

Non-Differentiable Generalized Convex Functions

2.1 Introduction

In several economic models convexity appears to be a restrictive condition. For instance, classical assumptions in Economics include the convexity of the production set in producer theory and the convexity of the upper level sets of the utility function in consumer theory.

On the other hand, in Optimization it is important to know when a local minimum (maximum) is also global. Such a useful property is not exclusive to convexity (concavity).

All this has led to the introduction of new classes of functions which have convex lower/upper level sets and which verify the local-global property, starting with the pioneer work of Arrow–Enthoven [7].

In this chapter we shall introduce the class of quasiconvex, strictly quasiconvex, and semistrictly quasiconvex functions and the inclusion relationships between them are studied. For functions in one variable, a complete characterization of the new classes is established. Examples of the most important generalized convex functions in Economics are given.

We shall focus our attention on generalized convexity since a function f is generalized concave if and only if $-f$ is generalized convex, so that every result for a generalized convex function can be easily re-stated in terms of generalized quasiconcave functions. For the sake of completeness in Appendix B we shall give a summary of the main properties of generalized concave functions.

2.2 Quasiconvexity and Strict Quasiconvexity

One way to generalize the definition of a convex function is to relax the convexity condition, and require, from a geometrical point of view, that the restriction of the function along a line joining any two points in the domain lies under at least one of the endpoints. A function that verifies such a condition is called quasiconvex. Figure 2.1 shows some examples of quasiconvex functions.

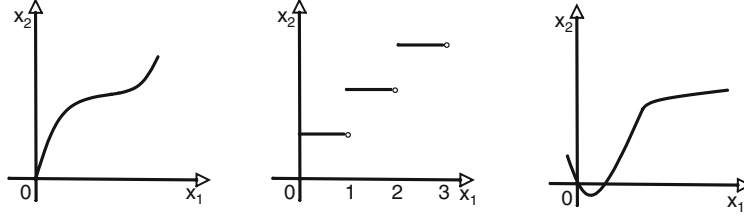


Fig. 2.1. Quasiconvex functions

From an analytical point of view, we have the following definition.

Definition 2.2.1. Let f be defined on a convex set $S \subseteq \mathbb{R}^n$. The function f is said to be quasiconvex on S if

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \max\{f(x_1), f(x_2)\} \quad (2.1)$$

for every $x_1, x_2 \in S$ and for every $\lambda \in [0, 1]$ or, equivalently,

$$f(x_1) \geq f(x_2) \text{ implies that } f(x_1) \geq f(x_1 + \lambda(x_2 - x_1)) \quad (2.2)$$

for every $x_1, x_2 \in S$ and for every $\lambda \in [0, 1]$.

If the inequality in (2.1) is strict, the function is called strictly quasiconvex. Formally:

Definition 2.2.2. A function f defined on a convex set $S \subseteq \mathbb{R}^n$ is said to be strictly quasiconvex if

$$f(\lambda x_1 + (1 - \lambda)x_2) < \max\{f(x_1), f(x_2)\} \quad (2.3)$$

for every $x_1, x_2 \in S$, $x_1 \neq x_2$, and for every $\lambda \in (0, 1)$ or, equivalently,

$$f(x_1) \geq f(x_2) \text{ implies that } f(x_1) > f(x_1 + \lambda(x_2 - x_1)) \quad (2.4)$$

for every $x_1, x_2 \in S$, $x_1 \neq x_2$, and for every $\lambda \in (0, 1)$.

It follows immediately, from the given definitions, that a strictly quasiconvex function is also quasiconvex; the converse is not true as is shown in the following example.

Example 2.2.1.

1. The function $f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is quasiconvex but not strictly quasiconvex;
2. Every monotone function of one variable is quasiconvex and every increasing or decreasing function of one variable is strictly quasiconvex; for instance, the concave function $f(x) = \log x$, $x > 0$ is strictly quasiconvex.

As we have just pointed out, quasiconvexity may be viewed as a relaxation of convexity. Unfortunately, in the new larger class of functions some properties of convex functions are lost. For instance, as we can deduce from (1) in Example 2.2.1, a quasiconvex function may have interior discontinuity points, a local minimum which is not global and an interior global maximum. The relationships between convexity, strict convexity, quasiconvexity and strict quasiconvexity are stated in the following theorem.

Theorem 2.2.1. *Let $S \subseteq \mathbb{R}^n$ be a convex set.*

- (i) *If f is convex on S , then f is quasiconvex on S .*
- (ii) *If f is strictly convex on S , then f is strictly quasiconvex on S .*
- (iii) *If f is strictly quasiconvex on S , then f is quasiconvex on S .*

Proof. (i) We have:

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \leq \lambda \max\{f(x_1), f(x_2)\} + (1 - \lambda) \max\{f(x_1), f(x_2)\} = \max\{f(x_1), f(x_2)\}.$$

Similarly we can prove (ii), while (iii) follows directly by definition. \square

Example 2.2.1 shows that the class of quasiconvex functions contains properly the class of convex functions and the class of strictly quasiconvex functions. Furthermore, there is not any inclusion relationship between the class of convex functions and the class of strictly quasiconvex ones. In fact, a convex function may have constant restrictions and a strictly increasing function of one variable is strictly quasiconvex but not necessarily convex.

The relationships between convexity, strict convexity, quasiconvexity and strict quasiconvexity are illustrated in Fig.2.2 where the arrow (\rightarrow) reads “implies”.

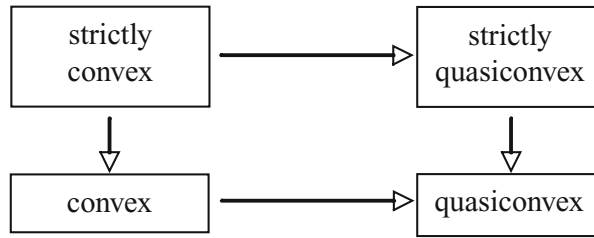


Fig. 2.2. Relationships between various types of convexity

An important case where quasiconvexity reduces to convexity is related to homogeneity. More exactly, the homogeneity assumption combined with quasiconvexity produces convexity as is shown in the following theorem.

Theorem 2.2.2. *Let f be a homogeneous function of degree $\alpha = 1$ defined on a convex set $S \subseteq \mathbb{R}^n$.*

If $f(x) > 0$ for all $x \in S$, then f is quasiconvex if and only if it is convex.

