Chapter 3
First Examples

3.1 A case study: hyperbolic Julia sets

Unless otherwise specified, in this section "dist" will stand for the distance in the spherical metric in $\hat{\mathbb{C}}$.
A rational mapping $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is called *hyperbolic* if the orbit of every critical point of R is either periodic, or converges to an attracting (or a super-attracting) cycle.
As easily follows from Implicit Function Theorem and considerations of local dynamics of an attracting orbit, hyperbolicity is an open property in the space of coefficients of rational mappings of degree $d \geq 2$. Fatou has conjectured that hyperbolic parameters are also dense in this space. This conjecture, known as the Density of Hyperbolicity Conjecture, forms the central open question in Complex Dynamics.
Considered as a rational mapping of the Riemann sphere, a quadratic polynomial $f_c(z) = z^2 + c$ has two critical points: the origin, and the super-attracting fixed point at ∞. In the case when $c \notin \mathscr{M}$, the orbit of the former converges to the latter, and thus f_c is hyperbolic. Proposition 2.13 implies that whenever f_c has an attracting orbit in $\mathbb{C}$, it is a hyperbolic mapping and $c \in \mathscr{M}$. In the quadratic case, the Density of Hyperbolicity Conjecture thus becomes:

Conjecture (Density of Hyperbolicity in the Quadratic Family). *Hyperbolic parameters are dense in* $\mathscr{M}$.

How accurate is the picture of $\mathscr{M}$ in Figure 2.6? Indeed, our ability to produce accurate images of $\mathscr{M}$ hinges on this set being computable. Peter Hertling [Her05] has shown that Density of Hyperbolicity in the quadratic family implies computability of $\mathscr{M}$.

As an example of a hyperbolic Julia set, consider the quadratic polynomial $f = f_c$ with $c = -0.12 + 0.665i$. This map has a periodic orbit

$$z_0 \approx -0.15 + 0.19i \mapsto z_1 \approx -0.13 + 0.61i \mapsto z_2 \approx -0.47 + 0.5i \mapsto z_0.$$

Its multiplier $\lambda = (f^3)'(z_0)$ satisfies $|\lambda| \approx 0.84 < 1$. We can select a small enough disk $D = B(z_0, \varepsilon)$ so that

$$f^3(z) - z_0 \approx \lambda(z - z_0) \text{ for all } z \in D,$$

and hence $f^3(D) \Subset D$. Using the Fatou-Sullivan Classification together with the Fatou-Shishikura bound, we see that every component of the interior of $K(f)$ belongs to the basin of the attracting orbit. In particular, the orbit of every point in $\mathring{K}(f)$ eventually passes through D, after which it becomes trapped in $D \cup f(D) \cup f^2(D)$.

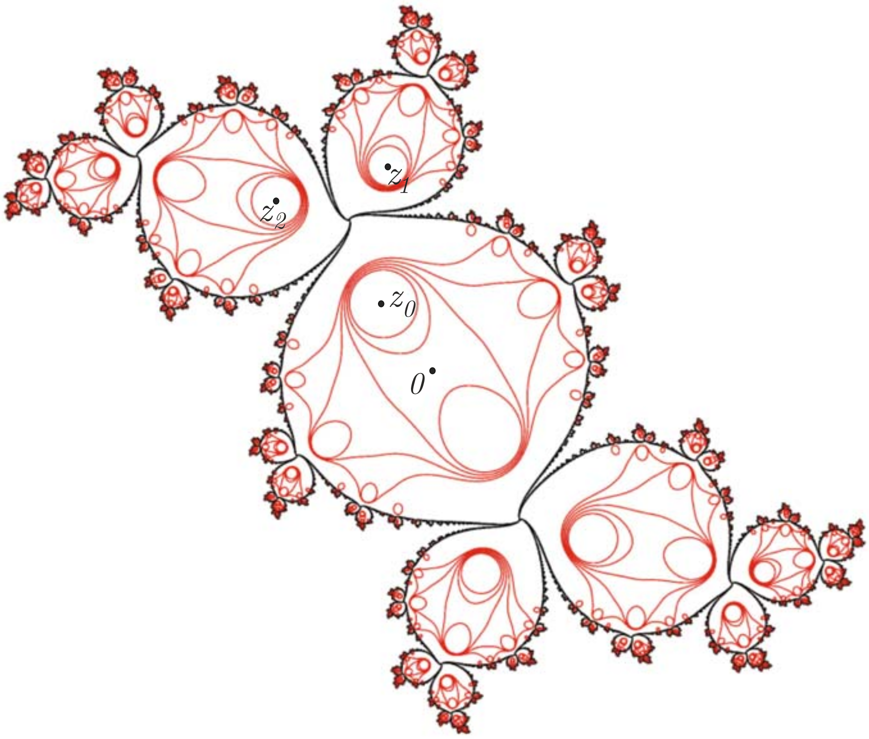

Fig. 3.1 The Julia set of f_c with $c = -0.12 + 0.665i$. A disk D around an attracting periodic point z_0 is shown together with its first fourteen preimages.

The union of the inverse images

$$\cdots f^{-m}(D) \Supset f^{-(m-1)}(D) \Supset \cdots \Supset f^{-1}(D) \Supset D$$

exhausts the interior of $K(f)$. By Proposition 2.13, one of the sets in this sequence captures the critical point 0 of f. As seen from Figure 3.1 for our choice of D it is the set $f^{-6}(D)$.

Note that the three components of the basin which contain the points of the attracting orbit meet at a "corner" point $p \approx -0.25 + 0.44i$. The point p itself is fixed under f. The shape of this Julia set is known under the name of a *Douady's rabbit*.

Computability of hyperbolic Julia sets

For a hyperbolic rational map it is easy to account for the points in the complement of the Julia set: they converge to one of the (finitely many) attracting orbits. This fact is key to proving the following.

Theorem 3.1 *Fix $d \geq 2$. There exists a TM M^ϕ with oracle access to the coefficients of a rational mapping of degree d which computes the Julia set of every hyperbolic rational map of degree d.*

In preparation to proving the theorem, let us first formulate a general fact:

Proposition 3.2 *Let $Q(z)$ be a complex polynomial. Then there exists a Turing Machine M^ϕ with an oracle input for the coefficients of $Q(z)$ such that the following holds. Consider any dyadic ball $B = B(\bar{x}, r) \subset \mathbb{C}$, $\bar{x} \in \mathbb{D}^2$, $r \in \mathbb{D}$, and let $\alpha_1, \ldots, \alpha_m$ be the roots of $Q(z)$ contained in B. For any natural number n, the machine M^ϕ will take n, r, and $\bar{x}$ as inputs, and will output a finite sequence of complex numbers $\beta_1, \ldots, \beta_k$ with dyadic rational real and imaginary parts for which:*

- *$\beta_i \in B(\bar{x}, r + 2^{-n})$;*
- *each β_i lies at a distance not more than 2^{-n} from some root of $Q(z)$;*
- *for every α_j there exists β_i with $|\alpha_j - \beta_i| < 2^{-n}$.*

For a classical reference, see [Wey24]; a review of modern approaches to iterative root-finding algorithms may be found in [BCSS98].

To understand how the Weyl's root-finding algorithm works, consider first how a square of side L in the complex plane can be tested for the presence of zeros of a polynomial Q. Not the most efficient, but a rather straightforward test, is given by the Argument Principle:

$$\frac{1}{2\pi i}\oint_{\partial G}\frac{Q'(z)}{Q(z)}dz = \text{number of zeros inside of } G$$

for any domain G in the plane whose boundary does not contain a zero. One may thus verify whether there is an approximate zero inside the square. If so, one proceeds to subdivide the original square into four congruent squares, and apply the test in each of those. The procedure is then repeated with the squares which tested positive, until the size of a square becomes smaller than the desired precision of the approximation.

Next, we show that if we *know* that a rational map R is hyperbolic, then given enough time we will find all of its attracting periodic orbits. We can actually construct *trapping discs* around the attracting orbits, such as the disc D and its preimages in the example above.

Proposition 3.3 *There is a TM M^{ϕ} that, given oracle access to the coefficients of R, outputs a finite sequence of discs $B_i = B(c_i, r_i)$ on $\hat{\mathbb{C}}$ with dyadic centers and dyadic radii such that*

- *all the attracting and super-attracting orbits of R belong to $B \equiv \cup B_i$,*
- *all orbits under R originating in B converge to an attracting periodic orbit, and*
- *$R(B) \Subset B$.*

Proof. Let $m \in \mathbb{N}$. By Proposition 3.2, it is possible to constructively approximate all periodic points of R in $\hat{\mathbb{C}}$ whose periods are at most m with precision $2^{-(m+3)}$ in the spherical metric.

