

2 The Mathematical Models

We shall review the four main formalisms proposed for transportation networks in a discrete and continuous framework. Section 2.1 is dedicated to the Monge-Kantorovich model. Section 2.2 describes the Gilbert-Steiner discrete irrigation model. Section 2.3 is devoted to the notation of the three continuous mathematical models (transport paths, patterns and traffic plans). Section 2.4.1 lists the mathematical questions and tells where they will be solved in the book. Section 2.5 discusses several extensions and related models in urban optimization.

2.1 The Monge-Kantorovich Problem

The seminal Monge problem [62] was to move a pile of sand from a place to another with minimal effort. In the Kantorovich [50] formalization of this problem, μ^+ and μ^- are positive measures on \mathbb{R}^N which model respectively the supply and demand mass distributions. A transport scheme from μ^+ onto μ^- is described by telling where each piece of supplied mass is sent. In the Kantorovich formalism this information is encoded in a positive measure π on $\mathbb{R}^N \times \mathbb{R}^N$ where $\pi(A \times B)$ represents the amount of mass going from A to B . This measure π will be called a *transference plan* (also called *transport plan* in the literature). To evaluate the efficiency of a transference plan, a cost function $c : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^+$ is considered where $c(x, y)$ is the cost of transporting a unit mass from x to y . The cost associated with a transference plan is $\int_{\mathbb{R}^N \times \mathbb{R}^N} c(x, y) d\pi(x, y)$. The minimization of this functional is the Monge-Kantorovich problem. This problem has attracted many mathematicians and is extremely well documented, as apparent in the recent and excellent treatises and reviews [2, 37, 38, 42, 86].

As an example, consider the cost function $c(x, y) = |x - y|^2$, and the supply and demand measures $\mu^+ = \delta_x$ and $\mu^- = \frac{1}{2}(\delta_{y_1} + \delta_{y_2})$, where δ_x stands for the Dirac mass at x . The minimizer π is the measure on $\mathbb{R}^N \times \mathbb{R}^N$ such that $\pi(\{x\} \times \{y_1\}) = \frac{1}{2}$ and $\pi(\{x\} \times \{y_2\}) = \frac{1}{2}$. The optimal transportation is achieved along geodesics between x , y_1 and y_2 as represented in Figure 2.1.

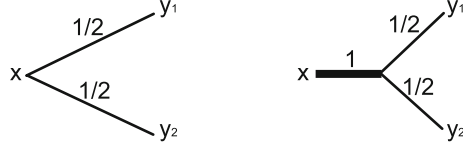


Fig. 2.1. The transport from δ_x to $\frac{1}{2}(\delta_{y_1} + \delta_{y_2})$. Monge-Kantorovich straight solution (left) versus Gilbert's branching one (right).

2.2 The Gilbert-Steiner Problem

In the Monge-Kantorovich framework, the cost of the structure achieving the transport is not modeled. Indeed, with this formulation, the cost behaves as if every single particle of sand went straight from its starting to its ending point. In the case of real supply-demand distribution problems, achieving this kind of single particle transport would be very costly. Thus from the economical viewpoint the Monge-Kantorovich problem is unrealistic. In most transportation networks, the aggregation of particles on common routes is preferable to individual straight ones. The local structure of human-designed distribution systems doesn't look as a set of straight wires but rather like a tree.

The Steiner problem which consists in minimizing the total length of a network connecting a given set of points could be such a transportation model. Yet this cost is not realistic because it does not discriminate the cost of high or low capacity edges (a road has not the same cost as a highway). The first model taking into account capacities of edges was proposed by Gilbert [44] in the case of communication networks. This author models the network as a graph such that each edge e is associated with a flow (or capacity) $\varphi(e)$. Let $f(\varphi)$ denote the cost per unit length of an edge with flow (or capacity) φ . It is assumed that $f(\varphi)$ is subadditive and increasing, i.e., $f(\varphi_1) + f(\varphi_2) \geq f(\varphi_1 + \varphi_2) \geq \max(f(\varphi_1), f(\varphi_2))$. Gilbert then considers the problem of minimizing the cost of networks supporting a given set of flows between terminals. The subadditivity of the cost f translates the fact that it is more advantageous to transport flows together. Thus, it leads to delay bifurcations. In the fluid mechanics context, this subadditivity follows from Poiseuille's law, according to which the resistance of a tube increases when it gets thinner (we refer to [12, 27] for a study of irrigation trees in this context). The simplest model of this kind is to take $f(\varphi) = \varphi^\alpha$ with $0 < \alpha < 1$.