Proof. From (i) of Theorem 2.2.1, convexity implies quasiconvexity without any other assumption. Now we will prove, firstly, that homogeneity combined with quasiconvexity implies subadditivity, i.e., for every $x_1, x_2 \in S$ we have $f(x_1 + x_2) \leq f(x_1) + f(x_2)$.

Let $y_1 = f(x_1) > 0$, $y_2 = f(x_2) > 0$. Since f is homogeneous of degree one, i.e., $f(tx) = tf(x)$, $t > 0$, we have $f(\frac{x_1}{y_1}) = f(\frac{x_2}{y_2}) = 1$, so that the quasiconvexity of f implies $f((1-t)\frac{x_1}{y_1} + t\frac{x_2}{y_2}) \leq 1 \quad \forall t \in (0, 1)$. By choosing $t = \frac{y_2}{y_1 + y_2}$, we have $1 - t = \frac{y_1}{y_1 + y_2}$, and $f(\frac{x_1}{y_1 + y_2} + \frac{x_2}{y_1 + y_2}) = \frac{1}{y_1 + y_2}f(x_1 + x_2) \leq 1$. Consequently, $f(x_1 + x_2) \leq (y_1 + y_2) = f(x_1) + f(x_2)$.

It remains to be proven that subadditivity and homogeneity imply convexity. We have $f(\lambda x_1 + (1-\lambda)x_2) \leq f(\lambda x_1) + f((1-\lambda)x_2) = \lambda f(x_1) + (1-\lambda)f(x_2)$. \square

While a convex function may be characterized by the convexity of its epigraph, a quasiconvex function may be characterized by the convexity of its lower level sets, as is shown in the following theorem.

Theorem 2.2.3. *A function f defined on a convex set $S \subseteq \mathbb{R}^n$ is quasiconvex on S if and only if the lower level set $L_{\leq \alpha} = \{z \in S : f(z) \leq \alpha\}$ is convex for every $\alpha \in \mathbb{R}$.*

Proof. Assume that f is quasiconvex and let $x, y \in L_{\leq \alpha}$. Since $f(x) \leq \alpha$, $f(y) \leq \alpha$, we have $f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\} \leq \alpha$, so that $\lambda x + (1-\lambda)y \in L_{\leq \alpha}$.

In order to prove the converse statement, assume without loss of generality, that $\max\{f(x), f(y)\} = f(x)$ and consider the lower level set $L_{\leq \alpha}$ with $\alpha = f(x)$, that is $L_{\leq f(x)} = \{z \in S : f(z) \leq f(x)\}$; obviously $y \in L_{\leq f(x)}$. Since $L_{\leq f(x)}$ is convex, $x + \lambda(y - x) \in L_{\leq f(x)}$ for every $\lambda \in [0, 1]$, i.e., $f(x + \lambda(y - x)) \leq f(x) = \max\{f(x), f(y)\}$. \square

By means of a simple application of the previous Theorem, we obtain the following result.

Theorem 2.2.4. *Let f be a quasiconvex function defined on a convex set $S \subseteq \mathbb{R}^n$ and let S^* be the set of all global minimum points of f . Then, S^* is convex.*

Proof. If $S^* = \emptyset$ the thesis follows by convention. Taking into account that $S^* = \{x \in S : f(x) = m\} = \{x \in S : f(x) \leq m\}$, where m is the minimum value of f on S , the convexity of S^* follows from the convexity of the lower level sets of a quasiconvex function. \square

As happens for a convex function and for a strictly convex function, (2.1) and (2.3) may be extended to every convex combination of a finite number of points.

Theorem 2.2.5. *f is quasiconvex on a convex set $S \subseteq \mathbb{R}^n$ if and only if, for $x^i \in S$, $i = 1, \dots, p$, we have*

$$f\left(\sum_{i=1}^p \lambda_i x^i\right) \leq \max_{i \in \{1, \dots, p\}} f(x^i), \quad \sum_{i=1}^p \lambda_i = 1, \quad \lambda_i \geq 0, \quad i = 1, \dots, p. \quad (2.5)$$

Furthermore, f is strictly quasiconvex on S if and only if the inequality in (2.5) is strict.

Proof. The validity of (2.5) for $p = 2$ is equivalent to the definition of a quasiconvex function.

Assume now that f is quasiconvex. We will prove that (2.5) holds by induction. Since (2.5) is true for $p = 2$, we must prove that the validity of (2.5) for every p elements implies that

$$f(\lambda_1 x^1 + \dots + \lambda_p x^p + \lambda_{p+1} x^{p+1}) \leq \max_{i \in \{1, \dots, p+1\}} f(x^i)$$

with $\sum_{i=1}^{p+1} \lambda_i = 1$, $\lambda_i \geq 0$, $x^i \in S$, $i = 1, \dots, p+1$.

If $\lambda_{p+1} = 0$, the thesis follows by means of the induction assumption, otherwise, by setting $\lambda_0 = \lambda_1 + \dots + \lambda_p$, we have $\lambda_0 + \lambda_{p+1} = 1$ so that

$y = \frac{\lambda_1}{\lambda_0} x^1 + \dots + \frac{\lambda_p}{\lambda_0} x^p$ is a convex combination of p points since $\sum_{i=1}^p \frac{\lambda_i}{\lambda_0} = 1$.

As a result, $y \in S$. On the other hand $\sum_{i=1}^{p+1} \lambda_i x^i = \lambda_0 y + \lambda_{p+1} x^{p+1}$; the thesis is reached by applying quasiconvexity to the points y, x^{p+1} .

The last statement follows similarly. \square

Another main difference between convex functions and quasiconvex functions is related to the algebraic structure. The class of convex functions is closed with respect to the addition while the sum of quasiconvex (strictly quasiconvex) functions is not in general quasiconvex (strictly quasiconvex). For instance, the functions $f(x) = x^3$, $g(x) = -3x$ are quasiconvex and strictly quasiconvex in \mathbb{R} since they are strictly monotone, but their sum $h(x) = f(x) + g(x) = x^3 - 3x$ is neither quasiconvex, nor strictly quasiconvex since $h(-2) = -2$, $h(0) = 0$, and $h(-1) = 2 > 0$.

Fortunately, in contrast to the convex case, increasing functions combined with quasiconvex functions produce quasiconvex functions, as is stated in the following theorems.

Theorem 2.2.6. *Let f be a quasiconvex function defined on a convex set $S \subseteq \mathbb{R}^n$ and let $g : A \rightarrow \mathbb{R}$ be a non-decreasing function, with $f(S) \subseteq A$. Then:*

- (i) kf , $k > 0$ is quasiconvex on S ;
- (ii) $g \circ f$ is quasiconvex on S .