For each such a periodic point z_i we will thus obtain its approximate position p_i together with a positive integer k_i such that $R^{k_i}(z_i) = z_i$. We will now approximate the image $R^{k_i}(B(p_i, 2^{-m/2}))$ of a ball around p_i. In other words, we will compute a set $W \in \mathscr{C}$ such that

$$\mathrm{dist}_H(W, R^{k_i}(B(p_i, 2^{-m/2}))) < 2^{-(m+1)},$$

and will attempt to verify that

$$B(W, 2^{-m}) \subset B(p_i, 2^{-m/2}). \tag{3.1.1}$$

This would imply

$$R^{k_i}(B(p_i, 2^{-m/2})) \subset B(W, 2^{-(m+1)}) \subset B(p_i, 2^{-m/2} - 2^{-(m+1)}). \tag{3.1.2}$$

Note that if z_i is an attracting point, then the equation (3.1.1) will hold for any sufficiently large value of m. On the other hand, by the Schwarz Lemma, equation (3.1.2) implies the existence of an attracting orbit in $B(p_i, 2^{-m/2})$, whose basin contains $B(p_i, 2^{-m/2})$. Once m and W satisfying (3.1.1) are found, we can also compute dyadic balls B_j containing each point $R^j(z_i)$ of the cycle

$$z_i \mapsto R(z_i) \mapsto \cdots \mapsto R^{k_i}(z_i) = z_i$$

so that, for each $j = 0, \ldots, k_i - 1$, $R(B_j) \Subset B_{j+1}$, where $j+1$ is taken modulo k_i.

All such B_j's will *eventually* be found, and will satisfy the conditions of the proposition. How will we know when to stop? To this end, we also compute at the m-th step a finite set C_m which is a $2^{-(m+3)}$-approximation of the m-th image $R^m(\mathrm{Crit}(R))$, and attempt to verify that $B(C_m, 2^{-(m+3)})$ is contained in the union

$\cup B_i$ of the balls we have already found. We terminate when this is the case, and output the balls B_i that we have found.

By Fatou's result 2.13, we know that, for each attracting periodic orbit, the orbit of at least one critical point converges to it. Since in our case R is hyperbolic, we know that the orbit of each critical point converges to an attracting periodic orbit, and the algorithm is guaranteed to terminate. $\square$

We are now ready to prove computability of hyperbolic Julia sets.

Proof (Theorem 3.1). As a first step, by looking at one of the balls B_j found in Proposition 3.3, we can compute a dyadic $a \in B_j$ that converges to an attracting orbit of R. Thus $a \notin J(R)$. By conjugating R by the fractional-linear map

$$f_a(z) = \frac{1}{z-a},$$

we obtain a rational map R' with $J(R') = f_a(J(R))$. The effect of the conjugation on the Julia set is to send the point a to ∞. In particular, $\infty \notin J(R')$. Thus, by a simple change of coordinates, we may assume without loss of generality that $\infty \notin J(R)$. By Proposition 2.1 it suffices to prove the computability of $J(R)$ as a subset of $\mathbb{C}$.

Informally, the idea of the argument is to estimate the Julia set from "above" and from "below". On the one hand, we know that the orbit of every point outside $J(R)$ eventually reaches the set B from Proposition 3.3. This allows us to exclude points that are far away from $J(R)$. On the other hand, we know that repelling periodic orbits are dense in $J(R)$, which permits us to eventually identify every point which is close to $J(R)$.

More formally, let $n \in \mathbb{N}$ be the input specifying the required degree of the approximation. The algorithm, which computes a set $J_n \in \mathscr{C}$ with

$$\mathrm{dist}_H(J_n, J(R)) < 2^{-n},$$

works as follows. Denote by U the complement $(\cup B_i)^c$ of the dyadic balls B_i that have been found in Proposition 3.3.

(1) Set $m := 1$.
(2) Compute a set $U_m \in \mathscr{C}$ such that $\mathrm{dist}_H(U_m, R^{-m}(U)) < 2^{-(n+3)}$.
(3) Compute a finite set L_m which approximates with precision $2^{-(n+3)}$ all periodic points of R in U, whose periods are at most m. This is possible by Proposition 3.2.
(4) Check the inclusion $B(L_m, 2^{-(n+1)}) \supset U_m$. If the inclusion holds, output the set $J_n \equiv L_m$ and exit. If not, go to step (5).
(5) Increment $m \leftarrow m+1$ and go to step (2).

Denote by O_m the set of all periodic points of R which are contained in U, and whose periods are at most m. All periodic orbits in U must be repelling. We thus have

$$O_m \subset J(R) = \overline{\cup O_m}. \tag{3.1.3}$$

On the other hand, by the Fatou-Sullivan classification, we have

$$\cap R^{-m}(U) = J(R). \tag{3.1.4}$$

For all m greater than some large enough m_0, the open neighborhood

$$B(O_m, 2^{-(n+3)}) \supset J(R),$$

as seen from the right-hand side of (3.1.3).

On the other hand, for all m greater than some large enough m_1, the open neighborhood

$$B(J(R), 2^{-(n+3)}) \supset R^{-m}(U).$$

Therefore, for each $m \geq \max(m_0, m_1)$ we have

$$B(L_m, 2^{-(n+1)}) \supset B(O_m, 2^{-(n+1)} - 2^{-(n+3)}) \supset B(J(R), 2^{-(n+2)}) \supset B(R^{-m}(U), 2^{-(n+2)} - 2^{-(n+3)}) \supset U_m, \tag{3.1.5}$$

and so the algorithm is guaranteed to terminate. When that happens, by the left-hand side of (3.1.3) we have

$$\text{dist}(z, J(R)) < 2^{-n}$$

for all $z \in L_m$. On the other hand, by (3.1.4) and (3.1.5) we have

$$\text{dist}(z, L_m) < 2^{-n}$$

for all $z \in J(R)$. □

The idea of approximating the Julia set from "above" and "below" which is featured in the above algorithm will be very useful for us in proving positive results. As far as we could tell, its first appearance in the theoretical literature is in the work of Zhong [Zho98]. Its practical applications are, however, rather limited. Of course, one can always attempt to generate images of a Julia set by computing the periodic orbits of periods at most m (or, alternatively, the first m preimages of a single point in $J(R)$). Apart from the fact that the picture may be rather far from the true image of $J(R)$, it will also generally require exponential time to generate.

On the other hand, for a polynomial mapping P, it is easy to determine a domain $U \in \hat{\mathbb{C}}$ whose preimages shrink to the *filled* Julia set $K(P)$. Indeed, any large enough disk around the origin would do. Algorithms approximating $K(P)$ by $P^{-m}(U)$ are perhaps the most widely used. They are known as the *escape-time* algorithms. Their obvious Achilles' heel is the general absence of an estimate on the distance $\text{dist}_H(K(P), P^{-m}(U))$ in terms of m.

Obtaining an estimate on the distance to $J(R)$ in polynomial time requires another idea, which is a key in the proof of the following:

Theorem 3.4 (Cf. [Bra04, Ret05]) *Fix $d \geq 2$. There exists a TM M^ϕ with oracle access to the coefficients of a rational mapping of degree d which computes the Julia set of every hyperbolic rational map of degree d with a polynomial complexity bound.*

We begin with the following standard fact:

Proposition 3.5 *A rational mapping R of degree $d \geq 2$ is hyperbolic if and only if there exists a smooth metric μ defined on an open neighborhood of $J(R)$ and a constant $\lambda > 1$ such that the derivative*

$$||DR^n(z)||_\mu > \lambda^n \text{ for every } z \in J(R), n \in \mathbb{N},$$

as long as the image $R^n(z)$ stays in the domain of definition of μ.

Note that the term "hyperbolic" has an established meaning in dynamics. In the context of one dimensional dynamical systems it means "uniformly expanding (or contracting)". Thus Proposition 3.5 justifies the use of the word in our case.

By compactness of $J(R)$, *in the spherical metric*, we will have

$$||DR^n(z)|| > C\lambda^n, \tag{3.1.6}$$

for $C > 0$ independent of n.

The proof of the existence of a metric μ for a hyperbolic mapping R is both instructive and useful for our purposes, and so we outline it below.

Proof (Proposition 3.5.). Let $\{B_i\}$ be the finite collection of dyadic balls around the attracting periodic orbits as in Proposition 3.3. Consider the union $B = \cup B_i$. By Proposition 3.3, the sequence of preimages of B grows successively larger:

$$B \subset R^{-1}(B) \subset R^{-2}(B) \subset \cdots, \text{ and } J(R) \subset \hat{\mathbb{C}} \setminus R^{-k}(B) \text{ for all } k \in \mathbb{N}.$$

By Fatou's result 2.13, there exists $k \in \mathbb{N}$ such that $R^{-k}(B)$ contains the entire postcritical set of R. Setting $V = \hat{\mathbb{C}} \setminus R^{-k}(B)$ and $U = R^{-1}(V)$, we see that $U \subsetneq V$, and

$$R : U \to V$$

is an unbranched analytic covering. By the Schwarz-Pick Theorem, it is an isometry of the hyperbolic metrics of U and V. On the other hand, by the same theorem, the proper inclusion $\iota : U \hookrightarrow V$ is a contraction of the hyperbolic metric. By the Chain Rule, for $z \in U$, we have

$$||DR(z)||_{V,V} = ||D\iota^{-1}(z)||_{V,U} ||DR(z)||_{U,V} = ||D\iota^{-1}(z)||_{V,U} > 1.$$

Note that the Julia set $J(R) \Subset U$ and, selecting a neighborhood $W \subset U$ of $J(R)$ which is compactly contained in V, we have

$$||DR(z)||_\mu > \lambda > 1 \text{ for all } z \in W,$$

where μ denotes the hyperbolic metric ρ_V. By the Chain Rule, the derivative of the n-th iterate

$$||DR^n(z)||_\mu > \lambda^n \text{ for } z \in J(R),$$

which concludes our proof. □

Let us make a useful note:

Proposition 3.6 *The constants λ and C of (3.1.6) can be estimated constructively.*

Proof. The algorithm for estimating C is easily derived from Proposition 2.4.

