

Preface

If \mathcal{H} is a Hilbert space and $T : \mathcal{H} \rightarrow \mathcal{H}$ is a continuous linear operator, a natural question to ask is: What are the closed subspaces \mathcal{M} of \mathcal{H} for which $T\mathcal{M} \subset \mathcal{M}$? Of course the famous invariant subspace problem asks whether or not T has *any* non-trivial invariant subspaces. This monograph is part of a long line of study of the invariant subspaces of the operator $T = M_z$ (multiplication by the independent variable z , i.e., $M_z f = zf$) on a Hilbert space of analytic functions on a bounded domain G in \mathbb{C} . The characterization of these M_z -invariant subspaces is particularly interesting since it entails both the properties of the functions inside the domain G , their zero sets for example, as well as the behavior of the functions near the boundary of G . The operator M_z is not only interesting in its own right but often serves as a model operator for certain classes of linear operators. By this we mean that given an operator T on \mathcal{H} with certain properties (certain subnormal operators or two-isometric operators with the right spectral properties, etc.), there is a Hilbert space of analytic functions on a domain G for which T is unitarity equivalent to M_z .

Probably the first to successfully study these types of problems was Beurling [13] who gave a complete characterization of the M_z -invariant subspaces of the Hardy space of the unit disk. These are the functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which are analytic on the open unit disk $\mathbb{D} := \{|z| < 1\}$ for which $\sum_{n \geq 0} |a_n|^2 < \infty$. Many others followed with a discussion, often a complete characterization, of the M_z -invariant subspaces where the Hardy space is replaced by the space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on \mathbb{D} satisfying $\sum_{n \geq 0} w_n |a_n|^2 < \infty$, where $(w_n)_{n \geq 0}$ is a sequence of positive weights. For example, when $w_n = n$, we get the classical Dirichlet space where the M_z -invariant subspaces were discussed in [60, 61, 62]. When $w_n = n^\alpha$ and $\alpha > 1$, we get certain weighted Dirichlet spaces where the M_z -invariant subspaces were completely characterized in [69]. See [52, 53] for some related results. When $w_n = n^{-1}$ (or more generally $w_n = n^\alpha$, $\alpha < 0$), we get the Bergman (weighted Bergman) spaces where the M_z -invariant subspaces were discussed in [8, 68]. See also [30, 42].

In Beurling's seminal paper, and the ones that followed, notice how the underlying domain of analyticity is kept fixed to be the unit disk \mathbb{D} , but the Hilbert space of analytic functions is changed by varying the weights w_n . In a series of papers beginning with Sarason [65], the basic type of Hilbert space is fixed but the domain of analyticity is changed. To see what we mean here, the condition $f(z) = \sum_{n \geq 0} a_n z^n$ is analytic on \mathbb{D} and $\sum_{n \geq 0} |a_n|^2 < \infty$, the definition of the Hardy space of \mathbb{D} , can be equivalently restated as

f is analytic on \mathbb{D} and there is a harmonic function U on \mathbb{D} for which $|f|^2 \leq U$ on \mathbb{D} . Such a function U is called a harmonic majorant for $|f|^2$. For a general bounded domain $G \subset \mathbb{C}$, one can define the Hardy space of G to be the analytic functions f on G for which $|f|^2$ has a harmonic majorant on G . Beginning with Sarason's paper, there were several authors [6, 7, 37, 44, 64, 76, 77, 78] who characterized the M_z -invariant subspaces of the Hardy space of annular-type domains, which include an annulus, a disk with several holes removed, and a crescent domain (the region between two internally tangent circles).

Conspicuously missing from this list of domains are slit domains, for example $G = \mathbb{D} \setminus [0, 1)$. In this monograph, we obtain a complete characterization of the M_z -invariant subspaces of the Hardy space of slit domains. Along the way, we give a thorough exposition of the Hardy space, and even the Hardy-Smirnov space, of a slit domain as well as several applications of our results to de Branges-type spaces and the classical backward shift operator of the Hardy space of \mathbb{D} . We also discuss several aspects of the operator $M_z|_{\mathcal{M}}$, where \mathcal{M} is an M_z -invariant subspace of the Hardy space of G . In particular, we explore questions about cyclicity, the spectrum, and the essential spectrum for $M_z|_{\mathcal{M}}$.



<http://www.springer.com/978-3-0346-0097-2>

The Hardy Space of a Slit Domain

Aleman, A.; Feldman, N.S.; Ross, W.T.

2009, 144 p., Softcover

ISBN: 978-3-0346-0097-2

A product of Birkhäuser Basel