Chapter 2
Newton and Leibniz on Time, Space and Forces

Absolute, true, and mathematical time, of itself, and from its own nature flows equably without regard to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.

Newton, Principia; Definitions

Si plures ponantur existere rerum status, nihil oppositum involventes, dicentur existere simul. Et ideo quicquid existit alteri existenti aut simul est aut prius aut posterius. Si eorum quae non sunt simul unum rationem alterius involvat, illud prius, hoc posterius habetur.

Leibniz, Initia

Further: Space and Time are Quantities: which Situation and Order are not.

Leibniz Clarke (Alexander), Clarke, 3rd Letter to Leibniz

The contemporary 20th century physics is based on a foundation which had been mainly invented by two scholars living 300 years ago in the 17th century, Newton and Leibniz, who assembled new rules for mathematics and mechanics being disparat from those known before them. Newton and Leibniz surpassed their famous predecessors and contemporaries Galileo, Huygens and Descartes. Both scholars composed an impressive and long-lasting system of concepts which survived all changes and modifications introduced later by their followers. It took over two centuries until a change comparable in significance and explanatory power had been invented in the beginning of the 20th century by Planck, Einstein, Bohr, Heisenberg and Schrödinger. Although only their contributions to mathematics and physics formed a reliable basis for the development in the 18th and 19th centuries, their theories of time, space and motion were thought to be essentially different from each other from the very beginning until present time.[1] Hence, the picture generated in

[1] Compare the summaries by Clarke [Leibniz Clarke], Mach [Mach, Mechanik], Reichenbach [Reichenbach, Space and Time].

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the 18th century was transmitted through the centuries. It is dominated by the opposition and difference and only step by step it had been discovered that Newton's and Leibniz's methods are grounded on a hidden relationship of common methods of thinking which had been only differently represented. This hidden conjunction and entanglement is clearly demonstrated by the controversy about the invention of the calculus [Newton, Principia, Book II, Sect. II, Lemma, pp. 250–254],[2] [Newton (Collins), Commercium].

Nowadays, the result of retrospection is summarized in terms of *equivalence* and *priority*, these words Meli had chosen for entitling his treatise on the Newton-Leibniz debate concerning the invention of the calculus [Meli]. Obviously, the well-known battles on the priority could only be set into scene because there was an equivalent approach and nobody else except Newton and Leibniz created equivalent alternative theories or foundations. A new approach had been only invented in the 18th century by Taylor, MacLaurin, Euler, d'Alembert and Lagrange (compare Chaps. 3 and 5).[3]

Newton and Leibniz almost simultaneously created the calculus between 1665 and 1675 with a negligible delay of the years compared to the delay in publication of the results published 20 and even 40 years later entitled *Nova methodus* [Leibniz, Nova Methodus] in 1684 and the preliminary announcement of the *Method of Fluxions* [Newton, Opticks] in 1704, respectively. Even in 1684, the impact of the new method was negligible and only understood by distinguished scholars Jakob and Johann Bernoulli. However, also those distinguished scholars like Jakob and Johann Bernoulli and later L'Hospital, Varignon and other mathematicians were asking for explanation by Leibniz [Bernoulli, Letter to Leibniz]. Only, as it could be expected, Newton had no problems of understanding, but supported Leibniz's approach by his authority. In 1687, Newton commented on the new method in *Principia* [Newton, Principia, Book II, Sect. II, pp. 250–253] although most of the results were geometrically demonstrated.

The same simultaneity in creation of new approaches appeared even more pronounced in making public the new foundation of *mechanics* by Leibniz [Leibniz, Brevis] and Newton [Newton, Principia] in 1686 and 1687, respectively. The common origin of the short communication by Leibniz entitled *Brevis demonstratio erroris memorabilis Cartesii et aliorum* [Leibniz, Brevis] and the comprehensive treatise by Newton entitled *Philosophiae naturalis principia mathematica* [Newton, Principia] is the criticism of Descartes's principles (compare Chap. 1). The main result of this exceptional breakthrough is that Newton and Leibniz did not only invent new programs for mechanics and mathematics, but completed also essential parts of their programs demonstrating the power of the new methods in solving problems which had been unavailingly attacked before. Even at

[2] "In literis quae mihi cum Geometra peritissimo G. G. Leibnitio annis abhinc decem intercedebant (...) rescripsit Vir Clarissimus se quoq; in ejusmodi methodum incidisse, & methodum suam communicavit a mea vix abludentem praeter quam in verborum & notarum formulis." [Newton, Principia, Book II, Sect. II, Lemma II, Scholion, pp. 253–254] (compare Chap. 3) Newton confirmed the equivalence of the approaches without emphasizing the differences.

[3] For the criticism by Nieuwentijt and Berkeley compare Chaps. 3 and 5.

present, in the 21st century, Newton's axioms of mechanics, Leibniz's living forces (presently known as kinetic energy) and the representation of calculus are still in power forming the indispensable basis of contemporary science. Also classical mechanics survived the advent of quantum mechanics [Planck 1900], [Einstein, Heuristisch], [Bohr 1 to 4] and becomes an essential part of Bohr's correspondence principle [Bohr, Correspondence] and had been accepted as a basic part of physics.[4]

The same story had to be told for Newton's and Leibniz's calculus. Leibniz's formalism and algorithmic representation of the calculus survived all attacks over centuries which had been initiated by Nieuwentijt [Nieuwentijt, Analysis] in 1695 and in 1696 [Nieuwentijt, Considerationes] and renewed by Berkeley [Berkeley, Analyst] in 1734. Even Leibniz's own interpretation of differentials as fictitious quantities [Leibniz (1712)] did not reduce the importance and reliability of the method. Finally, the foundation in terms of limits initiated by d'Alembert [d'Alembert, Encyclopédie, Limit], completed by Cauchy [Cauchy] and Weierstraß [Weierstraß] or Lagrange's alternative algebraic approach [Lagrange, Fonctions] confirmed the *validity* of the *algorithms* without changing the basic rules of the generation of derivatives and integration of functions (compare Chaps. 3 and 5). Unavoidably and mainly caused by the incomplete and retarded publication of essential papers, the reception of the legacy of Newton and Leibniz was unintentionally selective and unfinished over a long period.

Nowadays, in view of the progress in science and the diversity of problems appearing due to the variety of approaches and proposed models especially in the basic research in physics, people intended to complete the reception by analyzing the other parts of the legacy [Keynes], [Keynes (Reagle)], [Truesdell], [Dijksterhuis], [Westfall, Never], [Wilczek 2004a],[5] [Smolin][6] or even reconsidering of the well-known parts [Chandrasekhar] which may be regarded as a renewing of the development which also took place in the beginning of the 18th century.

Leibniz stated that the "ancients had the science of equilibrium", but it is necessary to have a "science of motion" [Leibniz, Specimen, I (8)] where "motion" is related to the "phenomena". The world is full of phenomena experienced by men who

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[4] Classical mechanics is necessary for the formulation of quantum mechanics (compare Chap. 8). "It is in principle impossible, however, to formulate the basic concepts of quantum mechanics without using classical mechanics." [Landau/Lifschitz, Quantum]

[5] "When I was a student, the subject that gave me the most trouble was classical mechanics. That always struck me as peculiar, because I had no trouble learning more advanced subjects, which were supposed to be harder. (...) Coming from mathematics, I was expecting an algorithm. (...) To anyone who reflects on it, it soon becomes clear that $F = ma$ by itself does not provide an algorithm for constructing the mechanics of the world." [Wilczek 2004a] As is will be demonstrated, the missing algorithm had already been established by Euler (compare Chap. 4).

[6] "This essay is written with the hope that perhaps some who have avoided thinking about background independent theories might consider doing so now. To aid those who might be so inclined, in the next section I give a sketch of how the absolute/relational debate has shaped the history of physics since before the time of Newton. Then, in Section 3 I explain precisely what is meant by relational and absolute theories. Section 4 asks whether general relativity is a relational theory and explains why the answer is: partly. We then describe, in Section 5, several relational approaches to quantum gravity" [Smolin].

intended to discover the principles behind the phenomena, i.e. the true principles the construction of the world is based on which are ruling the phenomena. The common name of these principles was chosen to be "forces of nature". In the following decades in the 17th century and subsequently over three centuries until today there is a long-lasting debate about the *nature*, the *origin* and the *effects* of these forces which causes motions and the change of motion of the bodies. The debate can be traced back to the ancient science. Aristotle invented a correlation between motion described in terms of velocity and forces. Archimedes considered the equilibrium of forces for the model of the lever. Galileo invented an alternative model and discovered that uniform motion takes place without a moving force behind the body causing a change of position [Galileo, Discorsi].

Following Newton, the phenomena and the forces are related to each other [Newton, Principa]. Following Leibniz, the phenomena are described in terms of relative motion whereas the forces have to be related to substances [Leibniz, Specimen]. Leibniz developed a relational theory of time and space and a non-relational theory of forces.[7] However, Leibniz assumed a close correlation between motion (velocity) and forces. As a consequence, the "living forces" of bodies are assigned to moving bodies independently of the type of motion. Only Euler based mechanics on a force-free uniform motion and excluded the "force of inertia" [Euler E842] (compare Sect. 4). In the 19th century, the triumvirate of Descartes, Newton and Leibniz, who represented different scientific disciplines including theology in personal union, had been separated into three parts of philosophy (called metaphysics including theology), mathematics and physics. People demonstrated the errors of Descartes which had been already analyzed by Newton without acknowledging his merits in mathematics and mechanics. Moreover, they did not mention the life-long battle of Newton against the Cartesian methods in philosophy and mathematics (compare Westfall [Westfall, Never]). In the second half of the 19th century, this commonly accepted picture established by Mach and other authors was questioned by Russell [Russell, Western] and Couturat [Couturat, Leibniz] who discovered Leibniz's contribution to logics and by Helmholtz [Helmholtz, Vorlesungen] who acknowledged Leibniz's contributions to mechanics as far as the conservation of living forces and, generally, the energy is concerned. In view of the Leibniz's manuscripts published by Couturat, Russell distinguished between the "exoteric" and the "esoteric" Leibniz to demonstrate that Leibniz was not only the author of the idea of the "best of all possible worlds", but a thinker who based his metaphysics on logics [Russell, Western]. Therefore, Voltaire in the 18th century and Mach in the 19th century had been misled themselves because they did not recognize the complete Leibnizian system foreshadowed behind its

[7] Only Euler based mechanics on a *relational theory of motion and* a *relational theory of forces* [Euler E177], [Euler E181], [Euler E842] (compare Chap. 4). Obviously, although Euler assumed *relative motion*, the theory of time and space is mainly that of Newton's absolute time and space. In the non-relativistic version of mechanics, r*elative* motion is always possible if the frame of reference is formed by two bodies which are localized in an immobile space or moving uniformly in one direction. Then, the transformation is the Galileo transformation. This kind of motion had been analyzed by Euler (compare Sect. 6).

official representation. In the 20th century, an analysis following in goal and spirit Russell's attempts had been given by Keynes for Newton's writings and activities [Keynes].[8]

Upon this background which was extended in the last decades by studying of Newton's complete writings,[9] the traditionally presupposed opposition and antagonism between the antipodes Newton and Leibniz optimally represented in the letters exchanged in the famous Leibniz-Clarke debate [Leibniz Clarke],[10] may be partially or even completely lifted taking into account the "esoteric components" in the work of Newton and Leibniz.

Newton's basic papers on motion and forces of bodies are: *De motu corporum* [Newton, De motu] written in 1685, *Philosophiae naturalis principia mathematica* [Newton, Principia] published 1687, the invention of the method of fluxions 1665–1671 [Newton, Method of Fluxions], letters to Leibniz 1665 and 1669

[8] "Magic was quite forgotten. He has become the Sage and Monarch of the Age of Reason. The Sir Isaac Newton of orthodox tradition – the eighteenth-century Sir Isaac, so remote from the child magician born in the first half of the seventeenth century – was being built up. Voltaire returning from his trip to London was able to report of Sir Isaac – 'it was his peculiar felicity, not only to be born in a country of liberty, but in an Age when all scholastic impertinences were banished from the World. Reason alone was cultivated and Mankind could only be his Pupil, not his Enemy.' Newton, whose secret heresies and scholastic superstitions it had been the study of a lifetime to conceal!" [Keynes] [http://www-history.mcs.st-and.ac.uk/Extras/Keynes_Newton.html] The text was written in 1936.

[9] See The Newton Project [http://www.newtonproject.sussex.ac.uk/links.html] and [http://www.isaac-newton.org] and the links therein.