To estimate λ, note that the contraction coefficient of the inclusion $||\iota'(z)||_{U,V}$ can be bounded by a constant depending only on the value of

$$d = \mathrm{dist}_V(z, V \setminus U)$$

(the distance measured in the hyperbolic metric of V). Indeed, let us lift the inclusion $z \in U \hookrightarrow V$ to $z' \in U' \hookrightarrow \mathbb{U}$. Denote by v any of the points of $\mathbb{U} \setminus U'$ for which

$$\mathrm{dist}_{\mathbb{U}}(z', v) = d.$$

By applying a suitable fractional-linear transformation, send v to 0, and z' to $x \in (0,1)$. An explicit computation gives

$$d = \log \frac{1+x}{1-x}, \text{ so that } x = \frac{e^d - 1}{e^d + 1}.$$

By the Schwarz-Pick Theorem, the hyperbolic derivative of the inclusion $U' \hookrightarrow \mathbb{U}$ will become larger, if we make $\mathbb{U} \setminus U'$ smaller. More specifically, let us consider the domain $W = \mathbb{U} \setminus \{0\}$. Then, comparing the inclusions

$$\iota_1 : U' \hookrightarrow \mathbb{U},\ \iota_2 : W \hookrightarrow \mathbb{U},$$

we have

$$||D\iota(z)||_{U,V} = ||D\iota_1(x)||_{U',\mathbb{U}} \leq ||D\iota_2(x)||_{W,\mathbb{U}}.$$

The expression on the right can be estimated explicitly. It is equal to

$$a(x) = \frac{2|x \log x|}{1 - x^2} < 1.$$

We obtain a lower bound on the expansion factor λ_z at the point z as

$$\lambda(d) = 1/a(x) \text{ for } x = \frac{e^d - 1}{e^d + 1}.$$

Note that this bound decreases with d.

From Proposition 2.4, we can constructively obtain a uniform lower bound

$$d_l \leq \sup_{z \in U} \mathrm{dist}_V(z, V \setminus U).$$

The value of $\lambda(d_l) > 1$ is thus a constructive estimate for the expanding factor λ.
$\square$

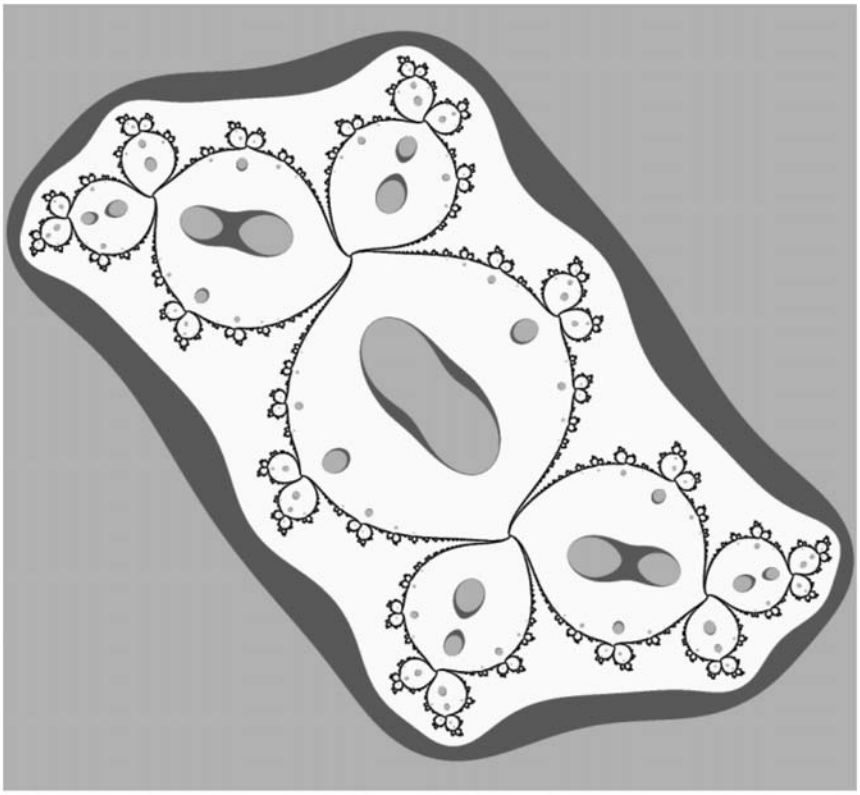

Fig. 3.2 Domains U (the complement of darker gray) and V (the complement of lighter gray) for a rabbit from the previous figure. The rabbit has two attracting orbits in $\hat{\mathbb{C}}$, the fixed point at ∞, and the period-3 cycle.

Proposition 3.7 (Preparatory step in the construction of M^{ϕ}) *There exists an algorithm which, given the coefficients of a hyperbolic rational map R of degree $d \geq 2$, outputs a planar domain $U \in \mathscr{C}$ such that:*

(I) $U \subsetneq R(U)$,
(II) $R(U) \cap \mathrm{Postcrit}(R) = \emptyset$,
(III) $J(R) \Subset U$.

Proof. We use the balls B_i around the attracting periodic orbits we found in Proposition 3.3. Let $\Omega_0 := \hat{\mathbb{C}} \setminus \cup B_i$. Define $\Omega_{i+1} := R^{-1}(\Omega_i)$ for all $i \geq 0$. By the properties of B_i, we have $\Omega_1 \Subset \Omega_0$. If we let $U_0 \in \mathscr{C}$ be any set such that $\Omega_1 \subset U_0 \subset \Omega_0$, U_0 will satisfy properties (I) and (III) above. To see that (I) holds observe that

$$U_0 \subset \Omega_0 = R(\Omega_1) \subset R(U_0).$$

(III) holds because $J_R \subset \Omega_1 \subset U_0$. Similarly, for any k we can compute $U_k \in \mathscr{C}$ such that $\Omega_{k+1} \subset U_k \subset \Omega_k$. For each such U_k conditions (I) and (III) hold just as they do for U_0.

Note that, if for some k

$$U_{k-2} \cap \mathrm{Postcrit}(R) = \emptyset, \tag{3.1.7}$$

we will be able to verify this. In this case $\Omega_{k-1} \cap \mathrm{Postcrit}(R) = \emptyset$, and thus $R(U_k) \cap \mathrm{Postcrit}(R) = \emptyset$, and $U = U_k$ satisfies condition (II). It remains to see that there is a k such that (3.1.7) holds. To this end, we use Fatou's result 2.13, which guarantees that all postcritical orbits leave Ω_0 in finitely many steps. Hence there is an ℓ such that $\Omega_\ell \cap \mathrm{Postcrit}(R) = \emptyset$, and $k := \ell + 2$ satisfies (3.1.7). □

We are now ready to compute $J(R)$ in polynomial time.

Proof (Theorem 3.4). At the preparatory stage of the computation, we obtain the domain U as in Proposition 3.7. Since the closure of the domain U does not intersect the postcritical set of R, we can compute a lower bound $s > 0$ on the distance from U to $\mathrm{Postcrit}(R)$.

Let us now run the algorithm of Theorem 3.1 to obtain a set $L \in \mathscr{C}$ with

$$\mathrm{dist}_H(L, J(R)) < s/8.$$

Computing several further preimages of U with sufficient precision, we can obtain a smaller domain $\widetilde{W} \Supset J(R)$ and such that:

- $R^2(\widetilde{W}) \Subset U$, and
- $\mathrm{dist}(z, L) < s/4$ for each $z \in \widetilde{W} \cup R(\widetilde{W})$;

and compute a dyadic number $\ell > 0$ such that

$$\mathrm{dist}(z, J(R)) > \ell \text{ for all } z \notin \widetilde{W}.$$

Set $V = R(U)$. Compute a dyadic $\varepsilon > 0$ such that $B(\widetilde{W}, \varepsilon) \Subset U$.

From Proposition 3.6 we find a lower bound $\lambda > 1$ on the expansion $||R'(z)||$ in the hyperbolic metric in V for $z \in R(\widetilde{W})$. We also construct a constant C from (3.1.6) as per Proposition 2.4. Thus, we have

$$||DR^n(z)|| > C\lambda^n,$$

for as long as the orbit of z stays in $R(\widetilde{W})$.

Suppose now that we are given a dyadic point $x \in \hat{\mathbb{C}}$, and a parameter m. Our goal is to output 1 if $d(x, J(R)) < 2^{-m}$, and 0 if $d(x, J(R)) > 2 \cdot 2^{-m}$. All the preliminary steps take time that depends on the hyperbolic function R but not on the precision parameter m. Consider the following subprogram; the logs are all base-2:

$i := 1$

while $i \leq m/\log\lambda - \log C/\log\lambda + 1$ **do**

(1) Compute the values of

$$p_i \approx R^i(x) = R(R^{i-1}(x)) \text{ and } d_i \approx DR^i(x) = DR^{i-1}(x) \cdot DR(R^{i-1}(x))$$

with precision $\min(2^{-(m+3)}, \varepsilon/4)$.

