [30, 45, 46]. It has been shown that every

rational nonlinear system possesses a linear-fractional representation [46]. Notice that when  $J = \mathbf{0}$ , the linear-fractional form uncertainties reduce to the norm-bounded ones [21, 25, 31, 37]. Notice also that the condition (2.5) guarantees that  $\mathbf{I}_{s_2} - J\Lambda(t)$  is invertible for all  $\Lambda(t)$  satisfying (2.6).

Let  $\mu_l[\zeta(t)]$  be the normalized fuzzy-basis function of the inferred fuzzy set  $F^l$  where  $F^l := \bigcap_{q=1}^g F_q^l$  and

$$\mu_l[\zeta(t)] := \frac{\prod_{q=1}^g \mu_{lq}[\zeta_q(t)]}{\sum_{p=1}^r \prod_{q=1}^g \mu_{pq}[\zeta_q(t)]} \geq 0, \quad \sum_{l=1}^r \mu_l[\zeta(t)] = 1 \quad (2.7)$$

where  $\mu_{lq}[\zeta_q(t)]$  is the grade of membership of  $\zeta_q(t)$  in  $F_q^l$ . In the sequel, we will drop the argument of  $\mu_l[\zeta(t)]$  for clarity, i.e., denote  $\mu_l$  as  $\mu_l[\zeta(t)]$ .

By using a center-average defuzzifier, product fuzzy inference and singleton fuzzifier, the following global T-S fuzzy dynamic model can be obtained

$$\begin{cases} x(t+1) = A(\mu, t)x(t) + A_d(\mu, t)x(t - \tau(t)) + B_1(\mu, t)u(t) + D_1(\mu, t)w(t) \\ z(t) = C(\mu, t)x(t) + C_d(\mu, t)x(t - \tau(t)) + B_2(\mu, t)u(t) + D_2(\mu, t)w(t) \\ x(t) = \phi(t), -\tau_2 \leq t \leq 0 \end{cases} \quad (2.8)$$

where

$$\begin{aligned} A(\mu, t) &:= \sum_{l=1}^r \mu_l A_l(t), A_d(\mu, t) := \sum_{l=1}^r \mu_l A_{dl}(t), B_1(\mu, t) := \sum_{l=1}^r \mu_l B_{1l}(t), \\ B_2(\mu, t) &:= \sum_{l=1}^r \mu_l B_{2l}(t), C(\mu, t) := \sum_{l=1}^r \mu_l C_l(t), C_d(\mu, t) := \sum_{l=1}^r \mu_l C_{dl}(t), \\ D_1(\mu, t) &:= \sum_{l=1}^r \mu_l D_{1l}(t), D_2(\mu, t) := \sum_{l=1}^r \mu_l D_{2l}(t). \end{aligned} \quad (2.9)$$

In this chapter, we consider the robust  $\mathcal{H}_\infty$  control problem for the uncertain fuzzy dynamic model (2.8) based on piecewise Lyapunov-Krasovskii functionals. In order to facilitate the piecewise controller design, we partition the premise variable space  $S \subseteq \mathbb{R}^s$  by the boundaries

$$\partial S_l^v := \{\zeta(t) | \mu_l[\zeta(t)] = 1, 0 \leq \mu_l[\zeta(t) + \delta] < 1, \forall 0 < |\delta| \ll 1, l \in \mathcal{L}\} \quad (2.10)$$

where  $v$  is the set of the face indexes of the polyhedral hull satisfying  $\partial S_l = \cup_v(\partial S_l^v), l \in \mathcal{L}$ .

Then, based on the boundaries (2.10), we can get the induced close polyhedral regions  $\bar{S}_i, i \in \mathcal{I}$  satisfying

$$\bar{S}_i \cap \bar{S}_j = \partial S_l^v, i \neq j, i, j \in \mathcal{I}, l \in \mathcal{L} \quad (2.11)$$

where  $\mathcal{I}$  denotes the set of region indexes. The corresponding open regions are defined as  $S_i, i \in \mathcal{I}$ . It is noted that there are no disjointed interiors in  $S$  and  $\cup_{i \in \mathcal{I}} \bar{S}_i$ .

In each region  $S_i, i \in \mathcal{I}$ , define

$$\mathcal{K}(i) := \{k | \mu_k[\zeta(t)] > 0, k \in \mathcal{L}, \zeta(t) \in \bar{S}_i, i \in \mathcal{I}\} \quad (2.12)$$

and then the global system (2.8) can be expressed by a blending of  $k \in \mathcal{K}(i)$  subsystems

$$\begin{cases} x(t+1) = \mathcal{A}_i(t)x(t) + \mathcal{A}_{di}(t)x(t-\tau(t)) + \mathcal{B}_{1i}(t)u(t) + \mathcal{D}_{1i}(t)w(t) \\ z(t) = \mathcal{C}_i(t)x(t) + \mathcal{C}_{di}(t)x(t-\tau(t)) + \mathcal{B}_{2i}(t)u(t) + \mathcal{D}_{2i}(t)w(t) \\ x(t) = \phi(t), -\tau_2 \leq t \leq 0, \zeta(t) \in S_i, i \in \mathcal{I} \end{cases} \quad (2.13)$$

where

$$\begin{aligned} \mathcal{A}_i(t) &:= \sum_{k \in \mathcal{K}(i)} \mu_k A_k(t), \mathcal{A}_{di}(t) := \sum_{k \in \mathcal{K}(i)} \mu_k A_{dk}(t), \mathcal{B}_{1i}(t) := \sum_{k \in \mathcal{K}(i)} \mu_k B_{1k}(t), \\ \mathcal{D}_{1i}(t) &:= \sum_{k \in \mathcal{K}(i)} \mu_k D_{1k}(t), \mathcal{C}_i(t) := \sum_{k \in \mathcal{K}(i)} \mu_k C_k(t), \mathcal{C}_{di}(t) := \sum_{k \in \mathcal{K}(i)} \mu_k C_{dk}(t), \\ \mathcal{B}_{2i}(t) &:= \sum_{k \in \mathcal{K}(i)} \mu_k B_{2k}(t), \mathcal{D}_{2i}(t) := \sum_{k \in \mathcal{K}(i)} \mu_k D_{2k}(t) \end{aligned} \quad (2.14)$$

with  $0 \leq \mu_k[\zeta(k)] \leq 1, \sum_{k \in \mathcal{K}(i)} \mu_k[\zeta(k)] = 1$ . For each region  $\bar{S}_i$ , the set  $\mathcal{K}(i)$  contains the indexes for the system matrices used in the interpolation within that subspace. For a crisp subspace,  $\mathcal{K}(i)$  contains a single element.

In this chapter, we address the piecewise controller design of fuzzy system (2.13), thus it is noted that when the state of the system transits from the region  $S_i$  to  $S_j$  at the time  $t$ , the dynamics of the system is governed by the dynamics of the region model  $S_i$  at that time. For future use, we also define a new set  $\Omega$  to represent all possible region transitions

$$\Omega := \{(i, j) | \zeta(t) \in \bar{S}_i, \zeta(t+1) \in \bar{S}_j, i, j \in \mathcal{I}\} \quad (2.15)$$

where  $j = i$  when  $\zeta(t)$  stays in the same region  $\bar{S}_i$  and  $j \neq i$  when  $\zeta(t)$  transits from the region  $S_i$  to  $S_j$ .

Now, we consider the following piecewise control law for the fuzzy system (2.13)

$$u(t) = K_i x(t), \zeta(t) \in \bar{S}_i, i \in \mathcal{I}. \quad (2.16)$$

Substituting (2.16) into system (2.13) leads to the following closed-loop dynamics

$$\begin{cases} x(t+1) = \bar{\mathcal{A}}_i(t)x(t) + \bar{\mathcal{A}}_{di}(t)x(t-\tau(t)) + \mathcal{D}_{1i}(t)w(t) \\ z(t) = \bar{\mathcal{C}}_i(t)x(t) + \bar{\mathcal{C}}_{di}(t)x(t-\tau(t)) + \mathcal{D}_{2i}(t)w(t) \\ x(t) = \phi(t), -\tau_2 \leq t \leq 0, \zeta(t) \in S_i, i \in \mathcal{I} \end{cases} \quad (2.17)$$

where

$$\bar{\mathcal{A}}_i(t) := \mathcal{A}_i(t) + \mathcal{B}_{1i}(t)K_i, \bar{\mathcal{C}}_i(t) := \mathcal{C}_i(t) + \mathcal{B}_{2i}(t)K_i. \quad (2.18)$$

The robust  $\mathcal{H}_\infty$  piecewise control problem to be investigated in this chapter is stated as follows.