Proof. (i) For the positivity of k and the quasiconvexity of f we have $(kf)(x_1 + \lambda(x_2 - x_1)) = kf(x_1 + \lambda(x_2 - x_1)) \leq k \max\{f(x_1), f(x_2)\} = \max\{kf(x_1), kf(x_2)\}$.

(ii) Let $x_1, x_2 \in S$ with $f(x_1) \geq f(x_2)$. Then, $f(x_1 + \lambda(x_2 - x_1)) \leq f(x_1)$, for all $\lambda \in [0, 1]$. Since g is a non-decreasing function, $g(f(x_1)) \geq g(f(x_2))$ implies $g(f(x_1 + \lambda(x_2 - x_1))) \leq g(f(x_1))$, $\forall \lambda \in [0, 1]$. \square

Theorem 2.2.7. *Let f be a strictly quasiconvex function defined on a convex set $S \subseteq \mathbb{R}^n$ and let $g : A \rightarrow \mathbb{R}$ be an increasing function, with $f(S) \subseteq A$. Then:*

- (i) kf , $k > 0$ is strictly quasiconvex on S ;
- (ii) $g \circ f$ is strictly quasiconvex on S .

Proof. Similar to the proof given in Theorem 2.2.6. \square

Another useful composition theorem is the following.

Theorem 2.2.8. *Let $g(x) = Ax + b$ where A is an $m \times n$ matrix, $b \in \mathbb{R}^m$, and let f be a quasiconvex function on a convex set $S \subseteq g(\mathbb{R}^n)$. Then, $z(x) = f(Ax + b)$ is quasiconvex on S .*

Proof. We have $z(\lambda x_1 + (1 - \lambda)x_2) = f(\lambda(Ax_1 + b) + (1 - \lambda)(Ax_2 + b)) \leq \max\{f(Ax_1 + b), f(Ax_2 + b)\} = \max\{z(x_1), z(x_2)\}$, $\forall \lambda \in [0, 1]$. \square

Property (ii) of Theorem 2.2.6 leads to the following result which extends Theorem 2.2.2.

Theorem 2.2.9. *Let f be a homogeneous function of degree $\alpha \geq 1$ defined on a convex set $S \subseteq \mathbb{R}^n$.*

If $f(x) > 0$ for all $x \in S$, then f is quasiconvex if and only if it is convex.

Proof. Taking into account Theorem 2.2.2, case $\alpha > 1$ remains to be considered. The function $g(x) = [f(x)]^{\frac{1}{\alpha}}$ is linearly homogeneous and quasiconvex since it is the composite function $g = z \circ f$, where $z(y) = y^{\frac{1}{\alpha}}$ is increasing and f is quasiconvex. It follows that $f(x) = [g(x)]^\alpha$ is convex as the composite function of a convex function and an increasing convex function. \square

Remark 2.2.1. Following the same line given in the proof of the previous theorem, it is easy to prove the following result.

Let f be a homogeneous function of degree α with $0 < \alpha \leq 1$, defined on a convex set $S \subseteq \mathbb{R}^n$.

If $f(x) > 0$ for all $x \in S$, then f is quasiconcave if and only if it is concave.

Example 2.2.2. The function $f(x_1, \dots, x_n) = (\sum_{i=1}^n x_i^2)^\beta$ is convex for $\beta \geq \frac{1}{2}$.

In fact, $\sum_{i=1}^n x_i^2$ is convex so that, from Theorem 2.2.6, f is quasiconvex; on the other hand f is homogeneous of degree $\alpha = 2\beta$ and consequently, from Theorem 2.2.9, f is convex for $\beta \geq \frac{1}{2}$.

Remark 2.2.2. Theorems 2.2.3 and 2.2.6 are sometimes useful in identifying quasiconvex functions or in constructing new quasiconvex functions from existing ones. Examples will be given in Sect. 2.3.

The given definitions of generalized convexity point out that the behaviour of the function is strictly related to the behaviour of its restriction on every line segment. This connection is expressed in the following theorem.

Theorem 2.2.10. *Let f be a function defined on a convex set $S \subseteq \mathbb{R}^n$. Then, f is quasiconvex (strictly quasiconvex) on S if and only if the restriction of f on each line segment contained in S is a quasiconvex (strictly quasiconvex) function.*

Proof. See Exercise 2.18. □

Remark 2.2.3. As we shall see, by means of Theorem 2.2.10, several results regarding generalized convexity of functions of several variables may be derived from the corresponding results for functions of one variable. For this reason we shall devote Sect. 2.5 to the study of generalized convex functions of one variable.

The non-constancy of a strictly quasiconvex function along a line allows us to characterize strictly quasiconvexity within quasiconvexity.

Theorem 2.2.11. *Let f be a function defined on a convex set $S \subseteq \mathbb{R}^n$. Then, f is strictly quasiconvex on S if and only if (i) and (ii) hold:*

- (i) f is quasiconvex on S ;
- (ii) Every restriction on a line segment is not constant.

Proof. (i) This follows from (iii) of Theorem 2.2.1 while (ii) follows directly from (2.3).

Assume now that (i) and (ii) hold. If f is not strictly quasiconvex, there exist $x_1, x_2 \in S$, $\bar{\lambda} \in (0, 1)$, such that $f(x_1) \geq f(x_2)$ and $f(\bar{x}) = f(x_1 + \bar{\lambda}(x_2 - x_1)) = f(x_1)$; since f is not constant in $[x_1, \bar{x}]$, there exists $x_0 \in (x_1, \bar{x})$ such that $f(x_0) < f(x_1) = f(\bar{x})$. If $f(\bar{x}) = f(x_1) > f(x_2)$, the quasiconvexity of f on the line segment $[x_0, x_2]$ is contradicted. If $f(\bar{x}) = f(x_1) = f(x_2)$, there exists $x^* \in (\bar{x}, x_2)$ such that $f(x^*) < f(\bar{x}) = f(x_2)$ and this contradicts the quasiconvexity of f on the line segment $[x_0, x^*]$. □

As we have pointed out by means of Example 2.2.1, a quasiconvex function is not necessarily continuous. Under a continuity assumption, we have the useful property that quasiconvexity on an open convex set S is preserved on the closure of S . More precisely, we have the following result.