(2) Check the inclusions $p_i \in \widetilde{W}$ and $p_i \in R(\widetilde{W})$:

- if $p_i \in \widetilde{W}$, go to step (5),
- if $p_i \notin R(\widetilde{W})$, proceed to step (3),
- if neither holds either option is fine.

(3) Check the inequality

$$\frac{\ell}{d_i} > 2^{-m}$$

with precision $2^{-(m+1)}$. If true, output 0 and exit the subprogram, else

(4) output 1 and exit the subprogram.

(5) $i \leftarrow i + 1$

end while

(6) Output 1 and exit.

end

The program runs for at most $m/\log\lambda - \log C/\log\lambda + 1 = O(m)$ iterations each of which consists of a constant number of arithmetic operations with $O(m)$ bits of precision. Hence the running time of the program can be bounded by $O(m^2 \log m \log\log m)$ using efficient multiplication (even slightly faster, see [Fur07]).

Suppose the subprogram outputs 0 and exits on line (3). This case is illustrated in Figure 3.3(A). The fact that the subprogram has reached line (3) means that the ball $B(p_i, l)$ is disjoint from $J(R)$. Also by the construction of $\widetilde{W}$ this ball contains no postcritical points, and hence there is a neighborhood N_0 of x that maps conformally to $B(p_i, l)$ under R^i. By the invariance of $J(R)$, N_0 is disjoint from $J(R)$. By the Koebe One-Quarter Theorem, the distance from x to $J(R)$ is at least

$$\text{dist}(x, J(R)) \geq \ell' = \frac{1}{4} \cdot \frac{\ell}{DR^i(x)} \geq 2^{-(m+3)}.$$

On the other hand, suppose the subprogram exits on line (4), a case illustrated in Figure 3.3(B). If this is true, surround the point $R^i(x)$ with the disks $B = B(R^i(x), s/2)$, and $\hat{B} = B(R^i(x), 3s/4) \Supset B$. By construction, $B \cap J(R) \neq \emptyset$.

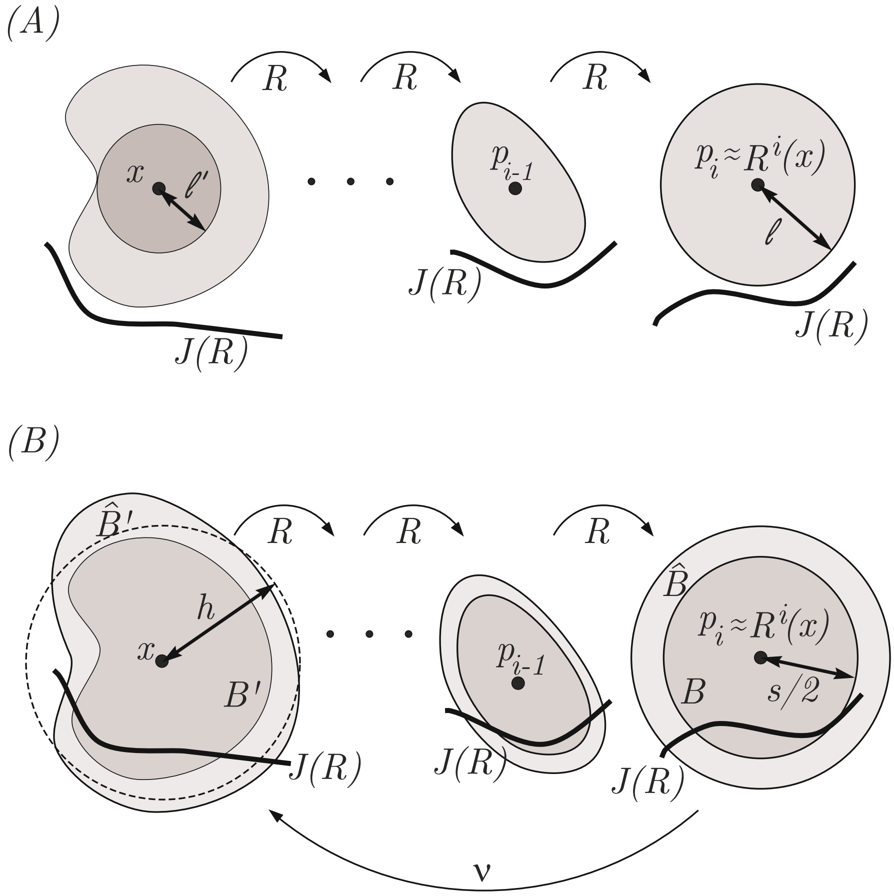

Fig. 3.3 A schematic figure illustrating the proof of correctness of the algorithm. Figure (A) illustrates exit on line (3) of the algorithm. Figure (B) illustrates exit on line (4).

On the other hand, as $R^2(\widetilde{W}) \subset U$, the disk $\hat{B}$ does not intersect with Postcrit(R). Hence there exists a well-defined branch of the inverse $\nu = R^{-i} : \hat{B} \mapsto \hat{B}' \ni x$. Denote by $B' \Subset \hat{B}'$ the image of B by this branch. Note that $B' \cap J(R) \neq \emptyset$.

We will now apply the Koebe Distortion Theorem to the restriction of ν from the larger disk $\hat{B}$ to the smaller one B. Namely, set $M(r) = (1+r)/(1-r)^3$. Note that the ratio of the radii of B and $\hat{B}$ is $r = 2/3$. By the Koebe Distortion Theorem

$$B' \subset B(x,h), \text{ where } h = \frac{s}{2DR^i(x)} \cdot M(2/3).$$

Putting this together with the negation of the inequality from line (3), we have

$$\mathrm{dist}(x, J(R)) < K \cdot 2^{-(m+3)}, \text{ where } K = \frac{8sM(2/3)}{\ell}. \tag{3.1.8}$$

Finally, suppose the sub-program exits on the last instruction. In this case, $x \in R^{-(i-1)}(\widetilde{W})$. On the other hand,

$$\mathrm{dist}(R^{-(i-1)}(\widetilde{W}), J(R)) < C^{-1}\lambda^{-m/\log\lambda + \log C/\log\lambda} = 2^{-m}.$$

In summary, converting all exponential estimates to base 2, there exists $M \in \mathbb{N}$ such that, for every $j \in \mathbb{N}$ sufficiently large, the subprogram can be used to distinguish between the cases:

- $\mathrm{dist}(x, J(R)) > K \cdot 2^{-j}$ (outputs 0), and
- $\mathrm{dist}(x, J(R)) < 2^{-j}$ (outputs 1).

This is not quite what we need, as we would like to distinguish the cases when this distance is $> 2^{-(m-1)}$ from when it is $< 2^{-m}$. To this end, we simply need to partition each pixel with side 2^{-n} into sub-pixels of size $2^{-n}/K$ and run the subprogram in the center of each of the sub-pixels. This only introduces a constant multiplicative overhead into the algorithm. □

The algorithms which use the estimate on the derivative of an iterate $R^i(z)$ to get an upper and lower bounds on the distance to $J(R)$ through the considerations of the Koebe One-Quarter Theorem and the Koebe Distortion Theorem, are known as *Distance Estimators*. They were first proposed by Milnor [Mil89] and Fisher [Fis89]. This approach can be very useful but, however, it has several obvious limitations. Firstly, a domain U whose preimages shrink to $J(R)$ cannot always be constructed (and indeed, does not always exist). But even when this obstacle can be overcome, the time bound on the rate of convergence of $R^{-m}(U)$ to $J(R)$ may be impractical. In the next section we will discuss a simple family of examples for which this bound becomes exponential.

3.2 Maps with parabolic orbits

Local dynamics of a parabolic orbit

We will describe here briefly the local dynamics of a rational mapping R with a parabolic periodic point p. By replacing R with its iterate, if needed, we may assume that $R(p) = p$, and $R'(p) = 1$. The map R then can be written as

$$R(z) = z + a(z-p)^{n+1} + O((z-p)^{n+2}), \tag{3.2.1}$$

for some $n \in \mathbb{N}$ and $a \neq 0$. Note that the integer $n+1$ is the local *multiplicity* of p as the solution of $R(z) = z$.

A complex number $\nu \in \mathbb{T}$ is called an *attracting direction* for p if the product $a\nu^n < 0$, and a *repelling direction* if the same product is positive. For each infinite orbit $\{R^k(z)\}$ which converges to the parabolic point, there is one of the n attracting directions ν for which the unit vectors

$$(R^k(z) - p)/|R^k(z) - p| \underset{k\to\infty}{\longrightarrow} \nu.$$

We say in this case that the orbit converges to p in the direction of ν. For each attracting direction ν, we say that a topological disk U is an *attracting petal* of R at p if the following properties hold:

- $\overline{U} \ni \{p\}$;
- $R^n(\overline{U}) \subset U \cup \{p\}$;
- an infinite orbit $\{R^k(z)\}$ is eventually contained in U if and only if it converges to p in the direction of ν.

Similarly, U is a *repelling petal* for R if it is an attracting petal for the local branch of R^{-1} which fixes p.