Following [44], consider atomic sources $\mu^+ = \sum_{i=1}^k a_i \delta_{x_i}$ and sinks $\mu^- = \sum_{j=1}^l b_j \delta_{y_j}$ with $\sum_i a_i = \sum_j b_j$, $a_i, b_j \geq 0$. An irrigation graph G is a weighted directed graph with straight edges $E(G)$ and a flow function $w : E(G) \rightarrow (0, \infty)$ satisfying Kirchhoff's law. Observe that G can be written as a vector measure

$$G = \sum_{e \in E(G)} \varphi(e) \mathcal{H}^1|_e \mathbf{e} \quad (2.1)$$

where \mathbf{e} denotes the unit vector in the direction of e and \mathcal{H}^1 the Hausdorff one-dimensional measure. We say that G irrigates (μ^+, μ^-) if its distributional derivative ∂G satisfies

$$\partial G = \mu^- - \mu^+. \quad (2.2)$$

The Gilbert energy of G is defined by

$$M^\alpha(G) = \sum_{e \in E(G)} \varphi(e)^\alpha \mathcal{H}^1(e). \quad (2.3)$$

We call the problem of minimizing $M^\alpha(G)$ among all finite graphs irrigating (μ^+, μ^-) the Gilbert-Steiner problem. The Monge-Kantorovich model corresponds to the limit case $\alpha = 1$ and the Steiner problem to $\alpha = 0$. The structure of the minimizer of (2.3) with $\mu^+ = \delta_x$ and $\mu^- = \frac{1}{2}(\delta_{y_1} + \delta_{y_2})$ is shown in Figure 2.1. Let us finally mention that the Gilbert model is adopted in [16, 52] to study optimal pipeline and drainage networks. From a numerical point of view, a backtrack algorithm exploring relevant Steiner topologies is proposed in [98] to solve a problem of water treatment network. A different algorithmic approach can be found in [101].

2.3 Three Continuous Extensions of the Gilbert-Steiner Problem

As we mentioned in the preface, the discrete Gilbert-Steiner model was only recently set in the Kantorovich continuous framework [94], [59] where the wells and sources are arbitrary measures, instead of finite sums of Dirac masses.

2.3.1 Xia's Transport Paths

Let μ^+, μ^- be two positive Radon measures with equal mass in a compact convex set $X \subset \mathbb{R}^N$. A vector measure T on X with values in \mathbb{R}^N is called by Xia [94] a *transport path* from μ^+ to μ^- if there exist two sequences μ_i^-, μ_i^+ of finite atomic measures with equal mass and a sequence of finite graphs G_i irrigating (μ_i^+, μ_i^-) such that $\mu_i^+ \rightarrow \mu^+$, $\mu_i^- \rightarrow \mu^-$ weakly as measures and $G_i \rightarrow T$ as vector measures. The energy of T is defined by

$$M^\alpha(T) := \inf \liminf_{i \rightarrow \infty} M^\alpha(G_i), \quad (2.4)$$

where the infimum is taken over the set of all possible approximating sequences $\{\mu_i^+, \mu_i^-, G_i\}$ to T . Denote

$$M^\alpha(\mu^+, \mu^-) := \inf \{M^\alpha(T) : T \text{ is a transport path from } \mu^+ \text{ to } \mu^-\}.$$

If $\alpha \in (1 - \frac{1}{N}, 1]$, by Theorem 3.1 in [94], the above infimum is attained and finite for any pair (μ^+, μ^-) . Xia shows in a series of papers several structure and regularity properties of optimal transport paths which we shall comment later on.