Theorem 2.2.12. *Let f be a continuous function on the closure of a convex set $S \subseteq \mathbb{R}^n$ and quasiconvex on the interior of S . Then, f is quasiconvex on the closure of S .*

Proof. Let $x, y \in clS$; we must prove that $f(x + \lambda(y - x)) \leq \max\{f(x), f(y)\}$, $\forall \lambda \in [0, 1]$. If $x, y \in intS$ the inequality is true by assumption. Let $\{x_n\} \subset intS$, $\{y_n\} \subset intS$ be sequences converging to x and y , respectively, with the convention $x_n = x$ for all n ($y_n = y$ for all n) if $x \in intS$ ($y \in intS$). We have $f(x_n + \lambda(y_n - x_n)) \leq \max\{f(x_n), f(y_n)\}$, $\forall \lambda \in [0, 1]$, so that the thesis follows by taking the limit for $n \rightarrow +\infty$ and taking into account the continuity of f . \square

2.3 Semistrict Quasiconvexity

In this section we shall introduce a class of generalized convex functions which is intermediate between convex and quasiconvex ones. This class is obtained by strengthening quasiconvexity.

Definition 2.3.1. A function f defined on a convex set $S \subseteq \mathbb{R}^n$ is said to be *semistrictly quasiconvex* if

$$f(\lambda x_1 + (1 - \lambda)x_2) < \max\{f(x_1), f(x_2)\} \quad (2.6)$$

for every $x_1, x_2 \in S$, with $f(x_1) \neq f(x_2)$ and for every $\lambda \in (0, 1)$ or, equivalently,

$$f(x_1) > f(x_2) \text{ implies that } f(x_1) > f(x_1 + \lambda(x_2 - x_1)) \quad (2.7)$$

for every $x_1, x_2 \in S$, $\lambda \in (0, 1)$.

As a direct consequence of the given definition, we have the following theorem.

Theorem 2.3.1. Let f be a function defined on a convex set $S \subseteq \mathbb{R}^n$.

- (i) If f is strictly quasiconvex on S , then f is semistrictly quasiconvex on S ;
- (ii) If f is convex on S , then f is semistrictly quasiconvex on S .

Since a constant function is semistrictly quasiconvex but not strictly quasiconvex, the class of strictly quasiconvex functions is properly contained in the class of semistrictly quasiconvex ones.

The following examples point out that there is not any inclusion relationship between the class of semistrictly quasiconvex functions and the one of quasiconvex functions.

Example 2.3.1. Consider the function $f(x) = \begin{cases} 1 & -1 \leq x \leq 1, x \neq 0 \\ 2 & x = 0 \end{cases}$

f is not quasiconvex since we have $f(0) = 2 > \max\{f(1), f(-1)\} = 1$; on the other hand, f is semistrictly quasiconvex, since, in order to apply the definition, we must necessarily consider the points $x_1 = 0$, $x_2 \neq 0$, so that $f(x_1 + \lambda(x_2 - x_1)) = f(\lambda x_2) = 1 < f(x_1) = 2$.

Example 2.3.2. Consider the function $f(x) = \begin{cases} x & 0 \leq x \leq 1, \\ 1 & 1 < x \leq 2 \end{cases}$

f is a non-decreasing function and hence quasiconvex, but it is not semistrictly quasiconvex since we have $f(0) = 0 < f(2) = 1$ and $f(\frac{3}{2}) = 1 = f(2)$.

Let us note that in Example 2.3.1, the function f is not lower semicontinuous at $x_0 = 0$. We shall prove that a sufficient condition for a semistrictly quasiconvex function to be quasiconvex is the lower semicontinuity of the function.

Theorem 2.3.2. *If f is a lower semicontinuous and semistrictly quasiconvex function on a convex set S , then f is quasiconvex on S .*

Proof. Let $x, y \in S$ such that $f(x) \geq f(y)$. If $f(x) > f(y)$, the thesis follows from (2.7). Let $f(x) = f(y)$ and assume the existence of $\bar{\lambda} \in (0, 1)$ such that $f(\bar{x}) = f(x + \bar{\lambda}(y - x)) > f(x)$. Since f is lower semicontinuous there exist $\epsilon > 0$ and a neighbourhood I of \bar{x} such that $f(z) \geq f(\bar{x}) - \epsilon$, $\forall z \in I$. Since $f(\bar{x}) > f(x)$, by choosing $\epsilon < f(\bar{x}) - f(x)$, we have $f(z) > f(x)$, $\forall z \in I$ and, in particular, $\forall z \in I \cap [x, y]$. Let $z \in I \cap [x, y]$. If $z \in (x, \bar{x})$, then the semistrict quasiconvexity of f is contradicted on the segment $[z, y]$ if $f(z) < f(\bar{x})$ or in the segment $[x, \bar{x}]$ if $f(z) \geq f(\bar{x})$.

If $z \notin (x, \bar{x})$, the proof is analogous. □

We can summarize the inclusion relationships between the various classes of convex and generalized convex functions by means of the diagram of Fig. 2.3 which assumes lower semicontinuity. All inclusions are proper.

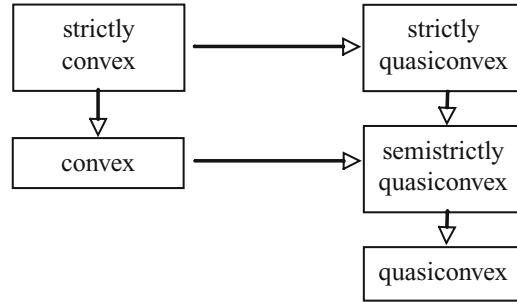


Fig. 2.3. Relationships between various types of convexity under lower semicontinuity

As happens in the quasiconvex case, the behaviour of a semistrictly quasiconvex function may be characterized by means of the behaviour of its restrictions on every line segment. Such a property allows us to characterize semistrict quasiconvexity within quasiconvexity and also to characterize strict quasiconvexity within semistrict quasiconvexity. More precisely, we have the following results.

Theorem 2.3.3. *Let f be a function defined on a convex set $S \subseteq \mathbb{R}^n$. Then, f is semistrictly quasiconvex on S if and only if the restriction of f to each line segment contained in S is a semistrictly quasiconvex function.*

Theorem 2.3.4. *Let f be a lower semicontinuous function defined on a convex set $S \subseteq \mathbb{R}^n$. Then, f is semistrictly quasiconvex if and only if (i) and (ii) hold:*

- (i) f is quasiconvex;
- (ii) Every local minimum is also global for each restriction on a line segment.