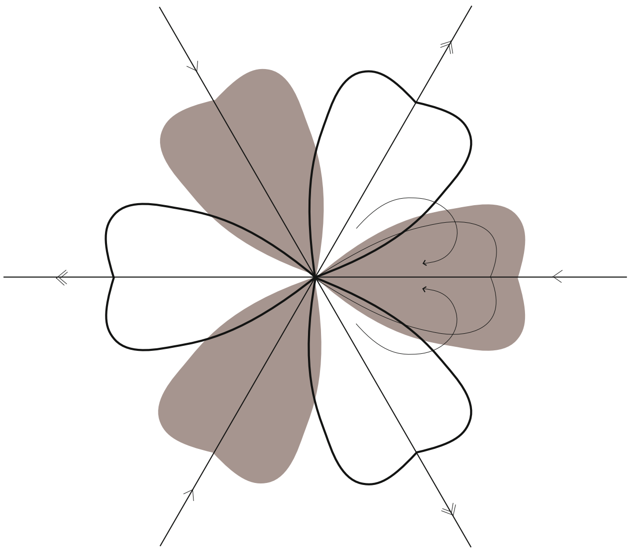

Fig. 3.4 A Leau-Fatou flower with three attracting petals (shaded) and three repelling petals (emphasized). The attracting and repelling directions are also indicated. The arrows show the direction of the orbits in one of the petals; the image of this petal is also indicated.

The petals form a *Leau-Fatou Flower* at p:

Theorem 3.8 *There exists a collection of n attracting petals P_i^a, and n repelling petals P_j^r such that the following holds. Any two repelling petals do not intersect,* *and every repelling petal intersects exactly two attracting petals. Similar properties hold for attracting petals. The union*

$$(\cup P_i^a) \cup (\cup P_j^r) \cup \{p\}$$

forms an open simply-connected neighborhood of p.

The proof of this statement is based on a multivalued change of coordinates

$$w = \kappa(z) = \frac{c}{(z-p)^n}, \text{ where } c = -\frac{1}{na}.$$

The map κ conformally transforms the infinite sector between two repelling directions into the plane with the negative real axis removed. In this sector, it changes the map R into

$$F(w) = w + 1 + O(1/\sqrt[n]{|w|}), \text{ as } w \to \infty.$$

Selecting a right half-plane $H_r = \{\mathrm{Re}\, z > r\}$ for a sufficiently large $r > 0$, we have

$$\mathrm{Re}\, F(w) > \mathrm{Re}\, w + 1/2, \text{ and hence } F(H) \subset H.$$

The corresponding attracting petal can then be chosen as the domain $\kappa^{-1}(H)$, using the appropriate branch of the inverse. Note that, given the coefficients of the rational mapping R, the description of the petal is constructive. Let us formulate this last statement in a language suitable for later references:

Lemma 3.9 *For each degree $d \geq 2$ there exists an oracle Turing Machine M^ϕ such that the following holds. Let R be a rational mapping of degree d with a parabolic periodic point p, with period m and multiplier* 1. *Let n be as in (3.2.1). The machine M^ϕ takes as input the values of m, n and a natural number k; it is given oracle access to the coefficients of R and the value of p. It outputs a set $L_k \in \mathscr{C}$ such that the following is true:*

- *$L_{k+1} \supset L_k$ and $\cup L_k = P$ is the union of attracting petals of R at p, covering all the attracting directions;*
- $\mathrm{dist}_H(L_k, P) < 2^{-k}$.

The dynamics inside a petal is described by the following:

Proposition 3.10 *Let P be an attracting or repelling petal of R. Then there exists a conformal change of coordinates Φ inside P, transforming $R(z)$ into the unit translation $z \mapsto z+1$. The image $\Phi(P)$ covers a right half-plane.*

The function Φ is called the *Fatou coordinate* , with the prefix *attracting* or *repelling* depending on the type of the petal P.

As an implication of Proposition 3.10, consider the quotient manifold $C_A \equiv P/_{z\sim R(z)}$, which parametrizes the *orbits* converging to the parabolic point through P. Then C_A is conformally isomorphic to the quotient of a right half-plane by the unit translation, which is the cylinder $\mathbb{C}/\mathbb{Z}$.

Suppose now that the multiplier of the fixed point p is a q-th root of unity, $R'(p) = e^{2\pi ip/q}$, where $(p,q) = 1$. A fixed petal for the iterate R^q corresponds to a cycle of q petals for R. It thus follows that q divides the number n of attracting/repelling directions of p as a fixed point of R^q. We make note of the following proposition, due to Fatou:

Proposition 3.11 *Each cycle of attracting petals of a rational mapping R captures an orbit of a critical point of R.*

This implies, in particular, that a quadratic polynomial f_c with a parabolic periodic point ζ with multiplier $e^{2\pi ip/q}$ has a Leau-Fatou flower at ζ with a single cycle of q attracting petals.

Computability of Julia sets in the presence of a parabolic orbit

A hyperbolic Julia set is computable (cf. Theorem 3.1) because it is easy to verify that an orbit belongs to the Fatou set of a hyperbolic rational mapping. A trapping neighborhood around every attracting orbit of such a mapping can be constructed algorithmically, and only those orbits which enter one of these neighborhoods do not lie in the Julia set.

An analogous approach in the presence of a parabolic cycle would require us to construct attracting petals, to detect the orbits which converge to the cycle. This construction cannot be made fully automated. Some non-uniform information will be required by the algorithm. For simplicity, let us formulate the computability statement only for Julia sets of parabolic quadratics. A more general theorem on the Julia set of a rational map whose Fatou set consists only of parabolic and attracting basins is easily obtained along the same lines.

Theorem 3.12 *There exists a Turing Machine M^ϕ with an oracle for a complex parameter c which computes the Julia set J_c of every parabolic quadratic polynomial f_c, given the following non-uniform information:*

- *the period m of the unique parabolic orbit of f_c;*
- *positive integers p and q with $(p,q) = 1$ such that the multiplier of the parabolic orbit of f_c is equal $e^{2\pi ip/q}$.*

Proof. Denote the parabolic orbit of f_c by

$$p_1 \mapsto p_2 \mapsto \cdots \mapsto p_m.$$

Note that the Taylor's expansion of $f_c^{q\cdot m}$ near each of the points p_i has the form

$$f_c^{q\cdot m}(z) = p_i + (z - p_i) + \alpha_{n+1}(z - p_i)^{n+1} + \ldots \alpha_{n+2}(z - p_i)^{n+2} + \ldots$$

Here n is the number of attracting (or repelling) directions. As we are in the quadratic case, there are exactly q attracting petals in the Leau-Fatou flower, so that $n = q$.

By Proposition 3.2, the roots of

$$f_c^m(z) = z$$

can be determined with an arbitrary accuracy. Among these solutions, repelling periodic points can be identified and excluded algorithmically. Thus the points $p_1, \dots, p_m$ can be identified with any desired precision. Hence, we can construct a sequence of domains L_k for the iterate f_c^{qm} provided by Lemma 3.9.

Now the proof of the theorem proceeds similarly to that of Theorem 3.1. By Proposition 2.16, $|c| \leq 2$. Hence the ball $D = B(0, 4.1)$ has the property $f_c^{-1}(D) \Subset D$, and all orbits which originate outside of D converge to ∞. Fixing k, we obtain the picture of J_c with precision 2^{-k} as follows:

1. Set $t = 1$.
2. Compute $U_t \in \mathscr{C}$ which approximates $f_c^{-t}(D) \setminus f_c^{-t}(L_t)$ up to an error of $2^{-(k+3)}$ in Hausdorff metric.
3. Compute $V_t \in \mathscr{C}$ which approximates $\cup_{s=1}^{t} f_c^{-s}(p_1)$ up to an error of $2^{-(k+3)}$ in Hausdorff metric.
4. Check the inclusion $U_t \subset B(V_t, 2^{-(k+1)})$. If true, output V_t and halt. If false, increment $t \mapsto t+1$ and go to step 2.

The verification of the algorithm is straightforward, and is left to the reader. Note that by Proposition 2.9 the set $\cup_{s=1}^{\infty} f_c^{-s}(p_1)$ is dense in J_c, and thus the sequence $\{V_t\}$ approximates J_c well from below. □

3.3 Computing Julia sets with parabolic orbits efficiently

3.3.1 The Distance Estimator in the parabolic case

Julia sets with parabolic orbits are well-known examples for which the Distance Estimator algorithm of §3.1 becomes impractical (cf. the discussion in [Mil06]). As we have already noted, a key to the successful application of the algorithm in the case of a hyperbolic rational map R is that, for a point z which lies at a distance 2^{-n} of the Julia set $J(R)$, it would only take $O(n)$ iterates to magnify this distance to the order of 1. The situation becomes very different if there is a parabolic orbit in $J(R)$.

To fix the ideas, let us consider a very simple example – a quadratic polynomial $f(z) = z + z^2$ with a parabolic fixed point at the origin. Take a point $z_0 = 2^{-n}$ for some large value of n. On the one hand, $z_0 \notin J(f)$. Indeed, if we denote $z_k = f^k(z_0)$, then an easy induction shows that

$$z_k \geq 2^{-n} + k \cdot 2^{-2n} \longrightarrow \infty.$$

In fact, z_0 lies at a distance of approximately 2^{-2n} from $J(f)$.

