Theorem 8.7.3. *Let $\nabla_{m-1-j}\varphi_j \in MW_p^l(\mathbb{R}^n)$, where $0 < l < 1$, $1 \le p \le \infty$. Then there exists one and only one solution of the Dirichlet problem such that*

$$\nabla_{m-1}U \in MW^k_{p,k-l-1/p}(\mathbb{R}^{n+1}_+), \qquad k \ge 1.$$

This solution satisfies the estimate

$$\|\nabla_{m-1}U; \mathbb{R}^{n+1}_+\|_{MW^k_{p,k-l-1/p}} \le K \sum_{j=0}^{m-1} \|\nabla_{m-1-j}\varphi_j; \mathbb{R}^n\|_{MW^l_p},$$

where K is a constant which depends on L and n, p, m, k, l.

Proof. If $U \in MW^k_{p,k-l-1/p}(\mathbb{R}^{n+1}_+)$ is a solution of the homogeneous problem, then

$$\|\nabla_{m-l}U; \mathbb{R}^{n+1}_+\|_{L_\infty} < \infty$$

and hence $U = 0$ (see, for instance, [ADN1], Ch. I, §2).

According to the same reference, the existence of a solution follows from the assumption

$$\nabla_{m-1-j}\varphi_j \in L_\infty(\mathbb{R}^n), \qquad j = 0, 1, \dots, m-1,$$

and the solution satisfies the equality

$$D_x^\alpha \frac{\partial^i}{\partial y^i} U(x,y) = \sum_{j=0}^{m-1} \sum_{|\beta|=m-1-j} \int_{\mathbb{R}^n} K_{i,j,\beta}(x-\xi, y) D_\xi^\beta \varphi_j(\xi)\, d\xi,$$

where $0 \le i \le m-1$, α is any multi-index of order $m-1-i$ and $K_{i,j,\beta}(z)$ are positive homogeneous functions of order $-n$, smooth in $\mathbb{R}^{n+1}\backslash\{0\}$ and such that $K_{i,j,\beta}(x,0) = 0$ for $x \ne 0$. These conditions imply the estimate

$$(|x|^2 + y^2)^{1/2} |\nabla_x K_{i,j,\beta}(x,y)| + |K_{i,j,\beta}(x,y)| \le c\, y(|x|^2 + y^2)^{-(n+1)/2}$$

which shows that the function $\zeta(x) = K_{i,j,\beta}(x,1)$ satisfies (8.7.10) for $s = 0$, $0 < l < 1$. It remains to make use of Theorem 8.7.2. □

8.8 Traces of Functions in $MW_p^l(\mathbb{R}^{n+m})$ on $\mathbb{R}^n$

In this section we show that the space of restrictions of functions in $MW_p^l(\mathbb{R}^{n+m})$ ($lp > n$, $l - m/p$ is a noninteger) to $\mathbb{R}^n$ coincides with $MW_p^{l-m/p}(\mathbb{R}^n)$.

8.8.1 Auxiliary Assertions

We use the extension operator $\mathcal{T}$ defined in Sect. 8.7.2. Let us assume that the conditions (8.7.10) and (8.7.11) for $k = [l]$, $s = 1$ are fulfilled.

Lemma 8.8.1. *Let $\sigma \in (0, l]$ and let $p \in [1, \infty)$. Then*

$$\Big(\int_{2|\eta|<|y|} |\nabla_{[\sigma],y}(\mathcal{T}\gamma)(x, y+\eta) - \nabla_{[\sigma],y}(\mathcal{T}\gamma)(x,y)|^p |\eta|^{-m-p\{\sigma\}}\, d\eta\Big)^{1/p}$$
$$\le c\,\mathcal{C}_{[l]+1,l+1}\, |y|^{-\sigma} \|\gamma; \mathbb{R}^n\|_{L_\infty}\,, \tag{8.8.1}$$

and

$$\Big(\int_{\mathbb{R}^n} |\nabla_{[\sigma],x}(\mathcal{T}\gamma)(x+h, y) - \nabla_{[\sigma],x}(\mathcal{T}\gamma)(x,y)|^p |h|^{-n-p\{\sigma\}}\, dh\Big)^{1/p}$$
$$\le c\,\mathcal{C}_{[l]+1,l+1} |y|^{-\sigma} \|\gamma; \mathbb{R}^n\|_{L_\infty}\,, \tag{8.8.2}$$

where $\mathcal{C}_{[l]+1,l+1}$ is defined by (8.7.10).