Proof. Assume that (i) and (ii) hold. If f is quasiconvex but not semistrictly quasiconvex, then there exist $x_1, x_2 \in S$, $\bar{\lambda} \in (0, 1)$ such that $f(x_1) > f(x_2)$ and $f(\bar{x}) = f(x_1 + \bar{\lambda}(x_2 - x_1)) = f(x_1)$. From (ii), f is not constant in $[x_1, \bar{x}]$, otherwise each point $\tilde{x} \in (x_1, \bar{x})$ is a local minimum which is not global for the restriction $\varphi(t) = f(x_1 + t(x_2 - x_1))$, $t \in [0, 1]$. Consequently, there exists $x_0 \in (x_1, \bar{x})$ such that $f(x_0) < f(x_1) = f(\bar{x})$ and this contradicts the quasiconvexity of f on the segment $[x_0, x_2]$.

With respect to the converse statement, (i) follows from Theorem 2.3.2 while (ii) is obvious. \square

The following theorem, whose proof can be found in [104, 209], shows that the class of continuous semistrictly quasiconvex functions is the wider class for which every local minimum is also global.

Theorem 2.3.5. *Let f be a continuous quasiconvex function defined on a convex set $S \subseteq \mathbb{R}^n$. Then, f is semistrictly quasiconvex if and only if every local minimum point is also global for f on S .*

Theorem 2.3.6. *Let f be a lower semicontinuous quasiconvex function defined on a convex set $S \subseteq \mathbb{R}^n$. Then, f is strictly quasiconvex if and only if the following conditions hold:*

- (i) f is semistrictly quasiconvex;
- (ii) Any restriction on a line segment attains its minimum at no more than one point.

Proof. See Exercise 2.19. \square

The composition Theorem 2.2.6 may easily be extended to the semistrictly quasiconvex case.

Theorem 2.3.7. *Let f be a semistrictly quasiconvex function defined on a convex set $S \subseteq \mathbb{R}^n$ and let $g : A \rightarrow \mathbb{R}$ be an increasing function, with $f(S) \subseteq A$. Then:*

- (i) kf , $k > 0$ is semistrictly quasiconvex on S ;
- (ii) $g \circ f$ is semistrictly quasiconvex on S .

Composition theorems related to generalized convex functions are useful tools in identifying or in constructing generalized quasiconvex functions. Some examples are given below.

Example 2.3.3. Let f be a positive and convex function defined on a convex set $S \subseteq \mathbb{R}^n$ and consider the increasing functions $h_1(y) = \log y$, $y > 0$, $h_2(y) = y^\alpha$, $y > 0$, $\alpha > 0$. From Theorem 2.3.7 and taking into account the inclusion relationships, we deduce the semistrict quasiconvexity of the functions $z(x) = \log f(x)$, and $z(x) = (f(x))^\alpha$, $\alpha > 0$. Furthermore if f is positive and strictly convex, then $z(x) = \log f(x)$, and $z(x) = (f(x))^\alpha$, $\alpha > 0$ are strictly quasiconvex.

Example 2.3.4. Let f be a quasiconcave (strictly quasiconcave, semistrictly quasiconcave, respectively) function constant in sign defined on a convex set $S \subseteq \mathbb{R}^n$. Then, the reciprocal function $z(x) = \frac{1}{f(x)}$ is quasiconvex (strictly quasiconvex, semistrictly quasiconvex, respectively).

In fact, since $h(t) = -\frac{1}{t}$ is an increasing function, the composite function $h(f(x)) = -\frac{1}{f(x)}$ is quasiconcave (strictly quasiconcave, semistrictly quasiconcave, respectively), so that $z(x) = \frac{1}{f(x)}$ is quasiconvex (strictly quasiconvex, semistrictly quasiconvex, respectively).

Note that the reciprocal of a constant in sign concave function is not convex but it is quasiconvex.

As is known, the class of convex functions is not closed with respect to the ratio. In the following theorem generalized convexity of a ratio is investigated.

Theorem 2.3.8. *Let f and g be functions defined on a convex set $S \subseteq \mathbb{R}^n$, and let*

$$z(x) = \frac{f(x)}{g(x)}$$

Then, the following properties hold:

- (i) *If f is non-negative and convex, and g is positive and concave, then z is semistrictly quasiconvex;*
- (ii) *If f is non-positive and convex, and g is positive and convex, then z is semistrictly quasiconvex;*
- (iii) *If f is convex, and g is positive and affine, then z is semistrictly quasiconvex.*

Proof. (i) We must prove that $z(x) = \frac{f(x)}{g(x)} < \frac{f(x_0)}{g(x_0)} = z(x_0)$ implies that $z((1-\lambda)x_0 + \lambda x) < z(x_0)$, $\lambda \in (0, 1)$.

Taking into account the convexity of f and the concavity of g , together with their sign, we have $f((1-\lambda)x_0 + \lambda x) \leq (1-\lambda)f(x_0) + \lambda f(x) < (1-\lambda)f(x_0) + \lambda \frac{f(x_0)}{g(x_0)}g(x) = \frac{f(x_0)}{g(x_0)}((1-\lambda)g(x_0) + \lambda g(x)) \leq \frac{f(x_0)}{g(x_0)}g((1-\lambda)x_0 + \lambda x)$. It follows that $\frac{f((1-\lambda)x_0 + \lambda x)}{g((1-\lambda)x_0 + \lambda x)} < \frac{f(x_0)}{g(x_0)}$, i.e., $z((1-\lambda)x_0 + \lambda x) < z(x_0)$.

(ii) This can be proven similarly.

(iii) This follows from (i) and (ii) by noting that an affine function is both convex and concave. \square

2.4 Generalized Convexity of Some Homogeneous Functions

In this section we shall characterize the quasiconcavity of some classes of homogeneous functions which appear frequently in Economics.

2.4.1 The Cobb–Douglas Function

One of the most important production or utility functions is the Cobb–Douglas function defined as:

$$f(x) = Ax_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}, \quad A > 0, \quad x_i > 0, \quad \alpha_i > 0, \quad i = 1, \dots, n. \quad (2.8)$$

The main properties of the Cobb–Douglas function, which is homogeneous of degree $\alpha = \sum_{i=1}^n \alpha_i$ (see Theorem 1.4.1), are stated in the following theorem.

