
Preface

When I was first approached for the 2005 edition of the Saint-Flour Probability Summer School, I was intrigued, flattered and scared.¹ Apart from the challenge posed by the teaching of a rather analytical subject to a probabilistic audience, there was the danger of producing a remake of my recent book *Topics in Optimal Transportation*.

However, I gradually realized that I was being offered a unique opportunity to rewrite the whole theory from a different perspective, with alternative proofs and a different focus, and a more probabilistic presentation; plus the incorporation of recent progress. Among the most striking of these recent advances, there was the rising awareness that John Mather's minimal measures had a lot to do with optimal transport, and that both theories could actually be embedded within a single framework. There was also the discovery that optimal transport could provide a robust synthetic approach to Ricci curvature bounds. These links with dynamical systems on one hand, differential geometry on the other hand, were only briefly alluded to in my first book; here on the contrary they will be at the basis of the presentation. To summarize: more probability, more geometry, and more dynamical systems. Of course there cannot be more of everything, so in some sense there is less analysis and less physics, and also there are fewer digressions.

So the present course is by no means a reduction or an expansion of my previous book, but should be regarded as a complementary reading. Both sources can be read independently, or together, and hopefully the complementarity of points of view will have pedagogical value.

¹ Fans of Tom Waits may have identified this quotation.

Throughout the book I have tried to optimize the results and the presentation, to provide complete and self-contained proofs of the most important results, and comprehensive bibliographical notes — a dauntingly difficult task in view of the rapid expansion of the literature. Many statements and theorems have been written specifically for this course, and many results appear in rather sharp form for the first time. I also added several appendices, either to present some domains of mathematics to non-experts, or to provide proofs of important auxiliary results. All this has resulted in a rapid growth of the document, which in the end is about six times (!) the size that I had planned initially. So **the non-expert reader is advised to skip long proofs at first reading**, and concentrate on explanations, statements, examples and sketches of proofs when they are available.

About terminology: For some reason I decided to switch from “transportation” to “transport”, but this really is a matter of taste.

For people who are already familiar with the theory of optimal transport, here are some more serious changes.

Part I is devoted to a qualitative description of optimal transport. The dynamical point of view is given a prominent role from the beginning, with Robert McCann’s concept of displacement interpolation. This notion is discussed before any theorem about the solvability of the Monge problem, in an abstract setting of “Lagrangian action” which generalizes the notion of length space. This provides a unified picture of recent developments dealing with various classes of cost functions, in a smooth or nonsmooth context.

I also wrote down in detail some important estimates by John Mather, well-known in certain circles, and made extensive use of them, in particular to prove the Lipschitz regularity of “intermediate” transport maps (starting from some intermediate time, rather than from initial time). Then the absolute continuity of displacement interpolants comes for free, and this gives a more unified picture of the Mather and Monge–Kantorovich theories. I rewrote in this way the classical theorems of solvability of the Monge problem for quadratic cost in Euclidean space. Finally, this approach allows one to treat change of variables formulas associated with optimal transport by means of changes of variables that are Lipschitz, and not just with bounded variation.

Part II discusses optimal transport in Riemannian geometry, a line of research which started around 2000; I have rewritten all these applications in terms of Ricci curvature, or more precisely curvature-

dimension bounds. This part opens with an introduction to Ricci curvature, hopefully readable without any prior knowledge of this notion.

Part III presents a synthetic treatment of Ricci curvature bounds in metric-measure spaces. It starts with a presentation of the theory of Gromov–Hausdorff convergence; all the rest is based on recent research papers mainly due to John Lott, Karl-Theodor Sturm and myself.

In all three parts, noncompact situations will be systematically treated, either by limiting processes, or by restriction arguments (the restriction of an optimal transport is still optimal; this is a simple but powerful principle). The notion of approximate differentiability, introduced in the field by Luigi Ambrosio, appears to be particularly handy in the study of optimal transport in noncompact Riemannian manifolds.

Several parts of the subject are not developed as much as they would deserve. Numerical simulation is not addressed at all, except for a few comments in the concluding part. The regularity theory of optimal transport is described in Chapter 12 (including the remarkable recent works of Xu-Jia Wang, Neil Trudinger and Grégoire Loeper), but without the core proofs and latest developments; this is not only because of the technicality of the subject, but also because smoothness is not needed in the rest of the book. Still another poorly developed subject is the Monge–Mather–Mañé problem arising in dynamical systems, and including as a variant the optimal transport problem when the cost function is a distance. This topic is discussed in several treatises, such as Albert Fathi’s monograph, *Weak KAM theorem in Lagrangian dynamics*; but now it would be desirable to rewrite everything in a framework that also encompasses the optimal transport problem. An important step in this direction was recently performed by Patrick Bernard and Boris Buffoni. In Chapter 8 I shall provide an introduction to Mather’s theory, but there would be much more to say.

The treatment of Chapter 22 (concentration of measure) is strongly influenced by Michel Ledoux’s book, *The Concentration of Measure Phenomenon*; while the results of Chapters 23 to 25 owe a lot to the monograph by Luigi Ambrosio, Nicola Gigli and Giuseppe Savaré, *Gradient flows in metric spaces and in the space of probability measures*. Both references are warmly recommended complementary reading. One can also consult the two-volume treatise by Svetlozar Rachev and Ludger Rüschendorf, *Mass Transportation Problems*, for many applications of optimal transport to various fields of probability theory.

While writing this text I asked for help from a number of friends and collaborators. Among them, Luigi Ambrosio and John Lott are the ones whom I requested most to contribute; this book owes a lot to their detailed comments and suggestions. Most of Part III, but also significant portions of Parts I and II, are made up with ideas taken from my collaborations with John, which started in 2004 as I was enjoying the hospitality of the Miller Institute in Berkeley. Frequent discussions with Patrick Bernard and Albert Fathi allowed me to get the links between optimal transport and John Mather's theory, which were a key to the presentation in Part I; John himself gave precious hints about the history of the subject. Neil Trudinger and Xu-Jia Wang spent vast amounts of time teaching me the regularity theory of Monge–Ampère equations. Alessio Figalli took up the dreadful challenge to check the entire set of notes from first to last page. Apart from these people, I got valuable help from Stefano Bianchini, François Bolley, Yann Brenier, Xavier Cabré, Vincent Calvez, José Antonio Carrillo, Dario Cordero-Erausquin, Denis Feyel, Sylvain Gallot, Wilfrid Gangbo, Diogo Aguiar Gomes, Nathaël Gozlan, Arnaud Guillin, Nicolas Juillet, Kazuhiro Kuwae, Michel Ledoux, Grégoire Loeper, Francesco Maggi, Robert McCann, Shin-ichi Ohta, Vladimir Ollier, Yann Ollivier, Felix Otto, Ludger Rüschendorf, Giuseppe Savaré, Walter Schachermayer, Benedikt Schulte, Theo Sturm, Josef Teichmann, Anthon Thalmaier, Hermann Thorisson, Süleyman Üstünel, Anatoly Vershik, and others.

Short versions of this course were tried on mixed audiences in the Universities of Bonn, Dortmund, Grenoble and Orléans, as well as the Borel seminar in Leysin and the IHES in Bures-sur-Yvette. Part of the writing was done during stays at the marvelous MFO Institute in Oberwolfach, the CIRM in Luminy, and the Australian National University in Canberra. All these institutions are warmly thanked.

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As usual, I encourage all readers to report mistakes and misprints. **I will maintain a list of errata, accessible from my Web page.**

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Cédric Villani



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Villani, C.

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