Proof. Let A_y and B_x denote the left-hand sides of (8.8.1) and (8.8.2). One verifies directly that

$$\begin{aligned}A_y \le \|\gamma; \mathbb{R}^n\|_{L_\infty} \Big\{ \int_{2|\eta|<|y|} \Big(\int_{\mathbb{R}^n} \Big| \nabla_{[\sigma],y} \Big[\zeta\Big(\frac{\xi - x}{|y+\eta|}\Big) |y+\eta|^{-n} \\ - \zeta\Big(\frac{\xi - x}{|y|}\Big) |y|^{-n} \Big] \Big| \, d\xi \Big)^p |\eta|^{-m-p\{\sigma\}}\, d\eta \Big\} \\ \le \|\gamma; \mathbb{R}^n\|_{L_\infty} \Big\{ \int_{2|\eta|<|y|} \frac{|\eta|^p\, d\eta}{|\eta|^{m+p\{\sigma\}}} \\ \times \Big(\int_0^1 dz \int_{\mathbb{R}^n} \big| \varphi^{(\sigma+1)}_{\xi - x}[|y| + z(|y+\eta| - |y|)] \big|\, d\xi \Big)^p \Big\}^{1/p},\end{aligned}$$

where
$$\varphi^{(l)}_{\xi - x}(t) = -\frac{\partial^{[l]}}{\partial t^{[l]}} \Big(t^{-n} \zeta\Big(\frac{\xi - x}{t}\Big)\Big).$$

Therefore,

$$\begin{aligned}A_y &\le c\,\mathcal{C}_{[l]+1,l+1} \|\gamma; \mathbb{R}^n\|_{L_\infty} \\ &\times \Big\{ \int_{2|\eta|<|y|} |\eta|^{p(1-\{\sigma\})-m}\, d\eta \Big(\int_0^1 \big(|y| + z(|y+\eta| - |y|)\big)^{-[\sigma]-1}\, dz \Big)^p \Big\}^{1/p} \\ &\le c\,\mathcal{C}_{[l]+1,l+1} \|\gamma; \mathbb{R}^n\|_{L_\infty} |y|^{-\sigma}\,.\end{aligned}$$

The inequality (8.8.1) is proved.

Making the change of variables

$$\xi - x = |y|\Xi, \qquad h = |y|H,$$

we obtain

$$B_x \le |y|^{-\sigma}\Big\{\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\big|\nabla_{[\sigma],\Xi}[\zeta(\Xi+H)-\zeta(\Xi)]\big|\,d\Xi\Big)^p |H|^{-n-p\{\sigma\}}\,dH\Big\}^{1/p}.$$

Now we divide the exterior integral into two integrals, the first of which is over the ball $\mathcal{B}_1$. We have

$$\begin{aligned}\int_{\mathcal{B}_1} &\le \int_{\mathcal{B}_1}\Big(\int_{\mathbb{R}^n}\Big|\sum_{i=1}^n H_i\int_0^1 \frac{\partial}{\partial \Xi_i}\nabla_{[\sigma],\Xi}\zeta(\Xi+zH)\,dz\Big|\,d\Xi\Big)^p|H|^{-n-p\{\sigma\}}\,dH\\ &\le \Big(\int_{\mathbb{R}^n}|\nabla_{[\sigma]+1}\zeta(\Xi)|\,d\Xi\Big)^p\int_{\mathcal{B}_1}|H|^{-n+(1-\{\sigma\})p}dH \le c\,\mathcal{C}^p_{[l]+1,l+1}\,.\end{aligned}$$