[10] Assuming Keynes' terminology, the published text may be considered as a debate between the "exoteric Newton" represented by Clarke and the "exoteric Leibniz". However, following Keynes, the debate was running on a *hidden subtext* both the opponents were acquainted with. A similar procedure was already observed for the invention of the calculus. The technical procedures and algorithms were of the same algebraic structure, but the foundations of the calculus were quite different and even in direct opposition to each other as far as the distinction between "sums and differences" (Cavalieri, Leibniz) and "fluents, moments and fluxions" are concerned (compare Chap. 3). In the debate on the priority in invention of the mathematical method, both authors were forced to disclose much more of their intentions as they primarily intended to do (compare Chap. 3). It should be stressed that both authors published the method only with a time delay of *several decades*, Leibniz in 1684 and Newton not until 1704. The delay may be explained by the common practice of publication, but may be also explained by the severe consequences for the interpretation of mechanics as the basic science for the understanding of the construction of the bodies and the ruling principles of the construction of universe. The intrinsic capability of *destructive* as well as *constructive* power of the Newton-Leibniz calculus was readily detected by critics who shortly after appeared [Nieuwentijt, Analysis], [Berkeley, Analyst] (compare Sect. 2.3). In 1734 for the first time and without any restraints, Euler stressed the *constructive* power [Euler E015/016], [Euler E212] whereas, in the same year in the treatise *The Analyst a Discourse addressed to an Infidel Mathematician*, Berkeley attacked the *destructive* potential [Berkeley, Analyst]. The debate had been continued in the 19th century where Cantor [Cantor] referred to the "exoteric Leibnizian interpretation" and in the 20th century where Robinson [Keisler], like Euler in the 18th century [Euler E212, § 85], recovered the power of Leibniz's principle of continuity as a ruling methodological principle (see Chaps. 3 and 5).

[Newton, Letter to Leibniz]. Newton published extended papers on the calculus only twenty year after Leibniz's *Nova Methodus* in 1704.[11]

Leibniz's basic papers on the motion and forces of bodies are: *Hypothesis physica nova* 1671 [Leibniz, Physica nova], comprising *Theoria motus abstracti* and *Theoria motus concreti*, the early manuscripts on the calculus [Gerhardt, Leibniz], [Gerhardt, Historia], [Child], *Tentamen anagogicum* [Leibniz, Tentamen], *Pacidius Philalethi* [Leibniz, Pacidius], *Nova Methodus* 1684 [Leibniz, Nova Methodus], *Brevis demonstratio* 1686 [Leibniz, Brevis], *Phoranomus* 1689 [Leibniz, Phoranomus], *Specimen dynamicum* 1695 and 1695 [Leibniz, Specimen], *De Ipsa natura* 1698 [Leibniz, De ipsa], Response to Nieuwentijt 1695 [Leibniz, Responsio], *Historia et Origo* [Leibniz, Historia] and *Initia rerum mathematicarum* 1715 [Leibniz, Initia].

Although there is a close relation between the treatises on mathematics (mainly the calculus) and physics (mainly the theory of motion), the latter were separately discussed in Chap. 3. This procedure follows the historical development since the exclusively and direct application of the calculus for solving mechanical problems appeared only in the beginning of the 18th century and was due to Varignon, the Bernoulli brothers, Daniel Bernoulli, Euler and d'Alembert. The first comprehensive treatise exclusively based on Leibniz's version of the calculus was written by Euler between 1734 and 1736 [Euler E015/016] and on earlier papers where Euler used the two main representations of the mechanical equation of motion [Euler E069] and the calculus of differences [Euler 1727]. Although Newton invented the calculus two decades before he published the final version of the *Principia*, he did not explicitly made of his new method for the calculation of the relation between fluents and fluxions or, the relation between "the phenomena and the forces". The reason was Newton's aversion to the unification of "arithmetic of variable and geometry" by Descartes and his adherents and the application of the new method to mechanics.[12]

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[11] "Newton had been scooped but even this event [Leibniz's publications on the calculus in 1684 and 1686] did not trigger him to go into print himself; although he did give a hint of his calculus in his *Principia* and John Wallis mentions it in 1693, but the first, main reference to it was in an appendix to Newton's other great book, *Optiks*, which was published in 1704, 20 years after Leibniz's publication." From "Man on the Moon" on the "Automatic for the People CD", 1992: [http://courses.science.fau.edu/~rjordan/phy1931/NEWTON/newton.htm]

[12] "I have often heard him censure the handling geometrical subjects by algebraic calculations; and his book of Algebra he called by the name of Universal Arithmetic, in opposition to the injudicious title of Geometry, which *Des Cartes* had given to the treatise, wherein he shews, how the geometer may assist his invention by such kind of computations. He frequently praised *Slusius, Barrow* and *Huygens* for not being influenced by the false taste, which then began to prevail. He used to commend the laudable attempt of *Hugo de Omerique* to restore the ancient analysis, and very much esteemed Apollonius's book *De sectione rationis* for giving us a clearer notion of that analysis than we had before. (...) He thought him [Hyugens] the most elegant of any mathematical writer of modern times, and the most just imitator of the ancients. Of their taste, and form of demonstration, Sir ISAAC always professed himself a great admirer: I have heard him even censure himself for not following them yet more closely than he did; and speak with regret of his mistake at the beginning of his mathematical studies, in applying himself to the work of *Des Cartes* and other algebraic writers, before he had considered the elements of *Euclide* with that attention, which so excellent a writer deserves" [Pemberton].

> Men of recent times, eager to add to the discoveries of the Ancients, have united the arithmetic of variable with geometry. Benefiting from that, progress has been broad and far-reaching if your eye is on the profuseness of output, but the advance is less of a blessing if you look at the complexity of the conclusions. For these computations, progressing by means of arithmetical operations alone, very often express in an intolerably roundabout way quantities which in geometry were designated by the drawing of a single line." [Newton, Math 4:421]

Although Newton condemend the arithemetical method, he made use of the calculus for the discovery of the laws of motion [Newton, Principia, Book II, Sect. II, Lemma II], but presented the results in another frame of reference based on geometry. Similarly to Archimedes who practiced the method of exhaustion in order to *present* the result which had been obtained by application of other methods.[13]

From Newton's statement it follows, that after the invention of the calculus two mathematical formalisms (languages) based either on geometry or arithmetic (algebra) were available for the representation of the relation between bodies, motions and forces. This development had been initiated by Descartes who represented geometrical relations by algebraic expressions. After Newton and Leibniz had presented the corner stones of the new science of motion between 1684[14] and 1687, their contemporaries and followers had to struggle with the problems appearing in mechanics and mathematics to attain the rigour in demonstrations known from geometry and the ancients.[15]

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[13] "Der wesentliche Unterschied gegen die moderne Auffassung besteht darin, daß die Existenz einer Inhaltszahl des Kreises als etwas ganz Selbstverständliches stillschweigend angenommen wird, während die moderne Infinitesimalrechnung auf diese anschauliche Evidenz verzichtet und vielmehr auf Grund des abstrakten Grenzbegriffes die Inhaltszahl als Grenzwert der Maßzahlen eingeschriebener Polygone definiert. (…) Eine von H. Heiberg 1906 entdeckte Schrift des Archimedes zeigt nun in der Tat, daß dieser bei seiner Forschung das Exhaustionsverfahren gar nicht anwandte. Erst nachdem er seine Resultate anderweitig gefunden hatte, bildete er hinterher, um den damaligen Anforderungen an Strenge zu genügen, den Exhaustionsbeweis aus. Zur Entdeckung seiner Sätze benutzte er jedoch eine Methode, die Schwerpunktbetrachtungen [und] den Hebelsatz (…) zu Hilfe nahm (…)" [Klein, Elementarmathematik, p. 225].

[14] "Among his achievements in all areas of learning, Leibniz's contributions to the development of European mathematics stand out as especially influential. His idiosyncratic metaphysics may have won few adherents, but his place in the history of mathematics is sufficiently secure that historians of mathematics speak of the 'Leibnizian school' of analysis and delineate a 'Leibnizian tradition' in mathematics that extends well past the death of its founder. This great reputation rests almost entirely on Leibniz's contributions to the calculus. Whether he is granted the status of inventor or co-inventor, there is no question that Leibniz was instrumental in instituting a new method, and his contributions opened up a vast new field of mathematical research." [Jesseph, Leibniz] Compare also Euler [Euler E212, Preface].

[15] "I recognize (…) that you have written some profound and ingenious things concerning various infinite bodies [de corporibus varie infinitis]. I think that I understand your meaning, and I have often thought about these things, but have not yet dared to pronounce upon them. For perhaps the infinite, such as we conceive it, and the infinitely small, are imaginary, and yet apt for determining real things, just as imaginary roots are customarily supposed to be. These things are among the ideal reasons by which, as it were, things are ruled, although they are not in the parts of matter. For if we admit real lines infinitely small, it follows also that lines are to be admitted which are terminated at either end, but which nevertheless are to our ordinary lines, as an infinite to a finite. Which things being posited, it follows that there is a point in space which can not be reached in an

2.1 Newton's Program for Mechanics

In 1687, Newton summarized the content of his long-lasting investigations of the motion and interaction of bodies in three axioms which form the basis of all developments of mechanics in the following centuries including quantum mechanics. The development of Newton's ideas from the beginning to the final results had been thoroughly analyzed and reconstructed only in the 20th century [Keynes], [Truesdell], [Westfall, Never].[16] In the 19th century, a thorough and comprehensive, but nevertheless critical reconsideration of Newton's basic concepts of space and time and the theory of motion had been performed by Mach [Mach, Mechanik]. Mach accentuated that Newton encountered enormous difficulties in solving the problem to generalize the conceptual basis known from static in order to establish the science of motion.[17] Mach rejected Newton's concept of absolute space and time [Mach, Mechanik], but ignored Leibniz's attempts of a foundation of mechanics and condemned Leibniz for his metaphysical and theological thinking [Mach, Mechanik]. Contrary to the picture drawn by Mach, current investigations of Newton's work reveal the same tight correlation between physics and theology in Leibniz and Newton [Snobelen, Newton].[18] This revised opinion about Newton had already been established by Keynes in the 1930's [Keynes][19] (for current investigations compare Snobelen [Snobelen]).

assignable time by uniform motion. And it will similarly be required to conceive a time terminated on both sides, which nevertheless is infinite, and even that there can be given a certain kind of eternity (as I may express myself) which is terminated. Or further that something can live so as not to die in any assignable number of years, and nevertheless die at some time. All which things I dare not admit, unless I am compelled by indubitable demonstrations." [Leibniz, GM III, pp. 499–500] Quoted from [Jesseph, Leibniz].

[16] For the present state of art see [http://www.newtonproject.sussex.ac.uk].

[17] "Das Pleonastische, Tautologische, Abundante der Newtonschen Aufstellungen wird übrigens psychologisch verständlich, wenn man sich einen Forscher vorstellt, der, von den ihm geläufigen Vorstellungen der Statik ausgehend, im Begriff ist, die Grundsätze der Dynamik aufzustellen." [Mach, Mechanik]

[18] "Newton's integrated programme for science and religion. The foregoing must not be taken to mean that the influence only flowed from Newton's theology to his natural philosophy. The same considerations that explain this direction of influence also make the reverse direction reasonable. Thus, Newton's methodological approach to the interpretation of prophecy may owe something to his satisfaction with the results of mathematics. It is also clear that Newton's conception of God was in part based on a possibly unconscious desire to create God in his own image. And so in his letters to Bentley Newton spoke of the 'cause' of the solar system being not 'blind & fortuitous, but very well skilled in Mechanicks & Geometry.' Newton's published and unpublished writings demonstrate that his religion interacted with his science at a high level. Newtonian physics cannot be disentangled from Newtonian theology. The lack of firm barriers within Newton's intellectual life suggests that it may even be problematic to speak in terms of 'influence' of one sphere on another. Instead, Newton's lifework evinces one grand project of uncovering God's Truth. Science and religion for Newton were not two distinct programmes, but two aspects of an integrated whole. For Newton, the unity of Truth meant that there was one culture, not two." [Snobelen, Newton]

[19] After reading Newton's manuscripts on alchemy, Keynes stated: "Newton was not the first of the age of reason. He was the last of the magicians, the last of the Babylonians and Sumerians, the

In 1687, Newton presented the three basic laws of mechanics in the *Principa* [Newton, Principia] after having probed other axiomatic foundations in the 1680's years [Westfall, Never]. The basic laws are formulated for a moving body whose state of uniform motion in the same direction is changed by a moving force impressed upon the body.