2.3.2 Maddalena-Solimini's Patterns

The Gilbert problem was given by Maddalena and Solimini [59] a different (Lagrangian) formulation in the case of a single source supply $\mu^+ = \delta_S$. The authors model the transportation network as a “pattern”, or set of “fibers” $\chi(\omega, \cdot)$, where $\chi(\omega, t)$ represents the location of a particle $\omega \in \Omega$ at time t . The set Ω is an abstract probability space indexing all fibers. Without loss of generality one can take $\Omega = [0, 1]$ endowed with the Lebesgue measure on borelians E denoted by $\lambda(E) = |E|$. All the fibers are required to stop at some time $T(\omega)$ and satisfy $\chi(\omega, 0) = S$ for all ω , which means that all fibers start at the same source S . The set of fibers is given a structure corresponding to the intuitive notion of branches. Two fibers ω and ω' belong to the same branch at time t if $\chi(\omega, s) = \chi(\omega', s)$ for all $s \leq t$. Then the partition of Ω given by the branches at time t yields a time filtration. The branch of ω at time t is denoted by $[\omega]_t$ and its measure by $|[\omega]_t|$. The energy of the pattern is defined by

$$\tilde{E}^\alpha(\chi) = \int_{\Omega} \int_0^{T(\omega)} |[\omega]_t|^{\alpha-1} dt d\omega. \quad (2.5)$$

It is easily checked that this definition extends the Gilbert energy (2.3) when the discrete graph is a finite tree. The measure irrigated by the Maddalena et al. pattern is defined for every set A as the measure in Ω of the set of fibers stopping in A , $\mu^-(A) = |\{\omega : \chi(\omega, T(\omega)) \in A\}|$. Both the formulation of the energy and the irrigation constraints are very handy in this model and we shall retain its essential features. We can illustrate the Lagrangian formalism with the simplest non trivial example of transportation from one Dirac mass to two Dirac masses (see Figure 2.1). Maddalena-Solimini's solution is given by the set of fibers $\chi : [0, 1] \times [0, \infty) \rightarrow \mathbb{R}^2$, where $\chi(\omega, t)$ is either the path from x to y_1 (if $\omega \in [0, 1/2]$), or the path from x to y_2 (if $\omega \in (1/2, 1]$). This solution can also be written as the sum of two paths with flow $\frac{1}{2}$ each. Thus the solution is formalized either as a parameterized set of fibers or as the sum of two Dirac masses on the set of paths. This later point of view leads to the notion of *traffic plans* as developed in the next section.

2.3.3 Traffic Plans

In [11] the authors of the present book extended the pattern formalism to the case where the source is any Radon measure. They called “traffic plan” any positive measure on the set of Lipschitz paths (see Chapter 3 for a more accurate definition). This model allows one to add *who goes where* constraints to the optimization problem. Thus, it also gives a framework to the *mailing problem*.

The who goes where (or mailing) problem has the same energy but a different boundary condition. Indeed, it specifies which part of μ^+ goes to a given part of μ^- , instead of just requiring μ^+ to go globally onto μ^- . Thus it

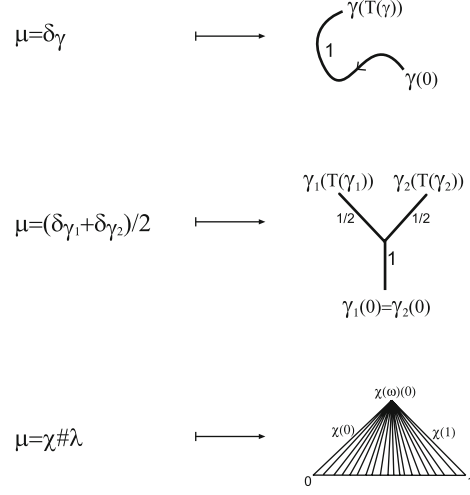


Fig. 2.2. Three traffic plans and their associated embedding : a Dirac measure on γ , a tree with one bifurcation, a spread tree irrigating Lebesgue's measure on the segment $[0, 1] \times \{0\}$ of the plane. In the bottom example, to $\omega \in [0, 1]$ corresponds $\chi(\omega) \in K$, the straight path from $(1/2, 1)$ to $(\omega, 0)$.