Given an uncertain state-delayed fuzzy system (2.8) and a scalar  $\gamma > 0$ , design a state-feedback piecewise controller of the form (2.16) such that the closed-loop system (2.17) is robustly asymptotically stable for any fuzzy-basis functions  $\mu[\zeta(t)]$  satisfying (2.7) when  $w(t) = 0$ , and under zero initial conditions, the  $l_2$ -gain between the exogenous input  $w(t)$  and the regulated output  $z(t)$  is less than  $\gamma$ , i.e.,  $\|z\|_2 \leq \gamma \|w\|_2$ , for any nonzero  $w \in l_2[0, \infty)$  and all admissible uncertainties. In this case, the system is said to be robustly asymptotically stable with  $\mathcal{H}_\infty$  performance gamma.

Before ending this section, we introduce the following well-known lemmas, which will be used in the derivation of our main results.

**Lemma 2.1.** [45] (*S-procedure*). Suppose that  $\Delta(t)$  is given by (2.3)-(2.6), with matrices  $M = M^T$ ,  $S$  and  $N$  of appropriate dimensions, the inequality

$$M + \text{Sym}\{S\Delta(t)N\} < 0$$

holds if and only if for some scalar  $\varepsilon > 0$

$$M + [\varepsilon^{-1}N^T \quad \varepsilon S] \begin{bmatrix} \mathbf{I} & -J \\ -J^T & \mathbf{I} \end{bmatrix}^{-1} [\varepsilon^{-1}N^T \quad \varepsilon S]^T < 0.$$

**Lemma 2.2.** [45] (*Schur Complements*). Given constant matrices  $\Sigma_1, \Sigma_2, \Sigma_3$  with  $\Sigma_1 = \Sigma_1^T$  and  $0 < \Sigma_2 = \Sigma_2^T$ , then  $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$  if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.$$

## 2.3 Piecewise $\mathcal{H}_\infty$ Control of T-S Fuzzy Systems with Time-delay

In this section, based on two novel piecewise Lyapunov-Krasovskii functionals combined with some matrix inequality convexifying techniques, both delay-independent and delay-dependent approaches will be developed to solve the robust  $\mathcal{H}_\infty$  control problem formulated in the previous section. It is shown that the controller gains can be obtained by solving a set of LMIs.

### 2.3.1 Delay-independent $\mathcal{H}_\infty$ Controller Design

**Theorem 2.1.** The closed-loop system (2.17) is robustly asymptotically stable with  $\mathcal{H}_\infty$  performance  $\gamma$  if there exist matrices  $0 < U_i = U_i^T \in \mathbb{R}^{n_x \times n_x}$ ,  $G_i \in \mathbb{R}^{n_x \times n_x}$ ,  $K_i \in \mathbb{R}^{n_u \times n_x}$ ,  $i \in \mathcal{I}$ ,  $0 < \bar{Q}_1 = \bar{Q}_1^T \in \mathbb{R}^{n_x \times n_x}$ , and a set of positive scalars  $\varepsilon_{kij} > 0$ ,  $k \in \mathcal{K}(i)$ ,  $(i, j) \in \Omega$  such that the following LMIs hold

$$\begin{bmatrix}
-U_j & \mathbf{0} & A_k G_i + B_{1k} \bar{K}_i & A_{dk} \bar{Q}_1 & D_{1k} & \mathbf{0} & \mathbf{0} & \varepsilon_{kij} W_{1k} \\
\star & -\mathbf{I}_{n_z} & C_k G_i + B_{2k} \bar{K}_i & C_{dk} \bar{Q}_1 & D_{2k} & \mathbf{0} & \mathbf{0} & \varepsilon_{kij} W_{2k} \\
\star & \star & U_i - G_i - G_i^T & \mathbf{0} & \mathbf{0} & G_i^T & G_i^T E_{1k}^T + \bar{K}_i^T E_{3k}^T & \mathbf{0} \\
\star & \star & \star & -\bar{Q}_1 & \mathbf{0} & \mathbf{0} & Q_1 E_{2k}^T & \mathbf{0} \\
\star & \star & \star & \star & -\gamma^2 \mathbf{I}_{n_w} & \mathbf{0} & E_{4k}^T & \mathbf{0} \\
\star & \star & \star & \star & \star & \frac{-\bar{Q}_1}{\tau_2 - \tau_1 + 1} & \mathbf{0} & \mathbf{0} \\
\star & \star & \star & \star & \star & \star & -\varepsilon_{kij} \mathbf{I}_{s_2} & \varepsilon_{kij} J \\
\star & \star & \star & \star & \star & \star & \star & -\varepsilon_{kij} \mathbf{I}_{s_1}
\end{bmatrix} < 0.$$

$k \in \mathcal{K}(i), (i, j) \in \Omega$  (2.19)

Moreover, the controller gain for each local region is given by

$$K_i = \bar{K}_i G_i^{-1}, i \in \mathcal{I}. \quad (2.20)$$

*Proof.* It is well-known that it suffices to find a Lyapunov function candidate  $V(t, x(t)) > 0, \forall x(t) \neq 0$  satisfying the following inequality

$$V(t+1, x(t+1)) - V(t, x(t)) + \|z(t)\|_2^2 - \gamma^2 \|w(t)\|_2^2 < 0 \quad (2.21)$$

to prove that the closed-loop system (2.17) is asymptotically stable with  $\mathcal{H}_\infty$  performance  $\gamma$  under zero initial conditions for any nonzero  $w \in l_2[0, \infty)$  and all admissible uncertainties.

Consider the following piecewise Lyapunov-Krasovskii functional

$$\begin{cases}
V(t) := V_1(t) + V_2(t) \\
V_1(t) := x^T(t) P_i x(t), \zeta(t) \in \bar{S}_i, i \in \mathcal{I} \\
V_2(t) := \sum_{s=-\tau_1}^{-\tau_2} \sum_{v=t+s}^{t-1} x^T(v) Q_1 x(v)
\end{cases} \quad (2.22)$$

where  $P_i = P_i^T > 0, i \in \mathcal{I}, Q_1 = Q_1^T > 0$  are Lyapunov matrices to be determined.

Define  $\Delta V(t) := V(t+1) - V(t)$  and along the trajectory of system (2.17), one has

$$\Delta V_1(t) = x^T(t+1) P_j x(t+1) - x^T(t) P_i x(t), (i, j) \in \Omega, \quad (2.23)$$

$$\begin{aligned}
\Delta V_2(t) &= (\tau_2 - \tau_1 + 1) x^T(t) Q_1 x(t) - \sum_{s=t-\tau_2}^{t-\tau_1} x^T(s) Q_1 x(s) \\
&\leq (\tau_2 - \tau_1 + 1) x^T(t) Q_1 x(t) - x^T(t - \tau(t)) Q_1 x(t - \tau(t)).
\end{aligned} \quad (2.24)$$

In addition, it follows from (2.17)

$$x(t+1) = \tilde{\mathcal{A}}_i(t) \xi_1(t), z(t) = \tilde{\mathcal{C}}_i(t) \xi_1(t) \quad (2.25)$$

where



$$\begin{aligned}
\xi_1(t) &:= [x^T(t) \ x^T(t - \tau(t)) \ w^T(t)]^T, \\
\tilde{\mathcal{A}}_i(t) &:= [\bar{\mathcal{A}}_i(t) \ \mathcal{A}_{di}(t) \ \mathcal{D}_{1i}(t)], \\
\tilde{\mathcal{C}}_i(t) &:= [\bar{\mathcal{C}}_i(t) \ \mathcal{C}_{di}(t) \ \mathcal{D}_{2i}(t)].
\end{aligned} \tag{2.26}$$

Then, based on the piecewise Lyapunov-Krasovskii functional defined in (2.22) together with consideration of (2.23)-(2.26), it is easy to see that the following inequality implies (2.21),

$$\xi_1^T \Theta_{ij}(t) \xi_1(t) < 0, \xi_1(t) \neq 0, (i, j) \in \Omega \tag{2.27}$$

where

$$\begin{aligned}
\Theta_{ij}(t) &:= \tilde{\mathcal{A}}_i^T(t) P_j \tilde{\mathcal{A}}_i(t) + \tilde{\mathcal{C}}_i^T(t) \tilde{\mathcal{C}}_i(t) \\
&\quad + \text{diag} \{-P_i + (\tau_2 - \tau_1 + 1) Q_1, -Q_1, -\gamma^2 \mathbf{I}_{n_w}\}.
\end{aligned} \tag{2.28}$$

Thus, if one can show

$$\Theta_{ij}(t) < 0, (i, j) \in \Omega \tag{2.29}$$

then the claimed result follows. To this end, by Schur complements, (2.29) is equivalent to

$$\begin{bmatrix}
-P_j^{-1} & \mathbf{0} & \bar{\mathcal{A}}_i(t) & \mathcal{A}_{di}(t) & \mathcal{D}_{1i}(t) & \mathbf{0} \\
\star & -\mathbf{I}_{n_z} & \bar{\mathcal{C}}_i(t) & \mathcal{C}_{di}(t) & \mathcal{D}_{2i}(t) & \mathbf{0} \\
\star & \star & -P_i & \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_x} \\
\star & \star & \star & -Q_1 & \mathbf{0} & \mathbf{0} \\
\star & \star & \star & \star & -\gamma^2 \mathbf{I}_{n_w} & \mathbf{0} \\
\star & \star & \star & \star & \star & \frac{-Q_1^{-1}}{\tau_2 - \tau_1 + 1}
\end{bmatrix} < 0, (i, j) \in \Omega. \tag{2.30}$$