Finally,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\mathcal{B}_1} &\le \int_{\mathbb{R}^n\setminus\mathcal{B}_1}\Big(\int_{\mathbb{R}^n}(|\nabla_{[\sigma]}\zeta(\Xi+H)|+|\nabla_{[\sigma]}\zeta(\Xi)|)\,d\Xi\Big)^p|H|^{-n-p\{\sigma\}}\,dH\\ &= 2^p\Big(\int_{\mathbb{R}^n}|\nabla_{[\sigma]}\zeta(\Xi)|\,d\Xi\Big)^p\int_{\mathbb{R}^n\setminus\mathcal{B}_1}|H|^{-n-p\{\sigma\}}\,dH \le c\,\mathcal{C}^p_{[l]+1,l+1}\,.\end{aligned}$$

The proof is complete. □

Let d_j denote the number of all derivatives of order j with respect to the variables $y_1,\dots,y_m$ and let $[W_p^\sigma(\mathbb{R}^n)]^{d_j}$ be the Cartesian product of d_j copies of the space $W_p^\sigma(\mathbb{R}^n)$. It is known (see [Usp] that there exists an extension operator E defined on vector-functions $(\boldsymbol{\varphi}_0,\boldsymbol{\varphi}_1,\dots,\boldsymbol{\varphi}_{[l-m/p]})$, where $\boldsymbol{\varphi}_j$ is the d_j-tuple vector-function. This operator maps into a space of scalar functions and has the following properties.

(i) E is a continuous operator:

$$\prod_{j=0}^{[l-m/p]}[W_p^{[l-j-m/p]}(\mathbb{R}^n)]^{d_j} \to W^k_{p,k-l}(\mathbb{R}^{n+m})\,. \tag{8.8.3}$$

(ii) The relation

$$(\nabla_j E\boldsymbol{\varphi})(x,0) = \boldsymbol{\varphi}_j(x) \qquad \text{with} \quad j=0,1,\dots,[l-m/p]$$

holds.

In what follows we use the fact that $W^{[l]+1}_{p,1-\{l\}}(\mathbb{R}^{n+m})$ is imbedded into $W_p^l(\mathbb{R}^{n+m})$ (see [Usp]). We also apply the Hardy-type inequality

$$\int_{\mathbb{R}^m} |y|^{-pl}|V(x,y)|^p\,dy \le c\,\|V(x,\cdot);\mathbb{R}^m\|^p_{W_p^l}\,, \tag{8.8.4}$$

where V is a function in $W_p^l(\mathbb{R}^m)$ subject to the conditions

$$(\nabla_j V)(x,0) = 0\,, \qquad j = 0,1,\ldots,[l-m/p]\,. \tag{8.8.5}$$

Moreover, we make use of the following norm in $W_p^l(\mathbb{R}^{n+m})$:

$$\begin{aligned}
&\|U;\mathbb{R}^{n+m}\|_{W_p^l}\\
&\sim \Big(\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^m} dy \int_{\mathbb{R}^m} |\nabla_{[l],y}U(x,y+\eta) - \nabla_{[l],y}U(x,y)|^p|\eta|^{-m-p\{l\}}\,d\eta\Big)^{1/p}\\
&+\Big(\int_{\mathbb{R}^m} dy \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} |\nabla_{[l],x}U(x+h,y) - \nabla_{[l],x}U(x,y)|^p|h|^{-n-p\{l\}}\,dh\Big)^{1/p}\\
&+\|U;\mathbb{R}^{n+m}\|_{L_p}
\end{aligned} \tag{8.8.6}$$

(see Proposition 4.2.3).