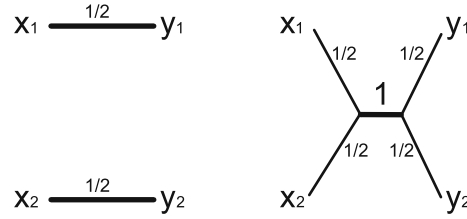


Fig. 2.3. Irrigation problem minimizer versus traffic plan minimizer with who goes where constraint. The energy is the same and the irrigating and irrigated measures are the same. In the second case, the who goes where constraint prescribes that the mass in x_1 must go y_2 and the mass in x_2 to y_1 . The solution is quite different.

incorporates a *transference plan constraint* which we shall call the “who goes where” constraint. For example, consider $\mu^+ = \delta_{x_1} + \delta_{x_2}$ and $\mu^- = \delta_{y_1} + \delta_{y_2}$ as in Figure 2.3. If we want to find the best transportation network with the constraint that all the mass in x_1 is sent to y_2 , and all the mass in x_2 is sent to y_1 , we need another formalism, since the graph approach does not convey this kind of information. The Lagrangian formalism introduced in [59] is adapted to that problem, since it keeps track of every mass particle.

Figure 2.2 shows three examples of traffic plans: a Dirac mass on a finite length path γ (which means that a unit mass is transported from $\gamma(0)$ to $\gamma(L)$), a traffic plan with “Y” shape, and a traffic plan transporting a Dirac mass to the Lebesgue measure on a segment of the plane. As we mentioned in

the preface, any graph with a flow satisfying Kirchhoff's law can be modeled by a finite atomic measure on the space of paths in the graph. Classical graph theory ensures that one can decompose the graph into a sum of paths starting on sources and ending on wells.

2.4 Questions and Answers

Thus, we have three different formalisms having as common denominator a cost per flow s proportional to s^α : the original discrete Gilbert model defined on finite graphs, a generalization by Xia's *transport path* model and a generalization by the traffic plan (measure on the set of paths) model. The irrigation pattern model [59] is a particular instance of traffic plan. Of course the rather intuitive equivalence of the continuous models and their consistency with the original Gilbert model will have to be checked. To be more precise:

1. Consistency problem: when the irrigated measures μ^+ and μ^- are finite atomic, do the minimizers for all continuous models coincide with the Gilbert minimizers?
2. For two general positive measures μ^+ and μ^- with equal mass, are the Xia's minimizers also optimal traffic plans and conversely?
3. When $\mu^+ = \delta_S$ is a single source, are optimal patterns and optimal traffic plans equivalent notions?

The answer to questions 1, 2 and 3 will be an easy consequence of a main structure property of optimal traffic plans: they have a "single path" structure, that is, contain at most one path joining any two points, as observable in most irrigation networks. In the case of optimal traffic plans for the irrigation problem, this result is easily extended to show that the network is a tree. This result was proven in the case of finite atomic measures by Gilbert. It is shown, again for discrete graphs in [94], and for irrigation patterns in [28, 79]. The equivalence of all models was proven in [58] and [13] and is the object of Chapter 9.

Do optimal networks look like (infinite) graphs and what is their structure? A first natural question is whether optimal irrigation networks are countably rectifiable. This property has been proven in [96] for transport paths, in [59] for irrigation trees and in [11] for traffic plans. The next question towards a graph structure is to control the branching number. Gilbert [44] proved that there is a universal bound for the branching number of any minimal finite graph, as observable in Figure 1.1. We shall show the same property in general following [95] and [13]. Another strong result, proven in various forms in [13, 29, 95] is that any optimal irrigation traffic plan can be monotonically approximated by finite irrigation trees.