Now, by introducing a nonsingular matrix  $G_i \in \mathbb{R}^{n_x \times n_x}$  and pre- and post-multiplying (2.30) by  $\text{diag} \{\mathbf{I}_{(n_x+n_z)}, G_i^T, Q_1^{-1}, \mathbf{I}_{(n_w+n_x)}\}$  and its transpose, together with consideration of (2.14), yields

$$\begin{bmatrix}
-U_j & \mathbf{0} & A_k(t)G_i + B_{1k}(t)\bar{K}_i & A_{dk}(t)\bar{Q}_1 & D_{1k}(t) & \mathbf{0} \\
\star & -\mathbf{I}_{n_z} & C_k(t)G_i + B_{2k}(t)\bar{K}_i & C_{dk}(t)\bar{Q}_1 & D_{2k}(t) & \mathbf{0} \\
\star & \star & -G_i^T U_i^{-1} G_i & \mathbf{0} & \mathbf{0} & G_i^T \\
\star & \star & \star & -\bar{Q}_1 & \mathbf{0} & \mathbf{0} \\
\star & \star & \star & \star & -\gamma^2 \mathbf{I}_{n_w} & \mathbf{0} \\
\star & \star & \star & \star & \star & \frac{-\bar{Q}_1}{\tau_2 - \tau_1 + 1}
\end{bmatrix} < 0,$$

$$k \in \mathcal{K}(i), (i, j) \in \Omega \tag{2.31}$$

where  $U_i := P_i^{-1}$ ,  $U_j := P_j^{-1}$ ,  $\bar{Q}_1 := Q_1^{-1}$ ,  $\bar{K}_i := K_i G_i$ .

Note that

$$U_i - G_i - G_i^T + G_i^T U_i^{-1} G_i = (U_i - G_i)^T U_i^{-1} (U_i - G_i) \geq 0 \tag{2.32}$$

which implies

$$-G_i^T U_i^{-1} G_i \leq U_i - G_i - G_i^T, i \in \mathcal{I}. \quad (2.33)$$

There, by (2.33), the following inequality implies (2.31)

$$\begin{bmatrix} -U_j & \mathbf{0} & A_k(t)G_i + B_{1k}(t)\bar{K}_i & A_{dk}(t)\bar{Q}_1 & D_{1k}(t) & \mathbf{0} \\ \star & -\mathbf{I}_{n_z} & C_k(t)G_i + B_{2k}(t)\bar{K}_i & C_{dk}(t)\bar{Q}_1 & D_{2k}(t) & \mathbf{0} \\ \star & \star & U_i - G_i - G_i^T & \mathbf{0} & \mathbf{0} & G_i^T \\ \star & \star & \star & -\bar{Q}_1 & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -\gamma^2 \mathbf{I}_{n_w} & \mathbf{0} \\ \star & \star & \star & \star & \star & \frac{-\bar{Q}_1}{\tau_2 - \tau_1 + 1} \end{bmatrix} < 0, \quad (2.34)$$

$k \in \mathcal{K}(i), (i, j) \in \Omega$

and thus it suffices to show (2.34) instead of (2.31).

On the other hand, using the parameter uncertainty relationships (2.3)-(2.6), one has

$$\begin{aligned} & \begin{bmatrix} -U_j & \mathbf{0} & A_k(t)G_i + B_{1k}(t)\bar{K}_i & A_{dk}(t)\bar{Q}_1 & D_{1k}(t) & \mathbf{0} \\ \star & -\mathbf{I}_{n_z} & C_k(t)G_i + B_{2k}(t)\bar{K}_i & C_{dk}(t)\bar{Q}_1 & D_{2k}(t) & \mathbf{0} \\ \star & \star & U_i - G_i - G_i^T & \mathbf{0} & \mathbf{0} & G_i^T \\ \star & \star & \star & -\bar{Q}_1 & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -\gamma^2 \mathbf{I}_{n_w} & \mathbf{0} \\ \star & \star & \star & \star & \star & \frac{-\bar{Q}_1}{\tau_2 - \tau_1 + 1} \end{bmatrix} \\ &= \begin{bmatrix} -U_j & \mathbf{0} & A_k G_i + B_{1k} \bar{K}_i & A_{dk} \bar{Q}_1 & D_{1k} & \mathbf{0} \\ \star & -\mathbf{I}_{n_z} & C_k(t) G_i + B_{2k} \bar{K}_i & C_{dk} \bar{Q}_1 & D_{2k} & \mathbf{0} \\ \star & \star & U_i - G_i - G_i^T & \mathbf{0} & \mathbf{0} & G_i^T \\ \star & \star & \star & -\bar{Q}_1 & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -\gamma^2 \mathbf{I}_{n_w} & \mathbf{0} \\ \star & \star & \star & \star & \star & \frac{-\bar{Q}_1}{\tau_2 - \tau_1 + 1} \end{bmatrix} \\ &+ \text{Sym} \left\{ \begin{bmatrix} W_{1k} \\ W_{2k} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \Delta(t) \begin{bmatrix} \mathbf{0} & \mathbf{0} & E_{1k} G_i + E_{3k} \bar{K}_i & E_{2k} \bar{Q}_1 & E_{4k} & \mathbf{0} \end{bmatrix} \right\}. \quad (2.35) \end{aligned}$$

Now, by Lemma 2.1, it is easy to see that (2.19) implies (2.35). The proof is thus completed.  $\square$

It is noted that in Theorem 2.1, if we set  $U_i \equiv U$ ,  $i \in \mathcal{I}$ , we will get the corresponding controller design results based on a common Lyapunov-Krasovskii functional, which is summarized in the following corollary.

**Corollary 2.1.** *The closed-loop system (2.17) is robustly asymptotically stable with  $\mathcal{H}_\infty$  performance  $\gamma$  if the LMIs (2.19) hold with  $U_i \equiv U$ ,  $i \in \mathcal{I}$ .*

### 2.3.2 Delay-dependent $\mathcal{H}_\infty$ Controller Design

For the case of time-invariant delay, i.e.,  $\tau_1 = \tau_2$ , it is noted that the results given in Theorem 2.1 are independent of the delay size. Thus, Theorem 2.1 can be applicable to the situations when no *a priori* knowledge about the size of time-delay is available. On the other hand, it is also well known that the delay-independent results for time-delay systems are usually more conservative than the delay-dependent ones especially when the size of time-delays is small. In this subsection, we consider the delay-dependent robust  $\mathcal{H}_\infty$  control of fuzzy system (2.8) based on control law (2.16) by constructing a new piecewise Lyapunov-Krasovskii functional. The corresponding result is summarized in the following theorem.

**Theorem 2.2.** *The closed-loop system (2.17) is robustly asymptotically stable with  $\mathcal{H}_\infty$  performance  $\gamma$  if there exist matrices  $0 < U_i = U_i^T \in \mathbb{R}^{n_x \times n_x}$ ,  $\bar{M}_i = \begin{bmatrix} \bar{M}_{1i} \\ \bar{M}_{2i} \end{bmatrix} \in$*

$$\mathbb{R}^{2n_x \times n_x}, \bar{N}_i = \begin{bmatrix} \bar{N}_{1i} \\ \bar{N}_{2i} \end{bmatrix} \in \mathbb{R}^{2n_x \times n_x}, \bar{R}_i = \begin{bmatrix} \bar{R}_{1i} \\ \bar{R}_{2i} \end{bmatrix} \in \mathbb{R}^{2n_x \times n_x}, \bar{X}_i = \bar{X}_i^T = \begin{bmatrix} \bar{X}_{11i} & \bar{X}_{12i} \\ \star & \bar{X}_{22i} \end{bmatrix} \in$$

$$\mathbb{R}^{2n_x \times 2n_x}, \bar{Y}_i = \bar{Y}_i^T = \begin{bmatrix} \bar{Y}_{11i} & \bar{Y}_{12i} \\ \star & \bar{Y}_{22i} \end{bmatrix} \in \mathbb{R}^{2n_x \times 2n_x}, \bar{K}_i \in \mathbb{R}^{n_u \times n_x}, i \in \mathcal{I}, G \in \mathbb{R}^{n_x \times n_x}, 0 < \bar{Q}_\alpha = \bar{Q}_\alpha^T \in \mathbb{R}^{n_x \times n_x}, \alpha \in \{1, 2, 3\}, 0 < \bar{Z}_\beta = \bar{Z}_\beta^T \in \mathbb{R}^{n_x \times n_x}, \beta \in \{1, 2\}, \text{ and a set of positive scalars } \varepsilon_{kij} > 0, k \in \mathcal{K}(i), (i, j) \in \Omega \text{ such that the following LMIs hold}$$