Theorem 2.4.1. *The Cobb–Douglas function (2.8) is quasiconcave and is concave if and only if $\alpha = \sum_{i=1}^n \alpha_i \leq 1$.*

Proof. Since $\log f(x) = \log A + \sum_{i=1}^n \alpha_i \log x_i$ is a concave function as a positive linear combination of concave functions, the function $f(x) = e^{\log f(x)}$ is quasiconcave. The last statement follows by means of the property that a positive homogeneous function of degree $\alpha \leq 1$ is concave if and only if it is quasiconcave. \square

Remark 2.4.1. If $U : \mathbb{R}_+^n \rightarrow \mathbb{R}$ is a utility function which represents the preferences of the consumer and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function, then the monotone transformation $\varphi \circ U$ is a utility function which represents the same preferences. It follows that the Cobb–Douglas utility function

$$U(x) = Ax_1^{\gamma_1} x_2^{\gamma_2} \dots x_n^{\gamma_n}, \quad A > 0, \quad x_i > 0, \quad \gamma_i > 0, \quad i = 1, \dots, n, \quad \sum_{i=1}^n \gamma_i = 1 \quad (2.9)$$

represents the same preferences as the Cobb–Douglas function (2.8). In fact, by setting $\beta = \frac{1}{\alpha_1 + \dots + \alpha_n}$, we have that $\varphi(z) = z^\beta$ is an increasing function and furthermore $\varphi(f(x)) = (Ax_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n})^\beta = A^\beta x_1^{\alpha_1 \beta} x_2^{\alpha_2 \beta} \dots x_n^{\alpha_n \beta} = A^\beta x_1^{\gamma_1} x_2^{\gamma_2} \dots x_n^{\gamma_n}$ with $\sum_{i=1}^n \gamma_i = \beta \sum_{i=1}^n \alpha_i = 1$. This justifies the use of utility functions of the kind (2.9) found in consumer theory.

2.4.2 The Constant Elasticity of Substitution (C.E.S.) Function

Another important function in Economics is the C.E.S. function defined by

$$f(x) = (a_1x_1^\beta + a_2x_2^\beta + \dots + a_nx_n^\beta)^{\frac{1}{\beta}}, \quad a_i > 0, \quad x_i > 0, \quad i = 1, \dots, n, \quad \beta \neq 0 \quad (2.10)$$

The main properties of the C.E.S. function, which is linearly homogeneous, are stated in the following theorem.

Theorem 2.4.2. *The C.E.S. function (2.10) is quasiconcave if and only if $\beta \leq 1$ and it is convex if and only if $\beta \geq 1$.*

Proof. By setting $g(x) = a_1x_1^\beta + a_2x_2^\beta + \dots + a_nx_n^\beta$, we have $f(x) = (g(x))^{\frac{1}{\beta}}$. When $\beta < 0$, g is convex as a linear combination of positive convex functions; it follows that $\frac{1}{g}$ is a quasiconcave function (see Example 2.3.4) so that f is quasiconcave as an increasing transformation of $\frac{1}{g}$. When $0 < \beta \leq 1$, g is concave as a positive linear combination of concave functions, so that f is quasiconcave as an increasing transformation of g . Finally, when $\beta \geq 1$, g is convex as a linear combination of convex functions, so that f is quasiconvex as an increasing transformation of g . The thesis follows from Theorem 2.2.2 since the C.E.S. function is linearly homogeneous. \square

2.4.3 The Leontief Production Function

Another important homogeneous function of degree α is the Leontief production function defined by

$$f(x) = \left(\min_i \left\{ \frac{x_i}{a_i} \right\} \right)^\alpha, \quad x_i > 0, \quad a_i > 0, \quad i = 1, \dots, n, \quad \alpha > 0. \quad (2.11)$$

The following theorem holds.

Theorem 2.4.3. *The Leontief production function (2.11) is quasiconcave and it is concave if and only if $\alpha \leq 1$.*

Proof. The function $g(x) = \min_i \left\{ \frac{x_i}{a_i} \right\}$ is concave as the minimum of a finite number of concave functions, so that f is an increasing transformation of g and thus it is quasiconcave. The last statement follows from Remark 2.2.1. \square

2.4.4 A Generalized Cobb–Douglas Function

Consider the function $z(x) = \prod_{i=1}^k (f_i(x))^{\alpha_i}$, $\alpha_i > 0$, where $f_i(x)$, $i = 1, \dots, k$, are positive concave functions on a convex set $S \subseteq \mathbb{R}^n$. Since $\log z(x) = \sum_{i=1}^k \alpha_i \log f_i(x)$ is a concave function (as a positive linear combination of concave functions), the function $z(x) = e^{\log z(x)}$ is quasiconcave.

2.5 Generalized Quasiconvex Functions in One Variable

Theorems 2.2.10 and 2.3.3 suggest carrying on the study of generalized convexity for one real variable function with the aim of extending the obtained results for functions of several variables.

A complete characterization of quasiconvexity is given in the following theorem where the parenthesis \rangle or \langle indicates that the corresponding end point can or cannot belong to the interval.

Theorem 2.5.1. *Let φ be a function defined on the interval $[a, b] \subseteq \mathbb{R}$. Then φ is quasiconvex if and only if one of the following conditions is verified:*

- (i) φ is non-increasing or non-decreasing in $[a, b]$;
- (ii) φ is non-increasing in $[a, b)$ but not in $[a, b]$ or φ is non-decreasing in $(a, b]$ but not in $[a, b]$;
- (iii) there exists $t_0 \in (a, b)$ such that φ is non-increasing in $[a, t_0\rangle$ and non-decreasing in $\langle t_0, b]$, where at least one of the two intervals is closed.

Proof. It is easy to verify that the validity of (i) or (ii) or (iii) implies the quasiconvexity of φ . Assume now the quasiconvexity of φ . Set $\ell = \inf\{\varphi(t), t \in [a, b]\}$ (note that ℓ may be $-\infty$) and let $\{t_n\} \subset [a, b]$ be such that $\varphi(t_n) \rightarrow \ell$ when $t_n \rightarrow t_0$. The following exhaustive cases occur: $t_0 = a$, $t_0 = b$, $t_0 \in (a, b)$.
case $t_0 = a$.

We will prove that φ is non-decreasing in $(a, b]$. If not, there exist $t_1, t_2 \in (a, b]$ with $t_1 < t_2$, such that $\varphi(t_1) > \varphi(t_2) \geq \ell$. Since ℓ is the infimum value of φ , there exists \bar{n} such that $a < t_{\bar{n}} < t_1 < t_2$ with $\varphi(t_{\bar{n}}) < \varphi(t_1)$ and this contradicts the quasiconvexity of φ applied to the interval $[t_{\bar{n}}, t_2]$.

