

Chapter 1

Introduction

The aim of the present book is a unified representation of some recent results in geometric function theory together with a consideration of their historical sources. These results are concerned with functions f , holomorphic or meromorphic in a domain Ω in the extended complex plane $\overline{\mathbb{C}}$. The only additional condition we impose on these functions is the condition that the range $f(\Omega)$ is contained in a given domain $\Pi \subset \overline{\mathbb{C}}$. This fact will be denoted by $f \in A(\Omega, \Pi)$. We shall describe how one may get estimates for the derivatives $|f^{(n)}(z_0)|, n \in \mathbb{N}, f \in A(\Omega, \Pi)$, dependent on the position of z_0 in Ω and $f(z_0)$ in Π .

1.1 Historical remarks

The beginning of this program may be found in the famous article [125] of G. Pick. There, he discusses estimates for the MacLaurin coefficients of functions with positive real part in the unit disc found by C. Carathéodory in [52]. Pick tells his readers that he wants to generalize Carathéodory's estimates such that the special role of the expansion point at the origin is no longer important. For the convenience of our readers we quote this sentence in the original language:

Durch lineare Transformation von z oder, wie man sagen darf, durch kreisgeometrische Verallgemeinerung, kann man die Sonderstellung des Wertes $z = 0$ wegschaffen, so daß sich Relationen für die Differentialquotienten von w an beliebiger Stelle ergeben.

The first great success of this program was G. Pick's theorem, as it is called by Carathéodory himself, compare [54], vol II, §286–289.

If $z_0 \in \Delta = \{z \mid |z| < 1\}$, $f \in A(\Delta, \Delta)$, and $f(z_0) = w_0$, then the inequality

$$\left| \frac{w_0 - f(z)}{1 - \overline{w_0}f(z)} \right| \leq \left| \frac{z_0 - z}{1 - \overline{z_0}z} \right|, \quad z \in \Delta, \quad (1.1)$$

is valid.

This theorem follows immediately from Schwarz's Lemma using holomorphic automorphisms of the unit disc. A direct consequence of Pick's theorem is the inequality

$$|f'(z_0)| \leq \frac{1 - |f(z_0)|^2}{1 - |z_0|^2}, \quad f \in A(\Delta, \Delta), \quad z_0 \in \Delta. \quad (1.2)$$

To be more complete about the history of the inequality (1.2) we have to mention that O. Szász (see the footnote in [160], p.308) attributes it to E. Lindelöf [105] and indicates that (we quote once more):

Neuere Beweise dieser Relation gaben Carathéodory und Jensen. Herr Jensen zeigte, wie der Satz aus dem schon von Herrn Landau 1906 bewiesenen Spezialfall: $|c_1| \leq 1 - |c_0|^2$ leicht folgt.

As an oft-quoted proverb says "Success has many fathers, but failure is an orphan". For us, at this place the second part of this proverb hints of unsolved mathematical problems.

It is clear that Schwarz's lemma and its generalizations became widely known in the period of a systematic study of results which are closely connected with the proofs of the Riemann theorem on conformal mappings. By the way, the original version of the lemma may be found in [149], where H.A. Schwarz discussed the Riemann mapping theorem.

Now, it is well known that one may generalize the inequalities (1.1) and (1.2) to $f \in A(\Omega, \Pi)$, $z_0 \in \Omega$, and $f(z_0) = w_0$, where Ω and Π are domains that have at least three boundary points. This generalization is known as the principle of hyperbolic metric (see for instance R. Nevanlinna [120] and G. M. Goluzin [70]).

An important case is presented by simply connected domains. Let Ω and Π be simply connected proper subdomains of $\mathbb{C}$ and let Φ_{Ω, z_0} be the unique conformal map of Δ onto Ω such that $\Phi_{\Omega, z_0}(0) = z_0$ and $\Phi'_{\Omega, z_0}(0) > 0$. The existence of this map is proved by the Riemann mapping theorem. We define $\Phi_{\Pi, f(z_0)}$ analogously. Then the function

$$w = \Phi_{\Pi, f(z_0)}^{-1} \circ f \circ \Phi_{\Omega, z_0} \quad (1.3)$$

belongs to the family $A(\Delta, \Delta)$ and satisfies $w(0) = 0$. Hence, Schwarz's Lemma implies

$$|f'(z_0)| \leq \frac{\Phi'_{\Pi, f(z_0)}(0)}{\Phi'_{\Omega, z_0}(0)},$$

the generalized Schwarz-Pick inequality. The quantities

$$R(z_0, \Omega) := \Phi'_{\Omega, z_0}(0) \quad \text{and} \quad R(f(z_0), \Pi) := \Phi'_{\Pi, f(z_0)}(0)$$

are called the conformal radius of Ω at the point z_0 and of Π at $w_0 = f(z_0)$, respectively, and will be used here to describe the positions of the points z_0 in Ω and $f(z_0)$ in Π .

1.2 On inequalities for higher derivatives

We ask for inequalities of the form

$$\frac{|f^{(n)}(z_0)|}{n!} \leq M_n(z_0, \Omega, \Pi) \frac{R(f(z_0), \Pi)}{(R(z_0, \Omega))^n}, \quad n \in \mathbb{N}, \quad (1.4)$$

where f and z_0 are as above.