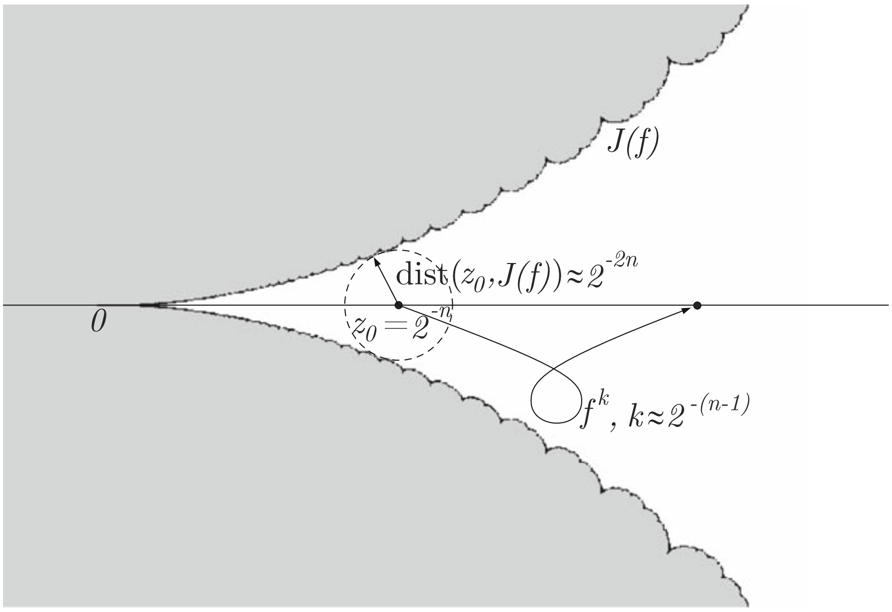

Fig. 3.5 Slow orbits in the neighborhood of the parabolic point.

On the other hand, if

$$z_k < 2^{-(n-1)}, \text{ then } z_{k+1} < z_k + 2^{-(2n-2)}.$$

Hence it will take the orbit of z_0 at least 2^{n-1} steps to reach $2 \cdot 2^{-n}$ (cf. Figure 3.5). We see that the orbit escapes to ∞, but it will take approximately 2^n steps to reach distance of order 1 from the origin. So if we apply the Distance Estimator algorithm to $f(z)$, it will become *exponential*- rather than polynomial-time.

Thus, a naïve approach to drawing a Julia set with parabolics leads to an impractical algorithm. To accelerate it, we will have to look at the dynamics of a rational map near a parabolic point more carefully. To avoid messy technicalities, let us again concentrate on the example of $f(z)$.

3.3.2 Accelerating the map $z \mapsto z + z^2$

Instead of iterating $f(z)$ starting at z_0 which is very near zero, let us write down the iterates of f on an arbitrary z symbolically. We only write the first 5 coefficients of each iteration.

$$\begin{cases} f^1(z) = z + 1\cdot z^2 + 0\cdot z^3 + 0\cdot z^4 + 0\cdot z^5 + \dots \\ f^2(z) = z + 2\cdot z^2 + 2\cdot z^3 + 1\cdot z^4 + 0\cdot z^5 + \dots \\ f^3(z) = z + 3\cdot z^2 + 6\cdot z^3 + 9\cdot z^4 + 10\cdot z^5 + \dots \\ f^4(z) = z + 4\cdot z^2 + 12\cdot z^3 + 30\cdot z^4 + 64\cdot z^5 + \dots \\ f^5(z) = z + 5\cdot z^2 + 20\cdot z^3 + 70\cdot z^4 + 220\cdot z^5 + \dots \\ \quad\vdots \end{cases}$$

We can see some patterns in the coefficients of $f^k(z)$. For example, the coefficient of z is always 1 and the coefficient of z^2 is k. Higher coefficients are given by

$$f^k(z) = z + k\cdot z^2 + k(k-1)\cdot z^3 + \frac{(2k-3)k(k-1)}{2}\cdot z^4 + \frac{(3k-4)k(k-1)(k-2)}{3}\cdot z^5 + \dots \tag{3.3.1}$$

A few observations can be made about the formula:

- The coefficient of z^r is a polynomial in k of degree $r-1$ with leading coefficient 1;
- the coefficient of z^r is always between 0 and k^{r-1}.

Denote the coefficient of z^r by $c_r(k)$. First, let us show how to compute $c_r(k)$ explicitly in general. We know that $c_0(k) = 0$ and $c_1(k) = 1$. For $r \geq 2$, we use the recurrence $f^{k+1}(z) = f(f^k(z))$ to obtain

$$c_r(k+1) = c_r(k) + \sum_{j=1}^{r-1} c_j(k)c_{r-j}(k). \tag{3.3.2}$$

Thus we obtain an explicit recurrence

$$c_r(k) = \sum_{t=1}^{k-1}\sum_{j=1}^{r-1} c_j(t)c_{r-j}(t). \tag{3.3.3}$$

By solving the recurrence we obtain the formulas for the coefficients in (3.3.1). Formulas for the first n coefficients can be obtained with sufficiently high precision in time polynomial in n.

We would like to use the newly obtained symbolic coefficients of the k-th iteration of f to make big "leaps" in the iterations of f for values of z that are very close to 0 (where the iteration takes a long time to converge). We have the formula

$$f^k(z) = z + c_2(k)z^2 + c_3(k)z^3 + \dots \tag{3.3.4}$$

For a small z with $|z| \approx 2^{-n}$, we would like to make $2^{n-1} = \Omega(1/|z|)$ iterations in one step. We do this by plugging in $k = 2^{n-1}$. We can afford to take poly(n) terms of the sum (3.3.4), and thus we need all the subsequent terms to be insignificant.

Proposition 3.13 *For each k and for each r, $c_r(k) \leq k^{r-1}$.*

Proposition 3.13 implies that as long as $k < \frac{1}{2|z|}$, we will have $c_r(k)z^r < 2^{-r}/k$, and thus n terms would suffice in order to maintain precision of 2^{-n}. The proposition is proved by induction.

Proof (Proposition 3.13). $c_1(k) = 1$ and $c_2(k) = k$, hence the statement is true for $r = 1,2$. For higher r's we prove it by induction. Assume it is true up to $r-1$ for some $r \geq 3$. By the induction hypothesis,

$$\begin{aligned} c_r(k) = \sum_{t=1}^{k-1}\sum_{j=1}^{r-1} c_j(t)c_{r-j}(t) \leq \sum_{t=1}^{k-1}\sum_{j=1}^{r-1} t^{j-1}t^{r-j-1} = \\ \sum_{t=1}^{k-1}\sum_{j=1}^{r-1} t^{r-2} = \sum_{t=1}^{k-1}(r-1)t^{r-2} \leq \\ \sum_{t=1}^{k-1}((t+1)^{r-1} - t^{r-1}) = k^{r-1} - 1 < k^{r-1}. \end{aligned}$$

□

Proposition 3.13 can now be used to compute a "long" iteration of a point z such that $|z|$ is small.

Proposition 3.14 *Suppose $|z| < 1/m$ for a sufficiently large m. Then we can compute the $\ell = \lfloor m/2 \rfloor$-th iterate of z and its derivative $df^\ell/dz(z)$ with a given precision 2^{-s} in time polynomial in s and $\log\, m$.*

Proof. We compute the first s coefficients $c_2(m), c_3(m), \ldots, c_{s+1}(m)$ with precision 2^{-s-1}. This can be done in time polynomial in s and $\log\, m$. Denote the approximate coefficients by $c'_2, c'_3, \ldots, c'_{s+1}$. We then approximate $f^m(z)$ by

$$f^m(z) \approx z + c'_2 z^2 + c'_3 z^3 + \ldots + c'_{s+1} z^{s+1}.$$

The error is bounded by

$$\begin{aligned} |c_2(m) - c'_2|z^2 + |c_3(m) - c'_3|z^3 + \ldots + |c_{s+1}(m) - c'_{s+1}|z^{s+1} \\ + |c_{s+2}(m)z^{s+2}| + |c_{s+3}(m)z^{s+3}| + \ldots \leq 2^{-s-1}(z^2 + z^3 + \ldots) \\ + m^{s+2}z^{s+2} + m^{s+3}z^{s+3} + \ldots \leq 2^{-s-1} + 2^{-s-1} = 2^{-s}. \end{aligned}$$

We use Proposition 3.13 here to bound the tail terms $c_r(m)z^r$. □

The algorithm now works similarly to the hyperbolic case (Theorem 3.4), occasionally using Proposition 3.14 to perform a long iteration when the orbit is close to 0. We first construct a domain U similar to the initial domain used in Proposition 3.7:

Proposition 3.15 *We can compute a planar domain $U \in \mathscr{C}$ such that:*

(I) $\overline{U} \subsetneq \overline{f(U)}$, with finitely many intersection points at preimages of the parabolic point 0,

(II) $f(U) \cap \mathrm{Postcrit}(f) = \emptyset$,
(III) $J(f) \subset \overline{U}$.