8.8.2 Trace and Extension Theorem

Theorem 8.8.1. *(i) Let $lp > n$, $1 \le p < \infty$ and $l - m/p$ be noninteger. Further, let $\Gamma \in MW_p^l(\mathbb{R}^{n+m})$ and $\gamma(x) = \Gamma(x,0)$. Then $\gamma \in MW_p^{l-m/p}(\mathbb{R}^n)$ and*

$$\|\gamma;\mathbb{R}^n\|_{MW_p^{l-m/p}} \le c\,\|\Gamma;\mathbb{R}^{n+m}\|_{MW_p^l}\,. \tag{8.8.7}$$

(ii) Let the kernel ζ satisfy (8.7.10) *and* (8.7.11). *If $\gamma \in MW_p^{l-m/p}(\mathbb{R}^n)$ with $lp > m$, $1 \le p < \infty$ and noninteger $l - m/p$, then $\mathcal{T}\gamma \in MW_p^l(\mathbb{R}^{n+m})$ and*

$$\|\mathcal{T}\gamma\colon \mathbb{R}^{n+m}\|_{MW_p^l} \le c\,\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{MW_p^{l-m/p}}\,. \tag{8.8.8}$$

Proof. (i) Let $U \in W_p^l(\mathbb{R}^{n+m})$, $U(x,0) = u(x)$. We have

$$\|\gamma u;\mathbb{R}^n\|_{W_p^{l-m/p}} \le c\,\|\Gamma U;\mathbb{R}^{n+m}\|_{W_p^l} \le c\,\|\Gamma;\mathbb{R}^{n+m}\|_{MW_p^l}\|U;\mathbb{R}^{n+m}\|_{W_p^l}$$

which implies (8.8.7).

(ii) It is sufficient to assume that l is a noninteger, since for integer l the result is contained in Theorem 8.7.1.

Let $U \in W_p^l(\mathbb{R}^{n+m})$ and

$$\boldsymbol{\varphi}(x) = (U(x,0), (\nabla_y(U)(x,0), \ldots, (\nabla_{[l-m/p],y}U)(x,0))\,.$$

We introduce the function $V = U - E\boldsymbol{\varphi}$, where E is the extension operator which was considered in Sect. 8.8.1. Then

$$\|U\mathcal{T}\gamma;\mathbb{R}^{n+m}\|_{W_p^l} \le \|(\mathcal{T}\gamma)E\boldsymbol{\varphi};\mathbb{R}^{n+m}\|_{W_p^l} + \|V\mathcal{T}\gamma;\mathbb{R}^{n+m}\|_{W_p^l}\,.$$

In view of the imbedding

$$W^{[l]+1}_{p,1-\{l\}}(\mathbb{R}^{n+m}) \subset W_p^l(\mathbb{R}^{n+m}),$$

the first term on the right-hand side does not exceed

$$c\,\|(\mathcal{T}\gamma)E\boldsymbol{\varphi};\mathbb{R}^{n+m}\|_{W^{[l]+1}_{p,1-\{l\}}}$$

which, by Theorem 8.7.1, is not greater than

$$c\,\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{MW_p^{l-m/p}}\|E\boldsymbol{\varphi};\mathbb{R}^{n+m}\|_{W^{[l]+1}_{p,1-\{l\}}}\,.$$

Since E performs the continuous mapping (8.8.3), it follows that

$$\begin{aligned}
&\|(\mathcal{T}\gamma)E\boldsymbol{\varphi};\mathbb{R}^{n+m}\|_{W_p^l}\\
&\quad\le c\,\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{MW_p^{l-m/p}}\sum_{j=0}^{[l-m/p]}\|(\nabla_{j,y}U)(\cdot,0);\mathbb{R}^n\|_{W_p^{l-m/p-j}}\,.
\end{aligned}$$

Consequently,

$$\|(\mathcal{T}\gamma)E\boldsymbol{\varphi};\mathbb{R}^{n+m}\|_{W_p^l} \le c\,\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{MW_p^{l-m/p}}\|U;\mathbb{R}^{n+m}\|_{W_p^l}. \tag{8.8.9}$$

Let us prove the inequality

$$\|V\mathcal{T}\gamma;\mathbb{R}^{n+m}\|_{W_p^l} \le c\,\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{L_\infty}\|V;\mathbb{R}^{n+m}\|_{W_p^l}\,. \tag{8.8.10}$$