Concerning the history of inequalities for higher derivatives, we should mention here that the interest of researchers in geometric function theory was concentrated for a long time after 1920s on the famous Bieberbach conjecture, i.e., the conjecture that

$$\frac{|f^{(n)}(0)|}{n!} \leq n|f'(0)| \quad (1.5)$$

for functions f holomorphic and injective on Δ . This may be seen as a special case of the above inequality, where f maps Δ conformally onto the simply connected domain $f(\Delta)$. Since it seemed very difficult to prove this conjecture, less attention was attracted by the generalized Bieberbach conjecture or Rogosinski conjecture, which in the above formulation means that

$$\frac{|f^{(n)}(0)|}{n!} \leq nR(f(0), \Pi) \quad (1.6)$$

for any simply connected proper subdomain Π of $\mathbb{C}$ and any $f \in A(\Delta, \Pi)$. An equivalent formulation of the Rogosinski conjecture is the following.

Let g be holomorphic and injective on Δ and let there exist a function $w : \Delta \rightarrow \Delta$ such that $w(0) = 0$ and $f = g \circ w$. Then (1.6) is valid.

As usual, we will abbreviate this relation between f and g by $f \prec g$ and say that under these circumstances f is subordinated to g .

It was a great surprise when de Branges succeeded in proving not only (1.5) but also (1.6) in 1985. Soon afterwards, it was recognized, especially by Yamashita, that one could use (1.5) to go further steps in Pick's program for functions injective on Δ . In fact, it had been seen earlier by Landau and Jakubowski that the validity of the Bieberbach conjecture would imply sharp bounds for $|f^{(n)}(z_0)|$, $z_0 \in \Delta$, and f injective on Δ . We will give an outline of these results in the present book. Further, we will speak on the ideas of Chua [56], who indicated that (1.5) could be used to derive similar bounds for functions f injective on simply connected domains Ω . In this paper, he considered in addition the case that Ω is convex and published the following conjectures.

Let f be holomorphic and injective on a proper convex subdomain Ω of $\mathbb{C}$. Then for any $z_0 \in \Omega$, and any $n \geq 2$ the inequality

$$\frac{|f^{(n)}(z_0)|}{n!} \leq (n+1)2^{n-2} \frac{|f'(z_0)|}{(R(z_0, \Omega))^{n-1}} \quad (1.7)$$

is valid. If in addition $f(\Omega)$ is convex, then

$$\frac{|f^{(n)}(z_0)|}{n!} \leq 2^{n-1} \frac{|f'(z_0)|}{(R(z_0, \Omega))^{n-1}} \quad (1.8)$$

is valid.

Motivated by the work of Ruscheweyh and Yamashita, who had proved estimates of the type (1.4) and similar ones, the authors of the present book concentrated research in the last years on formulas like (1.4). The results of this research together with known theorems in this direction form the content of this book. Especially, we shall show that Chua's conjectures are true. We will show even more, namely that

$$M_n(z_0, \Omega, \Pi) \leq (n+1)2^{n-2}$$

for $f \in A(\Omega, \Pi)$, Ω convex and Π simply connected and that

$$M_n(z_0, \Omega, \Pi) \leq 2^{n-1}$$

for $f \in A(\Omega, \Pi)$, Ω and Π convex. These results will be completed by several considerations for special cases of Ω and Π as well as by computations of the dependences of $M_n(z_0, \Omega, \Pi)$ of the variable z_0 .

One further implication of the validity of the Bieberbach conjecture is concerned with meromorphic functions univalent on Δ . In 1962, J. Jenkins considered functions f that in addition to the above properties satisfy the condition that the pole of f lies at a point $p \in (0, 1)$. He proved that for such functions the Bieberbach conjecture would imply

$$\frac{|f^{(n)}(0)|}{n!} \leq \frac{|f'(0)|}{p^{n-1}} \sum_{k=0}^{n-1} p^{2k}. \quad (1.9)$$

This result motivated us to generalize the above considerations to proper subdomains of $\overline{\mathbb{C}}$. It is natural that here the positions of z_0 in Ω and $f(z_0)$ in Π relative to the point at infinity enter the picture. We use the hyperbolic distances to characterize those items. We will prove results of the type (1.4) using a generalization of the conformal radius. These results are satisfying in the following cases:

- 1) Ω and Π are simply connected with respect to $\overline{\mathbb{C}}$.
- 2) Ω is simply connected with respect to $\overline{\mathbb{C}}$ and Π is convex.
- 3) Ω is convex and $\overline{\mathbb{C}} \setminus \Pi$ is compact and convex.

In the important case that $\Omega = \Delta$ or Ω convex and Π is simply connected with respect to $\overline{\mathbb{C}}$ we can present only partial results.

One of the chapters is devoted to the most difficult case, the case of multiply connected domains. Here, we define appropriate questions and give some answers that in most cases are far from the sharpness we achieved in many of the above mentioned theorems. Hence this chapter is more or less an impetus for further research.