$$k \in \mathcal{K}(i), (i, j) \in \Omega \quad (2.36)$$

$$\begin{bmatrix} \bar{Y}_i & \bar{R}_i \\ \star & \bar{Z}_2 \end{bmatrix} \geq 0, i \in \mathcal{I} \quad (2.39)$$

$$\begin{aligned} \Pi_{55} &:= U_i - G - G^T + (\tau_2 - \tau_1 + 1) \bar{Q}_1 + \bar{Q}_2 + \bar{Q}_3 \\ &\quad + \bar{M}_{1i} + \bar{M}_{1i}^T + \tau_2 \bar{X}_{11i} + (\tau_2 - \tau_1) \bar{Y}_{11i}, \\ \Pi_{56} &:= \bar{N}_{1i} - \bar{M}_{1i} - \bar{R}_{1i} + \bar{M}_{2i}^T + \tau_2 \bar{X}_{12i} + (\tau_2 - \tau_1) \bar{Y}_{12i}, \\ \Pi_{66} &:= -\bar{Q}_1 + \bar{N}_{2i} + \bar{N}_{2i}^T - \bar{M}_{2i} - \bar{M}_{2i}^T - \bar{R}_{2i} - \bar{R}_{2i}^T + \tau_2 \bar{X}_{22i} + (\tau_2 - \tau_1) Y_{22i}. \end{aligned}$$
$$K_i = \bar{K}_i G^{-1}, i \in \mathcal{I}. \quad (2.40)$$

*Proof.* Similar to the proof of Theorem 2.1, it suffices to find a Lyapunov function candidate  $V(t, x(t)) > 0$ ,  $\forall x(t) \neq 0$  satisfying (2.21) to prove that the closed-loop

system (2.17) is asymptotically stable with  $\mathcal{H}_\infty$  performance  $\gamma$  under zero initial conditions for any nonzero  $w \in l_2[0, \infty)$  and all admissible uncertainties.

Define  $e(t) := x(t+1) - x(t)$  and consider the following piecewise Lyapunov-Krasovskii functional

$$\begin{cases} V(t) := V_1(t) + V_2(t) + V_3(t) + V_4(t) \\ V_1(t) := x^T(t) P_i x(t), \zeta(t) \in S_i, i \in \mathcal{I} \\ V_2(t) := \sum_{s=-\tau_2}^{-\tau_1} \sum_{v=t+s}^{t-1} x^T(v) Q_1 x(v) \\ V_3(t) := \sum_{s=t-\tau_2}^{t-1} x^T(s) Q_2 x(s) + \sum_{s=t-\tau_1}^{t-1} x^T(s) Q_3 x(s) \\ V_4(t) := \sum_{s=-\tau_2}^{-1} \sum_{m=t+s}^{t-1} e^T(m) Z_1 e(m) + \sum_{s=-\tau_2}^{-\tau_1-1} \sum_{m=t+s}^{t-1} e^T(m) Z_2 e(m) \end{cases} \quad (2.41)$$

where  $P_i = P_i^T > 0, i \in \mathcal{I}, Q_\alpha = Q_\alpha^T > 0, \alpha \in \{1, 2, 3\}$  and  $Z_\beta = Z_\beta^T > 0, \beta \in \{1, 2\}$  are Lyapunov matrices to be determined.

Define  $\Delta V(t) := V(t+1) - V(t)$  and along the trajectory of system (2.17), one has

$$\Delta V_1(t) = x^T(t+1) P_j x(t+1) - x^T(t) P_i x(t), (i, j) \in \Omega \quad (2.42)$$

$$\Delta V_2(t) \leq (\tau_2 - \tau_1 + 1) x^T(t) Q_1 x(t) - x^T(t - \tau(t)) Q_1 x(t - \tau(t)) \quad (2.43)$$

$$\begin{aligned} \Delta V_3(t) &= x^T(t) (Q_2 + Q_3) x(t) \\ &\quad - x^T(t - \tau_2) Q_2 x(t - \tau_2) - x^T(t - \tau_1) Q_3 x(t - \tau_1), \end{aligned} \quad (2.44)$$

$$\begin{aligned} \Delta V_4(t) &= \tau_2 e^T(t) Z_1 e(t) + (\tau_2 - \tau_1) e^T(t) Z_2 e(t) - \sum_{m=t-\tau(t)}^{t-1} e^T(m) Z_1 e(m) \\ &\quad - \sum_{m=t-\tau_2}^{t-\tau(t)-1} e^T(m) (Z_1 + Z_2) e(m) - \sum_{m=t-\tau(t)}^{t-\tau_1-1} e^T(m) Z_2 e(m). \end{aligned} \quad (2.45)$$

In addition, define  $\xi_2(t) := [x^T(t) \ x^T(t - \tau(t))]^T$  and from the definition of  $e(t)$ , for any appropriately dimensioned matrices  $M_i, N_i, R_i, X_i, Y_i, i \in \mathcal{I}$  one has

$$0 \equiv 2\xi_2^T(t)M_i \left[ x(t) - x(t - \tau(t)) - \sum_{m=t-\tau(t)}^{t-1} e(m) \right], \quad (2.46)$$

$$0 \equiv 2\xi_2^T(t)N_i \left[ x(t - \tau(t)) - x(t - \tau_2) - \sum_{m=t-\tau_2}^{t-\tau(t)-1} e(m) \right], \quad (2.47)$$

$$0 \equiv 2\xi_2^T(t)R_i \left[ x(t - \tau_1) - x(t - \tau(t)) - \sum_{m=t-\tau(t)}^{t-\tau_1-1} e(m) \right], \quad (2.48)$$

$$\begin{aligned} 0 \equiv & \xi_2^T(t) [\tau_2 X_i + (\tau_2 - \tau_1) Y_i] \xi_2(t) - \sum_{m=t-\tau(t)}^{t-1} \xi_2^T(t) X_i \xi_2(t) \\ & - \sum_{m=t-\tau_2}^{t-\tau(t)-1} \xi_2^T(t) (X_i + Y_i) \xi_2(t) - \sum_{m=t-\tau(t)}^{t-\tau_1-1} \xi_2^T(t) Y_i \xi_2(t). \end{aligned} \quad (2.49)$$

On the other hand, under the following bounding conditions,

$$\begin{bmatrix} X_i & M_i \\ \star & Z_1 \end{bmatrix} \geq 0, i \in \mathcal{J} \quad (2.50)$$

$$\begin{bmatrix} X_i + Y_i & N_i \\ \star & Z_1 + Z_2 \end{bmatrix} \geq 0, i \in \mathcal{J} \quad (2.51)$$

$$\begin{bmatrix} Y_i & R_i \\ \star & Z_2 \end{bmatrix} \geq 0, i \in \mathcal{J} \quad (2.52)$$

one has

$$\begin{aligned} 0 & \leq \sum_{m=t-\tau(t)}^{t-1} \begin{bmatrix} \xi_2(t) \\ e(m) \end{bmatrix}^T \begin{bmatrix} X_i & M_i \\ \star & Z_1 \end{bmatrix} \begin{bmatrix} \xi_2(t) \\ e(m) \end{bmatrix} \\ & = \sum_{m=t-\tau(t)}^{t-1} \{ \xi_2^T(t) X_i \xi_2(t) + 2\xi_2^T(t) M_i e(m) + e^T(m) Z_1 e(m) \}, \end{aligned} \quad (2.53)$$

$$\begin{aligned} 0 & \leq \sum_{m=t-\tau_2}^{t-\tau(t)-1} \begin{bmatrix} \xi_2(t) \\ e(m) \end{bmatrix}^T \begin{bmatrix} X_i + Y_i & N_i \\ \star & Z_1 + Z_2 \end{bmatrix} \begin{bmatrix} \xi_2(t) \\ e(m) \end{bmatrix} \\ & = \sum_{m=t-\tau_2}^{t-\tau(t)-1} \{ \xi_2^T(t) (X_i + Y_i) \xi_2(t) + 2\xi_2^T(t) N_i e(m) + e^T(m) (Z_1 + Z_2) e(m) \}, \end{aligned} \quad (2.54)$$

$$\begin{aligned} 0 & \leq \sum_{m=t-\tau(t)}^{t-\tau_1-1} \begin{bmatrix} \xi_2(t) \\ e(m) \end{bmatrix}^T \begin{bmatrix} Y_i & R_i \\ \star & Z_2 \end{bmatrix} \begin{bmatrix} \xi_2(t) \\ e(m) \end{bmatrix} \\ & = \sum_{m=t-\tau(t)}^{t-\tau_1-1} \{ \xi_2^T(t) Y_i \xi_2(t) + 2\xi_2^T(t) R_i e(m) + e^T(m) Z_2 e(m) \}. \end{aligned} \quad (2.55)$$