If $\varphi(a) = \varphi(t_0) = \ell$ then φ is non-decreasing in $[a, b]$, otherwise φ is non-decreasing in $(a, b]$ but not in $[a, b]$ since $\varphi(a) > \ell$.

case $t_0 = b$.

By means of similar arguments, it is easy to prove that φ is non-increasing in $[a, b)$; furthermore, φ is non-increasing in $[a, b]$ if $\varphi(b) = \varphi(t_0) = \ell$.

case $t_0 \in (a, b)$.

We will prove that φ is non-increasing in $[a, t_0)$ and non-decreasing in $(t_0, b]$. If not, there exist $t_1, t_2, \hat{t}_1, \hat{t}_2 \in [a, b]$ such that $t_1 < t_2 < t_0 < \hat{t}_1 < \hat{t}_2$ with $\varphi(t_1) < \varphi(t_2)$ and $\varphi(\hat{t}_1) > \varphi(\hat{t}_2)$. Since ℓ is the infimum value of φ , there exists \bar{n} such that $t_2 < t_{\bar{n}} < \hat{t}_1$ with $\varphi(t_{\bar{n}}) < \varphi(t_2)$, $\varphi(t_{\bar{n}}) < \varphi(\hat{t}_1)$ and this contradicts the quasiconvexity of φ in the intervals $[t_1, t_{\bar{n}}]$, $[t_{\bar{n}}, \hat{t}_2]$, respectively.

It remains to be proven that at least one of the two intervals $[a, t_0)$, $\langle t_0, b]$ is closed.

Set $\ell_1 = \inf\{\varphi(t), t \in [a, t_0)\}$, $\ell_2 = \inf\{\varphi(t), t \in (t_0, b]\}$. Let us note that at least one of the two infimum is finite, otherwise there exist $t_1 < t_0$, $t_2 > t_0$, such that $\varphi(t_1) < \varphi(t_0)$ and $\varphi(t_2) < \varphi(t_0)$, and this contradicts the quasiconvexity of φ in the interval $[t_1, t_2]$.

Now we will prove that $\varphi(t_0) \leq \max\{\ell_1, \ell_2\}$; if not, taking into account that the function is non-increasing in $[a, t_0)$ and $\ell_1 < \varphi(t_0)$, there exists $t_1 < t_0$ such that $\varphi(t_1) < \varphi(t_0)$. In a similar way, since the function is non-decreasing

in $(t_0, b]$ and $\ell_2 < \varphi(t_0)$, there exists $t_2 > t_0$ such that $\varphi(t_2) < \varphi(t_0)$. It follows that the function φ is not quasiconvex on the interval $[t_1, t_2]$ and this is absurd.

Finally, let us note that $\ell = \min\{\ell_1, \ell_2, \varphi(t_0)\}$. If $\ell = \ell_1$ then $\varphi(t_0) \leq \ell_2$ and φ is non-decreasing in $[t_0, b]$. If $\ell = \ell_2$ then $\varphi(t_0) \leq \ell_1$ and φ is non-increasing in $[a, t_0]$. If $\ell = \varphi(t_0)$, then φ is non-increasing in $[a, t_0]$ and non-decreasing in $[t_0, b]$.

The proof is complete. \square

Theorem 2.5.1 may be specified in the case where φ is a lower semicontinuous function.

Theorem 2.5.2. *Let φ be a lower semicontinuous function defined on the interval $[a, b] \subseteq \mathbb{R}$. Then, φ is quasiconvex if and only if there exists $t_0 \in [a, b]$ such that φ is non-increasing in $[a, t_0]$ and non-decreasing in $[t_0, b]$, where one of the two subintervals may be reduced to a point.*

Proof. Assume that φ is quasiconvex. The lower semicontinuity of φ on $[a, b]$ implies the existence of its minimum value m . Set $A = \{t \in [a, b] : \varphi(t) = m\}$; A is a closed interval since φ is quasiconvex and lower semicontinuous. Let $t_0 \in A$. The function φ is non-increasing in $[a, t_0]$ when $t_0 \neq a$, since the existence of $t_1, t_2 \in [a, t_0)$, $t_1 < t_2$ with $\varphi(t_1) < \varphi(t_2)$ contradicts the quasiconvexity of φ in $[t_1, t_0]$.

In the same way it can be proven that φ is non-decreasing in $[t_0, b]$, $t_0 \neq b$, so that the thesis follows when $a < t_0 < b$.

When $a = t_0 = b$, φ is constant in $[a, b]$; when $a = t_0$ and $t_0 < b$, φ is constant in $[a, t_0]$ and non-decreasing in $[t_0, b]$.

When $a < t_0$ and $t_0 = b$, φ is non-increasing in $[a, t_0]$ and constant in $[t_0, b]$. In each case the thesis follows.

The converse statement follows immediately. \square

The given characterizations of quasiconvex functions can be specialized to the subclass of strictly quasiconvex functions as is stated in Corollary 2.5.1 and in Corollary 2.5.2.

Corollary 2.5.1. *Let $\varphi(t)$ be a function defined on the interval $[a, b] \subseteq \mathbb{R}$. Then, $\varphi(t)$ is strictly quasiconvex if and only if one of the following conditions holds:*

- (i) φ is increasing or decreasing in $[a, b]$;
- (ii) φ is decreasing in $[a, b]$ but not in $[a, b]$ or increasing in $(a, b]$ but not in $[a, b]$;
- (iii) there exists $t_0 \in (a, b)$ such that φ is decreasing in $[a, t_0)$ and increasing in $(t_0, b]$ where at least one of the two intervals is closed.

Corollary 2.5.2. *Let φ be a lower semicontinuous function defined on the interval $[a, b] \subseteq \mathbb{R}$. Then, φ is strictly quasiconvex if and only if there exists*

$t_0 \in [a, b]$ such that φ is decreasing in $[a, t_0]$ and increasing in $[t_0, b]$, where one of the two subintervals may be reduced to a point.

Since the class of semistrictly quasiconvex functions is contained in the class of quasiconvex ones under the lower semicontinuity assumption, Theorem 2.5.2 may be specified as follows.

