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A Survey on Covering Supermodular Functions

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Summary. In this survey we present recent advances on problems that can be described as the construction of graphs or hypergraphs that cover certain set functions with supermodular or related properties. These include a wide range of network design and connectivity augmentation and orientation problems, as well as some results on colourings and matchings.

In the first part of the paper we survey results that follow from the totally dual integral (TDI) property of various systems defined by supermodular-type set functions. One of the aims of the survey is to emphasize the importance of relaxing the supermodularity property to include a wider range of set functions. We show how these relaxations lead to a unified understanding of different types of applications.

The second part is devoted to results that, according to our current knowledge, cannot be explained using total dual integrality. We would like to demonstrate that an extensive theory independent of total dual integrality has been developed in the last 15 years, centered around various connectivity augmentation problems.

Our survey concentrates on the theoretical foundations, and does not include every detail on applications, since the majority of these applications are described in detail in another survey paper by the first author (Frank 2006). The comprehensive book "Combinatorial Optimization: Polyhedra and Efficiency" by Schrijver (2003) is also a rich resource of results related to submodular functions.

It should be noted that sub- and supermodularity have several applications in areas not discussed in this paper. In particular, we should mention the book "Submodular Functions and Optimization" by Fujishige (2005) and the book "Discrete Convex Analysis" by Murota (2003). The former explains the foundations of the theory of submodular functions and describes the methods of submodular analysis, while the latter presents a unified framework for nonlinear discrete optimization by extending submodular function theory using ideas from continuous optimization. Our survey focuses on topics not discussed in detail in those books.

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6.1 Introduction

The role of matroids in combinatorial optimization is well-known. Basic results like the greedy algorithm, the matroid intersection and partition theorems are discussed in standard textbooks (Cook et al. 1998; Korte and Vygen 2008). More general notions like polymatroids, submodular flows and other submodular frameworks, by uniting polyhedral techniques with powerful properties of submodularity, have been successful in solving graph problems related to paths, flows, trees, and cuts. Submodular flows proved to be particularly versatile as they unify apparently unrelated areas like graph orientation, (poly)matroid intersection, network flows or the famous Lucchesi–Younger theorem. A unifying feature of all these frameworks is that the describing linear system is totally dual integral. It is this property that makes tractable the weighted versions of the corresponding optimization problems, and indeed there is a rich literature of polynomial time algorithms.

Another general trend concerning submodular functions attempts to capture the special role played by parity considerations. The notion of odd components showed up first in Tutte's characterization of perfectly matchable graphs and was later extended by results of Mader on packing paths. A general theory, called matroid parity, has been founded by Lovász. Although in some special cases (e.g. non-bipartite matchings) the weighted version of these problems are tractable, no TDI-ness result is known for general matroid parity (not even for linear matroids).

These frameworks are traditionally formulated for submodular functions but one could speak of supermodular functions as well since the role of submodular and supermodular functions are symmetric. Therefore we use sometimes the term *semimodular* to refer to a function which is either submodular or supermodular.

There is a third direction of semimodular optimization. Interestingly, at about the same time as submodular flows were introduced by Edmonds and Giles (1977), Eswaran and Tarjan (1976) found a min-max theorem (and algorithm) on the minimum number of additional edges needed to make an initial digraph strongly connected. Though the proof is not difficult at all, this result is set apart by the fact that the minimum cost version is obviously NP-complete, so it cannot fit into the submodular flow (or any other TDI) framework. Another development that proved later crucial was the introduction of the splitting off operation by Lovász. These two results became the starting points of supermodular optimization, which mainly focuses on obtaining minimum size solutions for (di)graph and hypergraph problems where minimizing an arbitrary cost function is NP-hard. An interesting phenomenon in several problems of this type is that results about edge-connectivity and node-connectivity augmentation can be derived from abstract results about supermodular functions. This first appeared in Frank (1994) where edge-connectivity augmentation of digraphs was generalized to the covering of crossing supermodular functions by directed edges, and several results in the same vein followed.

The main objective of this survey is to present recent developments in this branch of supermodular optimization. We concentrate on the theoretical aspects, and show applications only as illustrations. A detailed account of possible applications concerning graph and hypergraph connectivity can be found in Frank (2006).

Though these results cannot be explained using the framework of submodular polyhedra and total dual integrality, there is a strong interplay between the two areas. Therefore we present a summary of polyhedral results and their relation to hypergraph problems in Sect. 6.2. We also include some results that are at the borderline of the two areas.

We step across the border in Sect. 6.3 and describe a series of problems where good characterizations exist but do not come from a polyhedral description. Most of these results are obtained using some variant of the splitting-off method, although some of them require quite different techniques. We include only a few short proofs which allow the reader to take a glimpse at the methods used.

Some explanation is due on the title of the paper. There are many possible ways to define a "covering" of a set function. For example, set functions may be covered by elements of the ground set (e.g. the generators of matroids), or by vectors (as in the case of contra-polymatroids). One may also consider set functions covered by colourings, for which a good example is the supermodular colouring theorem of Schrijver (1985).

The present survey deals with the covering of set functions by graphs, hypergraphs and directed hypergraphs. In the following we define precisely what is meant by this.

6.1.1 Notation for Hypergraphs

Let V be a finite ground set. A *hyperedge* is a non-empty subset of V; it is called an *r-hyperedge* if its size is r. A *hypergraph* $H = (V, \mathcal{E})$ consists of a family $\mathcal{E}$ of hyperedges on the ground set V. We allow the same subset to appear multiple times in $\mathcal{E}$. The *rank* of a hypergraph is the size of its largest hyperedge. In this paper a hypergraph of rank at most r will be called an *r-hypergraph* for short. A hypergraph is *uniform* if every hyperedge has the same size; it is *nearly uniform* if the difference between the sizes of two hyperedges is at most one.

For a hypergraph $H = (V, \mathcal{E})$ and a node set $X \subseteq V$ we introduce the following notation:

$$\begin{aligned}
\Delta_H(X) &= \{e \in \mathcal{E} : e \cap X \neq \emptyset, e - X \neq \emptyset\},\\
d_H(X) &= |\Delta_H(X)|,\\
i_H(X) &= |\{e \in \mathcal{E} : e \subseteq X\}|,\\
e_H(X) &= |\{e \in \mathcal{E} : e \cap X \neq \emptyset\}|.
\end{aligned}$$

For a partition $\mathcal{P}$ of V, let

$$e_H(\mathcal{P}) = |\mathcal{E}| - \sum_{X \in \mathcal{P}} i_H(X). \tag{1}$$

If $x : \mathcal{E} \to \mathbb{R}$ is a function on the hyperedge set, then we use the notation

$$d_x(Z) = \sum_{e \in \Delta_H(Z)} x(e) \quad \text{for } Z \subseteq V.$$

The set functions i_x and e_x are defined analogously. We say that a hypergraph $H = (V, \mathcal{E})$ *covers* a set function $p : 2^V \to \mathbb{R} \cup \{-\infty\}$ if $d_H(X) \geq p(X)$ for every $X \subseteq V$. A function $x : \mathcal{E} \to \mathbb{R}$ covers p if $d_x(Z) \geq p(Z)$ for every $Z \subseteq V$.

6.1.2 Notation for Directed Hypergraphs

A *hyperarc* is a pair (e, h_e) where e is a hyperedge and $h_e \in e$ is called the *head node*. With a slight abuse of notation, we will use e to denote both the hyperarc and the associated hyperedge. If $|e| = r$ then the hyperarc is called an *r-hyperarc*.

A hyperarc e *enters* a node-set X if $h_e \in X$ and $e \nsubseteq X$. A hyperarc e *leaves* a node-set X if $h_e \notin X$ and $e \cap X \neq \emptyset$.

A *directed hypergraph* $D = (V, \mathcal{A})$ consists of a node set V and a family of hyperarcs $\mathcal{A}$. We allow $\mathcal{A}$ to contain a hyperarc multiple times. The *rank* of a directed hypergraph is the size of its largest hyperarc. A directed hypergraph of rank at most r will be called a *directed r-hypergraph*.