> Lex. 1. Corpus omne preservare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum ille mutare.
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> Lex. 2. Mutationem motus proportionalem esse vi motrici impressae et fieri secundum lineam rectam, qua vis illa imprimitur.
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> Lex. 3. Actioni contrariam semper & aequalem esse reactionem: sive corporum duorum actiones semper esse aequales et in partes contraria dirigi.
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> Corol. I. Corpus viribus conjunctis diagonalem parallelogrammi eodem tempore describere, quo latera separatis. [Newton, Principia][20]

Motion is defined by the product of *mass* and *velocity*. Newton introduced motion as an extensive quantity. The motion of a whole is the sum of the motions all parts.[21] Following Galileo, the basic law for natural motion or falling bodies is *independently* of the *mass* and the *shape* of bodies. The motion of projectiles is also geometrically represented by a parabola. Newton made also use of this model, but interpreted the model differently to confirm the conditions for the *preservation of motion* which is only modified by the impact with other bodies.[22] Hence in Newton's interpretation, "uniform motion" means that the *product* of mass and velocity is constant whereas in Galileo's and later interpretations the *velocity* is the only quantity to describe the preservation and the change of the state. Here, the parameter which had been later called "heavy mass" is to be taken into account to obtain Galileo's result. The independence of the mass is correlated with the model where the force is represented by the homogeneous gravitation field. Nevertheless, motion

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last great mind which looked out on the visible and intellectual world with the same eyes as those who began to build our intellectual inheritance rather less than 10,000 years ago." [Keynes]

[20] "LAW I. Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.

LAW II. The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

LAW III. To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

COROLLARY I. A body by two forces conjoined will describe the diagonal of a parallelogram, in the same time that it would describe the sides, by those forces apart." [Newton, Principia (Motte)]

[21] "*The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjunctly*. The motion of the whole is the sum of the motions of all the parts; and therefore in a body double in quantity, with equal velocity, the motion is double; with twice the velocity, it is quadruple." [Newton, Principia, Definitions (Motte)]

[22] "PROJECTILES persevere in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are perpetually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in more free spaces, preserve their motions both progressive and circular for a much longer time." [Newton, Principia, Axioms (Motte)] Although Newton formulated the axiom for the "*preservation of state*", the illustration is for the "*preservation of motion*" where the bodies describe *curvilinear* paths like projectiles, tops, planets and comets and included the preservation of momentum.

is always also governed by the inert mass. Later, Euler accentuated the role of inert mass and invented for the first time an operational definition of mass which had been acknowledged by Jammer [Jammer, Mass] (compare Chap. 4). Furthermore, Euler distinguished between internal and external states [Euler E842, § 30].[23]

Newton's program is based on the earlier versions of *De gravitatione* [Newton, De gravitatione] and *De motu* [Newton, De motu][24] where Newton assumed that "force is the causal principle of rest and motion" [Newton, De gravitatione, Def. 5]. The inertia is defined as an "internal force" [Newton, De gravitation, Def. 8]. Newton preserved this definition in the *Principia*. In the 18th century, although Newton's assumption on the inertia was accepted at beginning, Euler, d'Alembert and other authors finally rigorously rejected the concept of a "force of inertia" [Euler E842].[25] In the treatise *De motu*, a preliminary version of the *Principia*, Newton summarized the suppositions for the new theory which had been later essentially modified and only partially preserved. The different stages of Newton's evolution of basic concepts had been analyzed by Westfall [Westfall, Never]. Newton assumed the notions of "impetus" and "conatus" (Definitions 6 and 7) which had been later also treated by Leibniz [Leibniz, Specimen].

In mechanics, Newton developed a method for the solution of two different, but correlated problems which are mechanically modelled by the distinction between *phenomena of motion* and *forces of nature*.[26] The phenomena are described in terms of geometry[27] and are caused by these forces.[28] Newton assumed that either the

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[23] "30. *Man sagt, ein Körper verbleibe in ebendemselben Zustande, wenn derselbe entweder in Ruhe verbleibt oder seine Bewegung nach ebenderselben Richtung mit einerlei Geschwindigkeit fortsetzet.*

Man kann sich in einem Körper einen doppelten Zustand vorstellen, den äusserlichen und den innerlichen. Dieser bestehet in der Art der Theile, aus welchen der Körper bestehet, und ihrer Zusammensetzung selbst; der äusserliche Zustand aber, von welchem allhier allein die Rede ist, besteht in den Verhältnissen des Körpers mit dem Raume." [Euler E842, § 30]

[24] "It is difficult, from our modern Olympian perspective, to understand the mindset of Newton's days. Even the concept of 'velocity' was relatively new at that time. While Newton was inventing mechanics, he was also inventing the very *language* in which it is expressed. And Newton's great nemesis, in all his ruminations, was the concept of infinitesimal. He constantly had to confront Zeno's paradox, and the many apparent contradictions arising therefrom, in all his considerations of fluxions and fluents and quadrature and acceleration." [http://www.ams.org/notices/200311/rev-krantz.pdf]

[25] The rejection of the force of inertia was an essential step in the development of mechanics by Euler since the origin of forces was exclusively related to the interaction of bodies, but not to an internal force which ensures the preservation of state (compare Chap. 4).

[26] Leibniz distinguished between corporeal phenomena and mechanical laws [Leibniz, Specimen, I (13)]. Châtelet translated: "En effet toute la difficulté de la Philosophie paroit consister à trouver les forces qu'employe la nature, par les Phénomenes du mouvement que nous connoissons, & à démontrer ensuite, par là, les autres Phénomens." [Principia, Newton (Châtelet)]

[27] "Lines are described, and by describing are generated, not by any apposition of parts, but by a continual motion of points. Surfaces are generated by the motion of Lines, Solids by the motion of Surfaces, (...). These Geneses are founded upon Nature, and are every Day seen in the motion of Bodies." [Newton, Quadrature, (Harris)]

[28] Newton solved the *inverse* problem for planetary motion. The direct problem had been treated by Hermann and Johann Bernoulli. "In 1710, Jean Bernoulli pointed out that Newton had not proved

phenomena or the forces are known. Thus, knowing the phenomena one has to derive the forces and, knowing the forces one has to calculate the geometrical representations in terms of phenomena being caused by the forces [Newton, Principia] (compare Chap. 1).

Leibniz did not essentially modify this program since he also related the *phenomena* to *geometry* and completed geometry by additional principles which are also related to the forces.[29] Following Leibniz, the motion as far as being separated from the forces is necessarily only *respective* or *local* motion [Leibniz, Specimen, I (2)]. Leibniz specified the "phenomena" to the "corporeal phenomena" which is the basis of his relational approach to time and space. As it follows from the title *Philosophiae naturalis principia mathematica*, Newton invented his program for mechanics in order to study mathematically and mechanically the relation between *phenomena* and *forces of nature*.

This subdivision into phenomena and forces results simultaneously in the definition of two main problems. Although the problems are of different type, there is a common methodological basis.[30] Later in 1736, Euler removed all metaphysical implications introduced by Newton and Leibniz and formulated the problem for the relative motion of bodies constituting a world which consists only of bodies and space.[31] As a consequence, Euler redefined the relation between phenomena and

Kepler's law of ellipses but only its converse and did so himself using calculus, solving 'the general problem by reducing it to the same integral that is used to solve it today' (Park 1990:416). (...) In 1742, Jean Bernoulli, in *Opera omnia*, proved that the orbits of objects bound by the inverse square force are conic sections." [Time Line] "Hermann was the first to study what, today, we call the direct Kepler problem: namely, given the central force inverse square law, determine the orbit. In spite of the pioneering excellence of this analysis, including the proof that in such a force-field all orbits are conic sections, it was immediately superseded by the more comprehensive treatment of Johann I Bernoulli. Although Hermann had priority in tackling the problem, he had not obtained the complete solution." [http://www.fyma.ucl.ac.be/∼gaino/Bernoulli/JacobHermann.html] "Proposition 1 of Book 1 of Newton's *Principia* (1687), which states that Kepler's area law holds for any central force, plays a fundamental role in the study of central force motion. Newton's geometric proof of this proposition is based on an intuitive theory of limits. In 1716–1717 the Swiss mathematician, Jakob Hermann, gave a proof of Proposition 1 based on infinitesimals. The present paper discusses both Newton's and Hermann's solutions. A comparison of the two gives us an insight into an episode of the process that led from the geometric style of Newton's *Principia* to the analytic style of Euler's *Mechanica* (1736)." [Guicciardini, Hermann]

[29] Leibniz called the "forces of nature" substances or "simple things" who are the true causes of the phenomena as the Newtonian forces of Nature are. "I am believe of the opinion that, to speak exactly, there is no need of extended substance. (...) True substances are only simple substances or what I call 'monads'. And I believe that there are only monads in nature, the rest being phenomena of them." [Leibniz, Letter to Dangicourt]

[30] Leibniz based mechanics upon the same distinction between corporeal phenomena and superordinate principles. Refuting the attempts of Henry Moore and Aristotle to relate the phenomena to a fundamental principle, Leibniz stated that the *corporeal phenomena* can be deduced from *mechanical causes*, but the *mechanical laws* have to be deduced from *superordinate principles*: "Optimum meo judicio temperamentum est, quo pietati et scientiae satisfit, ut omnia quidem phaenomena corporea a causis efficientibus mechanicis peti posse agnoscamus; sed ipsas leges mechanicas in universum a superioribus rationibus derivari intelligamus; atque ita causa efficiente altiore tantum in generalibus et remotis constituentis utamur." [Leibniz, Specimen, I (13)]

[31] "Hier werden diejenigen Veränderungen mit Fleiss ausgeschlossen, welche unmittelbar von Gott oder einem Geiste hervorgebracht werden. Wenn wir also in der Welt nichts als Körper betrachten,

forces in terms of the *paths* the bodies describe and the *forces* the bodies generate by their interaction. Euler's program is commensurable with the contemporary terminology (compare Chap. 4). Preserving the subdivision into internal and external principles, the basic intention is to define *algorithms* for reckoning of the paths based on the application of the calculus and mechanical principles which are as reliable as the laws of geometry: (N1) Calculate the paths if the forces are given, (N2) Calculate the forces if the paths are given (compare Chap. 4).[32]

Newton and Leibniz developed complementary programs accentuating the change of motion and the presevation of "living forces", i.e. the preservation of the quantity of motion, respecitvely. Hence, Leibniz mechanics includes implicitly a "hidden force model" whereas Newton's mechanics implies, for some special kinds of forces, a "hidden energy model" or "conservation of living forces model" represented by the expressions for the change of motion and the preservation of living forces.

> (...) nec plus minusve potentia in effectu quam in causa contineatur. [Leibniz, Specimen, I (11)]

Although Leibniz implicitly made use of the *inertia* of bodies and the invariance of the inert mass,[33] he added that this law is not derived from the notion of *mass*, but it has to be traced back to somewhat other, i.e. the *inherent* forces the bodies:

> Quae lex cum non derivetur ex notione molis, necesse est consequi eam ex alia re, quae corporibus insit, nempe ex ipsa vi, quae scilicet eandem semper quantitatem sui tuetur, licet a diversis corporibus exerceatur. [Leibniz, Specimen, I (11)]

This additional principle cannot be derived from mathematics, but it is only comprehensible to the reason and, consequently, it has to be formulated in terms of metaphysics such as cause and effect, action and suffering [Leibniz, Specimen, I (11)].

In the Leibnizian formula, there are no *explicit expressions* for Newtonian "impressed moving forces"[34] whereas in Newton's law there is no explicit expression for energy or "living forces". The step beyond the Newtonian program is performed

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so ist klar, dass ein jeder Körper so lange in seinem Zustande verbleiben muss, als sich von aussen keine Ursache ereignet, welche vermögend ist, in demselben eine Veränderung zu wirken." [Euler E842, § 49] (compare Chap. 4, Section *Euler's world models*).

