Preface

This book is based on the lectures given at the Oberwolfach Seminar on Tropical Algebraic Geometry in October 2004.

Tropical Geometry first appeared as a subject of its own in 2002, while its roots can be traced back at least to Bergman's work [1] on logarithmic limit sets. Tropical Geometry is now a rapidly developing area of mathematics. It is intertwined with algebraic and symplectic geometry, geometric combinatorics, integrable systems, and statistical physics. Tropical Geometry can be viewed as a sort of algebraic geometry with the underlying algebra based on the so-called tropical numbers. The tropical numbers (the term "tropical" comes from computer science and commemorates Brazil, in particular a contribution of the Brazilian school to the language recognition problem) are the real numbers enhanced with negative infinity and equipped with two arithmetic operations called tropical addition and tropical multiplication. The tropical addition is the operation of taking the maximum. The tropical multiplication is the conventional addition. These operations are commutative, associative and satisfy the distribution law. It turns out that such tropical algebra describes some meaningful geometric objects, namely, the Tropical Varieties. From the topological point of view the tropical varieties are piecewise-linear polyhedral complexes equipped with a particular geometric structure coming from tropical algebra. From the point of view of complex geometry this geometric structure is the worst possible degeneration of complex structure on a manifold. From the point of view of symplectic geometry the tropical variety is the result of the Lagrangian collapse of a symplectic manifold (along a singular fibration by Lagrangian tori).

The target audience of the Oberwolfach seminar was graduate students. The seminar was designed to introduce young mathematicians to this perspective research field, including presentation of basic notions and motivations for tropical algebraic geometry as well as demonstration of some of its striking applications. During the preparation of these lecture notes for publication, we adapted the notes to a wider audience, both beginners and established researchers. As a result, the discussions in this book are more detailed and contain some new results that were obtained after the seminar itself.

Besides a general introduction to tropical geometry, we discuss the concepts of complex and non-Archimedean amoebas, as well as the patchworking construction and enumerative tropical geometry. For a more advanced study of these topics, we recommend the articles [8, 22, 28, 39, 40, 41, 42, 48, 59].

We do not in this book attempt to cover all facets of tropical geometry. For instance, we do not discuss the combinatorial aspects of tropical varieties (see, for example, [31, 54, 62, 64]), or abstract tropical varieties of dimension greater than 1 [18, 34, 35]. Furthermore, we do not touch various other branches of tropical mathematics, but only recommend some references: [36, 64] (computational aspects), [5, 15, 53] (max-plus algebra), [9, 32, 37, 50, 52] (tropical mathematics in applied problems).

The book consists of three chapters. The first chapter, "Introduction to tropical geometry" by G. Mikhalkin, is a basic treatment of tropical varieties and their relation to classical geometry, in particular the theory of amoebae. Special emphasis is put on tropical curves. The second chapter, "Patchworking of algebraic varieties" by E. Shustin, deals with the patchworking construction in algebraic geometry, the link between real algebraic geometry and tropical geometry. The chapter starts with the original Viro method of gluing real algebraic hypersurfaces, then goes through various modifications and generalizations of the Viro method. In the final section the patchworking construction is used to prove Mikhalkin's correspondence theorem between real and complex algebraic curves on toric surfaces on one side and plane tropical curves on the other side. The third chapter, "Applications of tropical geometry to enumerative geometry" by I. Itenberg, presents various applications, based on Mikhalkin's correspondence theorem, of tropical geometry in real and complex enumerative geometry. These applications mostly concern calculation of Gromov–Witten invariants and Welschinger invariants (the latter invariants can be seen as real counterparts of genus zero Gromov–Witten invariants).

Each chapter is supplemented by exercises, most of which were proposed to and discussed by the participants of the seminar.

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Preface to the second edition

We are happy to observe that Tropical Geometry has become an even more popular subject. A number of new directions for tropical methods has emerged and developed. As a result the collection of new tropical research papers is too large to make an exhaustive list.

In this edition we have corrected some of the misprints from the first edition and added the references [2], [23], [43], [65] to new lecture notes similar in spirit to those from this book.

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