We use the following notation for directed hypergraphs:

$$\begin{aligned}\Delta_D^-(X) &= \{e \in \mathcal{A} :\ e \text{ enters } X\},\\ \Delta_D^+(X) &= \{e \in \mathcal{A} :\ e \text{ leaves } X\},\\ \varrho_D(X) &= |\Delta_D^-(X)|,\\ \delta_D(X) &= |\Delta_D^+(X)|.\end{aligned}$$

Analogously to undirected hypergraphs, a directed hypergraph $D = (V, \mathcal{A})$ is said to *cover* a set function $p : 2^V \to \mathbb{R} \cup \{-\infty\}$ if $\varrho_D(X) \geq p(X)$ for every $X \subseteq V$. If we have a function $x : \mathcal{A} \to \mathbb{R}$ on the hyperarc set, then we define

$$\varrho_x(Z) = \sum_{e \in \Delta_D^-(Z)} x(e) \quad \text{for } Z \subseteq V.$$

We say that x covers p if $\varrho_x(Z) \geq p(Z)$ for every $Z \subseteq V$.

Given an (undirected) hypergraph $H = (V, \mathcal{E})$, an *orientation* of H is a directed hypergraph obtained by assigning a head node to each hyperedge in $\mathcal{E}$.

6.2 Semimodular Frameworks with TDI Describing Systems

6.2.1 G-polymatroids

Let S be a finite ground set. A large part of the survey deals with the properties of set functions of type $f : 2^S \to \mathbb{R} \cup \{-\infty, +\infty\}$. Unless explicitly stated otherwise, we always assume that $f(\emptyset) = 0$.

Given a function $m : S \to \mathbb{R}$, we can define a set function by $m(X) = \sum_{v \in X} m(v)$ for $X \subseteq S$ (we denote this associated set function by the same letter as the original function). Set functions obtained this way are called *modular*.

A set function $b : 2^S \to \mathbb{R} \cup \{\infty\}$ is called *submodular* if the submodular inequality

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \tag{2}$$

holds for every pair $X, Y \subseteq S$ of subsets. A non-decreasing submodular function is called a *polymatroid function*. A set function $p : 2^S \to \mathbb{R} \cup \{-\infty\}$ is *supermodular* if $-p$ is submodular, i.e. the supermodular inequality

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \tag{3}$$

holds for every pair $X, Y \subseteq S$ of subsets. It is an easy observation that a finite-valued supermodular function is non-negative if and only if it is non-decreasing.

A pair (p, b) of set functions is called *paramodular* if p is supermodular, b is submodular, and they are *compliant* in the sense that the *cross-inequality*

$$b(X) - p(Y) \geq b(X - Y) - p(Y - X) \tag{4}$$

holds for every pair of subsets $X, Y \subseteq S$. Paramodular pairs appear naturally in many contexts. For example, the following is true.

Proposition 2.1. *If r is the rank function of a matroid and r' is its co-rank function, then (r', r) is a paramodular pair.*

We can state this a bit more generally. For a set function $f : 2^S \to \mathbb{R} \cup \{-\infty, +\infty\}$ with $f(S)$ finite, we define a kind of complementary set function $\overline{f}$ on the same ground set by $\overline{f}(X) = f(S) - f(S - X)$. If r is a matroid rank function, then $\overline{r}$ is the co-rank function. The above proposition can be generalized as follows:

Proposition 2.2. *If f is a supermodular function, then $\overline{f}$ is submodular, and $(f, \overline{f})$ is a paramodular pair.*

As an example, consider a hypergraph $H = (V, \mathcal{E})$. The set function $f(X) := i_H(X)$ is supermodular, and $\overline{f}$ is the set function $e_H(X)$. So (i_H, e_H) is a paramodular pair.

For set functions $p : 2^S \to \mathbb{R} \cup \{-\infty\}$ and $b : 2^S \to \mathbb{R} \cup \{\infty\}$ we define the following polyhedra:

$$\begin{aligned}
P(b) &:= \{x \in \mathbb{R}^S : x \geq 0, x(Z) \leq b(Z) \text{ for every } Z \subseteq S\},\\
S(b) &:= \{x \in \mathbb{R}^S : x(Z) \leq b(Z) \text{ for every } Z \subseteq S\},\\
B(b) &:= \{x \in \mathbb{R}^S : x(Z) \leq b(Z) \text{ for every } Z \subseteq S,\ x(S) = b(S)\},\\
C(p) &:= \{x \in \mathbb{R}^S : x \geq 0, x(Z) \geq p(Z) \text{ for every } Z \subseteq S\},\\
S'(p) &:= \{x \in \mathbb{R}^S : x(Z) \geq p(Z) \text{ for every } Z \subseteq S\},\\
B'(p) &:= \{x \in \mathbb{R}^S : x(Z) \geq p(Z) \text{ for every } Z \subseteq S,\ x(S) = p(S)\},\\
Q(p, b) &:= \{x \in \mathbb{R}^S : p(Z) \leq x(Z) \leq b(Z) \text{ for every } Z \subseteq S\}.
\end{aligned}$$

If b is a polymatroid function, then $P(b)$ is called a *polymatroid*, and b is its *border function*. For a submodular function b, the polyhedron $S(b)$ is the *submodular polyhedron* of b and $B(b)$ is the *base polyhedron* of b. If p is a non-negative supermodular function, then $C(p)$ is called a *contra-polymatroid*, and p its border function. If p, b is a paramodular pair then $Q(p.b)$ is called a *generalized polymatroid* or *g-polymatroid* for short, and p and b are respectively the *lower* and *upper border functions*.

Proposition 2.3. *If p is supermodular, then $B'(p) = B(\overline{p})$, so $B'(p)$ is the base polyhedron of the submodular function $\overline{p}$.*

For technical reasons, we extend the above classes of polyhedra to include the empty polyhedron. However, the following is true.

Lemma 2.4. *Polymatroids, contra-polymatroids, base polyhedra and g-polymatroids defined by border functions are never empty. Moreover, their defining border function (or pair of border functions) is unique, i.e. different border functions give different polyhedra.*

A crucial property of these polyhedra is that if the border functions are integer-valued then the polyhedron is automatically integer. The theoretical background is that the linear systems defining these polyhedra are totally dual integral.

Theorem 2.5 (Frank 1981; Frank and Tardos 1988). *A g-polymatroid is integer if and only if its upper and lower border functions are integer.*

As an example of g-polymatroids which are not base polyhedra, consider a hypergraph $H = (V, \mathcal{E})$, and a subset of nodes $U \subseteq V$. Let the polyhedron $P \subseteq \mathbb{R}^U$ be the convex hull of the in-degree vectors on U of all possible orientations of the hypergraph H.

Proposition 2.6. *The polyhedron P is a g-polymatroid. The lower border function of P is i_H restricted to U, and the upper border function of P is e_H restricted to U.*

Note that if $U = V$, then P is a base polyhedron.

Operations on g-Polymatroids

An important and extremely useful property of the class of g-polymatroids is that it is closed under several natural operations. Moreover, the subclass of integer g-polymatroids is also closed under these operations, with some obvious restrictions. Here we present a brief summary of these properties.

Let $f : S \to \mathbb{Z} \cup \{-\infty\}$ and $g : S \to \mathbb{Z} \cup \{\infty\}$ be functions with $f \leq g$. The polyhedron $T(f, g) := \{x \in \mathbb{R}^S : f \leq x \leq g\}$ is called a *box*. For two numbers $\alpha \leq \beta$, the polyhedron $K(\alpha, \beta) = \{x \in \mathbb{R}^S : \alpha \leq x(S) \leq \beta\}$ is called a *plank*.

Proposition 2.7. *Any box or plank is a g-polymatroid.*

We may define several operations on polyhedra, or specifically on g-polymatroids. The operations *direct sum*, *translation*, *reflection* and *face* have the usual meaning. The *projection* of a g-polymatroid along a subset $T \subsetneq S$ is a polyhedron in $\mathbb{R}^{S-T}$ obtained by removing the components corresponding to T from each vector in the g-polymatroid. The *sum* of g-polymatroids is the Minkowski sum.

Given a surjective function $\varphi : S \to S'$ and a vector $x \in \mathbb{R}^S$, we define a vector $x_\varphi \in \mathbb{R}^{S'}$ by $x_\varphi(s') = x(\varphi^{-1}(s'))$ $(s' \in S')$. The *aggregate* of a polyhedron $P \in \mathbb{R}^S$ with respect to φ is $P_\varphi = \{x_\varphi : x \in P\}$. We also define the *aggregate* of a set function $f : 2^S \to \mathbb{R} \cup \{-\infty, +\infty\}$ by $f_\varphi(X) = f(\varphi^{-1}(X))$.