A main question raised by Xia [96] is finally the interior regularity, according to which the structure of an irrigation tree away from the supports

of μ^+ and μ^- is a finite graph with straight edges and bounded branching number. In this book, we shall prove this regularity result for traffic plans when $\alpha > 1 - \frac{1}{N}$, using the results and techniques of [13, 95]. As a consequence of the equivalence of models mentioned above, all regularity results apply to all models as well.

Let us mention further graph structure properties of optimal traffic plans. Following [95] we shall prove (Proposition 7.16) that any path in the irrigation graph with flow larger than a constant is bi-Lipschitz, with explicit estimates on the Lipschitz constant depending on α and the minimal value of the flow. Another feature of optimal networks shown in [95] is the following. For any point x of the optimal irrigation network and for any $\varepsilon > 0$, there is a ball $B(x, r)$ such that the network inside $B(x, r)$ can be additively decomposed in two parts : a main network P made of a finite number of bi-Lipschitz graphs and a residual network R whose total flow is less than ε . In addition, all points in the ball where the flow is larger than ε are contained in the support of P . Actually, following [13, 28, 78], this kind of result will be made global by a pruning Lemma (see Proposition 7.14).

2.4.1 Plan

Chapter 3 is devoted to all detailed definitions of the main terms used throughout the book: traffic plan and its parameterization, fiber, irrigated and irrigating measure, transference plan, stopping time of each fiber, the multiplicity (or flow) at each point and finally the energy. General existence theorems are given for the existence of optimal traffic plans with prescribed irrigated and irrigating measure or with prescribed transference plan under the mere assumption that a feasible solution exists.

Chapter 4 gives a more geometric form to traffic plans and their energy. Its first important result is that traffic plans with finite energy are countably rectifiable. This first regularity result is used to obtain the consistency of the traffic plan energy E^α with the original Gilbert energy. This consistency reads

$$E^\alpha(P) = \int_{\mathbb{R}^N} |x|_\chi^\alpha d\mathcal{H}^1(x) \quad (2.6)$$

for loop-free traffic plans, where $|x|_\chi$ denotes the measure of the set of fibers passing by x .

Chapter 5 is devoted to a series of elementary operations permitting to combine traffic plans into new ones. New traffic plans can be built by restriction, concatenation, union, and hierarchical concatenation. This last operation gives a standard way to explicitly construct infinite irrigation trees, or patterns.

Chapter 6 uses these techniques to prove by explicit constructions that for $\alpha > 1 - \frac{1}{N}$ where N is the dimension of the ambient space, the optimal cost to transport μ^+ to μ^- is finite. Thus the energy of a minimal traffic plan

between μ^+ and μ^- , denoted $E^\alpha(\mu^+, \mu^-)$ turns out to define a metric on the space of probability measures. This distance will be compared with the classical Wasserstein distance $E^1(\mu^+, \mu^-)$. These results will be generalized in Chapter 10 which defines, following Devillanova and Solimini [78], an *irrigability dimension* and compares it with classical Hausdorff and Minkowski dimensions. These results imply in particular that the Lebesgue measure of a cube in \mathbb{R}^N is *not irrigable* for $\alpha \leq 1 - \frac{1}{N}$.

Chapter 7 proves several crucial structure properties for optimal traffic plans. First, the *single path property*, which implies that optimal traffic plan for the irrigation problem are trees. Second, a *monotone approximation of optimal traffic plans by finite graphs* irrigating atomic measures is constructed. The main results of this chapter have been proved in different contexts in [13, 58, 70, 79, 94, 95]. Our presentation follows [13].

Chapter 8 goes further and proves the finite graph structure of the irrigation network away from the irrigated measures. This is what Xia [96] calls “interior regularity”. Finally a “boundary regularity”, comes out, namely a universal bound on the number of branches at each point of the network. The main results of this chapter have been proved in different contexts in [13, 58, 94, 95] and again our presentation follows [13].

Chapter 9 uses the regularity results to prove the equivalence of all mentioned models: their minimizers are identical and their minima equal.