[32] "C'est aussi à quoi aboutissent toutes les recherches de la Mécanique, où l'on s'applique principalement à deux choses: (i) l'une, les forces qui agissent sur une corps étant données, déterminer le changement qui doit être produit dans son mouvement; (ii) l'autre, de trouver les forces mêmes, lorsque les changements, qui arrivent aux corps dans leur état, sont connus." [Euler E181, § 10]

[33] Leibniz claimed that the "earlier hypothesis", i.e. on inertia and impenetrability, on the bodies is "incomplete": "(...) vidi in quo consisteret systematica rerum explication, animadvertique hypothesin illam priorem notionis corporeae non esse completam; et cum aliis argumentis, tum etiam hoc ipso comprobari, quod in corpore praeter magnitudinem et impenetrabilitatem poni debeat aliquid, unde virium consideratio oriatur; (...)." [Leibniz, Specimen, I (11)]

[34] As reviewed by Voltaire, Newton claimed: "Let us listen to Newton and the experience and stop this metaphysical disputation. The motion, Newton said, is generated (produced) and lost. But, due to the tenacity of the fluids and the elasticity of the solids, is more motion lost as it is renewed in nature." [Voltaire, Élémens, Chap. IX]

by the assumption of *conservation law* for living forces which is related to the rejection of perpetual motion.[35] The assumption of living forces had been later acknowledged by Euler, d'Alembert, the Bernoullis and Châtelet [Châtelet, Institutions].[36]

2.2 Newton and Leibniz on Time, Space, Place and Motion

Leibniz's relational definition of time and space [Leibniz, Initia] is known to be essentially different and in strong contrast to Newton's theory of absolute time and absolute space [Newton, Principia]. In 19th century, Mach's well-known criticism ends up in the complete rejection of these basic notions of Newton's theory. However, Mach did not mention the contribution of Leibniz as a predecessor in the criticism of that part of Newton's theory. He also did not discuss the relation between absolute and relative times and spaces which have been introduced in the *Principia*. After the advent of Einstein's theory of relativity it was extremely difficult to reconstruct the sophisticated details of the Newton – Leibniz controversy since people declared Leibniz to be the winner. In 1924 Reichenbach stated that the success of Newton's *Principia* has hampered the development of mechanics for 200 years [Reichenbach].

The comparison between the ranking order of time and space found in the title of Reichenbach's book on *The philosophy of space and time* and that of the same concepts in Newton's *Principia* and Leibniz's *Initia* demonstrates the difference in the weighting in 17th and 20th centuries, however, it reveals also an astonishing agreement between Newton's *Principia* and Leibniz's *Initia*. Despite their controversy on the definition of time and space Newton and Leibniz agreed almost perfectly in the *order* of introduction of the concepts of time, space and motion. This order had been only called into question by Euler who based his considerations on the definition of *rest* and *motion* related to the place [Euler E015/016, § 1], [Euler E289, §§ 1–10] and the axiom of the uniform motion of bodies which is the basis for the definition of a *quantitative* relation between time and space intervals [Euler E149].

The difference and opposition between Newton and Leibniz are not due to the set of basic notions of time, space, place and motion, but due to the *logical structure* of

[35] The rejection of perpetual motion follows from Leibniz's statement: "(...) nec plus minusve potentiae in effectu quam in causa contineatur" [Leibniz, *Specimen*, I (11)], there is neither more nor less potency in the effect than in the cause, i.e. the conclusion (not the assumption) is that the only remaining case follows as the equality of cause and effect or the conservation of the potency. Mathematically, there are three cases, (i) effect is *less* than cause, (ii) effect is *larger* than cause and (iii) effects is *equal* to the cause. Newton did not make a choice, but stated only that there is a relation between "effect" and "cause", i.e. change in motion and impressed moving force, respectively. "Mutationem motus *proportionalem* esse vi motrici impressae, (...)." [Newton, Principia, Axioms] (compare also Chap. 7).

[36] In the translation of Newton's Principia, Châtelet replaced "forces of Nature" ("vires Naturae") with "forces which are employed by nature". "En effet toute la difficulté de la Philosophie paroit consister à trouver les forces qu'employe la nature, par les Phénomenes du mouvement que nous connoissons, & à démontrer ensuite, par là, les autres Phénomenes." [Newton, Principia (Châtelet)]

the assumed statements and the logical status of the notions. The basic difference for all notions is whether the objects are analyzed concerning their properties or their relations. Obviously, Newton's theory of absolute time is not based upon a relational concept of time since it is considered "without regard to anything external" [Newton, Principia, Definitions]. However, Newton completed the *non-relational* concept of *absolute* time (space and motion) with a *relational* concept of relative time, space, place and motion. All basic expressions appear doubly. No decision was made in favour of one of the two basic sets. Moreover, no decision can be made since the absolute quantities are related to mathematics and the relative quantities are related to measurement. As a consequence, all followers of Newton, his adherents and opponents, had to handle this problem and they did it differently. The same statement is true for Leibniz and the Leibniz-Wolffian school since the theory of Leibniz is based on a similar ambiguity, now related to forces. Also Euler developed the mechanics twofold and always compared the frame of relative motion to the frame of absolute motion. However, in contrast to most of his contemporaries who intended to merge both the topics, Euler made a clear decision on favor of *relative* motion (compare Chaps. 4 and 6). Instead of the classification according to absolute and relative concepts, Euler based the theory on two kinds of principles, *internal* and *external* principles of *motion*, where the internal principle are related to the isolated non-interacting body whereas the external principles are related to the interaction of bodies. Euler's mechanics is a consequent and true *relational theory of motion*, therefore, Euler may be considered as a predecessor of Einstein.

2.2.1 Newton and Leibniz on Time and Space

In the 20th century, the essential difference between the absolute and relative time and space had been accentuated [Reichenbach], [Reicehnbach, Space and Time].[37] Newton's and Leibniz's approaches to establish a general frame for time, space, place and motion are very similar and largely coinciding as far as their specification to absolute and relative quantities is not considered. Newton's frame reads as follows.

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[37] "Der Briefwechsel liest sich ähnlich wie eine moderne Diskussion über Relativitätstheorie; der Relativist sucht vergeblich einen Gegner zu überzeugen, der so in der absolutistischen Vorstellung befangen ist, dass er gar nicht merkt, wie sehr seine Argumente die Lehre voraussetzen, die sie erst beweisen wollen, und wie die vermeintlichen Widersprüche, die er dem Relativisten nachweisen will, eben nur auf einer ständigen Unterschiebung der absolutistischen Auffassung beruhen. (…) Es ist jedoch das seltsame Schicksal Newtons, dass er, der mit seinen physikalischen Entdeckungen die positive Wissenschaft reich befruchtete, zugleich die Entwicklung der begrifflichen Grundlagen dieser Wissenschaft weitgehend gehemmt hat. So fruchtbar seine optischen Entdeckungen waren – mit seiner Emissionstheorie des Lichtes hat er die Anerkennung der Wellentheorie (…) um ein Jahrhundert zurückgehalten. Und so weittragend seine Entdeckung des Gravitationsgesetzes war – die Analyse des Raum- Zeitproblems wurde durch seine Mechanik um mehr als zwei Jahrhunderte aufgehalten, nachdem sein Zeitgenosse Leibniz bereits wesentlich tiefere Einsichten in die Natur von Raum und Zeit gehabt hatte." [Reichenbach]

> I. Absolute, true, and mathematical time, of itself, and from its own nature flows equably without regard to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.
>
> II. Absolute space, in its own nature, without regard to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies. I say, a part of space; not a situation nor the external surface of a body.
>
> III. Place is a part of space which a body takes up, and is according to the space, either absolute or relative.
>
> IV. Absolute motion is the translation of a body from one absolute place into another; and relative motion, the translation from one relative place into another. [Newton, Principia (Motte)]. (comment (i) measurement,[38] (ii) fluxions[39] and (iii) absolute and relative[40])

In contrast to Newton who related only two different states, i.e. the states of "rest and uniform motion", to *bodies*, Leibniz generalized the procedure assuming a manifold of "states of things" which are either compatible or incompatible [Leibniz, Initia] without specifying these states as different states of bodies.[41]

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[38] Following Newton, the difference to the "*mathematical time*" is that the "*mechanically defined time*", although being also numerically represented by quantities, are *discrete* quantities. Beside the difference between mathematical and mechanical time Newton assumed implicitly the complementarily between "continual and discrete" mechanical quantities. This relation is excluded from mathematics since the geometrical line is not composed of parts. "I don't here consider Mathematical Quantities as composed of parts extremely small, but as generated by a continual motion." [Newton, Quadrature (Harris)] The "mechanical time" has to be related to measurement. The measured quantities are necessarily discrete, *one* hour, *one* day, *one* month, *one* year and so on, whose relation can be expressed in terms of *finite numbers* being either *integers* or *rational numbers*. Newton excluded *indivisible* time elements and indivisible space elements. The result is equivalent to and even stronger (more restrictive and more sophisticated) than Leibniz's assumption. Leibniz excluded the "existence" of parts whereas Newton excluded any "apposition of parts" ("not by any apposition of parts"). Thus, the "apposition of parts" is declared to be forbidden whereas the "decomposition into parts" is assumed to be allowed for the purpose of measurement.

[39] Mathematically, the "time" defined by Newton is related to the continuum and, mechanically, to a continual motion. The "time" does not consist of parts, neither of divisible nor indivisible parts. Leibniz's theory of time and motion is based on a similar assumption on the "parts". "Nam motus (perinde ac tempus) nunquam existit, si rem ad $\alpha\kappa\rho\iota\beta\varepsilon\iota\alpha\nu$ revoces, quia nunquam totus existit, quando partes coexistentes non habet." [Leibniz, Specimen, I (1)], but Leibniz accentuated the relation between "parts and whole" instead of the "generation" of quantities [Newton, Method of Fluxions].

[40] Euler argued in favour of Leibniz. "En ergo realem quietis definitionem nullis ideis vagis seu imaginariis implicatam, quae autem coniuncta est cum idea cuiuspiam corporis, cuius respectu punctum *O* quiescere dicitur; neque patet, quid sit quies absolute sic dicta separata a talis corporis notione" [Euler E289, § 8]. The absolute rest had been discussed *after* the relative rest (and motion) had been introduced and defined. In the *Mechanica*, Euler discussed absolute and relative rest (and motion) on an equal footing.

[41] Leibniz's goal is not only the foundation of physics, but also to find a method which answers the purpose of a foundation of mathematics: "(...) esse artem quandam Analyticam Mathematica ampliorem, ex qua Mathematica scientia pulcherrimas quasque sua Methodus mutuatur." [Leibniz, Initia, p. 353] This procedure corresponds to the difference in the foundation of the calculus by Newton and Leibniz where Newton preferred a *universal flux in time* and Leibniz time favoured *geometrical* and *algebraic methods* being alien to Newton's assumptions (compare

Referring to the primarily stipulated and distinguished states of the things, Leibniz defined *time* and *space* in one the same procedure simultaneously, i.e. as *correlated logical* operations[42] with respect to states, as expressing different relations between these states which are described as *different orders* of these things. Newton defined time as a permanent and uniform *flux* and *space* as an existing invariant thing.

In spite of these differences, Leibniz introduced the concepts in the same order as Newton. This almost perfect matching is remarkable since Leibniz has written his treatise after Newton has published the *Principia*. As Newton did, Leibniz defined firstly time (and duration), secondly, space. Time is the order of the non-coexisting things.

> Si plures ponantur existere rerum status, nihil oppositum involventes, dicentur existere simul. Et ideo quicquid existit alteri existenti aut simul est aut prius aut posterius.
>
> Si eorum quae non sunt simul unum rationem alterius involvat, illud prius, hoc posterius habetur.
>
> (I′) Tempus est ordo existendi eorum quae non sunt simul.
> (II′) Spatium est ordo coexistendi seu ordo existendi inter ea quae sunt simul.
> (III′) *Situs* est coexistentia modus.
> (IV′) *Motus* est mutatio situs. [Leibniz, Initia]

The only modification is that Leibniz replaced the place (*locus* (III)) with the situation (*situs* (III′)). In contrats to Newton's procedure,[43] an explicit definition of *rest* is missing. It is, however, included in the definition of situs. Beside the simultaneity related to the *states* of the things, which are either simultaneously or non-simultaneously, Leibniz defined for the measurement a special type of simultaneity, called *compraesentia*, using the notion of situs. *Situs* is a modus of coexisting things comprising quantity and quality. The quantity can be only known from the comparison of different things which form for a certain time a system or, are in a stable configuration, e.g. a body and the unit for measuring the length of the body. Leibniz called this *simultaneous existence* or to be *simultaneous present* (compraesentia) so that they are simultaneously *perceived* (seu perceptione simultanea) [Leibniz, Initia][44] by an observer. The procedure defines the conditions for measurement if one of the coexistent things which are compared to each other is chosen to be the gauge (unit).[45] However, Leibniz is aware of the problem to define spatial

Chap. 3). Although the resulting operations turned out to be independent of the foundation, the inherent advantage of the Leibnizian method is that time and space as variables may be treated on an equal footing (compare Euler's derivation of the two equations of motions [Euler E015/016, §§ 131–155]).