Theorem 2.8. *The class of g-polymatroids is closed under the following operations*:

- *direct sum*
- *translation*
- *reflection through a point*
- *projection along a subset*
- *aggregate*
- *sum*
- *face*
- *intersection with a box*
- *intersection with a plank*

In all of these cases, if the original g-polymatroid is integer and the operation involves integer numbers, then the resulting g-polymatroid is integer too.

The upper and lower border functions of sums and aggregates can be explicitly given.

Theorem 2.9. *Let (p, b) be a paramodular pair on S, and let $\varphi : S \to S'$ be a surjective function. Then (p_φ, b_φ) is a paramodular pair on S', and $Q(p, b)_\varphi = Q(p_\varphi, b_\varphi)$. If p and b are integral, then for each integer vector x in $Q(p, b)_\varphi$ there is an integer vector $y \in Q(p, b)$ such that $y_\varphi = x$.*

Theorem 2.10. *Let $(p_1, b_1), \dots, (p_k, b_k)$ be paramodular pairs. The sum of the g-polymatroids $Q(p_1, b_1), \dots, Q(p_k, b_k)$ is the g-polymatroid given by $Q(\sum_{i=1}^k p_i, \sum_{i=1}^k b_i)$. If all functions p_i and b_i are integral, then each integer vector x in $Q(\sum_{i=1}^k p_i, \sum_{i=1}^k b_i)$ can be written as $x = x^1 + \dots + x^k$, where x^i is an integer vector in $Q(p_i, b_i)$.*

Base polyhedra are special cases of g-polymatroids, so projections of base polyhedra are g-polymatroids. The following theorem states that every g-polymatroid arises this way.

Theorem 2.11. *A g-polymatroid $Q(p, b)$ defined by a paramodular pair (p, b) arises as the projection of a base polyhedron along one element, namely, $Q(p, b)$ is the projection of $B(b^*)$ along s^* where*

$$b^*(X) = \begin{cases} b(X) & \text{if } X \subseteq S, \\ -p(S - X) & \text{if } s^* \in X. \end{cases} \tag{5}$$

The intersection of a g-polymatroid with a box can of course be empty. Luckily, the characterization of non-emptiness is quite simple and elegant, and it implies the *linking property* that we will use several times in the rest of the paper.

Theorem 2.12. *Let (p, b) be a paramodular pair, and $f : S \to \mathbb{Z} \cup \{-\infty\}$, $g : S \to \mathbb{Z} \cup \{\infty\}$ such that $f \le g$. Then the g-polymatroid $Q(p, b) \cap T(f, g)$ is non-empty if and only if the following two conditions hold*:

$$f(X) \le b(X) \quad \textit{for every } X \subseteq S, \tag{6}$$

$$g(X) \ge p(X) \quad \textit{for every } X \subseteq S. \tag{7}$$

Corollary 2.13 (Linking property). *Let p, b, f, g be as in Theorem 2.12. If there is an element x of $Q(p, b)$ with $x \ge f$ and there is an element y of $Q(p, b)$ with $y \le g$, then there is an element z of $Q(p, b)$ with $f \le z \le g$. In addition, if p, b, f, g are all integer-valued, then z can be integer too.*

It is worth formulating Theorem 2.12 for the special case of base-polyhedra.

Theorem 2.14. *For a submodular function b for which $b(S)$ is finite, the intersection $B(b) \cap T(f, g)$ is non-empty if and only if*

$$f(Y) \le b(Y) \quad \textit{for every } Y \subseteq S \quad \textit{and} \tag{8}$$

$$g(Y) \ge \overline{b}(Y) \quad \textit{for every } Y \subseteq S. \tag{9}$$

A stronger theorem characterizing non-emptiness of the intersection of a g-polymatroid with a box and a plank can also be derived, leading to the *strong linking property*.

Theorem 2.15. *Let (p, b) be a paramodular pair, and $f : S \to \mathbb{Z} \cup \{-\infty\}$, $g : S \to \mathbb{Z} \cup \{\infty\}$ such that $f \le g$. In addition, let $\alpha \le \beta$ be two numbers. Then the g-polymatroid $Q(p, b) \cap T(f, g) \cap K(\alpha, \beta)$ is non-empty if and only if the following four conditions hold*:

$$f(X) \le \min\{b(X), \beta - p(S - X)\} \quad \textit{for every } X \subseteq S, \tag{10}$$

$$g(X) \ge \max\{p(X), \alpha - b(S - X)\} \quad \textit{for every } X \subseteq S. \tag{11}$$

Proof. Let $M = Q(p, b) \cap T(f, g) \cap K(\alpha, \beta)$. If $x \in M$, then $f(X) \le x(X) \le b(X)$ and $f(X) \le x(X) = x(S) - x(S - X) \le \beta - p(S - X)$, from which the necessity of (10) follows. Furthermore, $g(X) \ge x(X) \ge p(X)$ and $g(X) \ge x(X) = x(S) - x(S - X) \ge \alpha - b(S - X)$, that is, (11) is also necessary.

To prove the sufficiency, we invoke Theorem 2.11 stating that $Q(p, b)$ arises as the projection of a base-polyhedron $B(b^*)$ where b^* is defined on $S^* = S + s^*$ by (5). Let f^* denote the extension of the f to S^* where $f^*(s^*) := -\beta$, and let g^* be the extension of g to S^* where $g^*(s^*) := -\alpha$.

We claim that f^* and b^* meet (8). Indeed, for $Y \subseteq S$ (10) implies $f^*(Y) = f(Y) \le b(Y) = b^*(Y)$ while in case $s^* \in Y$ we have $f^*(Y) = f(X) - \beta \le -p(S - X) = b^*(Y)$ for $X := Y - s^*$. Similarly, g^* and b^* meet (9), since for

$Y \subseteq S$ (11) implies $g^*(Y) = g(Y) \geq p(Y) = -b^*(S^* - Y) = \overline{b^*}(Y)$, while in case $s^* \in Y$ we have $g^*(Y) = g(X) - \alpha \geq -b(S - X) = -b^*(S - X) = \overline{b^*}(Y)$ for $X := Y - s^*$. Therefore Theorem 2.14 implies that there is an element $x^* \in B(b^*)$ and the restriction of x^* to S is in M by the construction. □

Corollary 2.16 (Strong linking property). *Let $p, b, f, g, \alpha, \beta$ be as in Theorem 2.15. If there is an element x of $Q(p, b)$ with $x \geq f$ and $x(S) \leq \beta$, and there is an element y of $Q(p, b)$ with $y \leq g$ and $y(S) \geq \alpha$, then there is an element z of $Q(p, b)$ with $f \leq z \leq g$ and $\alpha \leq z(S) \leq \beta$. In addition, if $p, b, f, g, \alpha, \beta$ are all integer-valued, then z can be integer too.*

Relaxations of Submodularity and Supermodularity

The results of the previous paragraphs raise the question whether there is a common reason behind the fact that g-polymatroids are closed under all these different operations. The answer is yes: the deeper reason is that g-polymatroids may be defined using set functions with weaker properties. This fact is very important in applications, where set functions with only these weaker properties show up regularly.

Two subsets X and Y of the ground set S are called *intersecting* if $X \cap Y \neq \emptyset$, $X - Y \neq \emptyset$, and $Y - X \neq \emptyset$. If in addition $S - (X \cup Y) \neq \emptyset$, then they are called *crossing*. A set function b is called *intersecting (crossing) submodular* if the submodular inequality (2) holds whenever X and Y are intersecting (crossing). The definitions are analogous for intersecting and crossing supermodular functions.

Let b be a set function on the ground set S. A subset $X \subseteq S$ is called *b-separable from below* (or *separable* for short) if X can be partitioned into at least two non-empty disjoint subsets $X_1, \ldots, X_t$ for which $\sum_{i=1}^{t} b(X_i) \leq b(X)$. A set function b is *near submodular* if the submodular inequality holds for all intersecting pairs of non-separable subsets. A set function p is *near supermodular* if $-p$ is near submodular. A pair of set functions (p, b) is a *near paramodular pair* if p is near supermodular, b is near submodular, and the cross-inequality (4) holds for all intersecting pairs of subsets (X, Y) where X is not b-separable from below and Y is not p-separable from above.