Corollary 2.5.3. *Let $\varphi(t)$ be a lower semicontinuous function defined on the interval $[a, b] \subseteq \mathfrak{R}$. Then, $\varphi(t)$ is semistrictly quasiconvex if and only if there exist $\alpha, \beta \in [a, b]$ such that φ is decreasing in $[a, \alpha]$, constant in $[\alpha, \beta]$ and increasing in $[\beta, b]$, where one or two subintervals may be reduced to a point.*

Proof. By referring to the proof given in Theorem 2.5.2, consider the closed interval $A = \{t \in [a, b] : \varphi(t) = m\}$ and let $\alpha = \min A$, $\beta = \max A$. We must prove that $\varphi(t)$ is decreasing on the interval $[a, \alpha]$ and increasing on $[\beta, b]$. In this regard it is sufficient to note that the existence of $t_1, t_2 \in [a, \alpha]$ such that $t_1 < t_2 < \alpha$ and $\varphi(t_1) \geq \varphi(t_2)$, contradicts the semistrictly quasiconvexity of φ on the interval $[t_1, \alpha]$. Similarly, it can be proven that φ is increasing in $[\beta, b]$. \square

2.6 Exercises

2.1. Which of the following functions are quasiconvex, semistrictly quasiconvex or strictly quasiconvex?

(a) $f(x) = x |x|$; (b) $f(x) = x |x| - x^2$; (c) $f(x) = x |x| + x^2$;

(d) $f(x) = \begin{cases} \frac{1}{x-1} & 0 \leq x < 1 \\ 0 & x = 1 \\ \frac{1}{x+1} & x > 1 \end{cases}$

2.2. Sketch a graph of a continuous function and the graph of a discontinuous function of one variable which satisfies the following conditions: decreasing in $[0, 1]$, constant in $[1, 2]$, decreasing in $[2, 3]$, constant in $[3, 4]$, increasing in $[4, 5]$, constant in $[5, 6]$ and increasing in $[6, 7]$.

Is this function quasiconvex, strictly quasiconvex or semistrictly quasiconvex?

2.3. Let $f : [a, b] \rightarrow \mathfrak{R}$ and let $x_0 \in (a, b)$. Assume that $f(x_0) = \alpha$, $\lim_{x \rightarrow x_0^-} f(x) = -\infty$, $\lim_{x \rightarrow x_0^+} f(x) = -\infty$. Show that f is not quasiconvex.

2.4. Are the following functions quasiconvex, semistrictly quasiconvex or strictly quasiconvex?

(a) $f(x) = \log \sum_{i=1}^n x_i^2$; (b) $f(x) = \log \sum_{i=1}^n x_i$, $x_i > 0$, $i = 1, \dots, n$;

(c) $f(x) = \left(\sum_{i=1}^n e^{x_i} \right)^\beta$, $\beta > 0$.

2.5. Let f be a function defined on the convex set $S \subseteq \mathbb{R}^n$. Prove that:

- (a) if f is positive and convex, then $-\frac{1}{f}$ is quasiconvex;
- (b) if f is negative and quasiconvex (strictly quasiconvex) then $\frac{1}{f}$ is quasiconcave (strictly quasiconcave).

2.6. Let f, g be functions defined on a convex set $S \subseteq \mathbb{R}^n$. Using the characterization of quasiconvex functions in terms of its lower level sets, prove that the function $z(x) = \frac{f(x)}{g(x)}$ is quasiconvex if f is non-negative and convex and g is positive and concave.

2.7. Let f and g be functions defined on a convex set $S \subseteq \mathbb{R}^n$, and let $z(x) = \left(\frac{f(x)}{g(x)}\right)^\alpha$, $\alpha > 0$. If f is non-negative and convex, and g is positive and concave, then z is semistrictly quasiconvex.

2.8. Let f and g be functions defined on a convex set $S \subseteq \mathbb{R}^n$, and let $z(x) = \frac{f(x)}{g(x)}$. Prove that z is strictly quasiconvex if one of the following conditions holds:

- 1. f is non-negative and strictly convex and g is positive and concave;
- 2. f is non-negative and convex and g is positive and strictly concave;
- 3. f is non-positive and strictly convex and $g(x)$ is positive and convex.

2.9. Apply Theorem 2.2.3 to show that the linear fractional function $f(x) = \frac{a^T x + a_0}{b^T x + b_0}$, $b^T x + b_0 > 0$, is both quasiconvex and quasiconcave.

2.10. Show that the following functions are quasiconvex:

- (a) $f(x, y) = \log \frac{x^4}{y+1}$, $y > -1$;
- (b) $f(x, y) = \log \frac{x^2 - xy + y^2}{4 - y^2}$, $-2 < y < 2$;
- (c) $f(x, y) = \log(x^4 + y^2) - \log(y - 1)$, $y > 1$.

2.11. Show that the following functions are convex:

$$f(x_1, \dots, x_n) = \left(\sum_{i=1}^n x_i^4\right)^{\frac{1}{3}}; \quad g(x, y) = \frac{x^2}{y}; \quad h(x) = \frac{(a^T x)^2}{b^T x}.$$

2.12. Show that the function $f(x, y) = xy$, $x, y \geq 0$ is quasiconcave and that the function $f(x, y) = \sqrt{xy}$, $x, y \geq 0$ is concave.

2.13. Show that the ratio $z(x) = \frac{f(x)}{g(x)}$ is semistrictly quasiconcave when f is non-negative and concave and g is positive and convex.

2.14. Show that the function $z(x) = f(x) \cdot g(x)$ is semistrictly quasiconcave when f is non-negative and concave and g is positive and concave.

2.15. Show that the function $f(x) = (a^T x + a_0)(b^T x + b_0)$ is semistrictly quasiconcave if $a^T x + a_0 \geq 0$ and $b^T x + b_0 > 0$ while it is semistrictly quasiconvex if $a^T x + a_0 \leq 0$ and $b^T x + b_0 > 0$.

2.16. Let f be a quasiconvex function defined on the convex set $S \subseteq \mathbb{R}^n$. Prove that f cannot have an interior strict local maximum point.

2.17. Give an example which shows that a quasiconvex function may have an interior global maximum point which is not a local minimum.

2.18. Show Theorem 2.2.10.

2.19. Show Theorem 2.3.6.

2.20. Let f be a non-constant lower semicontinuous semistrictly quasiconvex function on the convex set $S \subseteq \mathbb{R}^n$. Show that f , in contrast to the quasiconvex case, cannot have an interior global maximum.

2.21. Give an example which shows that the assumption of lower semicontinuity in Exercise 2.20 cannot be relaxed.

2.22. Show that, in contrast to the quasiconvex case, a non-constant lower semicontinuous semistrictly quasiconvex function cannot have an interior local maximum point which is not a local minimum.

2.23. Give a direct proof of the statement: if f is convex then f is semistrictly quasiconvex.

2.7 References

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