Thus, based on the piecewise Lyapunov-Krasovskii functional defined in (2.41), together with consideration of the following relationships,

$$\begin{cases} x(t+1) = \hat{\mathcal{A}}_i(t)\xi_3(t) \\ z(t) = \hat{\mathcal{C}}_i(t)\xi_3(t) \\ e(t) = x(t+1) - x(t) = \check{\mathcal{A}}_i(t)\xi_3(t) \\ \xi_3(t) := [x^T(t) \ x^T(t - \tau(t)) \ w^T(t) \ x^T(t - \tau_2) \ x^T(t - \tau_1)]^T \\ \hat{\mathcal{A}}_i(t) := [\hat{\mathcal{A}}_i(t) \ \hat{\mathcal{A}}_{di}(t) \ \hat{\mathcal{D}}_{1i}(t) \ \mathbf{0}_{n_x \times n_x} \ \mathbf{0}_{n_x \times n_x}] \\ \hat{\mathcal{C}}_i(t) := [\hat{\mathcal{C}}_i(t) \ \hat{\mathcal{C}}_{di}(t) \ \hat{\mathcal{D}}_{2i}(t) \ \mathbf{0}_{n_z \times n_x} \ \mathbf{0}_{n_z \times n_x}] \\ \check{\mathcal{A}}_i(t) := [\hat{\mathcal{A}}_i(t) - \mathbf{I}_{n_x} \ \hat{\mathcal{A}}_{di}(t) \ \hat{\mathcal{D}}_{1i}(t) \ \mathbf{0}_{n_x \times n_x} \ \mathbf{0}_{n_x \times n_x}] \\ M_i := \begin{bmatrix} M_{1i} \\ M_{2i} \end{bmatrix} \ N_i := \begin{bmatrix} N_{1i} \\ N_{2i} \end{bmatrix} \ R_i := \begin{bmatrix} R_{1i} \\ R_{2i} \end{bmatrix} \\ X_i := \begin{bmatrix} X_{11i} & X_{12i} \\ \star & X_{12i} \end{bmatrix} \ Y_i := \begin{bmatrix} Y_{11i} & Y_{12i} \\ \star & Y_{12i} \end{bmatrix} \end{cases} \quad (2.56)$$

it is easy to see that the following inequality implies (2.21)

$$\xi_3^T(t) \Sigma_{ij}(t) \xi_3(t) < 0, \xi_3(t) \neq 0, (i, j) \in \Omega \quad (2.57)$$

where

$$\begin{aligned} \Sigma_{ij}(t) &:= \hat{\mathcal{A}}_i^T(t) P_j \hat{\mathcal{A}}_i(t) + \hat{\mathcal{C}}_i^T(t) \hat{\mathcal{C}}_i(t) + \tau_2 \check{\mathcal{A}}_i^T(t) Z_1 \check{\mathcal{A}}_i(t) \\ &\quad + (\tau_2 - \tau_1) \check{\mathcal{A}}_i^T(t) Z_2 \check{\mathcal{A}}_i(t) + \Phi, \\ \Phi &:= \begin{bmatrix} \Phi_{11} & \Phi_{12} & \mathbf{0} & -N_{1i} & R_{1i} \\ \star & \Phi_{22} & \mathbf{0} & -N_{2i} & R_{2i} \\ \star & \star & -\gamma^2 \mathbf{I}_{n_w} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -Q_2 & \mathbf{0} \\ \star & \star & \star & \star & -Q_3 \end{bmatrix}, \\ \Phi_{11} &:= -P_i + (\tau_2 - \tau_1 + 1)Q_1 + Q_2 + Q_3 + M_{1i} + M_{1i}^T + \tau_2 X_{11i} + (\tau_2 - \tau_1)Y_{11i}, \\ \Phi_{12} &:= N_{1i} - M_{1i} - R_{1i} + M_{2i}^T + \tau_2 X_{12i} + (\tau_2 - \tau_1)Y_{12i}, \\ \Phi_{22} &:= N_{2i} + N_{2i}^T - M_{2i} - M_{2i}^T - R_{2i} - R_{2i}^T + \tau_2 X_{22i} + (\tau_2 - \tau_1)Y_{22i}. \end{aligned} \quad (2.58)$$

Thus, if one can show (2.50)-(2.52) and

$$\Sigma_{ij}(t) < 0, (i, j) \in \Omega \quad (2.59)$$

then the claimed result follows. To this end, by Schur complements, (2.59) is equivalent to

$$\begin{bmatrix} -P_j^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\mathcal{A}}_i(t) \\ \star & -Z_1^{-1} & \mathbf{0} & \mathbf{0} & \sqrt{\tau_2} \check{\mathcal{A}}_i(t) \\ \star & \star & -Z_2^{-1} & \mathbf{0} & \sqrt{\tau_2 - \tau_1} \check{\mathcal{A}}_i(t) \\ \star & \star & \star & -\mathbf{I}_{n_z} & \hat{\mathcal{C}}_i(t) \\ \star & \star & \star & \star & \Phi \end{bmatrix} < 0, (i, j) \in \Omega. \quad (2.60)$$





$$-G^T P_i G \leq U_i - G - G^T, U_i = P_i^{-1} \quad (2.63)$$

$$-G\bar{Z}_1^{-1}G^T \leq \bar{Z}_1 - G - G^T, Z_1^{-1} = G\bar{Z}_1^{-1}G^T, \quad (2.64)$$

$$-G\bar{Z}_2^{-1}G^T \leq \bar{Z}_2 - G - G^T, Z_2^{-1} = G\bar{Z}_2^{-1}G^T. \quad (2.65)$$

Meanwhile, pre- and post-multiplying (2.50)-(2.52) by  $\text{diag}\{G^T, G^T, G^T\}$  and its transpose yields (2.37)-(2.39) directly. The proof is thus completed.  $\square$

*Remark 2.2.* It is noted that from the proof of Theorem 2.2, the delay-dependent criterion is realized by using a free-weighting matrix technique [22–25], which enables one to avoid performing any model transformation to the original system and thus no bounding technique is employed to estimate the inner product of the involved crossing terms [21]. Moreover, when treating the time-varying delay and estimating the upper bound of the difference of Lyapunov functional, some useful terms such as  $\sum_{m=t-\tau_2}^{t-\tau(t)-1} e^T(m)(\bullet)e(m)$  are fully utilized by introducing some additional terms into the proposed Lyapunov-Krasovskii functional. In addition, by using the relationships (2.53)-(2.55), the delay upper bound  $\tau_2$  is separated into two parts:  $\tau_2 = [\tau_2 - \tau(t)] + \tau(t)$  without ignoring any useful terms. Thus, compared with the existing delay-independent and delay-dependent approaches for discrete-time T-S fuzzy delay systems [28–33, 36, 37], these features have the potential to enable one to obtain less conservative results.

From the proof of Theorem 2.2, it is easy to see that the matrix inequality linearization procedure is based on the bounding inequalities given in (2.63)-(2.65), where all three different Lyapunov matrices are constrained by a same slack variable. This inevitably brings some degree of design conservatism. Another means to solve the nonlinear matrix inequality problem given in (2.61) is by utilizing the cone complementarity linearization algorithm [47]. To this end, we define variables  $U_i := P_i^{-1}$ ,  $S_1 := Z_1^{-1}$ ,  $S_2 := Z_2^{-1}$ , and by using a cone complementarity technique [47], the nonconvex feasibility problem given in (2.50)-(2.52), (2.61) is converted to the following nonlinear minimization problem involving LMI conditions:

$$\begin{aligned}
& \text{Minimize } \text{Tr} \left( \sum_{i \in \mathcal{I}} P_i U_i + \sum_{\beta=1}^2 Z_\beta S_\beta \right), \text{ subject to (2.50)-(2.52) and} \\
& \left[ \begin{array}{cccccc}
-U_j & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_k + B_{1k} K_i & A_{dk} \\
\star & -S_1 & \mathbf{0} & \mathbf{0} & \sqrt{\tau_2} (A_k + B_{1k} K_i - \mathbf{I}_{n_x}) & \sqrt{\tau_2} A_{dk} \\
\star & \star & -S_2 & \mathbf{0} & \sqrt{\tau_2 - \tau_1} (A_k + B_{1k} K_i - \mathbf{I}_{n_x}) & \sqrt{\tau_2 - \tau_1} A_{dk} \\
\star & \star & \star & -\mathbf{I}_{n_z} & C_k + B_{2k} K_i & C_{dk} \\
\star & \star & \star & \star & \Phi_{11} & \Phi_{12} \\
\star & \star & \star & \star & \star & \Phi_{22} \\
\star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star \\
\hline
D_{1k} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \varepsilon_{kij} W_{1k} & \\
\sqrt{\tau_2} D_{1k} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sqrt{\tau_2} \varepsilon_{kij} W_{1k} & \\
\sqrt{\tau_2 - \tau_1} D_{1k} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sqrt{\tau_2 - \tau_1} \varepsilon_{kij} W_{1k} & \\
D_{2k} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\
\mathbf{0} & -N_{1i} & R_{1i} & E_{1k}^T + K_i^T E_{3k}^T & \mathbf{0} & \\
\mathbf{0} & -N_{2i} & R_{2i} & E_{2k}^T & \mathbf{0} & \\
-\gamma^2 \mathbf{I}_{n_w} & \mathbf{0} & \mathbf{0} & E_{4k}^T & \mathbf{0} & \\
\star & -Q_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\
\star & \star & -Q_3 & \mathbf{0} & \mathbf{0} & \\
\star & \star & \star & -\varepsilon_{kij} \mathbf{I}_{s_2} & \varepsilon_{kij} J & \\
\star & \star & \star & \star & -\varepsilon_{kij} \mathbf{I}_{s_1} & 
\end{array} \right] < 0, \\
& k \in \mathcal{K}(i), (i, j) \in \Omega. \quad (2.66)
\end{aligned}$$