Theorem 2.17. *A polyhedron $Q = Q(p, b)$ defined by a near paramodular pair (p, b) is a g-polymatroid. If p and b are integral, then Q is an integer polyhedron.*

Theorem 2.18. *A g-polymatroid $Q = Q(p, b)$ defined by a near paramodular pair (p, b) is non-empty if and only if*

$$p\Big(\bigcup_{Z \in \mathcal{F}} Z\Big) \leq \sum_{Z \in \mathcal{F}} b(Z) \quad \textit{and} \quad \sum_{Z \in \mathcal{F}} p(Z) \leq b\Big(\bigcup_{Z \in \mathcal{F}} Z\Big) \tag{12}$$

for every sub-partition $\mathcal{F}$ of S.

As a special case, we may obtain Fujishige's theorem on the non-emptiness of base polyhedra.

Theorem 2.19 (Fujishige 1984). *Let b be a crossing submodular function with $b(S) = 0$. Then $B(b)$ is a base polyhedron, and it is non-empty if and only if*

$$\sum_{Z \in \mathcal{P}} b(Z) \geq 0 \quad \textit{and} \quad \sum_{Z \in \mathcal{P}} b(S - Z) \geq 0$$

for every partition $\mathcal{P}$ of S.

Proof. Let $s \in S$ be an arbitrary element, and let us define two new set functions:

$$b'(X) := \begin{cases} b(X) & \text{if } s \notin X, \\ 0 & \text{if } X = S, \\ \infty & \text{otherwise,} \end{cases}$$

$$p'(X) := \begin{cases} -b(S - X) & \text{if } s \notin X, \\ 0 & \text{if } X = S, \\ -\infty & \text{otherwise.} \end{cases}$$

It is easy to see that b' is intersecting submodular, p' is intersecting supermodular, and the cross-inequality holds for intersecting pairs. Furthermore, one can check that $B(b) = Q(p', b')$. By Theorem 2.18, we get exactly the non-emptiness conditions of the theorem. If $Q(p', b')$ is non-empty, then it has a lower border function p^* with $p^*(S) = 0$ and an upper border function b^* with $b^*(S) = 0$. It follows that $p^* \geq \overline{b^*}$ and $b^* \leq \overline{p^*}$, hence both inequalities must hold with equality, therefore $B(b) = B(b^*)$ is a base polyhedron. $\square$

Let $S(b)$ be a non-empty submodular polyhedron defined by a near submodular function b (this is a submodular polyhedron because of Theorem 2.17). We know that $S(b)$ is defined by a unique submodular border function b'. Similarly, if we consider the supermodular polyhedron $S'(p)$ for a near supermodular function p, it has a uniquely defined supermodular border function. The question arises whether we can define these two set functions in terms of p and b. This leads us to the notions of upper and lower truncation.

Given a near submodular function b, we define its *lower truncation* by

$$b^{\vee}(X) = \min\left\{ \sum_{Z \in \mathcal{P}} b(Z) : \mathcal{P} \text{ is a partition of } X \right\}.$$

Analogously, the *upper truncation* of a near supermodular function p is defined by

$$p^{\wedge}(X) = \max\left\{ \sum_{Z \in \mathcal{P}} p(Z) : \mathcal{P} \text{ is a partition of } X \right\}.$$

Theorem 2.20. *The lower truncation $b^{\vee}$ of a near submodular function is submodular, and the upper truncation $p^{\wedge}$ of a near supermodular function p is supermodular. Furthermore, $S(b) = S(b^{\vee})$ and $S'(p) = S'(p^{\wedge})$.*

We note that while $Q(p, b)$ is a g-polymatroid for a near paramodular pair (p, b), its lower and upper border functions are not necessarily $p^{\wedge}$ and $b^{\vee}$.

Orientations Covering Crossing Supermodular Functions

The first application we show concerns the orientation of hypergraphs. We assume that each hyperedge has size at least two. The following lemma establishes the link between orientations of a hypergraph and base polyhedra.

Lemma 2.21 (Orientation Lemma). *Given a hypergraph $H = (V, \mathcal{E})$ and an in-degree specification $m : V \to \mathbb{Z}_+$, there is an orientation D of H such that $\varrho_D(v) = m(v)$ for every $v \in V$ if and only if $m(V) = |\mathcal{E}|$ and $m(Z) \geq i_H(Z)$ for every $Z \subseteq V$.*

This lemma, which is relatively easy to prove, implies a rather strong result on orientations covering a crossing supermodular function.

Theorem 2.22 (Frank, Király and Király 2003a). *Let $H = (V, \mathcal{E})$ be a hypergraph, and p a non-negative crossing supermodular function on the node set with $p(V) = 0$. The in-degree vectors of orientations of H that cover p are the integer vectors in the base polyhedron $B'(p + i_H)$.*

Since $p + i_H$ is crossing supermodular, we can use Theorem 2.19 to derive a necessary and sufficient condition for the existence of an orientation covering p.

Theorem 2.23 (Frank, Király and Király 2003a). *Let $H = (V, \mathcal{E})$ be a hypergraph, and p a non-negative crossing supermodular function on the node set. There is an orientation of H covering p if and only if the following two conditions hold for every partition $\mathcal{P}$ of V:*

$$e_H(\mathcal{P}) \geq \sum_{Z \in \mathcal{P}} p(Z), \tag{13}$$

$$\sum_{e \in \mathcal{E}} (|\{Z \in \mathcal{P} : e \cap Z \neq \emptyset\}| - 1) \geq \sum_{Z \in \mathcal{P}} p(V - Z). \tag{14}$$

One can go even further by considering the intersection with a box.

Theorem 2.24. *Let $H = (V, \mathcal{E})$ be a hypergraph, p a non-negative crossing supermodular function on V, and $f : V \to \mathbb{Z}_+$, $g : V \to \mathbb{Z}_+$ lower and upper bounds such that $f \leq g$. There is an orientation of H covering p for which $f(v) \leq \varrho(v) \leq g(v)$ for every $v \in V$ if and only if the following two conditions hold for every sub-partition $\mathcal{F}$ of V:*

$$|\mathcal{E}| - \sum_{Z \in \mathcal{F}} i_H(Z) \geq f\Big(V - \bigcup_{Z \in \mathcal{F}} Z\Big) + \sum_{Z \in \mathcal{F}} p(Z), \tag{15}$$

$$\sum_{e \in \mathcal{E}} (|\{Z \in \mathcal{F} : e \cap Z \neq \emptyset\}| - 1) \geq \sum_{Z \in \mathcal{F}} p(V - Z) - g\Big(V - \bigcup_{Z \in \mathcal{F}} Z\Big). \tag{16}$$

Corollary 2.25 (Linking property for orientations). *Let $H = (V, \mathcal{E})$ be a hypergraph, p a non-negative crossing supermodular function on V, and $f : V \to \mathbb{Z}_+$, $g : V \to \mathbb{Z}_+$ lower and upper bounds such that $f \leq g$. If there is an orientation of H covering p for which $\varrho(v) \geq f(v)$ for every $v \in V$, and there is one for which $\varrho(v) \leq g(v)$ for every $v \in V$, then there is an orientation of H covering p for which $f(v) \leq \varrho(v) \leq g(v)$ for every $v \in V$.*

Packing Arborescences with Free Roots

One of the fundamental theorems in graph connectivity is Edmonds' disjoint arborescence theorem (Edmonds 1973). Here we state it in an equivalent form that allows multiple prescribed roots.

Theorem 2.26. *Let $D = (V, A)$ be a directed graph, and $m : V \to \mathbb{Z}_+$ an integer vector for which $m(V) = k$. There are k edge-disjoint spanning arborescences in D such that each $v \in V$ is the root of $m(v)$ arborescences if and only if*

$$\varrho_D(Z) \geq k - m(Z) \quad \textit{for every } \emptyset \neq Z \subseteq V.$$

By combining this theorem with the theory of submodular polyhedra, we can derive a generalization that allows lower and upper bounds on the number of roots at each node.