It is easy to see that

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n}dx\int_{\mathbb{R}^m}dy\int_{2|\eta|>|y|}\big|\nabla_{[l],y}(V\mathcal{T}\gamma))(x,y+\eta)\\
&\quad-(\nabla_{[l],y}(V\mathcal{T}\gamma))(x,y)\big|^p|\eta|^{-m-p\{l\}}\,d\eta\Big)^{1/p}\\
&\quad\le c\Big(\int_{\mathbb{R}^{n+m}}|\nabla_{[l],y}(V\mathcal{T}\gamma)|^p|y|^{-p\{l\}}\,dz\Big)^{1/p}
\end{aligned} \tag{8.8.11}$$

which, by (8.7.15) with $s=0$ and $\mu=0$, does not exceed

$$\begin{aligned}
&c\sum_{j=0}^{[l]}\Big(\int_{\mathbb{R}^{n+m}}|\nabla_{[l]-j,y}\mathcal{T}\gamma|^p|\nabla_{j,y}V|^p|y|^{-p\{l\}}\,dz\Big)^{1/p}\\
&\quad\le c\,\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{L_\infty}\sum_{j=0}^{[l]}\Big(\int_{\mathbb{R}^{n+m}}|\nabla_{j,y}V|^p|y|^{(j-l)p}\,dz\Big)^{1/p}.
\end{aligned}$$

This fact and (8.8.4) show that the left-hand side of (8.8.11) is dominated by

$$c\,\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{L_\infty}\|V;\mathbb{R}^{n+m}\|_{W_p^l}.$$

The expression

$$\Big(\int_{\mathbb{R}^n}dx\int_{\mathbb{R}^m}dy\int_{2|\eta|<|y|}\big|\nabla_{[l],y}[(V\mathcal{T}\gamma)(x,y+\eta)-(V\mathcal{T}\gamma)(x,y)]\big|^p|\eta|^{-m-p\{l\}}d\eta\Big)^{1/p}$$

is majorized by

$$\begin{aligned}
&c\sum_{j=0}^{[l]}\Big(\int_{\mathbb{R}^n}dx\int_{\mathbb{R}^m}dy\int_{2|\eta|<|y|}|(\nabla_{[l]-j,y}\mathcal{T}\gamma)(x,y+\eta)|^p\\
&\times|\nabla_j(V(x,y+\eta)-V(x,y))|^p|\eta|^{-m-p\{l\}}d\eta\Big)^{1/p}\\
&+c\sum_{j=0}^{[l]}\Big(\int_{\mathbb{R}^n}dx\int_{\mathbb{R}^m}dy|(\nabla_{j,y}V)(x,y)|^p\\
&\times\int_{2|\eta|<|y|}\big|\nabla_{[l]-j,y}\big(\mathcal{T}\gamma(x,y+\eta)-\mathcal{T}\gamma(x,y)\big)\big|^p|\eta|^{-m-p\{l\}}d\eta\Big)^{1/p}. \qquad (8.8.12)
\end{aligned}$$

Since

$$|(\nabla_{[l]-j,y}\mathcal{T}\gamma)(x,y+\eta)|\le c\,\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{L_\infty}\,|y|^{j-[l]},$$

the first sum does not exceed

$$\begin{aligned}
&c\,\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{L_\infty}\sum_{j=0}^{[l]-1}\Big(\int_{\mathbb{R}^n}dx\int_{\mathbb{R}^m}|y|^{p(j-[l])}\int_{2|\eta|<|y|}|\eta|^{-m+p(1-\{l\})}\\
&\times\Big(\int_0^1|\nabla_{j+1,y}V(x,y+t\eta)|dt\Big)^p d\eta\Big)^{1/p}+c\,\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{L_\infty}\|V;\mathbb{R}^{n+m}\|_{W_p^l}.
\end{aligned}$$