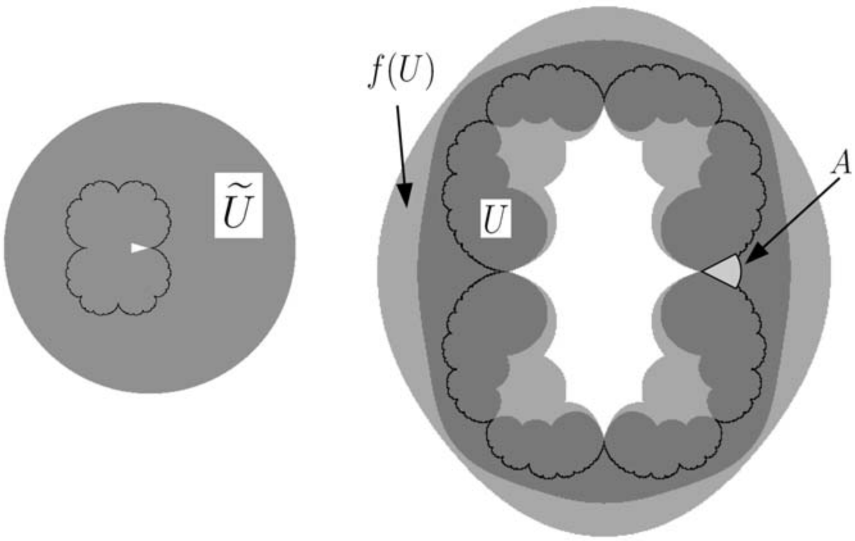

Fig. 3.6 The sets $\widetilde{U}$, $U \subset f(U)$ and A.

The set U is obtained by taking a domain $\widetilde{U}$ that is a sufficiently large disc with a wedge removed around the attracting direction of the parabolic point 0 (see Figure 3.6). All orbits originating in the wedge stay there and converge to the parabolic point 0. The orbit of the critical point $-1/2$ converges to 0 and eventually ends up in the wedge. Hence the inverse images of $\widetilde{U}$ will eventually consume the critical point. In our illustration, we take $U = f^{-3}(\widetilde{U})$. It then satisfies the requirements of Proposition 3.15. Furthermore, as in the hyperbolic case, by taking a few more inverse images under f, we can assure that every point in U is at least 32 times closer to $J(f)$ than to the postcritical set of f.

We can now apply a combination of the Distance Estimator algorithm with the "giant steps" from Proposition 3.14. For the algorithm, we will need to define a region A which is a wedge around the repelling direction of the map of some constant radius ε that contains $J(f) \cap B(0,\varepsilon)$ (see Figure 3.6 again). If a point z is ε-close to 0 ($|z| < \varepsilon$) but it is not in A, then we can estimate the distance $d(z, J(f))$ within a constant multiplicative error.

Suppose now that we are given a dyadic point $x \in \hat{\mathbb{C}}$, and a parameter m, and our goal is to output 1 if $d(x, J(f)) < 2^{-m}$, and 0 if $d(x, J(f)) > M \cdot 2^{-m}$ for some constant M. We use the following algorithm, where C is an appropriately chosen large constant (full details may be found in [Bra06]).

begin
$i=1, x_0=x, d_0=1$.
if $x_0 \notin U$, estimate $d(x_0, J(f))$ directly;
while $i \leq Cm$ **do**

(1) $x_i \leftarrow x_{i-1} + x_{i-1}^2$;
(2) $d_i \leftarrow 2x_{i-1} \cdot d_{i-1}$;
(3) Check the inequality $|x_i| < 2^{-Cm}$ with precision 2^{-Cm}; if the inequality holds halt and return 1, otherwise continue to (4);
(4) Check whether $x_i \in U$ with precision 2^{-2m}; if not, estimate $d(x, J(f))$ by $d(x_i, J(f))/d_i$, return the appropriate answer and halt;
(5) Check whether $|x_i| < \varepsilon$ with precision 2^{-2m}; if it is the case

(a) check whether $x_i \in A$ with precision 2^{-2m};
(b) if x_i is in A, make a "giant leap" of $\lfloor 1/(2|x_i|) \rfloor$ steps from x_i to obtain x_i' and d_i';
- if x_i' escapes U use binary search to find the smallest iterate $f^l(x_i)$ that escapes U; set $d_i \leftarrow Df^l(x_i)d_i$, $x_i \leftarrow f^l(x_i)$;
- otherwise, set $x_i \leftarrow x_i'$ and $d_i \leftarrow d_i'$;
- loop back to step (5);

(c) if x_i is not in A, estimate $d(x_i, J(f))$, estimate $d(x, J(f))$ by $d(x_i, J(f))/d_i$, return the appropriate answer and halt;

(6) $i \leftarrow i+1$

end while

Output 1 and exit.

end

The proof of the fact that $d(x_i, J(f))/d_i$ estimates $d(x_i, J(f))$ within a constant multiplicative error whenever x_i is in U is similar to the hyperbolic case. Also, outside of A and finitely many preimages of A, ∂U and $\partial f(U)$ are bounded away from each other, thus giving an expansion by some $c > 1$ in the hyperbolic metric of $f(U)$. This means that if $d(x_0, J(f)) > 2^{-m}$, then the main loop may be executed at most Cm times for some constant C. Note that step (5)(c) does not decrease the hyperbolic distance from x_i to $J(f)$ by the same argument, although we have no estimate on the factor by which it increases this distance (only that it is ≥ 1).

Evidently, if the algorithm exists on line (4), i.e. when x_{i+1} is *very* close to $J(f)$, then x_0 must have been closer than 2^{-m} to $J(f)$.

Finally, assuming that $|x_i| > 2^{-Cm}$, it will take $O(m)$ iterations of step (5)(b) to escape the set A, which means that the total number of jumps in the algorithm is bounded by $O(m^2)$, which is *polynomial* in m.

3.3.3 Computing parabolic Julia sets in polynomial time: the general case

The construction above generalizes to any rational map $R(z)$ that only has parabolic and attracting orbits. Full details may be found in [Bra06]. We state:

Theorem 3.16 *Given*

- *a rational function $R(z)$ such that every critical orbit of R converges to an attracting or a parabolic orbit; and*
- *some finite combinatorial information about the parabolic orbit of R;*

there is an algorithm M that produces an image of the Julia set $J(R)$. It takes M at most time $C_R \cdot n^c$ to decide one pixel in $J(R)$ with precision 2^{-n}. Here c is some (small) constant and C_R depends on R but not on n.

The combinatorial information is required to identify the parabolic orbits, for instance, by specifying their periods, and approximate locations. It should also allow us to present an iterate R^q of R near each parabolic point p_i in a canonical form

$$R^q : \; p_i + z \mapsto p_i + z + z^{u_i+1} + a_{u_i+2} z^{u_i+2} + \ldots;$$

to do this we need to know q and u_i.

The algorithm works exactly as in the example above. It starts by creating a domain U such that $U \subset R(U)$ with only finitely many intersection points between ∂U and $\partial R(U)$ (at some preimages of the parabolic points). The set U is selected so that $U \cap \mathrm{Postcrit}(R) = \emptyset$.

To find out whether a point x is 2^{-n}-close to $J(R)$, the algorithm iterates it until the orbit escapes U while keeping track of the derivative. As in the special case, if the orbit reaches a set A which is a collection of wedges around the repelling directions of the parabolic orbits, it applies one long iteration to accelerate the computation. If the orbit lands extremely close to one of the parabolic points, then x must have been 2^{-n}-close by a derivative argument. Otherwise, it will take $O(n)$ long steps to escape A. Using a hyperbolic metric argument as above, one shows that at most $O(n)$ steps may be made outside A, bringing the total number of iterations before the algorithm terminates to $O(n^2)$.

The only possible complication is in computing the long iteration. Note that if our map was $g(z) = z + z^3$ instead of $z + z^2$, it would take $\approx 2^{2n}$ iterations to escape from $x_0 = 2^{-n}$, rather than $\approx 2^n$ iterations. Thus we would need a more powerful acceleration (that jumps $(1/z)^2$ steps rather than $1/z$ steps) in this case. To justify plugging in $k = \lfloor 1/(2z^2) \rfloor$ into the formula for $g^k(z)$ we need a generalization of Proposition 3.13.

Proposition 3.17 (cf. Lemma 5 in [Bra06]) *Let $u \geq 1$ be an integer. Set $g(z) = z + z^{u+1}$. Let $\alpha = 2u^3$. Write the k-th iterate of g:*

$$g^k(z) = z + c_{u+1}(k) z^{u+1} + c_{u+2}(k) z^{u+2} + \ldots$$

Then $c_r(k) \leq (\alpha k)^{r/u}$.

In particular, for $g(z) = z + z^3$, $u = 2$ and $c_r(k) \leq (16k)^{r/2}$. Thus we can take $k = \lfloor 1/(32z^2) \rfloor$ and the series will still converge. This allows for a jump of $\Omega(1/z^2)$ in one step, as required. Proposition 3.17 allows us to take even bigger jumps for higher values of u.

3.4 Lack of uniform computability of Julia sets

Our first interesting result in the negative direction answers the following natural question:

Is it possible to compute all Julia sets, or in particular all quadratic Julia sets, with a single oracle Turing Machine $M^{\phi}(n)$*?*

This is ruled out by Theorem 1.12, as the dependence $c \mapsto J(f_c)$ is discontinuous in the Hausdorff distance. For an excellent survey of the continuity problem see the paper of Douady [Dou94].