Theorem 2.27 (Frank 1978; Cai 1983). *Let $D = (V, A)$ be a directed graph, $f : V \to \mathbb{Z}_+$ and $g : V \to \mathbb{Z}_+$ lower and upper bounds for which $f \leq g$. There exist k edge-disjoint spanning arborescences in D such that each $v \in V$ is the root of at least $f(v)$ and at most $g(v)$ arborescences if and only if*

$$f(V) \leq k, \tag{17}$$

$$g(Z) \geq k - \varrho_D(Z) \quad \textit{for every } \emptyset \neq Z \subseteq V, \tag{18}$$

$$\sum_{Z \in \mathcal{F}} \varrho_D(Z) \geq k(|\mathcal{F}| - 1) + f\left(V - \bigcup_{Z \in \mathcal{F}} Z\right) \quad \textit{for every subpartition } \mathcal{F} \textit{ of } V. \tag{19}$$

The proof uses the fact that the set function $p(X) := k - \varrho(X)$ $(\emptyset \neq X \subseteq V)$ is intersecting supermodular, so we can consider the g-polymatroid $C(p) \cap T(f, g)$ and characterize its non-emptiness using Theorem 2.12.

Corollary 2.28. *A digraph $D = (V, A)$ has k edge-disjoint spanning arborescences if and only if*

$$\sum_{Z \in \mathcal{F}} \varrho_D(Z) \geq k(|\mathcal{F}| - 1)$$

for every subpartition $\mathcal{F}$ of V.

As in the case of orientations, the presence of a g-polymatroid implies a kind of linking property.

Corollary 2.29 (Linking property for arborescences). *If a digraph $D = (V, A)$ has k edge-disjoint spanning arborescences such that each node $v \in V$ is the root of at least $f(v)$ of them, and it has k edge-disjoint spanning arborescences so that each node $v \in V$ is the root of at most $g(v)$ of them (where $f(v) \leq g(v)$), then D has k edge-disjoint spanning arborescences where the number of arborescences rooted at v is between $f(v)$ and $g(v)$.*

6.2.2 Intersection of Two g-Polymatroids

While the intersection of two g-polymatroids is not necessarily a g-polymatroid, it still has very nice properties. A central result of combinatorial optimization is Edmonds' (poly)matroid intersection theorem (Edmonds 1970), which can be extended to g-polymatroids.

Theorem 2.30. *Let (p_1, b_1) and (p_2, b_2) be paramodular pairs. The linear system $\{x \in \mathbb{R}^S : \max\{p_1(Z), p_2(Z)\} \leq x(Z) \leq \min\{b_1(Z), b_2(Z)\} \ \forall Z \subseteq V\}$ is totally dual integral. If p_1, b_1, p_2, b_2 are all integral, then $Q(p_1, b_1) \cap Q(p_2, b_2)$ is an integer polyhedron.*

Theorem 2.31. *Let (p_1, b_1) and (p_2, b_2) be paramodular pairs. The polyhedron $P = Q(p_1, b_1) \cap Q(p_2, b_2)$ is non-empty if and only if*

$$p_1 \leq b_2 \quad \textit{and} \quad p_2 \leq b_1.$$

A corollary is the following theorem, which is a kind of discrete analogue of the theorem that a convex and a concave function can be separated by a linear function.

Corollary 2.32 (Discrete separation theorem). *Let $p : 2^S \to \mathbb{R} \cup \{-\infty\}$ be a supermodular function and $b : 2^S \to \mathbb{R} \cup \{\infty\}$ be a submodular function such that $p \leq b$. Then there is a modular function m that satisfies $p \leq m \leq b$. In addition, if p and b are integral, then m can be chosen to be integral too.*

To show a simple application, we may consider the problem of finding an orientation of a graph that covers two crossing supermodular set functions h_1 and h_2 simultaneously. To get the intersection of two g-polymatroids, we have to assume that one of them is non-negative. We state the result only in the case when the set functions are symmetric, otherwise the necessary and sufficient condition is much more complicated.

Theorem 2.33. *Let h_1 and h_2 be symmetric crossing supermodular set functions, and let us assume that h_1 is non-negative. A graph $G = (V, E)$ has an orientation covering $\max\{h_1, h_2\}$ if and only if $d_G(X) \geq 2\max\{h_1(X), h_2(X)\}$ for every $X \subseteq V$.*

Supermodular Colourings

A more involved application of g-polymatroid intersection is the supermodular colouring theorem of Schrijver (1985). Here we state it in a slightly generalized form that involves skew supermodular functions. A set function $p : 2^S \to \mathbb{R} \cup \{-\infty\}$ is *skew supermodular* if for any two subsets $X \subseteq S$ and $Y \subseteq S$ at least one of the following two inequalities holds:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y),$$
$$p(X) + p(Y) \leq p(X - Y) + p(Y - X).$$

If p is a skew supermodular function, and we raise $p(\{v\})$ to 0 on all elements where it is negative, we get a near supermodular function. This is the basis of the following result.

Theorem 2.34. *If p is a skew supermodular function, then $C(p)$ is a contra-polymatroid.*

A *k-colouring* of a ground set S is a partition $S_1, \dots, S_k$ of S, where we allow some members of the partition to be empty. The sets S_i are called the *colour classes*. Given a set function $p : 2^S \to \mathbb{Z}$, we say that a k-colouring $\{S_1, \dots, S_k\}$ *dominates* p if $|\{i : S_i \cap Z \neq \emptyset\}| \geq p(Z)$ for every $Z \subseteq S$.

Lemma 2.35. *Let p be a skew supermodular set function that satisfies $p(X) \leq \min\{k, |X|\}$ for every $X \subseteq S$. Then the polyhedron*

$$\{x \in [0,1]^S : x(Z) \geq 1 \textit{ if } p(Z) = k,\ x(Z) \leq |Z| - p(Z) + 1 \textit{ for every } Z \subseteq S\} \tag{20}$$

is an integer g-polymatroid.

Corollary 2.36. *The polyhedron defined by (20) is the convex hull of the possible colour classes of k-colourings that dominate p.*

A k-colouring of S is called *equitable* if the size of each colour class is $\lfloor |S|/k \rfloor$ or $\lceil |S|/k \rceil$. The following skew supermodular colouring theorem is a slight extension of the supermodular colouring theorem of Schrijver, and it can be proved by combining the g-polymatroid intersection theorem with Corollary 2.36.

Theorem 2.37. *Let k be a positive integer, p_1 and p_2 two skew supermodular functions such that $\max\{p_1(X), p_2(X)\} \leq \min\{k, |X|\}$ for every $X \subseteq S$. Then there is a k-colouring of S that dominates $\max\{p_1, p_2\}$. Moreover, the colouring may be chosen to be equitable.*

Interestingly, this theorem can be used to prove a result on hypergraphs covering symmetric skew supermodular functions. The key to the connection between the two topics is the following lemma.

Lemma 2.38 (Bernáth and Király 2008). *Let p be a symmetric skew supermodular function such that $\max\{p(Z) : Z \subseteq S\} = k$. If $\{S_1, \dots, S_k\}$ is a k-colouring that dominates p, then the hypergraph with edge set $\{S_1, \dots, S_k\}$ covers p.*

This means that with the aid of the skew supermodular colouring theorem we can find a hypergraph covering two symmetric skew supermodular functions simultaneously, provided that their maximal value is the same.

Corollary 2.39. *Let $p_1 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ and $p_2 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be two symmetric skew supermodular set functions such that* $\max\{p_1(X) : X \subseteq V\} = \max\{p_2(X) : X \subseteq V\} = k$*, and let $m : V \to \mathbb{Z}_+$ be a degree prescription such that $m(v) \le k$ for any $v \in V$ and*

$$\sum_{v \in X} m(v) \ge \max\{p_1(X), p_2(X)\} \quad \text{for every } X \subseteq V.$$

Then there exists a nearly uniform hypergraph H of exactly k hyperedges such that $d_H(v) = m(v)$ for every $v \in V$ and H covers both p_1 and p_2.