The four last chapters establish crucial links with the former discrete and physical theories and give several applications.

Chapter 11 which follows the work of Santambrogio [75] establishes a crucial equivalence of the theory with the theory of Optimal Channel Networks. It proves that the link established in this discrete theory between optimal irrigation and optimal landscape is still valid in the continuous framework. There is for every optimal pattern (traffic plan starting from a point source) a unique landscape function $z(x)$ up to an additive constant. Then the fibers of the optimal traffic plan turn out to be the steepest descent curves of the landscape. Here one sees one of the advantages of the continuous framework, which permits to ask and answer regularity issues. It is indeed proven that the optimal landscape is Hölder. Figure 2.4 shows various results of landscape optimization: the resulting river network is a tree with regular branches.

Chapter 12 investigates particular examples considering only discrete measures and deriving the optimal configurations in the simplest case: one source and two wells. This study yields a geometric procedure to construct optimal configurations when the data are not too large.

Using these techniques, Chapter 13 investigates the structure of an optimal traffic plan irrigating the Lebesgue measure on a segment from one source. It demonstrates that the irrigation tree is infinite and gives a geometric procedure and heuristics to construct a global optimum.

Chapter 14 proves that all irrigation problems fit into the framework given by the Gilbert energy. Starting from the “naive” embedded tube models

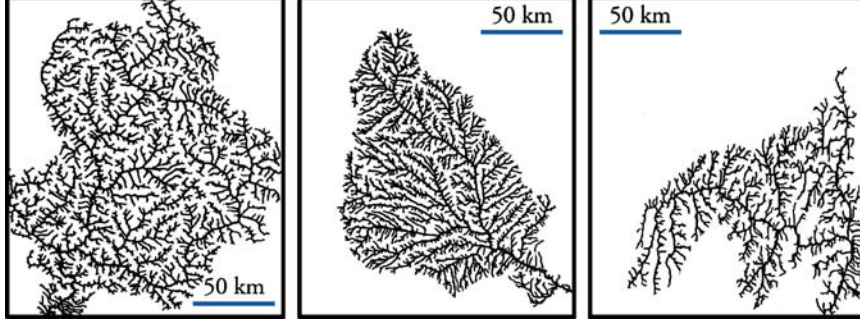


Fig. 2.4. Different river networks. These branched networks manage to collect the water raining over the whole territory they cover. According to [73], such a network evolves towards a locally minimal flow energy configuration.

and writing their physical Poiseuille energy, a simple Lagrangian argument shows that we are led back to the Gilbert energy. The main objection to the infinitesimal models is that Poiseuille law in 3D leads to $\alpha = \frac{2}{3} = 1 - \frac{1}{3}$ which is precisely the critical exponent, for which a Lebesgue measure can't be irrigated.

Finally Chapter 15 proposes a (certainly non-exhaustive) list of open problems.

2.5 Related Problems and Models

2.5.1 Measures on Sets of Paths

Although the name of traffic plan is new in the literature, there are several antecedents for the use of measures on sets of paths for transportation models. The idea of a probabilistic representation of sets of paths appears in many contexts (particularly for equations of diffusion type). The first reference in the context of conservation laws and fluid mechanics seems to be in Brenier [19], where it is used to describe particle trajectories for the incompressible Euler equation. Paolini and Stepanov [70] use normal one-dimensional currents but point out that they can be represented by measures over the metric set of Lipschitz paths in \mathbb{R}^N . Buttazzo, Pratelli and Stepanov [23] [22] used measures on paths in urban transportation models. They proposed to call them *transports*. Now this term along with the term *transport plan* and the term *transference plan* are being used with a different meaning in the context of the Monge-Kantorovich problem. Hence our choice of a new term for the object denoting a set of trajectories. Measures on set of paths are familiar in optimal transport theory, where transport maps and transference plans can be thought of in a natural way as measures in the space of minimizing geodesics [71]. See [7] for a similar approach within Mather's

theory. In a more general context, [4] and [3] use measures in the space of continuous maps to characterize a flow with BV vector field and prove its stability. The Lecture Notes [87] contain, among several other things, a comprehensive treatment of the topic of measures in the space of action-minimizing curves, including at the same time the optimal transport and the dynamical systems case. This unified treatment was inspired by [9].