[42] Coexistence (space): "State1 AND State2", Succession (time): "EITHER State1 OR State2" (State1 and State2 are abbreviations for the statement "The system is in State I", $\mathrm{I} = 1, 2$).

[43] "But real, absolute rest, is the continuance of the body in the same part of that immovable space, (...)." [Newton, Principa. Axioms]

[44] "*Quantitas* seu Magnitudo est, quod in rebus sola compraesentia (seu perceptione simultanea) cognosci potest." [Leibniz, Initia]

[45] Euler discussed the same procedure for the definition of units [Euler E289, § 6]. Having the unit defined the result of the measurement can be expressed in terms of numbers. This procedure is applicable to all physical quantities. The problem of the invariance of units had been already emphasized by Leibniz [Leibniz, Nouveaux Essais, Vol. I, Book II, Chap. XIII, § 4].

and temporal order by the same principle of "compraesentia" since two perceptions appearing in a temporal order can only be compared to each other if there is in fact a *transition* but really neither an *annihilation* of the first nor a *creation* of the second thing [Leibniz, Initia].[46] According to these principles, Leibniz defined the relative translation of a body A with respect to other bodies C, D, E and G whose relative positions are not changed by the translation[47] of the body A. The translation is not related to a definite time interval but is only described geometrically. Euler renewed this model for describing the position, or situation called *situs* [Euler E289, § 3]. As Leibniz, Euler considered relative translations, but extended the analysis by inclusion of relative *motion* [Euler E842][48] thereby considering translations which are performed in a finite or infinitesimal time interval.

Therefore, the distinction between basic quantities is now made predominantly between *absolute and relative motions* rather than between absolute and relative *space and time* (compare Chap. 6). Euler's approach allows for a new consideration of the whole problem since it contains implicitly the question whether or not the motion, described by the *velocity* of the body, is limited in its magnitude. In the case of time and space it was clear for Newton and Leibniz that both the quantities are not limited but are infinite. These assumptions have been introduced axiomatically without reference to motion. In contrast, the statement on the unlimited magnitude of velocity has to be deduced within a certain model. Leibniz considered the rotation of a disk and concluded that the velocity can be increased without limitation by the increase of the distance from the center of disk.

Leibniz compared the order between bodies and the order between numbers. The order is changed by motion. What about the rest?

> Et comme les corps passent d'un endroit de l'espace à l'autre [from one place to another place], c'est à dire qu'ils changent l'ordre entr'eux, les choses aussi passent d'un endroit de l'ordre ou d'un nombre à l'autre, lorsque par exemple le premier devient le second et le second devient le troisième etc. (...) En effect le temps et le lieu ne sont que des espèces d'ordre, et dans ces ordres la place vacante (qui s'appelle vuide à l'egard de l'espace) s'il y en avoit, marqueroit la possibilité seulement de ce qui manque avec son rapport à l'actuel. [Leibniz, Nouveaux Essais, Vol. I, Book II, Chap. IV, § 5, p. 138][49]

Leibniz assumed free and occupied places within the orders, but no empty space. What about vacant position in time? Leibniz did only treat time and space as orders on an equal footing, but did not demonstrate the equivalence for "empty places" in space and time.

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[46] "Coexistere autem cognoscimus non ea tantum quae simul percipiuntur, sed etiam quae successive percipimus, modo ponatur durante transitu a perceptione unius ad perceptionem alterius aut non interiisse prius, aut non natum esse posterius." [Leibniz, Initia, p. 368]

[47] Leibniz used a modified version of the model of the stadium of Zeno and Aristotle [Leibniz, Clarke]. In the original version the bodies represented by chariots are *moving* a certain distance in a certain time interval. Leibniz discussed a spatial *translation* instead of motion.

[48] In 20th century, Reichenbach interpreted Leibniz's theory of space and time as a relational theory which contrasted the notions of Newton [Reichenbach]. However, Reichenbach's interpretation of the Leibnizian theory is misleading since he identified the Leibnizian theory of relative *position* with the Einsteinian theory of relative *motion*.

[49] "Indeed, time and positions (lieu) are species of orders, and within these orders is a vacant place (which is called empty with respect to the space) if it exists, denoted the mere possibility [not the reality] of that what is missing with respect to the actual." [Leibniz, Nouveaux Essais]

2.2.2 Order and Quantification

The main objection of Clarke against Leibniz's arguments is the missing possibility to relate quantities or numbers to the order. Leibniz distinguished two kinds of order, first those of coexisting things and, second those of non-coexisting things. Following Leibniz two time intervals of different or equal lengths may be considered. Although the magnitude of two time intervals Δt_1 and Δt_2 may be given by comparison as either $\Delta t_1 < \Delta t_2$ or $\Delta t_1 > \Delta t_2$, the relations "before" and "after" are not determinate by these inequalities as long as a common frame of references had not been introduced. The "past" and the "present" year may be of arbitrary duration.

> Itaque quae anno praeterito et praesente facta sunt negamus esse simul, involvit enim oppositos ejusdem rei status. [Leibniz, Initia]

Newton obtained automatically a quantification of time and space (compare below the comment by Clarke). Hence, the shortage of Leibniz's theory is caused by the investigation of relative *translations* instead of relative *motions* of bodies in the same frame.

> 47. I will here show, how men come to form to themselves the notion of space. They consider that many things exist at once and they observe in them a certain order of co-existence, according to which the relation of one thing to another is more or less simple. This order is their *situation* or distance. When it happens that one of those co-existent things changes its relation to a multitude of others, which do not change their relation among themselves; and that another thing, newly come, acquires the same relation to the others, as the former had; we then say, it is come into the place of the former; and this change, we call a motion in that body, where in is the immediate cause of the change.
>
> And, to give a kind of a definition: *place* is that, which we say is the same to A and, to B, when the relation of the co-existence of B, with C, E, F, G etc. agrees perfectly with the relation of the co-existence, which A had with the same C, E, F, G, etc. (...). Lastly, *space* is that, which results from places taken together. [Leibniz Clarke (Alexander), Leibniz's 5th Letter]

This consideration is consequent as long as the change of distances and the spatial relations are considered to be independently of time. The velocity is indeterminate, i.e. the velocities of the bodies A and B are also indeterminate. However, in case of *two bodies* which are either resting or moving *relatively to each other* the velocity is not indeterminate. In the first case, the velocity is zero, in the second case it has a definite finite value.

The objection of Clarke [Leibniz Clarke] was not only directed against Leibniz's assumption of ordering, but it has been also reinforced by the true argument that time and space cannot be considered as mere orders.[50]

> Further: Space and Time are Quantities: which Situation and Order are not. [Leibniz Clarke (Alexander), 3rd Letter to Leibniz][51]

[50] This argument had been renewed by Euler [Euler E149, §§ 20–21]. Euler claimed that space and time are more than mere orders because the *equal* distances in *equal* time a body is travelling cannot be explained by a mere order of succession and coexistence.

[51] The foundation of the criticism of Clarke is due to Newton's basic assumptions on the generation of lines, surfaces and solids by a continual motion which is closely related to the foundation of the

Therefore, Leibniz's criticism of Newton's absolute space is as well justified as Clarke's criticism of the lack of quantities in Leibniz's relational theory. This point of view has been later stressed also by Euler. Finally, both the aspects have been brought together only by Einstein due to the introduction of light velocity into mechanics. Additionally, the assumption of an upper limit for all kinds of motion results in an order between the measurements performed by different observers (compare Chap. 6). The order can be also expressed in terms of Lorentz transformation. Moreover, assuming such ordering relations, it follows that the difference between Galileo and Lorentz transformations results only from the order in the relative motion of bodies which is established without any reference to electrodynamics.

After the invention of calculus, the notions of order and quantification were considerable modified and their mechanical meaning was expanded due to the distinction between finite and infinitesimal quantities or, in terms of mechanics, the motion of bodies with finite velocities and those mechanical quantities being related to the "beginning" and the "end" of motion. Before the invention of "infinitely little quantity" by Newton (compare Chap. 3), the beginning and the end of motion had been only qualitatively discussed by the construction of geometrical models of the continuum [Leibniz, Hypothesis] or by the construction of models for motion by transcreation [Leibniz, Pacidius].[52] Newton decided to concentrate the study of motion to the *very beginning* and the *end* of motion or the *nascent* and *evanescent* motion and to describe these special forms by the first and last ratios of fluents and fluxions (compare Chap. 3).

2.2.3 *The Very Beginning of Motion*

Following Newton, the very beginning of motion can be described in terms of nascent (or generated or produced [Newton, Principia, Book II, Lemma II, p. 256])

calculus (compare Chap. 3). The ordering is an inherent property of the method of generation since the lines, surface and solids are not generated at an *instant* (or, using Leibniz's terminology "tout d'un coup" [Leibniz, Monadology, § 6]), but an emerging and evolving continuous process whose *beginning* is described in terms of an infinitesimal quantity [Newton, Method of Fluxion]. The beginning cannot be separated from the continuation and a line does not consist of infinitesimal parts. Therefore, the continual motion is not described in terms of an ordered series of elementary steps, represented by "time intervals" o, $2\,o$, $3\,o$, ... and the independent variable is not represented by an arithmetical series (compare Chap. 3).

[52] "From among his early attempts on the continuum problem I distinguish four distinct phases in his interpretation of infinitesimals: (i) (1669) the continuum consists of assignable points separated by unassignable gaps; (ii) (1670–71) the continuum is composed of an infinity of indivisible points, or parts smaller than any assignable, with no gaps between them; (iii) (1672–75) a continuous line is composed not of points but of infinitely many infinitesimal ones, each of which is divisible and proportional to a generating motion at an instant (conatus); (iv) (1676 onward) infinitesimals are fictitious entities, which may be used as *compendia loquendi* to abbreviate mathematical reasonings." [Arthur, Fictions] Models of such type had been excluded by Newton who postulated the generation of paths by a continual flux [Newton, Quadrature (Harris)], i.e. a flux which is neither interrupted by gaps of finite extension nor by gaps having no extension.

quantities since this stage represents something in between rest and traversing a finite distance. The evanescent or disappearing quantities are suitable to describe the complementary process. Following Newton, both the processes are analytically represented by the first and ultimate ratios of the augments of fluents in time [Newton, Quadrature, Harris],[53] [Newton, Principia, Book I, Sect. I]. The increments of velocity and the errors are of infinitesimal magnitude. Although these increments of velocity cannot be measured, they are not merely fictitious quantities, but represent the change of the state of the body in the frame of a thought experiment.

> Cor. 1. (...) that the errors of bodies describing similar parts of similar figures proportional times, are nearly in the duplicate ratio of the times in which they are generated, if so be these errors are generated by any equal forces similarly applied to bodies, and measured by the distances of the bodies from those places of similar figures, at which, without the action of those forces, the bodies would have arrived in those proportional times.
>
> Cor. 2. But the errors ... are as the forces and the squares of the times conjunctly. ($s^{\text{errors}}_{\text{begin}} \sim K \cdot t^2_{\text{begin}}$) (which is equivalent to $\Delta\Delta s \sim K\Delta t^2$).[54]
>
> Cor. 3. The same thing is to be understood of any space whatsoever described by bodies urged with different forces. All which, in the very beginning of motion, are as the forces and the squares of the times conjunctly.
>
> Cor. 4. And therefore the forces are as the spaces described in the very beginning of the motion directly, and the squares of the time inversely. ($K \sim s^{\text{errors}}_{\text{begin}}/t^2_{\text{begin}}$).
>
> Cor. 5. And the squares of the times are as the spaces describ'd directly and the forces inversely. ($t^2_{\text{begin}} \sim s^{\text{errors}}_{\text{begin}}/K$). [Newton, Principia, Book I, Sect. I, Lemma IX][55]

Following Galileo and representing the "change in motion" in terms of finite quantities (compare Chap. 1), i.e. finite increments of motion in dependence on finite increments of time, the relations are given as follows where the *increments of velocity* and the *increments of the increments* of the path $\Delta(mv) \sim K$ or $m\Delta v \sim K \cdot \Delta t$ and $\Delta\Delta s \sim K\Delta t^2$ are proportional to the increment of time and the square of the increment of time, respectively. Following Newton, motion is quantitatively described by the product of mass and velocity. Newton called the increment $\Delta(mv)$ "change of motion" and described the change of the state by this quantity instead of making use of the "change of velocity" (for Euler's procedure compare Chap. 4).[56] However, analyzing the general relations between "errors, time and forces", the previously defined *change in motion* in terms of $\Delta(mv)$ had not been explicitly included.