Corollary 2.40. *Let $p_1 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ and $p_2 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be two symmetric skew supermodular set functions such that* $\max\{p_1(X) : X \subseteq V\} = \max\{p_2(X) : X \subseteq V\} = k$*. Given the right oracles for p_1 and p_2, it is possible to find in polynomial time a hypergraph of minimum total size that covers both p_1 and p_2. In addition, the hypergraph can be chosen to be nearly uniform.*

As an application let us consider the simultaneous local edge-connectivity augmentation of two hypergraphs. For a hypergraph $H = (V, \mathcal{E})$ and nodes $u, v \in V$, $\lambda_H(u, v)$ denotes the size of the minimum cut that separates u and v. Given two hypergraphs H_1, H_2 on the same ground set V, two symmetric requirement functions $r_1, r_2 : V \times V \to \mathbb{Z}_+$, we want to find a hypergraph H of minimum total size such that $\lambda_{H_i+H}(u, v) \ge r_i(u, v)$ for any $u, v \in V$ and $i \in \{1, 2\}$. Corollary 2.40 implies that if we assume that $\max\{r_1(u, v) - \lambda_{H_1}(u, v) : u, v \in V\} = \max\{r_2(u, v) - \lambda_{H_2}(u, v) : u, v \in V\}$, then we can solve the problem in polynomial time. Furthermore, we can even achieve that the hyperedges in H are of almost equal size. It should be mentioned that without assuming the equality of the maximum deficiencies the problem is NP-hard.

6.2.3 Submodular Flows

In this section we briefly describe submodular flows and present some applications related to hypergraph orientations that cover supermodular functions. Since we concentrate on supermodularity as opposed to submodularity, we will define submodular flows in terms of supermodular functions. Due to the symmetry involved (by Proposition 2.3, base polyhedra may be defined by supermodular as well as submodular functions) the definition is equivalent to the original one.

Let $D = (V, A)$ be a directed graph, $f : A \to \mathbb{Z} \cup \{-\infty\}$, $g : A \to \mathbb{Z} \cup \{+\infty\}$ lower and upper bounds for which $f \le g$. Let $p : 2^V \to \mathbb{R} \cup \{-\infty\}$ be a crossing supermodular function with $p(V) = 0$. A function $x : A \to \mathbb{R}$ is called a *submodular flow* if

$$\lambda_x(Z) := \varrho_x(Z) - \delta_x(Z) \ge p(Z) \quad \text{for every } Z \subseteq V. \tag{21}$$

Equivalently, the vector $\lambda_x \in \mathbb{R}^V$ must be in the base polyhedron $B'(p)$. A submodular flow x is *feasible* if $f \leq x \leq g$. The set $Q(f, g; p)$ of feasible submodular flows is called a *submodular flow polyhedron*. The fundamental result of Edmonds and Giles (1977) is that the natural linear system describing the submodular flow polyhedron is TDI. The characterization of non-emptiness is also simple if p is supermodular.

Theorem 2.41 (Frank 1982). *Let $D = (V, A)$ be a digraph, p a supermodular function for which $p(V) = 0$, and $f \leq g$ lower and upper bounds on the edges. There exists a feasible submodular flow if and only if*

$$\varrho_g(Z) - \delta_f(Z) \geq p(Z) \quad \textit{for every } Z \subseteq V. \tag{22}$$

If p, f, g are integral, then the submodular flow can be chosen to be integer too.

If the set function is only crossing supermodular, then the condition of the existence of a feasible submodular flow is much more complicated. To describe it, we have to introduce the notion of a tree-composition of a subset of V.

Let Z be a subset of the ground set V. A tree-composition of Z is given using a partition $\{Z_1, \ldots, Z_k\}$ of Z, a partition $\{U_1, \ldots, U_l\}$ of $V - Z$, and a directed tree T with node set $\{z_1, \ldots, z_k\} \cup \{u_1, \ldots, u_l\}$ where every edge of the tree is of type $u_i z_j$. The associated *tree-composition* is a family of subsets of V with one set corresponding to each edge of T. To obtain the set corresponding to an edge $u_i z_j$, we take the union of all partition members that correspond to nodes reachable from z_j in the underlying undirected graph of $T - \{u_i z_j\}$.

By convention, a tree-composition of V is either a partition or a co-partition of V.

Theorem 2.42 (Frank 1993). *Let $D = (V, A)$ be a digraph, p a crossing supermodular function for which $p(V) = 0$, and $f \leq g$ lower and upper bounds on the edges. There exists a feasible submodular flow if and only if*

$$\varrho_g(X) - \delta_f(X) \geq \sum_{Z \in \mathcal{F}} p(Z) \tag{23}$$

for every subset $X \subseteq V$ and every tree-composition $\mathcal{F}$ of X. If p, f, g are integral, then the flow can be chosen to be integer too.

Submodular flow polyhedra are closely related to the intersection of two g-polymatroids.

Theorem 2.43 (Frank and Tardos 1988). *A polyhedron P is a submodular flow polyhedron if and only if it is a projection of the intersection of two g-polymatroids.*

Applications

Our first application comes from the simple observation that if $\delta_D(X) = 0$ whenever $p(X)$ is finite, then the submodular flow problem corresponds to the covering of p by a vector $x : A \to \mathbb{R}$ with $f \leq x \leq g$.

Theorem 2.44. *Let $D = (V, A)$ be a directed graph, let $p : 2^V \to \mathbb{R} \cup \{-\infty\}$ be crossing supermodular set function such that $\delta_D(X) = 0$ whenever $p(X)$ is finite and $p(V) = 0$. Let furthermore $g : A \to \mathbb{R}_+$ be an upper bound function. Then the linear system*

$$\{x \in \mathbb{R}^A : \ 0 \le x \le g, \ \varrho_x(Z) \ge p(Z) \text{ for every } Z \subseteq V\}$$

is totally dual integral. If p and g are integral, then the existence of a solution implies the existence of an integer solution.

Corollary 2.45. *Let $D = (V, A)$ be a directed graph whose underlying undirected graph is connected, and let $\mathcal{C}$ be a crossing family of directed cuts. The minimum number of edges in A that cover every cut in $\mathcal{C}$ equals the maximum number of edge-disjoint dicuts in $\mathcal{C}$.*

The next application concerns the orientation of hypergraphs. A *mixed hypergraph* is a triple $M = (V; \mathcal{E}, \mathcal{A})$, where $\mathcal{E}$ is a set of hyperedges and $\mathcal{A}$ is a set of hyperarcs. We can extend Theorem 2.24 to mixed hypergraphs using submodular flows.

Theorem 2.46. *Let $M = (V; \mathcal{E}, \mathcal{A})$ be a mixed hypergraph, and $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ a crossing supermodular set function. Suppose that a cost is assigned to each possible orientation of every hyperedge in $\mathcal{E}$. Then the problem of finding a minimum cost orientation of M covering p can be formulated as a submodular flow problem, solvable in polynomial time.*

6.2.4 Covering Intersecting Supermodular Functions

We start with a special case of Theorem 2.54 which was proved in Frank (1979).

Theorem 2.47. *Let $D = (V, A)$ be a digraph, p an intersecting supermodular function on V, and $g : A \to \mathbb{Z}_+$ a function on the edges for which $\varrho_g(X) \ge p(X)$ for every $X \subseteq V$. Then the linear system*

$$\{x \in \mathbb{R}^A : \ 0 \le x \le g, \ \varrho_x(Z) \ge p(Z) \text{ for every } Z \subseteq V\}$$

is totally dual integral.

Corollary 2.48. *The minimum number of (not necessarily distinct) edges of D covering an intersecting supermodular function p is equal to the maximum of $\sum_{X \in \mathcal{F}} p(X)$ over laminar families $\mathcal{F}$ which are independent in the sense that no edge of D enters more than one member of $\mathcal{F}$.*

This implies the Lucchesi–Younger theorem on dicut covers (Lucchesi and Younger 1978) if we consider the set function

$$p(X) = \begin{cases} \sigma(X) & \text{if } \emptyset \ne X \subseteq V \text{ and } \delta_D(X) = 0, \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma(X)$ denotes the number of undirected components of $V - X$.

Corollary 2.49 (Lucchesi–Younger). *In a digraph the minimum size of a dicut cover equals the maximum number of disjoint dicuts.*

We now show a generalization where upper bounds are given on the in-degrees of the nodes. In a digraph $D = (V, A)$ the *entrance* $\Gamma^-(X)$ of a set $X \subseteq V$ is the set

$$\{v \in X : \exists uv \in A,\ u \in V - X\}.$$

For a function $g : V \to \mathbb{Z}_+$ let $\beta_g(X) := \sum_{v \in \Gamma^-(X)} g(v)$. It can be shown that the function β_g is submodular, which is the basis of the following result, proved in Frank and Tardos (1989).