2.5.2 Urban Transportation Models with more than One Transportation Means

A series of papers by Buttazzo, Brancolini, Pratelli, Santambrogio, Solimini, and Stepanov has considered recently urban transportation models where two means for transportation with different cost structure or more compete.

The seminal papers seem to be [17], [24], [25] and [21], where a rather natural extension of the Monge-Kantorovich problem is being proposed to model urban transportation network. Given an initial and a final measure representing for example the homes and work places of citizens, it is assumed that the citizens dispose of two transportation means, namely a high speed and cheap network (typically the underground or highway network, see Figures 2.5 and 2.6). This network is modeled as a connected rectifiable set with finite length. Another low speed pedestrian network permits to access



Fig. 2.5. An old map of France's main roads. In contrast with the tree structure of hydrographic maps, the road network is a general network containing cycles. This network approximately solves another problem, the mailing, or “who goes where” problem. For this problem existence and structure results will be given in the traffic plan framework. However, these results are by far less complete than the results on the irrigation problem because of the lack of a tree structure. In particular the regularity theory is fully open (see Chapter 15 on the open problems). Image from <http://www.france-property-and-information.com/>

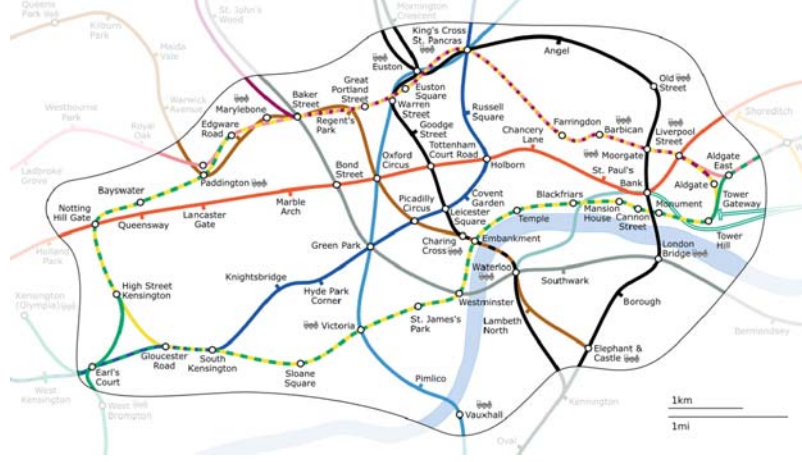


Fig. 2.6. A London subway map. According to the works of Brancolini, Buttazzo, Paolini, Pratelli, Santambrogio, Stepanov and Solimini described in Section 2.5.2, the urban transportation model involves two or more competing transportation means. In the basic model, the subway, or any fast network, is modeled as a connected set with finite length and very low cost for the users. Users walk to this network and then use it. Thus the problem is a mixed problem involving a Monge-Kantorovich individual transport to and from the fast network ($\alpha = 1$). The fast transportation means has instead a Steiner energy (the length, $\alpha = 0$).

the high speed network. The cost of the high speed network is proportional to its length and almost free to the users. The low speed network is more costly and is modeled by a Monge-Kantorovich transportation: Thus the users are led to take the shortest path to and from the transportation network. The whole problem can be viewed as a Monge-Kantorovich problem with the high-speed network as free boundary. A notion of Wasserstein distance between measures, relative to a given transportation network, is therefore introduced. The cost to be minimized is the overall transportation time for all users. The authors prove existence of high speed networks ensuring a minimal cost transportation for all users.