$$P_i > 0, U_i > 0, \begin{bmatrix} P_i & \mathbf{I}_{n_x} \\ \star & U_i \end{bmatrix} \geq 0, i \in \mathcal{I} \quad (2.67)$$

$$Z_\beta > 0, S_\beta > 0, \begin{bmatrix} Z_\beta & \mathbf{I}_{n_x} \\ \star & S_\beta \end{bmatrix} \geq 0, \beta \in \{1, 2\} \quad (2.68)$$

$$Q_\alpha \geq 0, \alpha \in \{1, 2, 3\}. \quad (2.69)$$

Then, for given delay bounds  $\tau_1$  and  $\tau_2$ , the suboptimal performance of  $\gamma$  can be found by the following algorithm. The convergence of this algorithm is guaranteed in terms of similar results in [21, 47].

**Algorithm 2.1** Suboptimal performance of  $\gamma$

*Step 1.* Choose a sufficiently large initial  $\gamma > 0$  such that there exists a feasible solution to (2.50)-(2.52) and (2.66)-(2.69). Set  $\gamma_0 = \gamma$ .

*Step 2.* Find a feasible set

$$\left\{ \begin{array}{l} P_{i0}, U_{i0}, X_{i0}, Y_{i0}, M_{i0}, N_{i0}, R_{i0}, K_{i0}, i \in \mathcal{I}, \varepsilon_{kij0}, k \in \mathcal{K}(i), (i, j) \in \Omega, \\ Q_{\alpha 0}, Z_{\beta 0}, S_{\beta 0}, \alpha \in \{1, 2, 3\}, \beta \in \{1, 2\} \end{array} \right\}$$

satisfying (2.50)-(2.52) and (2.66)-(2.69). Set  $\sigma = 0$ .

*Step 3.* Solving the following LMIs problem for the variables  $P_i, U_i, X_i, Y_i, M_i, N_i, R_i, K_i, i \in \mathcal{I}, Q_\alpha, \alpha \in \{1, 2, 3\}, Z_\beta, S_\beta, \beta \in \{1, 2\}, \varepsilon_{kij}, k \in \mathcal{K}(i), (i, j) \in \Omega$ :

$$\text{Minimize } \text{Tr} \left( \sum_{i \in \mathcal{I}} (P_{i\sigma} U_i + P_i U_{i\sigma}) + \sum_{\beta=1}^2 (Z_{\beta\sigma} S_\beta + Z_\beta S_{\beta\sigma}) \right)$$

$$\text{Subject to (2.58)-(2.62) and (2.65)-(2.67).}$$

Set  $P_{i(\sigma+1)} = P_i$ ,  $U_{i(\sigma+1)} = U_i$ ,  $i \in \mathcal{I}$ ,  $Z_{\beta(\sigma+1)} = Z_\beta$ ,  $S_{\beta(\sigma+1)} = S_\beta$ ,  $\beta \in \{1, 2\}$ .

**Step 4.** Substituting the controller gains  $K_i$  obtained in Step 3 into (2.66) and by some simple algebraic manipulations yield (2.70). Then, if the LMIs (2.50)-(2.52) and (2.70) are feasible with respect to the variables  $P_i = P_i^T > 0$ ,  $X_i$ ,  $Y_i$ ,  $M_i$ ,  $N_i$ ,  $R_i$ ,  $\bar{\varepsilon}_{kij}$ ,  $k \in \mathcal{K}(i)$ ,  $(i, j) \in \Omega$ ,  $Q_\alpha = Q_\alpha^T \geq 0$ ,  $Z_\beta = Z_\beta^T > 0$ ,  $\alpha \in \{1, 2, 3\}$ ,  $\beta \in \{1, 2\}$ , then set  $\gamma_0 = \gamma$  and return to Step 2 after decreasing  $\gamma$  to some extent. If the conditions (2.50)-(2.52) and (2.70) are infeasible within the maximum number of iterations allowed, then exit. Otherwise, set  $\sigma = \sigma + 1$  and go to Step 3.

$$\begin{bmatrix} -P_j & \mathbf{0} & \mathbf{0} & \mathbf{0} & P_j(A_k + B_{1k}K_i) & P_j A_{dk} \\ \star & -Z_1 & \mathbf{0} & \mathbf{0} & \sqrt{\tau_2} Z_1(A_k + B_{1k}K_i - \mathbf{I}_{n_x}) & \sqrt{\tau_2} Z_1 A_{dk} \\ \star & \star & -Z_2 & \mathbf{0} & \sqrt{\tau_2 - \tau_1} Z_2(A_k + B_{1k}K_i - \mathbf{I}_{n_x}) & \sqrt{\tau_2 - \tau_1} Z_2 A_{dk} \\ \star & \star & \star & -\mathbf{I}_{n_z} & C_k + B_{2k}K_i & C_{dk} \\ \star & \star & \star & \star & \Phi_{11} & \Phi_{12} \\ \star & \star & \star & \star & \star & \Phi_{22} \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{bmatrix} \begin{bmatrix} D_{1k} & \mathbf{0} & \mathbf{0} & \mathbf{0} & W_{1k} \\ \sqrt{\tau_2} D_{1k} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sqrt{\tau_2} W_{1k} \\ \sqrt{\tau_2 - \tau_1} D_{1k} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sqrt{\tau_2 - \tau_1} W_{1k} \\ D_{2k} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -N_{1i} & R_{1i} & \bar{\varepsilon}_{kij}(E_{1k}^T + K_i^T E_{3k}^T) & \mathbf{0} \\ \mathbf{0} & -N_{2i} & R_{2i} & \bar{\varepsilon}_{kij} E_{2k}^T & \mathbf{0} \\ -\gamma^2 \mathbf{I}_{n_w} & \mathbf{0} & \mathbf{0} & \bar{\varepsilon}_{kij} E_{4k}^T & \mathbf{0} \\ \star & -Q_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -Q_3 & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -\bar{\varepsilon}_{kij} \mathbf{I}_{s_2} & \bar{\varepsilon}_{kij} J \\ \star & \star & \star & \star & -\bar{\varepsilon}_{kij} \mathbf{I}_{s_1} \end{bmatrix} < 0, \quad k \in \mathcal{K}(i), (i, j) \in \Omega. \quad (2.70)$$

**Remark 2.3.** It is noted that the results given in Theorem 2.2 and Algorithm 2.1 do not encompass each other. The results obtained in Theorem 2.2 are very nice in the sense that these conditions are convex, and thus can be readily solved with commercially available software. The design conservatism of Theorem 2.2 mainly comes from the bounding inequalities given in (2.67)-(2.69). In the iterative approach, the conditions given in (2.66)-(2.68) are equivalent to the corresponding performance analysis results presented in (2.70). This is the main advantage of Algorithm 2.1 over Theorem 2.2. However, the numerical computation cost involved in Algorithm

2.1 is much larger than that involved in Theorem 2.2, especially when the number of iterations increases.

*Remark 2.4.* The results presented in this chapter are obtained by using piecewise Lyapunov functions [12, 14, 15, 19, 32, 33]. However, it is noted that the extension of the proposed design techniques based on fuzzy Lyapunov functions is straightforward. A piecewise Lyapunov function is defined along the partitioned premise variable space [12, 14, 15, 19, 32, 33], while a fuzzy-basis-dependent Lyapunov function is based on a mapping from fuzzy-basis functions to a Lyapunov matrix [13, 16–18, 30]. It is hard to theoretically explain which kind of Lyapunov function is better. However, for discrete-time fuzzy systems, the controller design procedure based on these two kinds of Lyapunov functions are very similar once the Lyapunov matrices are separated from the system matrices. The controller designs for continuous-time fuzzy systems based on piecewise/fuzzy Lyapunov functions are much more complicated.

## 2.4 Simulation Examples

In this section, we use two examples to demonstrate the advantages and less conservatism of the controller design methods proposed in this chapter.