Theorem 2.50. *Let $D = (V, A)$ be a digraph, $g : V \to \mathbb{Z}_+$ a function on the nodes, and p an intersecting supermodular function. There is an integer-valued function $x : A \to \mathbb{Z}_+$ for which $\varrho_x(Z) \geq p(Z)$ for every $Z \subseteq V$ and $\varrho_x(v) \leq g(v)$ for every $v \in V$ if and only if*

$$p(X) \leq \beta_g(X) \tag{24}$$

for every $X \subseteq V$.

It should be noted that if we have bounds on the out-degrees of the nodes instead of the in-degrees, then the problem becomes NP-complete, since the Hamiltonian path problem can be formulated this way.

Applications

By combining Theorem 2.50 with Edmonds' disjoint arborescence theorem (Theorem 2.26) it is possible to prove the following result of Vidyasankar (1978).

Theorem 2.51 (Vidyasankar 1978). *Let s be a node of in-degree 0 in the digraph $D = (V, A)$. The edges of D can be covered by k arborescences of root s if and only if $\varrho(v) \leq k$ for every $v \in V - s$ and*

$$k - \varrho(X) \leq \sum_{v \in \Gamma^-(X)} (k - \varrho(v)) \tag{25}$$

for every $X \subseteq V - s$.

Another application of Theorem 2.50 is that we can add in-degree bounds in the Lucchesi–Younger theorem. A *branching* is an acyclic digraph where every node has in-degree at most one.

Theorem 2.52. *Let $D = (V, A)$ be a digraph. There is a branching whose edges cover every directed cut if and only if the deletion of any nonempty subset $X \subset V$ results in at most $|X|$ components with in-degree 0.*

The proof uses the fact that the function $\sigma(X)$ defined on the kernels (which denotes the number of components of $D - X$ in the undirected sense) is intersecting supermodular, so Theorem 2.50 can be applied with this and $g \equiv 1$.

The next application is a common generalization of Rado's theorem on matroids (Rado 1942) and a theorem of Lovász on superadditive intersecting supermodular functions (Lovász 1970). For an edge set F and a node set $X \subseteq V$, let $\Gamma_F(X)$ denote the set of nodes in $V - X$ that are joined by an edge in F to a node in X.

Theorem 2.53. *Let $G = (S, T; E)$ be a simple bipartite graph, $p : 2^S \to \mathbb{Z} \cup \{-\infty\}$ an intersecting supermodular function, and $g : S \to \mathbb{Z}_+$ an upper bound function on the nodes. Let furthermore $M = (T, r)$ be a matroid of rank function r on the ground set T. There exists an edge set $F \subseteq E$ for which*

$$r(\Gamma_F(X)) \geq p(X) \quad \textit{for every } X \subseteq S \quad \textit{and} \tag{26}$$

$$d_F(v) \leq g(v) \quad \textit{for every } v \in S \tag{27}$$

if and only if

$$p(X) \leq r(\Gamma_E(Y)) + g(X - Y) \quad \textit{for every } Y \subseteq X \subseteq S. \tag{28}$$

6.2.5 Oriented Subgraphs of Mixed Hypergraphs

It was observed by Khanna et al. (2005) (for graphs, extended to hypergraphs in Frank et al. 2003a) that if p is intersecting supermodular, then the mixed (hyper)graph orientation problem remains tractable if we allow not only the orientation of the undirected hyperedges, but also their deletion, or their replacement by multiple hyperarcs. The next theorem formulates this result in the most general setting, and then we present two special cases.

Theorem 2.54 (Khanna, Naor and Shepherd 2005; Frank, Király and Király 2003a). *Let $D = (V, \mathcal{A})$ be a directed hypergraph for which $\mathcal{A}$ can be partitioned into hyperarc sets $\mathcal{A}_1, \dots, \mathcal{A}_k$ where the underlying hyperedge of the hyperarcs in $\mathcal{A}_i$ is the same hyperedge e_i. Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be an intersecting supermodular set function. Let furthermore $f : \mathcal{A} \to \mathbb{Z} \cup \{-\infty\}$, $g : \mathcal{A} \to \mathbb{Z} \cup \{+\infty\}$ be lower and upper bounds for which $f \leq g$, and let $l_i \leq u_i$ $(i = 1, \dots, k)$. The linear system*

$$\{x \in \mathbb{R}^{\mathcal{A}} : \ f \leq x \leq g, \ l_i \leq x(\mathcal{A}_i) \leq u_i, \ \varrho_x(Z) \geq p(Z) \ \forall Z \subseteq V\}$$

is totally dual integral. The optimization problem over this polyhedron can be solved using submodular flows.

An *oriented sub-hypergraph* of a mixed hypergraph M is a sub-hypergraph of a directed hypergraph obtained from M by orienting the hyperedges in $\mathcal{E}$.

Corollary 2.55. *Let $M = (V; \mathcal{E}, \mathcal{A})$ be a mixed hypergraph, and $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ an intersecting supermodular set function. Suppose that a cost is assigned to each hyperarc in $\mathcal{A}$, and to each possible orientation of every hyperedge in $\mathcal{E}$. Then the problem of finding a minimum cost oriented sub-hypergraph of M covering p can be formulated as a submodular flow problem, solvable in polynomial time.*

The next result considers only graphs. A *partial orientation* of a mixed graph $M = (V; E, A)$ is a mixed graph obtained by orienting some edges in E. If a weight is assigned to both orientations of every edge in E, then the weight of a partial orientation is the sum of the weights of the oriented edges (so the un-oriented edges do not count).

Corollary 2.56. *Let $M = (V; E, A)$ be a mixed graph, and $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ an intersecting supermodular set function. Suppose that M covers p, and a weight is assigned to both orientations of every edge in E. Then the problem of finding a maximum weight partial orientation of M covering p can be formulated as a submodular flow problem, solvable in polynomial time.*

For crossing supermodular functions the above problem is NP-hard, even if the aim is to maximize the number of oriented edges.

6.2.6 Covering Intersecting Bi-set Functions

In this last section discussing TDI problems, we introduce a concept that will have an important role in the second part of the survey. The idea is to consider bi-set functions instead of set functions.

Let V be a ground set. A *bi-set* is a pair (X_O, X_I) where $X_I \subseteq X_O \subseteq V$. The set X_O is called the *outer member* of the bi-set, while X_I is called the *inner member*. A bi-set is called *trivial* if $X_O = V$ or $X_I = \emptyset$.

We say that a directed edge uv *enters* the bi-set $X = (X_O, X_I)$ if $u \in V - X_O$ and $v \in X_I$. Given a digraph $D = (V, A)$, $\varrho_D(X)$ denotes the number of edges entering the bi-set X. Two bi-sets are *independent* if their inner members are disjoint or their outer members are co-disjoint, i.e. if no edge can enter both. A family $\mathcal{I}$ of bi-sets is independent if its members are pairwise independent.

The *intersection* of two bi-sets $X = (X_O, X_I)$ and $Y = (Y_O, Y_I)$ is defined by $X \cap Y = (X_O \cap Y_O, X_I \cap Y_I)$. Analogously, the *union* of the two bi-sets is $X \cup Y = (X_O \cup Y_O, X_I \cup Y_I)$. We can consider the supermodular inequality for a bi-set function p:

$$p(X) + p(Y) \le p(X \cap Y) + p(X \cup Y). \tag{29}$$

We say that a bi-set function p is *intersecting supermodular* if the supermodular inequality (29) holds whenever $X \cap Y$ is non-trivial. The bi-set function p is *crossing supermodular* if the supermodular inequality (29) holds whenever $X \cap Y$ and $X \cup Y$ are non-trivial.

It turned out that Theorem 2.47 can be extended to bi-set functions.

Theorem 2.57 (Frank 2008). *Let $D = (V, A)$ be a digraph and let $g : A \to \mathbb{Z}_+$ be a capacity function. Let p be an intersecting supermodular bi-set function for which $\varrho_g(X) \ge p(X)$ for every bi-set X. Then the linear system*

$$\{x \in \mathbb{R}^A : \ 0 \le x \le g, \ \varrho_x(Z) \ge p(Z) \text{ for every bi-set } Z\}$$

is totally dual integral. The polyhedron described by the system is a submodular flow polyhedron.