Here we have used the relation (8.8.6). Further, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}|y|^{(j-[l])p}\int_{2|\eta|<|y|}|\eta|^{-m+p(1-\{l\})}\Big(\int_0^1|\nabla_{j+1,y}V(x,y+t\eta)|\,dt\Big)^p d\eta\,dy\\
&\quad\le c\int_0^1 dt\int_{\mathbb{R}^m}|\eta|^{-m+p(1-\{l\})}\int_{2|\eta|<|y|}|y|^{(j-[l])p}|\nabla_{j+1,y}V(x,y+t\eta)|^p\,dy\,d\eta\\
&\quad\le c\int_{\mathbb{R}^m}|\eta|^{-m+p(1-\{l\})}\int_{|\chi|>|\eta|}|\chi|^{(j-[l])p}|\nabla_{j+1,\chi}V(x,\chi)|^p\,d\chi\,d\eta\\
&\quad= c\int_{\mathbb{R}^m}|\chi|^{p(j+1-l)}|\nabla_{j+1,\chi}V(x,\chi)|^p\,d\chi
\end{aligned}$$

which, according to (8.8.4), does not exceed $c\,\|V(x,\cdot);\mathbb{R}^m\|^p_{W^l_p}$ for almost all $x\in\mathbb{R}^n$. Thus, the first sum in (8.8.12) is not greater than

$$c\,\mathcal{C}_{[l]+1,l+1}\|V;\mathbb{R}^{n+m}\|_{W^l_p}.$$

Using (8.8.1), we find that the second sum in (8.8.12) has the majorant

$$c\,\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{L_\infty}\sum_{j=0}^{[l]}\Big(\int_{\mathbb{R}^{n+m}}|(\nabla_{j,y}V)(z)|^p|y|^{(j-l)p}\,dz\Big)^{1/p}$$

which, by (8.8.4), is dominated by

$$c\,\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{L_\infty}\|V;\mathbb{R}^{n+m}\|_{W^l_p}.$$

To obtain a bound for

$$\Big(\int_{\mathbb{R}^m}dy\int_{\mathbb{R}^n}dx\int_{\mathbb{R}^n}|\nabla_{[l],x}(V\mathcal{T}\gamma)(x+h,y)-(V\mathcal{T}\gamma)(x,y)]|^p|h|^{-n-p\{l\}}dh\Big)^{1/p},$$

it suffices to estimate the integrals

$$\int_{\mathbb{R}^m}dy\int_{\mathbb{R}^n}|\nabla_{j,x}V(x,y)|^p$$
$$\times\int_{\mathbb{R}^n}\big|\nabla_{[l]-j,x}[(\mathcal{T}\gamma)(x+h,y)-(\mathcal{T}\gamma)(x,y)]\big|^p|h|^{-n-p\{l\}}\,dh\,dx\,,$$

$$\int_{\mathbb{R}^m}dy\int_{\mathbb{R}^n}\big|[\nabla_{[l]-j,x}(\mathcal{T}\gamma)](x,y)|^p$$
$$\times\int_{\mathbb{R}^n}|\nabla_{j,x}(V(x+h,y)-V(x,y))\big|^p|h|^{-n-p\{l\}}\,dh\,dx\,.$$

The first integral is estimated by Lemma 8.8.1 and inequality (8.8.4). It does not exceed

$$c\,\big(\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{L_\infty}\|V;\mathbb{R}^{n+m}\|_{W^l_p}\big)^p\,.$$

The second integral is dominated by

$$c\,\mathcal{C}^p_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|^p_{L_\infty}\int_{\mathbb{R}^n}dx\int_{\mathbb{R}^n}|h|^{-n-p\{l\}}$$
$$\times\int_{\mathbb{R}^m}|y|^{p(j-\{l\})}|\nabla_{j,x}(V(x+h,y)-V(x,y))|^p dydh$$
$$\le c\,\mathcal{C}^p_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|^p_{L_\infty}\int_{\mathbb{R}^n}dx\int_{\mathbb{R}^n}|h|^{-n-p\{l\}}$$
$$\times\int_{\mathbb{R}^m}|\nabla_{[l],z}(V(x+h,y)-V(x,y))|^p\,dydh$$
$$\le c\,(\mathcal{C}_{[l]+1,l+1}\|\gamma;\mathbb{R}^n\|_{L_\infty}\|V;\mathbb{R}^{n+m}\|_{W^l_p})^p.$$

Here we have used (8.8.4) and (8.8.6). Thus inequality (8.8.10) is proved.