*Example 2.1.* Consider the following modified Henon mapping model with time-delay and external disturbance borrowed from [33] with some modifications.

$$\begin{cases} x_1(t+1) = -x_1^2(t) + 0.04x_1(t - \tau(t)) + 0.3x_2(t) + 1.4 + 0.5w(t) \\ x_2(t+1) = x_1(t) - 0.02x_1(t - \tau(t)) + 0.03x_2(t - \tau(t)) + 0.5w(t) \end{cases} \quad (2.71)$$

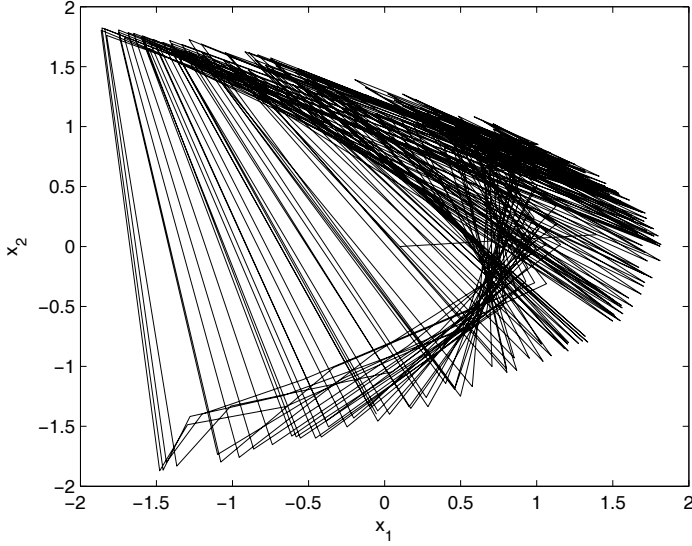
where the disturbance is given by  $w(t) = 0.1e^{-0.3t} \sin(0.02\pi t)$  and the time-delay  $\tau(t)$  is assumed to satisfy  $2 \leq \tau(t) \leq 5$ . With the initial condition  $\phi(t) = [0.1 \ 0]^T$ ,  $-5 \leq t \leq 0$ , the chaotic behavior of the above system is shown in Figure 2.1. It is observed from Figure 2.1 that the attractor region is  $\{x(t) | -2 \leq x_1(t) \leq 2\}$ .

Next, we consider the following time-delay T-S fuzzy system to represent the system (2.71).

**Plant Rule  $R^l$ :** IF  $\theta_1(t)$  is  $F^l$ , THEN

$$\begin{cases} \theta(t+1) = A_l\theta(t) + A_{dl}\theta(t - \tau(t)) + B_{1l}u(t) + D_{1l}w(t) \\ z(t) = C_l\theta(t) + C_{dl}\theta(t - \tau(t)) + B_{2l}u(t) \end{cases} \quad (2.72)$$

where  $z(t)$  is the regulated output,  $\theta(t) := x(t) - x_f$  and  $x_f = \begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix} = \begin{bmatrix} 0.9088 \\ 0.8995 \end{bmatrix}$  is one of the fixed points of system (2.71) embedded in the attractor region. The system parameters are given as follows.



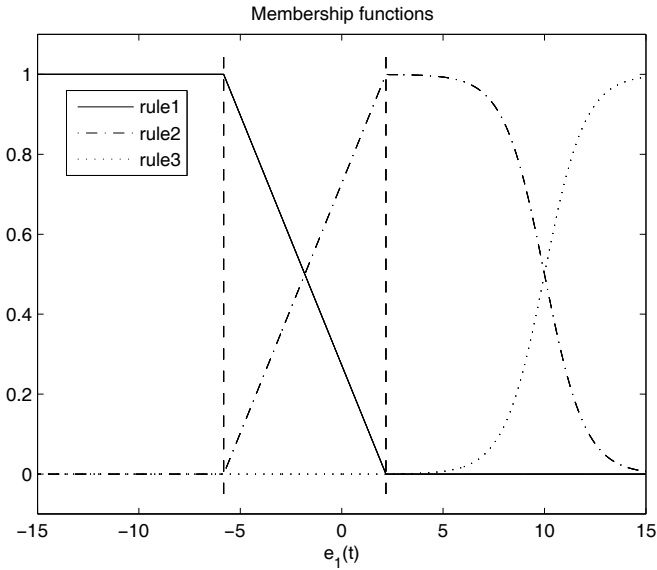
**Fig. 2.1** The chaotic behavior of system (2.71)

$$\begin{aligned}
 A_1 &= \begin{bmatrix} 4 & 0.3 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -4 & 0.3 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0.3 \\ -1 & 0 \end{bmatrix}, \\
 A_{d1} &= A_{d2} = \begin{bmatrix} 0.04 & 0 \\ -0.02 & 0.03 \end{bmatrix}, A_{d3} = \begin{bmatrix} 0.009 & 0 \\ 0.6 & 0 \end{bmatrix}, \\
 B_1 &= B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, C_1 = C_2 = C_3 = [0.2 \ -0.3], \\
 C_{d1} &= C_{d2} = C_{d3} = [0.03 \ 0.01], B_{21} = B_{22} = B_{23} = 0.1.
 \end{aligned}$$

In the region of  $\{\theta(t) | -15 \leq \theta_1(t) \leq 15\}$ , we define the following membership functions, which are also shown in Figure 2.2.

$$\begin{bmatrix} \mu_1[\theta_1(t)] \\ \mu_2[\theta_1(t)] \\ \mu_3[\theta_1(t)] \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & -15 \leq \theta_1(t) < -4 - 2x_{f1} \\ \begin{bmatrix} 0.5 \left( 1 - \frac{\theta_1(t) + 2x_{f1}}{4} \right) \\ 0.5 \left( 1 + \frac{\theta_1(t) + 2x_{f1}}{4} \right) \\ 0 \end{bmatrix}, & -4 - 2x_{f1} \leq \theta_1(t) < 4 - 2x_{f1} \\ \begin{bmatrix} 0 \\ 0 \\ 1 - \left( 1 + e^{10 - \theta_1(t)} \right)^{-1} \end{bmatrix}, & 4 - 2x_{f1} \leq \theta_1(t) \leq 15. \end{cases}$$

It is easy to see that in the region of  $\{\theta(t) | -4 - 2x_{f1} \leq \theta_1(t) \leq 4 - 2x_{f1}\}$ , system (2.72) describes the error system of (2.71) to the fixed point  $x_f$ .



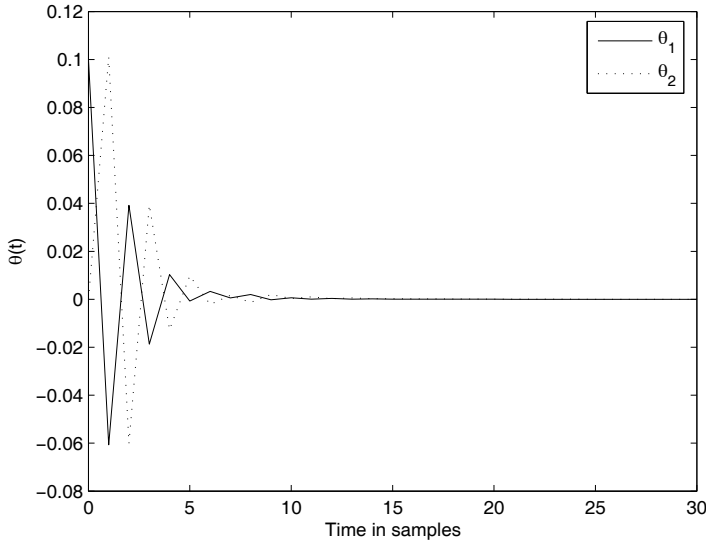
**Fig. 2.2** Membership functions for the fuzzy system in Example 2.1

The objective is then to design a piecewise controller to stabilize the system (2.72) to zero with a guaranteed disturbance attenuation level  $\gamma$ . Based on the partition method presented in Section 2.2, there are four boundaries. Thus, the premise space is divided into three regions as shown in Figure 2.2. Let the  $\mathcal{H}_\infty$  performance  $\gamma = 3.5$  and it has been found that there is no feasible solution based on the delay-independent method given in Theorem 2.1, or the delay-dependent meth-

ods by [32, 33, 36, 37] and Theorem 2.2 given in this chapter. However, by applying Algorithm 2.1 and after 37 iterations, one indeed obtains a feasible solution with controller gains given by

$$K_1 = [-3.4795 \ -0.2238], K_2 = [1.2401 \ -0.2879], K_3 = [2.9863 \ 0.1781].$$

With a randomly generated time-varying delay  $\tau(t)$  between  $\tau_1 = 2$  and  $\tau_2 = 5$ , and initial condition  $\theta(t) = [0.1 \ 0]^T$ ,  $-5 \leq t \leq 0$ , the state trajectories of the error system are shown in Figure 2.3. It can be observed that the performance of the resulting closed-loop system is satisfactory.



**Fig. 2.3** Time response of the error system in Example 2.1

*Example 2.2.* Consider the following uncertain discrete-time state-delayed T-S fuzzy system of the form (2.1) with three rules.