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[53] "Now let those Augments vanish and their ultimate Ratio will be the Ratio of *1* to nx^{n-1}." [Newton, Quadrature (Harris)] The ratio is made of finite quantities also it had been previously represented in terms of infinitesimal quantities. Here, Berkeley found an inconsistency in the foundation [Berkeley, Analyst] (compare Chap. 3).

[54] Comments in brackets by D.S.

[55] Later in the 19th century, only the relation derived in Corol. 2 had been accepted and called differential quotient $d^2s/dt^2 = K/m$. All other relations discussed by Newton had been either ignored or rejected. Obviously, although Newton later argued against differentials [Newton (Collins), Commercium], the assumed relations can be readily interpreted as those being valid for differentials.

[56] "30. Man sagt, ein Körper verbleibe in ebendemselben Zustande, wenn derselbe entweder in Ruhe verbleibt oder seine Bewegung nach ebenderselben Richtung mit einerlei Geschwindigkeit fortsetzet." [Euler E842, § 30]

On the contrary, the analysis is performed for the deviation of the *positions* the body arrives in *uniform motion* from those which are generated due the presences of forces (compare above Cor. 1). The positions and the errors are defined geometrically or "measured by the distances of the bodies from those places of similar figures". The *geometric* frame of reference is formed by the path the body describes in uniform motion. As a consequence, the analysis cannot be efficiently transferred to the *change of rest* since an appropriate frame of reference is missing. Analytically, the frame of reference is formed by the equations $v = 0$ or $v = \text{const}$. Then, the change of the state is always represented by the increment of velocity Δv independently of the initial state of the body being either the state of rest or the state of uniform motion, i.e. $v = 0$ or $v = \text{const}$, respectively (compare Chap. 4). Although the result is independent of the *velocity* of the body [Euler E015/016, § 131], the *magnitude* of the change depends on the inertia or the mass of the body which is even the missing parameter in the relation between errors, times and forces. Uniform motion is *mass independent* whereas the *change* of uniform motion is *not* mass independent. Therefore, the true values of errors are to be obtained by comparison of a mass independent and mass dependent trajectory described by the body.

Hence, there are complementary approaches, (i) Newton presented an explicit relation for the *change in motion* whereas the conservation of living forces was hidden, whereas Leibniz, on the contrary, (ii) presented explicitly the *conservation of living forces* whereas the change in motion by derivative forces was hidden.

Following Leibniz, the efficiency of the action of forces may be characterized by the magnitude of the change of velocity being either a maximum or a minimum. Obviously, the *most efficient* change is the change of the state of rest since there is a transition from rest to motion. Then, the very beginning of motion is described by the equation $\mathrm{d}v = \mathrm{d}v_{\max} = (K/m)\mathrm{d}t$. Consequently, the *most inefficient* change of rest is given by the relation $\mathrm{d}v = \mathrm{d}v_{\min} = (K/m)\mathrm{d}t$ where, obviously, the lower limit is represented by $\mathrm{d}v = \mathrm{d}v_{\min} = 0$, i.e. no change at all. This result can be obtained (i) either for $K = 0$ which is the trivial case or (ii) for $K \neq 0$ and $m \to \infty$ which is the non-trivial case. Following Eule and assuming constant forces of finite magnitude $K_{\text{finite}} = \text{const}$, it is impossible that there is no change of the state [Euler E015/016], [Euler E842], [Euler E289]. The change of the state is described by the increment of velocity $\mathrm{d}v$ or, following Newton, "the errors *dds* in the very beginning of motion". Euler claimed that, according to the principles of least action, the change of velocity is generated by the forces which are the least those ones to avoid the penetration of bodies [Euler E343, Lettre LXXVIII] (compare Chap. 4).

2.2.4 Polygon and Circle: Periodic Motion

Newton analyzed the motion of a body under the influence of external forces in a model based on periodic motion performed by the body along the sides of a polygon inscribed into a circle (see Fig. 2.1). The model system is composed of physical and non-physical components. The body is assumed to move (i) uniformly along the sides of the inscribed polygon, then, arriving at the intersection point of the side

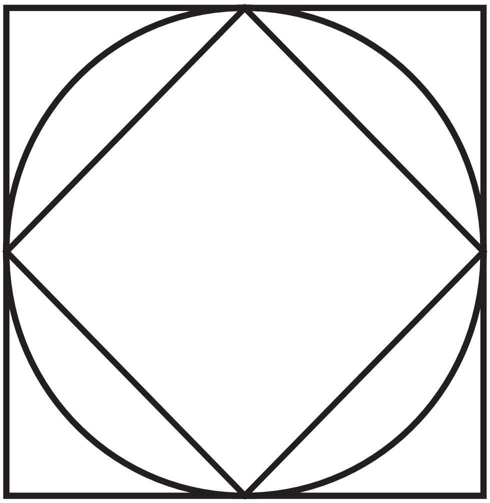

Fig. 2.1 Newton's geometrical model for the relation between the change of velocity caused by forces
Newton's derivation of the change of momentum of a moving body due to a central force [Westfall, Never]

and the circle (ii) the body is reflected at the side of the circumscribed polygon and (iii) continues the uniform motion with the same magnitude of velocity along the consecutive side as long as (iv) it returned after a finite time to its initial position.[57] Although the model comprises non-physical elements like the reflection at the circumscribed polygon, Newton obtained the correct results by an appropriate analytical representation of periodic motion [Westfall, Never, p. 149]. Moreover, Newton made use of the general relation between curved and straight lines represented by the relation between polygon and circle in a plane. This model had been later also use for the representation of the relation between finite and infinite quantities as well as the notion of limit. Lagrange referred to ancient science to establish an appropriate notion of limit. It is possible to approach to this quantity as close as possible, but never attain it or pass through the limit, in the present example from the inscribed to the circumscribed polygon and vice versa.[58] Later in 1695, the critics of the calculus also referred to the model of polygon and curved lines [Nieuwentijt, Analysis],[59] [Weissenborn].

[57] Periodic motions of such type had been analyzed by Leibniz [Leibniz, Brevis] and later by Helmholtz who referred to Leibniz [Helmholtz, Vorlesungen] (compare Chap. 7).

[58] "Les véritables limites, suivant les notions des anciens, son des quantités qu'on ne peut passer, quoiqu'on puisse s'en approcher aussi près que l'on veut; telle est, par exemple, le circonférence du cercle à l'égard des polygones inscrit et circonscrit, parce que, quelque grand que devienne le nombre des côtes, jamais le polygone intérieur ne sortira du cercle, ni l'extérieur n'y entrera." [Lagrange, Fonction]

[59] Nieuwentijt B (1695) Analysis infinitorum seu curvilineorum proprietates ex polygonorum natura deductae. Wolters, Amsterdam.

2.3 Leibniz's Program for Mechanics

In 1695, Leibniz commented on the early version of the theory of motion published in the treatise *Theoria motus abstracti*. "Mihi adhuc juveni, et corporis naturam cum *Democrito* et hujus ea in re sectatoribus *Gassendo* et *Cartesio*, in sola massa inerte tunc constituenti, (...)." [Leibniz, Specimen I (10)]. Later, Leibniz did not explain the conservation of state as Descartes and Newton by inertia. As a result, instead of simplyfing and generalizing the theory, Leibniz was forced to introduce a variety of different forces [Leibniz, Brevis], [Leibniz, Specimen] and run in trouble to define analytically the relation between forces. Nevertheless as in Newton's mechanics, the *increments* or *decrements* of velocity also played a crucial role in Leibniz's mechanics. These quantities introduced explicitly by Newton appeared in Leibniz's theory as *hidden variables* or parameters. Having rejected the idea of inertia, Leibniz was forced to replace the notion of *inert mass* by forces whose most important representative had been called "living forces" and determinate by the product of mass and the square of velocity $m \cdot v \cdot v$ [Leibniz, Brevis], [Leibniz, Specimen]. Then excluding the possibility of perpetual motion, Leibniz postulated that there "is neither more nor less portency in the effect than in the cause" [Leibniz, Specimen, I (11)],[60] [Leibniz, Specimen, UV (11)][61] and studied the redistribution of living forces among the interacting bodies. The postulate is analytically formulated as an equation comprising and connecting the living forces *before* and *after* the impact (interaction). The time of interaction is not taken into account, but indirectly included in the difference between the initial and final values of the velocity. The magnitude of difference should be related to the duration and the intensity of interaction. The intensity of the interaction is represented by the *force* and is independent of the mass of the bodies whereas the *duration* of interaction is either of infinitesimal or of finite magnitude. Hence, a finite change of velocity is necessarily generated in a finite time interval (of interaction) whereas an infinitesimal change of velocity (being either an increment or decrement) is as necessarily generated in an infinitesimal time interval. Postulating the equality

$$m_1 v_{1\,\mathrm{before}}^2 + m_2 v_{2\,\mathrm{before}}^2 = m_1 w_{1\,\mathrm{after}}^2 + m_2 w_{2\,\mathrm{after}}^2 = \mathrm{const}, \tag{2.1}$$

Leibniz had indirectly assumed that there was a beginning and an end of interaction whose magnitudes, following Newton, cannot be represented by finite increments, but only by the ratio of infinitesimal increments or first and last ratios of the "errors" and the square of augment of time, i.e. the ratio dds/dt^2. Hence, also Newton made use of infinitesimal quantities *ds, dds* and *dt* in advance, i.e. before their ratio had been finally taken into account.

Obviously, the increments/decrements of velocities or the differences "before interaction" and "after interaction" $|v_{\mathrm{before}} - w_{\mathrm{after}}| = \Delta v$ played the role of "hidden parameters". Moreover, although Leibniz introduced derivative forces being generated

[60] "(...) nec plus minusve potentiae in effectu quam in causa contineatur." [Leibniz, Specimen, I (11)]

[61] "(...) sed effectum plenum esse causae integrae aequalem." [Leibniz, Specimen, UV (11)]

by the impact of bodies, these forces are not explicitly represented. However, making use of the calculus of differences the relation is obtained $\Delta v_1 \sim K_{12} \cdot \Delta t_{\text{impact}}$ and it follows: (i) the forces are generated by interacting bodies, (ii) hence K_{12}, K_{21} are to be indicated by both the bodies involved in the interaction, (iii) the change of living forces is given by $\Delta(v \cdot v) = 2 \cdot v \cdot \Delta v$, (iv) hence, the change of living forces $v_1 \Delta v_1 \sim K_{12}$ can also be represented by the previously introduced forces. Hence, Newton and Leibniz invented different representation of one and the same quantity, the change in velocity Δv which is generated due to the interaction of bodies. These results were only analytically completely demonstrated by Euler [Euler E015/016, §§ 125–152].

2.3.1 Early Version

Leibniz's program for mechanics is embedded into a general program for a general science (scientia generalis). From the very beginning, mechanics is not only explicitly related to geometry, arithmetic, logic and metaphysics, but methodologically based on these disciplines.[62] The fundamental methodological relation is that physics had been assumed to be *subordinate* to mathematics, i.e. arithmetics or algebra and geometry, and metaphysics. The non-physical basic notions for physics are (i) magnitude, (ii) position (situs)[63] and (iii) resistance or action and suffering.

> Physica est Arithmetica sive Algebrae subordinata quatenus agit de magnitudine; Geometricae quatenus agit de situ; Metaphysica quatenus de resistentia sive actione et passione. [Leibniz, A VI, 267][64] (1679)

The foundation of mechanics by merging arithmetics, geometry and metaphysics had been preserved in the following decades. In the *Discours de métaphysique* Leibniz presented mechanics as a result of a unification of principles in terms of *fundamental* and *subordinate* rules [Leibniz, Discours],[65] later interpreted in terms of necessary and contingent truths [Leibniz, Monadology]. Neither time nor motion can ever exist as a whole,

[62] Although Leibniz invented the mathematical algorithm for differentials and differentio-differentials and higher order differentials, the definition of velocity by the ratio of differentials of space and time, $ds = v\,dt$ or $ds/dt = v$, had been not discussed by Leibniz, but only by Varignon in 1700 [Varignon 1700].