Putting $\gamma = 1$, $\mathcal{T}\gamma = 1$ in (8.8.9), we obtain the estimate

$$\|E\boldsymbol{\varphi}; \mathbb{R}^{n+m}\|_{W_p^l} \le c\,\|U; \mathbb{R}^{n+m}\|_{W_p^l}$$

which, together with (8.8.10) and the equality $V = U - E\boldsymbol{\varphi}$, shows that

$$\|V\mathcal{T}\gamma; \mathbb{R}^{n+m}\|_{W_p^l} \le c\,\mathcal{C}_{[l]+1,l+1}\|\gamma; \mathbb{R}^n\|_{L_\infty}\|U; \mathbb{R}^{n+m}\|W_p^l\,.$$

The proof is complete. □

8.9 Multipliers in the Space of Bessel Potentials as Traces of Multipliers

The goal of this section is to show that multipliers in the space $H_p^l(\mathbb{R}^n)$ are traces of multipliers in a certain class of differentiable functions in $\mathbb{R}^{n+m}$ with a weighted mixed norm.

8.9.1 Bessel Potentials as Traces

The space $L_{p,\beta}^k(\mathbb{R}^{n+m})$ is defined as the completion of $C_0^\infty(\mathbb{R}^{n+m})$ in the norm

$$\Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^m}|y|^{2\beta}|\nabla_{k,z}U|^2dy\Big)^{p/2}dx\Big)^{1/p} + \Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^m}|y|^{2\beta}|U|^2dy\Big)^{p/2}dx\Big)^{1/p}.$$

Let the first term be denoted by $< U >_{p,\beta,k}$. For $k > r$ and $\beta > r - m/p$, by Hardy's inequality one has

$$< U >_{p,\beta-r,k-r} \le c < U >_{p,\beta,k}\ . \tag{8.9.1}$$

The following assertion shows that elements of $H_p^l(\mathbb{R}^n)$ are traces on $\mathbb{R}^n$ of functions in $L_{p,\beta}^k(\mathbb{R}^{n+m})$ (see [Sh1] for $0 < l < 1$, the general case is treated in a similar way). Below we use the spherical coordinates (ρ, ω) in $\mathbb{R}^m$: $\rho = |y|$ and $\omega = y/|y|$.

Lemma 8.9.1. *(i) Let $U \in L_{p,\beta}^k(\mathbb{R}^{n+m})$, where k is an integer, $2\beta > -m$, and $1 < p < \infty$. Then, for almost all $x \in \mathbb{R}^n$, the limit*

$$U(x, 0) = \lim_{\rho\to+0}\int_{\partial\mathcal{B}_1^{(m)}} U(x; \rho, \omega)d\omega$$

exists. Moreover, $U(\cdot, 0) \in H_p^l(\mathbb{R}^n)$ with $l = k - \beta - m/2$, $\{l\} > 0$ and

$$\|U(\cdot, 0); \mathbb{R}^n\|_{H_p^l} \le c\,\|U; \mathbb{R}^{n+m}\|_{L_{p,\beta}^k}. \tag{8.9.2}$$

(ii) Let $u \in H_p^l(\mathbb{R}^n)$, $l > 0$, $1 < p < \infty$. There exists a linear continuous extension operator:

$$H_p^l(\mathbb{R}^n) \ni u \to U \in L_{p,\beta}^k(\mathbb{R}^{n+m}),$$

where k is an integer, $k > l$, and $\beta = k - l - m/2$.

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