**Plant Rules  $R^l$ :** IF  $x_1(t)$  is  $F^l$ , THEN

$$\begin{cases} x(t+1) = A_l(t)x(t) + A_{dl}(t)x(t-\tau(t)) + B_{1l}(t)u(t) + D_{1l}(t)w(t) \\ z(t) = C_l(t)x(t) + C_{dl}(t)x(t-\tau(t)) + B_{2l}(t)u(t) + D_{2l}(t)w(t) \end{cases}$$

where

$$\begin{aligned}
A_1 &= \rho \begin{bmatrix} 1 & 0.3 \\ 0.2 & 1 \end{bmatrix}, A_2 = \rho \begin{bmatrix} 1 & 0.3 \\ 0.4 & 1 \end{bmatrix}, A_3 = \rho \begin{bmatrix} 1.2 & 0.5 \\ 0.2 & 1 \end{bmatrix}, \\
A_{d1} &= A_{d2} = A_{d3} = \begin{bmatrix} -0.1 & 0.05 \\ 0 & 0.1 \end{bmatrix}, B_{11} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}, B_{12} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \\
B_{13} &= \begin{bmatrix} 0.6 \\ 1 \end{bmatrix}, D_{11} = D_{12} = D_{13} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, C_1 = C_3 = [0.3 \ 0], C_2 = [0.4 \ 0], \\
C_{d1} &= C_{d2} = C_{d3} = [0.2 \ 0.2], B_{21} = B_{22} = B_{23} = 0.1, D_{21} = D_{22} = D_{23} = 0, \\
W_{11} &= \begin{bmatrix} 0.1 \\ -0.22 \end{bmatrix}, W_{12} = \begin{bmatrix} -0.03 \\ 0.01 \end{bmatrix}, W_{13} = \begin{bmatrix} 0.05 \\ 0.01 \end{bmatrix}, W_{21} = W_{22} = W_{23} = 0, \\
E_{11} &= [0.1 \ 0.1], E_{12} = [0.01 \ 0.03], E_{13} = [0.03 \ 0.02], E_{21} = [0.01 \ 0.02], \\
E_{22} &= [0.1 \ 0.04], E_{23} = [0.01 \ 0.02], E_{31} = 0.1, E_{32} = 0.2, E_{33} = 0.05, \\
E_{41} &= E_{42} = E_{43} = 0.1, J = 0.2
\end{aligned}$$

and  $\rho$  is a scalar parameter. The normalized membership functions are shown in Figure 2.4. Then, based on the partition method presented in Section 2.2, the premise variable space can be partitioned into five regions:

$$\begin{aligned}
\mathcal{X}_1 &:= \{x_1 | -\infty < x_1 < 3\}, \mathcal{X}_2 := \{x_1 | -3 \leq x_1 < -1\}, \\
\mathcal{X}_3 &:= \{x_1 | -1 \leq x_1 < 1\}, \mathcal{X}_4 := \{x_1 | 1 \leq x_1 < 3\}, \mathcal{X}_5 := \{x_1 | 3 \leq x_1 < \infty\}.
\end{aligned}$$

The time-varying delay  $\tau(t)$  is assumed to satisfy (2.2) with  $\tau_1 = 3$ ,  $\tau_2 = 8$ . It is noted that the open loop system is unstable.

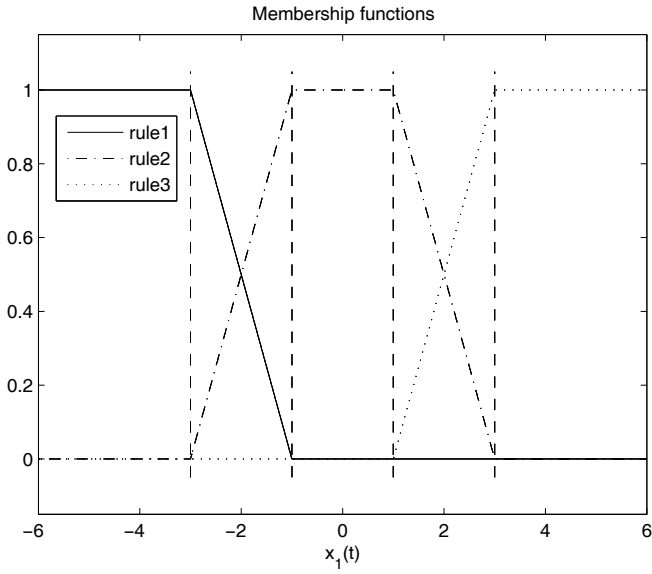
The objective is to design a piecewise controller of the form (2.16) such that the resulting closed-loop system (2.17) is robustly asymptotically stable with  $\mathcal{H}_\infty$  disturbance attenuation level  $\gamma$ . To this end, choosing the scalar  $\rho = 1.3$ , it has been found that there is no feasible solution based on the delay-dependent methods by [32, 33, 36, 37] and Theorem 2.2 given in this chapter. However, by applying Theorem 2.1 given in this chapter, one indeed obtains a set of feasible solutions with the optimal  $\mathcal{H}_\infty$  performance index  $\gamma_{\min} = 6.3827$ . The controller gains are given by

$$\begin{aligned}
K_1 &= [-1.9770 \ -0.7920], \\
K_2 &= [-1.9593 \ -0.8243], K_3 = [-2.3719 \ -0.8310], \\
K_4 &= [-0.9968 \ -1.0107], K_5 = [-4.9562 \ -0.7747].
\end{aligned}$$

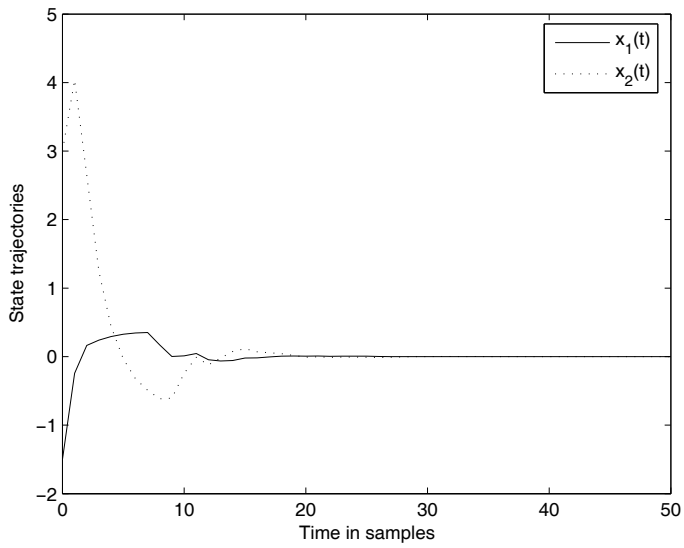
With a randomly generated time-varying delay  $\tau(t)$  between  $\tau_1 = 3$  and  $\tau_2 = 9$  and initial condition  $x(t) = [-1.5 \ 3]^T$ ,  $-8 \leq t \leq 0$ , the state trajectories of the closed-loop system are shown in Figure 2.5. It can be observed that the performance of the resulting closed-loop system is satisfactory.

In addition, we also try to apply Algorithm 2.1 to solve the piecewise control problem for the above system. Unfortunately, it is observed that there is no feasible solution based on Algorithm 2.1 even when the maximum number of iterations allowed is  $N_{\max} = 500$ .





**Fig. 2.4** Membership functions for the fuzzy system in Example 2.2



**Fig. 2.5** State-trajectories of the closed-loop system in Example 2.2

**Table 2.1**  $\mathcal{H}_\infty$  performance for different cases with  $\tau_1 = 2$ ,  $\tau_2 = 3$  of Example 2.2

methods	$\rho = 0.9$	$\rho = 1.1$	$\rho = 1.3$	$\rho = 1.5$
[37]	1.1092	1.7595	$\infty$	$\infty$
Theorem 2.2	1.0577	1.6258	$\infty$	$\infty$
Theorem 2.1	0.5792	0.7755	1.2684	2.6634
Corollary 2.1	0.5799	0.8028	1.4156	3.5608

If the time-delay interval is reduced, feasible solutions for the several results obtained in this chapter and [37] are obtained, a more detailed comparison of the minimum robust  $\mathcal{H}_\infty$  performance indexes  $\gamma_{\min}$  obtained based on these methods for variety of cases is summarized in Table 2.1. The results given in this example indicate that when treating the controller design problems for time-delay systems, the delay-independent method sometimes may be more effective than the corresponding delay-dependent ones, because no matrix bounding inequality constraints and/or numerical iteration procedures are involved in the delay-independent controller synthesis case. From the table, it can also be seen that the piecewise Lyapunov-Krasovskii functional-based approach produces less conservative results than the common Lyapunov-Krasovskii functional-based approach.

## 2.5 Conclusions

In this chapter, based on two novel piecewise Lyapunov-Krasovskii functionals combined with some matrix inequality convexifying techniques, both delay independent and delay-dependent approaches have been developed to study the robust  $\mathcal{H}_\infty$  piecewise state-feedback control problem for a class of uncertain discrete-time T-S fuzzy systems with interval-like time-varying state delay. Numerical examples are presented to demonstrate the applicability of the proposed approaches. It is also noted that the proposed methods can be extended to solve the output-feedback controller design problems for discrete-time state-delayed T-S fuzzy systems.

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