Theorem 3.18 ([Dou94]) *Denote by* $\mathbb{J}(c)$ *and* $\mathbb{K}(c)$ *the functions* $c \mapsto J_c$ *and* $c \mapsto K_c$ *respectively viewed as functions from* $\mathbb{C}$ *to* K_2^* *with the latter space equipped with Hausdorff distance. Then the following is true:*

(a) if c is Siegel then $\mathbb{J}(c)$ *is discontinuous at c, but* $\mathbb{K}(c)$ *is continuous at c;*
(b) if c is parabolic then both $\mathbb{J}(c)$ *and* $\mathbb{K}(c)$ *are discontinuous at c;*
(c) if c is neither Siegel nor parabolic, then both $\mathbb{J}(c)$ *and* $\mathbb{K}(c)$ *are continuous at c.*

The discontinuity of $\mathbb{J}$ at Siegel parameters is not difficult to prove:

Proposition 3.19 *Let* $c_* \in \mathscr{M}$ *be a parameter value for which* f_{c_*} *has a Siegel disk. Then the map* $\mathbb{J}(c)$ *is discontinuous at* c_**. More specifically, let* z_0 *be the center of the Siegel disk. For each* $s > 0$ *there exists* $\tilde{c} \in B(c_*, s)$ *such that* $f_{\tilde{c}}$ *has a parabolic periodic point in* $B(z_0, s)$*.*

Proof. Denote by Δ the Siegel disk around z_0, p its period, and θ the rotation angle. By the Implicit Function Theorem, for some $\varepsilon > 0$ there exists a holomorphic mapping $\zeta : B(c_*, \varepsilon) \to \mathbb{C}$ such that $\zeta(c_*) = z_0$ and $\zeta(c)$ is fixed under $(f_c)^p$. The mapping

$$\nu : c \mapsto D(f_c)^p(\zeta(c))$$

is holomorphic, and hence it is either constant or open.

If $\nu(c) \equiv d$ is constant, then there exists a maximal non-empty open set of parameters $A \ni c_*$ with a Siegel periodic point with the same period and multiplier. Since A is obviously closed in $\mathbb{C}$, it follows that every quadratic has a Siegel disk. This is not possible: for instance, $f_{1/4}$ has a parabolic fixed point, and thus no other non-repelling cycles, by the Fatou-Shishikura Bound. Therefore ν is open, and in particular there is a sequence of parameters $c_n \to c_*$ such that $\zeta(c_n)$ has multiplier $e^{2\pi i p_n/q_n}$. Since $\zeta(c_n)$ is parabolic, it lies in the Julia set of f_{c_n}. $\zeta(c_n) \to z_0$. Hence

$$\mathrm{dist}_H(J(f_{c_n}), J(f_{c_*})) \geq \mathrm{dist}(\zeta(c_n), \partial\Delta) > \mathrm{dist}(z_0, \partial\Delta)/2$$

for n large enough. □

Thus an arbitrarily small change of the multiplier of the Siegel point may lead to an implosion of the Siegel disk – its inner radius collapses to zero.

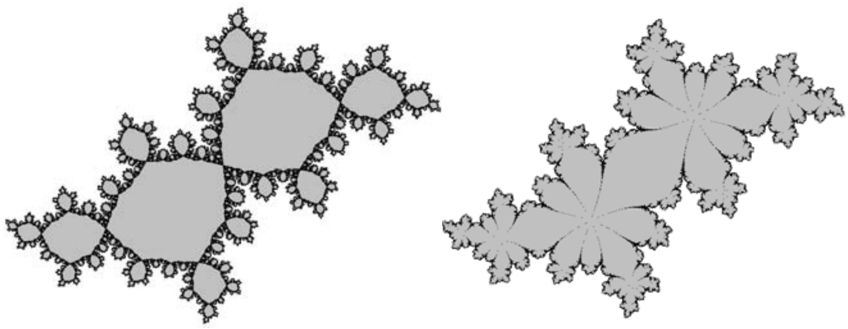

Fig. 3.7 An illustration of a Siegel implosion. On the left is the filled Julia set K_{c_*} (gray) and the Julia set J_{c_*} (black) of a quadratic polynomial with a Siegel fixed point ζ_0. The multiplier $f_{c_*}(\zeta_0) = e^{2\pi i\theta}$, where the rotation angle θ is the inverse golden mean, given by the infinite continued fraction $[1,1,1,1,1,\ldots]$. On the right is the filled Julia set of a nearby quadratic polynomial, whose fixed point is parabolic, with multiplier $[1,1,1,1,1] = 5/8$.

As an immediate consequence of Proposition 3.19 and Theorem 1.12 we have:

Proposition 3.20 *For any TM $M^\phi(n)$ with an oracle for $c \in \mathbb{C}$, denote by S_M the set of all values of c for which M^ϕ computes J_c. Then $S_M \neq \mathbb{C}$.*

In other words, a single algorithm for computing all quadratic Julia sets does not exist.

3.4.1 Discontinuity at a parabolic parameter

The discontinuity in $\mathbb{J}(c)$ which occurs at parabolic parameter values has found many interesting dynamical implications. The proof is very involved, and its outline may be found in [Dou94]. It is based on the Douady-Lavaurs theory of *parabolic implosion.* Let us briefly describe its mechanism for the case of a quadratic polynomial f_c.

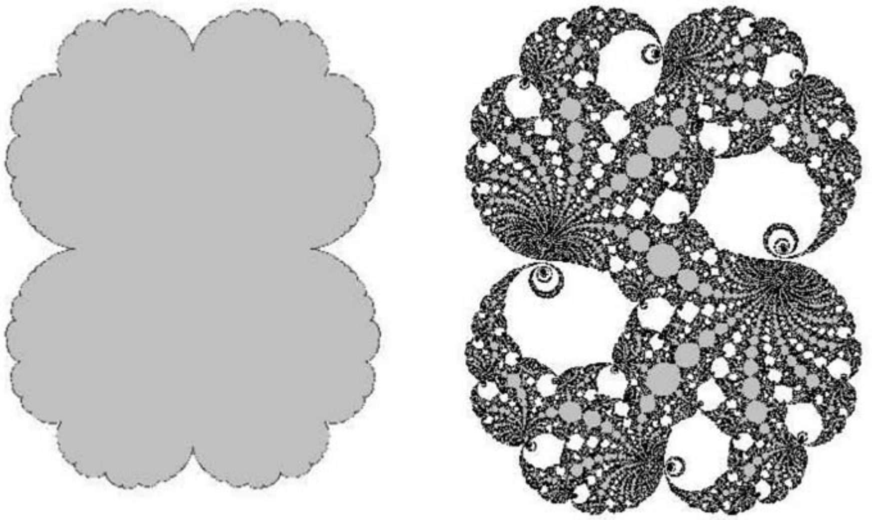

Fig. 3.8 Before and after a parabolic implosion. The Julia sets (black) and filled Julia sets (light gray) of a parabolic quadratic $f_{1/4}$ (left), and of $f_{1/4+\varepsilon}$ for a small complex ε.

Denote by ζ a parabolic periodic point of f_c with multiplier $e^{2\pi ip/q}$, and let $m \in \mathbb{N}$ be its period. Let P_A and P_R be attracting and repelling petals of f_c. Recall that, by Proposition 3.11, the cycle of images $f_c^{jm}(P_A \cup P_R)$, $j = 0, \dots, q-1$ forms a full Leau-Fatou flower at ζ.

By Proposition 3.10, the quotient

$$C_A = P_A / f_c^{mq} \simeq \mathbb{C}/\mathbb{Z}.$$

The quotient C_A, is sometimes called the *attracting Fatou cylinder*. It parametrizes the orbits converging under the dynamics of the iterate f_c^m to the point ζ. A repelling Fatou cylinder $C_R \simeq \mathbb{C}/\mathbb{Z}$ is defined similarly as the quotient of a repelling petal.

Let τ be any conformal isomorphism $C_A \to C_R$. After uniformization,

$$C_A \underset{\approx}{\mapsto} \mathbb{C}/\mathbb{Z},\ C_R \underset{\approx}{\mapsto} \mathbb{C}/\mathbb{Z}$$

$\tau(z) \equiv z + q \bmod \mathbb{Z}$ for some $q \in \mathbb{C}$. Let $g_\tau : P_A \to P_R$ be any lift of τ; it necessarily commutes with f_c^{mq}. Consider the semigroup G generated by the dynamics of the pair (f_c, g_τ). The orbit Gz of a point $z \in \mathbb{C}$ is independent of the choice of the lift g_τ and only depends on τ.

Set

$$J_{(c,\tau)} = \{z \in \mathbb{C} \text{ such that } Gz \cap J_c \neq \emptyset\}.$$

It can be shown that this set is the boundary of

$$K_{(c,\tau)} = \{z \in \mathbb{C} \text{ such that } Gz \text{ is bounded}\}.$$

Notice that $K_{(c,\tau)} \subsetneq K_c$: some of the orbits which converge to ζ under f_c are thrown into the complement $(\mathbb{C} \setminus K_c) \cap P_R$ by g_τ. Holes which thus open in the set K_c motivate the use of the term "implosion".

The Douady-Lavaurs theory postulates:

Theorem 3.21 *For every τ as above and every $s > 0$ there exists $\tilde{c} \in B(c,s)$ such that $B(J_{\tilde{c}}, s) \supset J_{(c,\tau)}$.*

Thus the Julia set of f_c "explodes" under the perturbation from c to $\tilde{c}$.

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