[63] Later in 1715, Leibniz defined "situs" as a "modus of coexistens" made up of magnitude and relative position. "Situs est coexistentia modus. Itaque non tantum quantitatem, sed et qualitatem involvit." [Leibniz, Initia, p. 354]

[64] "Physics is subordinated to Arithmetics or Algebra as far the magnitude is concerned and to Geometry as far as the position [defined topologically and metrically] is concerned further to Metaphysics as far as the resistance or action and suffering are concernd." [Leibniz, A VI, 267] (1679)

[65] "17. An example of the subordinate rule of natural law, which shows that God always systematically conserves the same *force*, but not (contrary to the Cartesians and others) the same quantity of *motion*." [Leibniz, Discours] Here, Leibniz stressed the difference between "motion" and "force". "17. Exemple d'une Maxime subalterne du loy de Nature, où il est monstré que Dieu conserve tousjours regulierement la même force, mais non pas la même quantité de mouvement, contre les Cartesiens et plusieurs autres." [Leibniz, Discours]

> (...) because a whole does not exist if it has no coexisting parts. Thus there is nothing real in motion itself except that momentaneous which must consist of a force striving toward change. From that originates also all this being in corporeal nature except the subject of geometry, i.e. the extension. By this argumentation, the truth and the doctrines of the ancients are simultaneously taken into account. [Leibniz, Specimen, I (1)][66]

Leibniz compared the saving of the ideas of Plato and the Stoics in the age of Democritus to the saving of the doctrines of peripatetics on the forms and entelechies in present times [Leibniz, Specimen, I (1)].[67]The main discrepancy between Newton's and Leibniz's mechanics is due to the different assumptions on the conservation of mechanical quantities. Following Descartes who considered the "quantity of motion" as being conserved in the world, Leibniz alternatively constructed the conservation of another quantity called "living forces" [Leibniz, Brevis].[68] This principal dissimilarity was well known and caused a long lasting debate on the true measure of forces originating from Leibniz's *Brevis memorabilis* [Leibniz, Brevis] published in 1686 one year before Newton's *Principia*. Leibniz' main intention was to demonstrate that perpetual motion is to be excluded [Leibniz, Specimen, I (11)]. In 1738, Voltaire commented:

> Descartes, sans faire mention de la force, avançait sans preuve qu'il y a toujours quantité égale de mouvement; et son opinion était d'autant moins fondée que les lois mêmes du mouvement lui étaient absolument inconnues.
>
> Leibnitz, venu dans un temps plus éclairé, a été obligé d'avouer, avec Newton, qu'il se perd du mouvement; mais il prétend que, quoique la même quantité de mouvement ne subsiste pas, la force subsiste toujours la même.
>
> Newton, au contraire, était persuadé qu'il implique contradiction que le mouvement ne soit pas proportionnel à la force. [Voltaire, Élémens, Chap. IX]

In 1686, Leibniz emphasized the relation between "forces" or "metaphysical principles" and the "phenomena of bodies" or the "phenomena in nature" described by geometry or the "science of extension". Leibniz claimed that geometry is necessary, but not sufficient to describe the interaction of bodies since except extension. There should be additional principles of action and suffering. These additional principles are represented by a variety of different forces among them those of purely metaphysical origin like the *primitive* forces [Leibniz, Specimen] and claimed that the

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66 "Nihilique adeo in ipso reale est, quam momentaneum illud, quo in vi ad mutationem nitente constitui debet. Huc igitur redit quicquid in natura corporea praeter Geometriae objectum seu extensionem. Eaque demum ratione simul et veritati et doctrinae veterum consulitur." [Leibniz, Specimen, I (1)]

67 Geometry is the science of extension. Following Newton who recovered the ancient prototype for the relation between points, lines, surfaces and solids (compare Chap. 3), extension (extended things) is generated by motion, hence, mechanics is the science how extension is generated by motion whereas, following Leibniz, mechanics is the science how extension is generated by forces [Leibniz, Specimen, I (1)]. Later in 1715, Leibniz adopted also the Newtonian model [Leibniz, Initia].

68 "18. La distinction de la force et de la quantité de mouvement est importante, entre autres pour juger qu'il faut recourir à des considérations métaphysiques séparées de l'étendue à fin d'expliquer les phénomènes des corps." [Leibniz, Discours]

"phenomena would be completely others" if "mechanical rules would only depend on geometry".[69]

Following Leibniz and applying the model of the infinity of possible worlds being different from each other [Leibniz, Monadology, §§ 53–58], the phenomena depend on the world where they appear, whereas, because of its necessity, the *extension* or the *geometry* should be the same in different worlds. The extension is an invariant property of all possible worlds. Hence, the extension is also a necessary property of the existing world.[70] The phenomena are generated by the motion of corporeal things. The derivative forces are related to those corporeal motions which result in an interaction between bodies [Leibniz, Specimen, I (4)].[71] One may ask whether there is one possible world where the phenomena are *independent* of metaphysics, but depend only on geometry. The answer may be obtained from Leibniz's model how the different places in the continuum or plenum may be distinguished from each other.[72]

In all later writings, Leibniz extended the frame of basic concepts which had been established very early between 1675 an 1679.[73] These attempts had been continued in the next decade until 1686. The program for mechanics is based on the investigation of the problems to be solved with respect to (i) composition of the continuum, (ii) time, (iii) place, (iv) motion, (v) atoms, (vi) indivisibles and (vii) infinity. Leibniz did not modify this program as far as the relation between geometry, arithmetic and metaphysics is concerned, but modified essentially the assumptions on the basic properties of bodies, i.e. mainly the concept of inertia [Leibniz, Specimen, I (10)].

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[69] . "21. Si les réglés mécaniques dépendoient de la seule Géométrie sans la métaphysique, les phénomènes seroient tout autres. " [Leibniz, Discourse]

[70] His assumption is not in contradiction with the model that the "forces is prior to extension" which is not related to the plenum, but to the corporeal things in the world, "in rebus corporeis". "In rebus corporeis esse aliquid praeter extensionem, imo extensione prius, alibi admonuimus, nempe ipsam vim naturae ubique ab Autore inditam." [Leibniz, *Specimen*, I (1)]

[71] "Vim ergo derivativam, qua scilicet corpora actu in se invicem agunt, aut a se invicem patiuntur, hoc loco non aliam intelligimus, quam quae motui (locali scilicet) cohaeret, et vicissim ad motum localem porro producendum tendit. Nam per motum localem caetera phenomena materialia explicari posse agnoscimus. Motus est continua loci mutatio, itaque tempore indiget." [Leibniz, *Specimen*, I (4)]. Relative motion is not something real or absolute, "quasi reale quiddam esset motus et absolutum", hence, the phenomena are described with respect to a frame of reference which is not "given", but "chosen" (therefore "contingent"). "Sic igitur habendum est, si corpora quotcunque sint in motu, ex phaenomenis non posse colligi in quo eorum sit motus absolutus determinatus vel quies, sed cuilibet ex iis assumpto posse attribui quietem ut tamen eadem phaenomena prodeant" [Leibniz, Specimen, II (2)].

[72] This "world" consists only of the plenum ("le plein étant supposé") or the "extended thing" without borders [Leibniz, Monadology, § 8]. The vacuum is automatically excluded. There is no empty space [Leibniz, Monadology, § 69]. Moreover, the plenum is completely free of monads. As a consequence, there are no different parts which can be distinguished from each other, i.e. the "world does not consist of coexisting parts". This property is also assigned to "motion" and "time" [Leibniz, Specimen, I (1)].

[73] Compare the analysis of Leibniz's interpretation of differentials by Arthur [Arthur, Syncategorematic].

In contrast to later version, extension appeared in the early theory of motion in two modifications, first as the extension of a body and second as the extension of vacuum.

> Corpus est extensum resistens. Extensum est quod habet magnitudinem et situm. Resistens est quod agit in id a quo patitur. Vacuum est extensum sine resistentia. [Leibniz, A VI, 267][74]

In the further development, the "resisting thing which acts in those others from whose it suffers" is specified in terms of "active and passive forces" [Leibniz, Specimen]. Unfortunately, the change in the terminology results in the decomposition of the previously correlated and complementary parts of the "action" which always simultaneously appear in the impact of bodies. The alternative possible interpretations are the following: (i) The bodies are *permanently acting* and *suffering* and (ii) the bodies are not permanently acting and suffering or not permanently resisting, but only in occasion of the impact or interaction. The theorem (i) had been developed by Leibniz in the *Specimen*. Guided by the idea, that the science of the ancients is the science of equilibrium or statics where permanently acting forces impressed upon bodies are compared to each other by joining the bodies in the lever, the science of motion should be necessarily also a science of forces. Following Leibniz: (i) body is the extended resisting thing, resisting is what acts upon that from which it suffers and (ii) a vacuum is an extended thing without resistance. The uniform motion is out of the scope of a theory where the velocity of a body is correlated with forces. The directions of forces are indeterminate. Newton assumed that the change in motion is determinate by the direction of the "impressed moving force". Following Euler, the directions of the forces in case of interaction are determinate with respect to the plane whose orientation geometrically represents the necessary conditions for the interaction of two bodies [Euler E842, §§ 69 and 71].

2.3.2 Later Version: Living Forces

In 1686, Leibniz published a criticism of Descartes' postulate on the equivalence of "moving force" and "quantitiy of motion" ("qui *vim motricem* et *quantitatem motus* pro re aequivalente habebat" [Leibniz, Brevis]). In 1686 and 1687, Leibniz published a short note on the errors of Descartes [Leibniz, Brevis][75] whereas Newton

[74] "A body is an extended resisting thing. An extended thing is that magnitude and place has. Resisting is that acts in those others from whose it suffers. Vacuum is an extended thing without having resistance." [Leibniz, A VI, 267] (1679–1681)

[75] The famous *relational theory of time and space* had been systematically developed by Leibniz only later in the treatise *Initia rerum mathematicarum metaphysia* in 1715 [Leibniz, Initia] and in the correspondence with Clark between 1715 and 1716 [Leibniz, Clarke]. In the 1715 paper on the foundation of mathematics, Leibniz did not consider bodies and forces as mechanical objects, but bodies only as geometrical, i.e. extended things. One of the main subjects is the relation between point, line, surfaces and solids (compare Chap. 3). Motion is treated as a *translation* of geometrical objects like *points* and *lines* which is performed with an *indeterminate* velocity.

published a comprehensive treatise on the *Mathematical principles of natural philosophy* which was also based on a carefully performed analysis and criticism of Descartes' principles, respectively. The main subject of both treatises is the relation between the motion of bodies and forces of nature where motion is considered as *phenomena* whereas forces are different from phenomena where the latter have to be investigated by studying the phenomena [Newton, Principia].

In both writings it was assumed that bodies move (i) either with constant velocity in one and the same direction which was already postulated by Descartes [Descartes, Principles] (ii) or, if the bodies perturb each other, they refuse to continue their previous motion due to the presence and action of "impressed moving forces", as Newton explained, or, as Leibniz explained,[76] due the forces which enable "mutual actions" of bodies during the impact,[77] or, due to the generation of forces by interacting bodies, as Euler explained later [Euler E842]. In 1686, Leibniz called the measure of the force of bodies "moving force" ("vis motrix") to distinguish this concept from the Cartesian "quantity of motion" ("quantitas motus"). Later in 1695, Leibniz renamed this force by "living force" ("vis viva") which may be generated by removal of an obstacle and whose appearance was observed in the up and down motion of a weight[78] [Leibniz, Specimen]. Following Leibniz, the idea of a living force is related to the "dead force" ("vis mortua") which had been studied in the theory of equilibrium by the ancients.[79] The "quantity of motion" in the whole nature can neither be increased nor decreased.

> Itaque cum rationi consentaneum sit, eandem motricis potentiae summam in natura conservari, et neque imminui, (...) neque augeri, quia vel ideo motus perpertuus mechanicus nuspiam succedit, (...). [Leibniz, Brevis] (A 1686)

The Leibnizian concept had been later developed by Châtelet [Châtelet, Institutions] and d'Alembert [d'Alembert, Traité] who explained the living forces by the ability of the moving body to remove partially or completely obstacles the body met whereas the Cartesian measure is appropriate for measuring the pressure which is due to an invincible obstacle. This distinction had been introduced by Leibniz who called the second type of forces "dead forces" to stress the difference to "living forces" [Leibniz, Specimen, I (6)].

> Hinc *Vis* quoque duplex; alia elementaris, quam et *mortuam* appello, quia in ea nondum existit motus, sed tantum solicitatio ad motum, (...) alia vero vis ordinaria est, cum motu actuali conjuncta, quam voco *vivam*. [Leibniz, Specimen, I (6)] (A 1695)

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[76] Leibniz based the theory on the *differences* between bodies, i.e. their different velocities. Hence, the *change* of motion is not regarded as independent of the velocities, but directly related to the *nisus*.

[77] Leibniz called these forces "derivative forces". "Vim ergo derivativam, qua scilicet corpora actu in se invicem agunt, aut a se invicem patiuntur, hoc loco non aliam intelligimus, quam quae motui (locali scilicet) cohaeret, et vicissim ad motum localem porro producendum tendit." [Leibniz, Specimen, I (4)]

[78] This construction was stimulated by Huygens' pendulum. Leibniz removed the horizontal motion and considered only the pure up and down motion of the weight.

