

Preface

The characterization of combinatorial or geometric structures in terms of their groups of automorphisms has attracted considerable interest in the last decades and is now commonly viewed as a natural generalization of Felix Klein's Erlangen program (1872). In addition, especially for finite structures, important applications to practical topics such as design theory, coding theory and cryptography have made the field even more attractive.

The subject matter of this research monograph is the study and classification of flag-transitive Steiner designs, that is, combinatorial t -($v, k, 1$) designs which admit a group of automorphisms acting transitively on incident point-block pairs. As a consequence of the classification of the finite simple groups, it has been possible in recent years to characterize Steiner t -designs, mainly for $t = 2$, admitting groups of automorphisms with sufficiently strong symmetry properties. For Steiner 2-designs, arguably the most general results have been the classification of all point 2-transitive Steiner 2-designs in 1985 by W. M. Kantor, and the almost complete determination of all flag-transitive Steiner 2-designs announced in 1990 by F. Buekenhout, A. Delandtsheer, J. Doyen, P. B. Kleidman, M. W. Liebeck, and J. Saxl.

However, despite the classification of the finite simple groups, for Steiner t -designs with $t > 2$ most of the characterizations of these types have remained long-standing challenging problems. Specifically, the determination of all flag-transitive Steiner t -designs with $3 \leq t \leq 6$ has been of particular interest and object of research for more than 40 years.

The main part of this monograph is devoted to the complete classification of all flag-transitive Steiner t -designs for each of the remaining parameters $t = 3, 4, 5, 6$. The obtained results generalize theorems of Jacques Tits (1964) and Heinz Lüneburg (1965). The primary objects that are characterized are the Mathieu-Witt designs associated with the five sporadic simple Mathieu groups; thus the results are also important for a future unified geometric theory of the sporadic simple groups. The proofs rely on the classification of the finite 2-transitive permutation groups, which itself depends on the finite simple group classification. Along with group theory, the proofs also involve incidence geometric, combinatorial and number theoretical arguments. Especially for the latter, the study of

Diophantine equations, in particular Thue-Mahler and generalized Ramanujan-Nagell equations, turns out to be helpful for crucial parts of the proofs. The main results have been published recently [59, 60, 61, 62, 63], and are presented in this treatment in a sufficiently self-contained and unified manner. Moreover, a broad introduction to the topic of flag-transitive Steiner designs is provided, along with illustrative examples.

Here is a brief chapter-by-chapter description of the contents; a more detailed one may be found at the beginning of each chapter.

Chapters 1–3 are of expository nature; Chapter 1 gives an introduction to the theory of incidence structures and combinatorial designs; Chapter 2 is on permutation groups and group actions, in particular the classification of the finite doubly transitive permutation groups is stated; Chapter 3 assembles number-theoretical tools like Zsigmondy’s theorem on primitive prime divisors and related issues. The advanced reader may skip these first three chapters.

In Chapter 4, we start to look at Steiner designs which admit a group of automorphisms with sufficiently strong symmetry properties. One of the reasons for this consideration of highly symmetric designs is a general view that, while the existence of combinatorial objects is of interest, they are even more fascinating when they have a rich group of symmetries. Various examples are illustrated, most of them arising from finite geometries. Among the highly symmetric properties of designs, flag-transitivity is certainly a particularly important and natural one, and hence will be of further central consideration. In particular, as the starting point of our examination of all flag-transitive Steiner designs, we derive the following result: For any non-trivial t -design $\mathcal{D}$ with $t \geq 3$, the flag-transitivity of a group $G \leq \text{Aut}(\mathcal{D})$ of automorphisms of $\mathcal{D}$ always implies its doubly transitivity on the points of $\mathcal{D}$. The proof involves Block’s Lemma, a well-known result which is also treated in detail in this chapter. In the next chapter, Chapter 5, the complete determination of all flag-transitive Steiner t -designs with $t \geq 3$ is stated. Moreover, a census of some of the most general results on highly symmetric Steiner t -designs is given.

The rest of the book is dedicated to the complete classification of all flag-transitive Steiner t -designs with $3 \leq t \leq 6$: First, in Chapter 6 we classify all flag-transitive Steiner quadruple systems, i.e., Steiner 3-designs with block size 4. The key ideas of the proof are presented at a level suitable for beginning graduate students. In a more rigorously mathematical way, these results are extended in Chapter 7 to arbitrary Steiner 3-designs. Chapter 8 deals with the determination of all flag-transitive Steiner 4-designs. We present the classification of all flag-transitive Steiner 5-designs in Chapter 9 and prove finally in Chapter 10 that there are no non-trivial flag-transitive Steiner 6-designs.

This book provides the first full discussion of flag-transitive Steiner designs, a central part of the study of highly symmetric combinatorial configurations at the interface of several mathematical disciplines. It is addressed to graduate students in mathematics or computer science with some familiarity with combinatorics and

basic group theory as well as to established researchers in design theory, finite or incidence geometry, coding theory, cryptography, algebraic combinatorics, and more generally, discrete mathematics.

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