The transportation network is modeled as a connected closed set Σ . The users can either walk or join and use Σ . Thus, the cost for going from x to y is $d_\Sigma(x, y) := d(x, y) \wedge (dist(x, \Sigma) + dist(y, \Sigma))$, i.e. the minimum between the Euclidean (walking) distance $d(x, y)$ and the sum of distances from x and y to the network. Notice that the distance d_Σ describes how the Euclidean distance is twisted by the network. Calling the population density μ^+ and the density of workplaces μ^- , the cost of this transportation network is given by the Monge-Kantorovich distance between μ^+ and μ^- , for the cost $d_\Sigma(x, y)$. The authors consider optimal transportation networks, i.e. transportation networks with a minimal cost among all transportation networks with length less than a prescribed length L . An existence theory is presented in [17] and in [24] while [81] studies the qualitative topological and geometrical properties

of optima. It is worthwhile noticing that the above problem combines the two extremal exponents for the Gilbert-Steiner energy, namely $\alpha = 1$ for the pedestrians (Monge-Kantorovich) and $\alpha = 0$ for the transportation network ($\alpha = 1$).

In [70] (see also [80]), a wide extension of the Gilbert model discussed in these notes was presented which also involves more than one transportation means. Taking the notations of these authors and of Xia [94], consider a one-dimensional real flat chain T such that $\partial T = \mu^+ - \mu^-$ and define its $M^\alpha(T)$ cost as the integral over the current of the multiplicity to the power α . Minimizing this energy only yields the generalized Gilbert energy used in [94]. Now there will be two transportation means. One of them is modeled as a first current S with a usage cost $M^\beta(S)$ and a construction cost $H(M^\delta(S))$ where H is itself concave increasing and unbounded, thus typically a power s^η , $0 < \eta < 1$. The second current T corresponds to another transportation means with no construction cost and usage cost $M^\alpha(T)$. The constraint on the irrigated measure reads $\partial(S + T) = \mu^+ - \mu^-$. The overall energy to minimize with this constraint is

$$E(S, T) = aM^\alpha(T) + bM^\beta(S) + H(M^\delta(S)).$$

The authors prove the existence of a pair of optimal flat chains S and T and the existence of a threshold θ_0 such that the multiplicity of S is above θ_0 and the multiplicity of T is below θ_0 . In addition, $S+T$ has no cycles. To prove the existence theorem, the one-dimensional currents are handled as measures on a space of Lipschitz paths, that is, traffic plans in the terminology of the present book.

In [18], an attempt was made to formulate transportation models as geodesic paths in the set of probability measures on a given open set U in Euclidean space. First, a cost is defined on the space of static probability measures. Such costs can lead to concentration or dispersion as well. In a setting very close to the Gilbert-Steiner energy, the measure under consideration is an atomic measure $\sum_k m_k \delta_{x_k}$ and the cost is defined as $J(\mu) = \sum_k m_k^\alpha$. Such a cost is finite only on discrete atomic masses. Then the cost of a path $\mu(t)$ in the set of measures is the integral of its speed multiplied by this cost, namely $\mathcal{J} = \int_0^1 J(\mu(t)) |\mu'(t)| dt$ where $|\mu'(t)|$ is the metric derivative of $t \mapsto \mu(t)$ in the Wasserstein space of measures $W_p(U)$. Such integral functionals permit to model the evolution of a measure which is forced to concentrate on a finite set of points at any time between arrival and departure. Thus, such costs can model the branching structure of transportation networks. Quite interesting in this aspect is the fact that the same model permits to model a diffusive propagation as well, forbidding any concentration, by taking $\alpha \geq 1$. In all cases an existence result for optimal paths has been given by the authors under conditions linking the dimension and the exponent in the cost functional. In the case $0 < \alpha < 1$ one can wonder whether the functional is a Gilbert energy or not. In fact, it turns out to be different and has the drawback that the transportation cost is maintained for stopped masses, until the

whole motion of all masses has stopped. Thus, the model suffers from excessive coordination between moving masses. See also [76], for an overview of variational problems on probability measures for functionals involving transport costs with extra terms encouraging or discouraging concentration. This paper looks for optimality conditions, regularity properties and explicit computations in the case where Wasserstein distances and interaction energies are considered.

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