[79] "Veteres, quantum constat, solius vis mortuae scientiam habuerunt, eaque est, quae vulgo dicitur Mechanica, agens de vecte, trochlea, plano inclinato." [Leibniz, Specimen, I (8)]

Comparing (A1686) to (A1695) a remarkable difference should be highlighted since in the earlier version the body is assumed to move in both cases whereas in the later version Leibniz compared a body which is only *prepared for motion*, but is not moving to a body which is really *moving*. In 1695, though the assumed forces cannot be represented in the intuition, Leibniz based the theory on the notion of an inherent force. In 1698, Leibniz commented on this topic in the writing *De ipsa natura sive de vi insita actionibusque creaturarum* [Leibniz, De ipsa].[80]

> Haec autem vis insita distincte quidem intellegi potest, sed non explicari imaginabiliter. [Leibniz, De ipsa, § 7]

The emergence of motion is regarded as criterion for the existence of such force when motion sets in if an external obstacle is removed which previously hindered the body to move. In the 1698 paper *De Ipsa* which may be read as a reaction to Newton's *Principia*, Leibniz reconsidered the concept of forces.

> (...) ita verbum benedictionis non minus mirificum aliquam post se in rebus reliquisse producendi actus suos operandique foecunditatem nisumve, ex quo operatio, si nihil obstet, consequatur. [Leibniz, Ipsa, § 8]

Following Kepler and substituting Newton's *vis inertia* with Kepler's concept of *inertia*, Leibniz assumed a natural inertia as an inherent force which made the body able to rest (vis passiva resistendi [Leibniz, De ipsa, § 11]) which had been later also assumed by Châtelet [Châtelet, Institutions]. Methodologically, a correlation should be established between (i) active (moving) – passive (resisting) and (ii) primitive and derivative forces.

Very early Leibniz rejected the idea that "the nature of bodies is solely determined by the inert mass" [Leibniz, Specimen, I (10)]. Instead of the inert mass Leibniz introduced a concept of *complementary* forces, mainly the "active and passive forces". Both the forces are further specified by the same subdivision into "primitive active" and "derivative actives" as well as "primitive passive" and "derivative passive" forces. The distinction is finally based on another distinction[81] which is due to "*internal* and *external*". The "primitive active" force is an inherent force whereas the "derivative active" forces results from the limitation of the first one appearing in the impact of bodies [Leibniz, Specimen, I (3)].

> Duplex autem est *vis Activa* (...) nempe aut *primitiva*, quae in omni substantia corporeaper se inest (...) aut *derivativa*, quae primitivae velut limitatione per corporum inter se conflictus resultans, varie exercetur. [Leibniz, Specimen, I, (3)]

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[80] For the development of Leibniz's concept of forces before 1695 compare Duchesneau [Duchesneau].

[81] Behind this distinction is the principle of sufficient reason based on the assumption of contingent truths [Leibniz, Monadology, §§ 31–35]. "Internal" and "external" are contingent relations of finite things. In contrast, rest and motion are not considered as basically contingent properties since "cum corpus omnimode quiescens a rerum natura abhorrere arbiter" [Leibniz, Specimen, I (3)]. Châtelet also assumed a similar difference between rest and motion [Châtelet, Institutions]. Therefore, following Descartes in assuming extension as basic property, Châtelet modified Descartes' assumption on rest and motion in favour of Leibniz.

Leibniz completed the Cartesian approach based solely on geometry by the introduction of forces. However, none of these forces can be considered as "necessary" since their generation is guided by contingency. The only candidate for being as necessary as the extension was the inertia which had been rejected by Leibniz. Being aware of this gap Leibniz replaced the previous concept of inertia by the conservation of "living forces" where the shadow of inertia is entering as the numerical value of the masses of bodies involved in the impact,[82] however, an explicit expression for the *change in motion* where the "derivative forces" should appear is missing. Hence, Leibniz represented the change in motion in terms of finite differences between the initial and the final velocities being assigned to the body before and after the impact, respectively. (Compare Eq. 2.1). The mass is represented by a numerical value [Couturat, Cassirer],[83] but it is not related to any of the forces Leibniz discussed. A direct relation between mass and forces is missing in Newton's 2nd Law (compare Corollaries 1 to 5 in Sect. 2.2.3). Newton assumed that the "inherent force of inertia" is activated by the "impressed moving force" [Newton, Principia, Definitions]. The quantitative definition of the mass is due to density times volume.[84]

In 1686, Leibniz distinguished between the Cartesian *quantity of motion* (quantitas motus) and the *force motion* (vis motrix) [Leibniz, Brevis]. It is quite clear whether the Cartesian concept is considered as invalid or as valid, it has another range of validity. Leibniz based his argumentation on the same principle as Descartes did.

> Itaque cum rationi consentaneum sit, eandem motricis potentiae summam in natura conservari, et neque imminui, quoniam videmus nullam vim ab uno corpore amitti, quin in aliud transferatur, neque augeri, quia vel ideo motus perpetuus mechanicus nuspiam succedit, quod nulla machina ac proinde ne integer quidem mundus suam vim intendere potest sine novo externo impulsu; inde factum est, ut Cartesius, qui *vim motricem* et *quantitatem motus* pro re aequivalente habebat, pronunciaverit eandem quantitatem motus a Deo in mundo conservari. [Leibniz, Brevis]

Here, Leibniz stressed the conformity with Descartes as far as the conservation of certain quantities is concerned. The difference to Newton's treatment of mechanics published only one year later in 1687 is already automatically included

[82] The impact is always considered as the mutual action of bodies [Leibniz, Specimen, I (10)]. "Atque hoc est quod experimur, eundem nos dolorem sensuros sive in lapidem quiescentem ex filo si placet suspensum incurrat manus nostra, sive eadem celeritate in manum quiescentem incurrat lapis." [Leibniz, Specimen, II (2)] Here, Leibniz assumed the universality of relative motion derived from the impact of bodies. The only missing conclusion is that the impact is only possible due to the *inertia* of both interacting bodies.

[83] "Pour définir la force vive, c'est-à-dire précisément la quantité qui se conserver dans le choc élastique et qui déterminer par suite la marche ultérieure des mobiles, il fallait tenir compté du facteur *masse*, sans lequel on ne peut l'équivalence des forces vives échangées par le choc. L'invention du concept de masse ne constituait pas seulement un progrès capital de la mécanique: elle permettait à Leibniz de dissocier complètement l'idée de matière de l'idée d'étendue, puisque le coefficient appelé masse est une quantité numérique, et non une grandeur spatiale." [Couturat, Cassirer]

[84] "*The quantity of matter is the measure of the same, arising from its density and bulk conjunctly.* Thus air of double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity." [Newton, Principia, Definitions]

since Newton claimed that neither of the mechanical quantities are conserved [Newton, Principia]. Therefore, Leibniz introduced the notions of (i) "force" and "moving force" before Newton invented the terminus (ii) "impressed moving force". The difference is striking since (i) it is assigned to a body moving with a certain velocity whereas (ii) it is explicitly related to the "change of velocity". In Leibniz's model based on Huygens' experiments the *change* of motion is treated as an *intermediate* step or time interval which connects (a) two states S1 and S2 of one and the same body A being essentially different from each other, an initial state and a final state, respectively, and, additionally, the states of two bodies, A and B, having different masses. Descartes' measure is also related to two bodies C and D set in motion and being in different states.

In 1695, Leibniz introduced a variety of different forces which are finally, in 1698, reduced to two basic types, internal and external forces. The external forces are represented by *obstacles* hindering motion whereas the internal forces are defined by the permanent tendency to *act*, subsequently but not primarily, divided into the ability to *act* and to *suffer*.

> (...) ita verbum *benedictionis* non minus *mirificum* aliquam post se in rebus reliquisse producendi actus suos operandique foecunditatem nisumve, ex quo operatio, si nihil obstet, consequatur. Quibus addi potest, quod alibi a me explicatum est, etsi nondum fortasse satis perspectum omnibus, ipsam rerum substantiam in agendi patiendique vi consistere: unde consequens est, ne res quidem durabiles produci posse, si nulla ipsis vis aliquamdiu permanens divina virtute imprimi potest. [Leibniz, De ipsa, § 8]

The latter statement is in complete opposition to Newton who introduced the inherent force, called "vis insita", being the cause of the *preservation* of state, i.e. representing the *absence* of any action or suffering. Moreover, this force is not permantenly present, but has to be activated by the force impressed upon the body. Nevertheless, despite the differences there is a common basis in Newton and Leibniz which is mainly caused by the common origin of the problems in matematics and mechanics. Although from the outer view Newton's and Leibniz's mechanics looked differently, the common indispensable basis of the theory remains to be the geometry.

This common origin had been preferentially noticed by those predecessors who belong to the young generation of scholars in the first half of the 18th century. Later, following in goal and spirit Newton and Leibniz and demonstrating the common basis instaed of the non-compatible parts of their theories, Châtelet is not ready to replace *completely* geometry with the calculus.[85] Here, Châtelet is following Descartes, but she does not follow the Cartesians who reduced all properties of

[85] Here, we find a controversy which can be also observed in 20th century physics. In contrast to Euler, the development of quantum mechanics had not been guided directly by mathematics. However, assisted by Klein, Schrödinger recalled into the memory the geometrical representation of motion by Hamilton. Even in 1891, Klein tried to stimulate physics community to make use of Hamilton's theory. The response was disappointing. Only Schrödinger was ready to accept the invitation and, not surprisingly, was successful in developing independently of Heisenberg a new theory, called quantum mechanics. "Eines genetischen Zusammenhangs mit Heisenberg bin ich mir durchaus nicht bewußt. Ich hatte von seiner Theorie natürlich Kenntnis, fühlte mich aber durch die mir sehr schwierig scheinenden Methoden der transzendenten

the body to extension. The consequence is that the *extension* is considered as basic property of *all bodies* and the difference between bodies and other extended things[86] has to be introduced by the assumption of forces [Châtelet, Institutions, §§ 138–147]. This is the procedure of all followers of Descartes. Newton intoduced the forces of inertia and impressed moving forces [Newton, Principia]. Leibniz introduced active and passive, primitive and derivative, living and dead, total and partial and respective and directive (common) forces [Leibniz, Specimen].

> 145. Tous les changements qui arrivent dans les Corps peuvent s'expliquer par ces trois principes, *l'étendue, la force résistante*, & *la force active*; (...). [Châtelet, Institutions, § 145]

The followers of Newton and Leibniz had different, but correlated approaches at their disposal:

A. Newton's body-space (absolute space, absolute rest and motion) approach,
B. Newton's body-body approach (relative rest and motion),
C. Newton's body-force approach,
D. Newton's force-force model, the inherent force is excited by the impressed moving force,
E. Leibniz's relational body-body approach (obtained from (A)),
F. Leibniz's original force-force approach from 1695 (also related to (A)),
G. The modification of primitive moving forces inherent the bodies by the interaction of bodies and
H. The combined body-body and force-force approach from 1698.

Euler based his mechanics on (A) and (B) rejecting Newton's force of inertia and preserving only Leibniz's derivative forces as the only kind of forces investigated in mechanics. From the assumed body-body relations it follows that all the forces are only generated due the interaction of bodies. The theory of relative motion and the theory of the changes in motion had to be developed in parallel. Euler developed a complete relativistic theory within the frame of classical mechanics (compare Chap. 6).

Châtelet based the *Institutions* on (A) as far as extension is concerned and on (C) and (D) as far as the relation between active and passive forces are concerned. Therefore, Châtelet concentrates on the relation between dead and living forces. The theory of motion and rest is treated separately where the relational part is dominating in comparison to the relativistic elements. The most important force is the inherent "primitive active force" [Leibniz, Specimen, I (2)].

Algebra und durch den Mangel an Anschaulichkeit abgeschreckt, um nicht zu sagen abgestoßen" [Schrödinger, Heisenberg]. However, the mathematician David Hilbert was aware of the common features of Heisenberg and Schrödinger's approaches in advance [Thall's history of quantum mechanisc. http://mooni.fccj.org/∼ethall/quantum/quant.htm]. However, Hilbert's message was not listened by Heisenberg.

[86] Euler distinguished between *space* and *bodies* as *extended* things claiming that this difference cannot be explained by forces since rest and uniform motion of bodies are not related to forces [Euler E842 Chap. 2] (compare Chap. 4).

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