1.3 On methods

What can be said about methods? Essentially, the proofs of Schwarz-Pick type inequalities are based on relationships of certain hyperbolic characteristics of domains with the following results in geometric function theory:

1. Littlewood's ideas on subordinate functions developed by Rogosinski, Goluzin, Clunie, Robertson, and Sheil-Small.
2. Coefficient estimates using Löwner's theory on parametric representation of univalent functions. Especially, Löwner's theorem on inverse coefficients and de Branges' proof of the Bieberbach conjecture.
3. Explicit representation of convex hulls for several families of analytic functions by the Herglotz formula and its generalizations.

Usually, coefficient estimates concern univalent or subordinate functions holomorphic in the unit disk. We prove and use several new versions of the known theorems for subordinate or quasi-subordinate functions which are holomorphic in a neighbourhood of the origin.

Also, to prove Schwarz-Pick type inequalities for higher derivatives one needs certain hyperbolic characteristics of plane domains. For instance, let $Q_n(z, w, \Omega, \Pi)$ be defined as the smallest possible value such that the inequality

$$\frac{|f^{(n)}(z)|}{n!} \leq Q_n(z, w, \Omega, \Pi) \frac{R(w, \Pi)}{R(z, \Omega)^n}$$

holds for all $f \in A(\Omega, \Pi)$, $f(z) = w$, and

$$C_n(\Omega, \Pi) = \sup\{Q_n(z, w, \Omega, \Pi) \mid (z, w) \in \Omega \times \Pi\}.$$

The principle of the hyperbolic metric implies that $Q_1(z, w, \Omega, \Pi) = 1$, and, in turn, that $C_1(\Omega, \Pi) = 1$ for any pair (Ω, Π) equipped with the Poincaré metric.

In the case $n \geq 2$ the quantity Q_n depends on the hyperbolic characteristics

$$R^k(z, \Omega) \frac{\partial^k \log R(z, \Omega)}{\partial z^k} \text{ and } R^k(w, \Pi) \frac{\partial^k \log R(w, \Pi)}{\partial w^k}, \quad 1 \leq k \leq n-1.$$

In particular, $Q_2(z, w, \Omega, \Pi)$ depends on z and w via the quantities $p = |\nabla R(z, \Omega)|$ and $q = |\nabla R(w, \Pi)|$, only. Consequently, explicit estimates of punishing factors $C_n(\Omega, \Pi)$ are closely connected with the behaviour of the above characteristics, and, roughly speaking, estimating of these quantities is related to certain coefficient problems of geometric function theory. Finally, we have to attract the reader's attention to the following fact which deserves to be widely known:

The quantity reciprocal to the conformal or hyperbolic radius is exactly the density of the Poincaré metric. More precisely, the equation

$$\lambda_\Omega(z) := \frac{1}{R(z, \Omega)}, \quad z \in \Omega,$$

defines the density of the hyperbolic metric in the domain Ω with Gaussian curvature $K = -4$.

1.4 Survey of the contents

In this section, we would like to give a short survey of the contents of the present book.

Chapters 2 and 3 have a preparatory character. There we will gather the materials from the work of many researchers in geometric function theory that we need for our results. Therefore we do not give all proofs in detail in these two chapters and we shortened and simplified old proofs. During these efforts we found “new looks through old holes” now and then.

Chapter 2 is dedicated to the most famous coefficient theorems of the last century, namely the Bieberbach conjecture and the conjecture for the coefficients of functions with bounded boundary rotation.

The third chapter discusses the second theme that is important for our generalizations of the Schwarz-Pick lemma; the Poincaré metric, its historical background, and well-known theorems concerning this metric in different circumstances. Among them are the famous theorems of Landau, Teichmüller, Beardon and Pommerenke.

Most of the material presented in the following chapters has been developed by us in the last ten years, some of it has been published, other results appear in this book for the first time.

Chapter 4 contains the most prominent members of the family of punishing factors, those for pairs of simply connected domains and those for pairs of convex domains. Moreover, these two are in some sense the most beautiful, namely 4^{n-1} and 2^{n-1} . We discuss the work of Ruscheweyh, Chua, and Yamashita on these questions. Further, we present the complete solution for the case $n = 2$ for any pair of domains.

The fifth chapter is devoted to more special results. The most prominent of them may be the proof of a far reaching generalization of Chua’s conjecture. We add the determination of punishing factors for pairs of simply connected domains in the extended plane, and we prove sharp lower bounds in some general circumstances.

Chapter 6 is concerned with some generalizations to multiply connected domains for arbitrary $n > 2$ and it has at some places a tentative character, since we are not sure that we have found the “best” way of generalization.

In the last two chapters, we present some material in the neighbourhood of our results. This is meant as the basis for further research on the many questions that are natural to pose here.

At this point, we want to thank the Deutsche Forschungsgemeinschaft for continued support of our research. Their many grants for F. G. Avkhadiev enabled us to do all the scientific work presented in this book.



<http://www.springer.com/978-3-7643-9999-3>

Schwarz-Pick Type Inequalities

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2009, VIII, 156 p., Softcover

ISBN: 978-3-7643-9999-3

A product of Birkhäuser Basel