Let us return for a moment to set functions. Let S be a ground set and $T \subseteq S$ a subset. A set function $p : 2^S \to \mathbb{Z} \cup \{-\infty\}$ is called *T-intersecting supermodular* if the supermodular inequality holds for any pair of subsets X, Y for which $X \cap Y \cap T \neq \emptyset$.

Given a T-intersecting supermodular function p, we can define an intersecting supermodular bi-set function by

$$p'(X_O, X_I) = \begin{cases} p(X_O) & \text{if } X_O \cap T \neq \emptyset \text{ and } X_I = X_O \cap T, \\ -\infty & \text{otherwise.} \end{cases}$$

Using this and Theorem 2.57 we can derive a theorem on covering T-intersecting supermodular functions.

Theorem 2.58. *Let $D = (V, A)$ be a digraph, let $g : A \to \mathbb{Z}_+$ be a capacity function, and let T be a subset of the nodes that contains the head of each edge in A. Let p be a T-intersecting supermodular function for which $\varrho_g(X) \geq p(X)$ for every set X. Then the linear system*

$$\{x \in \mathbb{R}^A : \ 0 \leq x \leq g, \ \varrho_x(Z) \geq p(Z) \textit{ for every } Z \subseteq V\}$$

is totally dual integral. The polyhedron described by the system is a submodular flow polyhedron.

Corollary 2.59. *Let $D = (V, A)$ be a digraph, and $r \in V$ a specified root node. Then the convex hull of the characteristic vectors of rooted k-connected subgraphs of D is a submodular flow polyhedron.*

6.3 Beyond Total Dual Integrality

In the second part of the survey, we discuss problems involving graphs and hypergraphs that do not fit into the TDI frameworks described in the previous sections. As it was mentioned in the Introduction, the origins of this area can be traced back to the paper of Eswaran and Tarjan on making a graph 2-edge-connected (Eswaran and Tarjan 1976), and the splitting-off theorem of Lovász (1979). In fact, the majority of the results presented here are some kind of generalization of the result of Eswaran and Tarjan, and the proofs rely on various extensions of the splitting-off method.

One characteristic that sets these problems apart from those discussed previously is that the minimum cost versions are hard to solve. Still, we will show that for some special types of cost functions the problems are tractable. The reason behind this phenomenon is that while the feasible graphs or hypergraphs cannot be described by polyhedral methods, their degree vectors often can be.

While the problems described in this part have some common features like the nice description of the feasible in-degree vectors, there is no unifying framework similar to total dual integrality for the problems in the previous part. One may hope that such a framework will be discovered in the future.

6.3.1 Covering Crossing Supermodular Bi-set Functions by Digraphs

Recall the definition of bi-sets in Sect. 6.2.6, where we have shown that the problem of covering an intersecting supermodular bi-set function with a directed subgraph of minimum cost is solvable. This is not true for crossing supermodular bi-set functions; however, it was proved in Frank and Jordán (1995) that it is still possible to find a minimum number of edges that cover the function.

Theorem 3.1 (Frank and Jordán 1995). *Let p be a crossing supermodular bi-set function that has value 0 on all trivial bi-sets. Then the minimum number of directed edges covering p equals to the maximum of $\sum_{X \in \mathcal{I}} p(X)$ on all possible independent families $\mathcal{I}$.*

Furthermore, while it is not possible to give a polyhedral description of the covering edge-sets, the in-degree vectors of these edge-sets form a familiar polyhedron.

Theorem 3.2. *Let p be a crossing supermodular bi-set function that has value 0 on all trivial bi-sets. The possible in-degree vectors of digraphs that cover p are the integer vectors in an integer contra-polymatroid.*

Another important property is that the in-degrees and the out-degrees can be chosen independently, which implies a linking theorem.

Theorem 3.3. *Let p be a crossing supermodular bi-set function that has value 0 on all trivial bi-sets, and let $g_{in} : V \to \mathbb{Z}_+$ and $g_{out} : V \to \mathbb{Z}_+$ be upper bounds of the in-degrees and out-degrees, respectively. If p can be covered by at most γ edges, it can be covered by a digraph whose in-degrees are bounded by g_{in}, and it can be covered by a digraph whose out-degrees are bounded by g_{out}, then p can be covered by a digraph that satisfies all these conditions simultaneously.*

Applications

First, let us consider the node-connectivity augmentation of directed graphs.

Lemma 3.4. *A digraph $D = (V, A)$ is k-connected if and only if $\varrho_D(X) + |X_O - X_I| \geq k$ for every non-trivial bi-set X.*

Let $D = (V, A)$ be a digraph, and let us define the bi-set function

$$p_D(X) = \begin{cases} k - \varrho_D(X) - |X_O - X_I| & \text{if } X \text{ is non-trivial,} \\ 0 & \text{otherwise.} \end{cases}$$

This bi-set function is crossing supermodular, which gives us the following theorem.

Theorem 3.5. *A directed graph $D = (V, A)$ can be made k-connected by adding at most γ new edges if and only if*

$$\sum_{X \in \mathcal{I}} p_D(X) \leq \gamma$$

holds for every family $\mathcal{I}$ of pairwise independent bi-sets.

The second application comes from a completely different area: matching theory. Given a bipartite graph $G = (S, T; E)$, a *t-matching* is a subgraph with maximum degree at most t. It is *$K_{t,t}$-free* if it does not contain $K_{t,t}$ as a subgraph. As a generalization of the square-free 2-matching problem, we would like to find a $K_{t,t}$-free t-matching with a maximum number of edges.

It is rather surprising that one can describe this problem as the covering of a bi-set function. The key idea is that the covering will contain the edges which are *not* in the t-matching.

Theorem 3.6 (Frank 2003). *The maximum number of edges in a $K_{t,t}$-free t-matching of a bipartite graph $G = (S, T; E)$ is equal to*

$$\min_{Z \subseteq V} (t|Z| + i_G(V - Z) - c_t(Z)),$$

where $c_t(Z)$ denotes the number of components of $G - Z$ that are isomorphic to $K_{t,t}$.

A different generalization of the square-free 2-matching problem can also be deduced from Theorem 3.1. A *bi-clique* in a bipartite graph $G = (S, T; E)$ is a subgraph that is a $K_{i,j}$ for some $i, j \geq 1$. The *size* of the bi-clique is $i + j$.

Theorem 3.7 (Frank 2003). *In a bipartite graph $G = (S, T; E)$ the maximum number of edges in a subgraph not containing bi-cliques of size more than t equals*

$$\min\left\{|E| - \sum_{B \in \mathcal{B}} (|V(B)| - t) : \ \mathcal{B} \textit{ is a family of edge-disjoint bi-cliques}\right\}.$$

The last application is the theorem of Győri on subpaths of a directed path (Győri 1984). Let P be a directed path, and let $\mathcal{P}$ be a family of subpaths of P. We say that a family of subpaths $\mathcal{G}$ is a *generator* of $\mathcal{P}$ if every member of $\mathcal{P}$ is the union of some members of $\mathcal{G}$. Let P_1 and P_2 be subpaths in $\mathcal{P}$, and let e_1 be an edge of P_1 and e_2 be an edge of P_2. The two path-edge pairs (P_1, e_1) and (P_2, e_2) are called *independent* if either there is no node that precedes e_1 in P_1 and precedes e_2 in P_2, or there is no node that follows e_1 in P_1 and follows e_2 in P_2.

Theorem 3.8 (Győri 1984). *Let P be a directed path, and let $\mathcal{P}$ be a family of subpaths of P. The minimum cardinality of a generator of $\mathcal{P}$ equals the maximum number of pairwise independent path-edge pairs in $\mathcal{P}$.*

This theorem follows from Theorem 3.1; in fact, that theorem implies that the characterization is also true for subpaths of a directed cycle, which does not follow from the original proof of Győri.

Theorem 3.8 has a surprising application in combinatorial geometry. Let us call a bounded region in the plane *angular* if its boundary is composed of a finite number of horizontal and vertical segments, and let us call it *vertically convex* if every vertical line intersects it in a segment.

Corollary 3.9. *Let R be an angular, vertically convex region of the plane. The minimum number of angular rectangles whose union is R equals the maximum number of points of R such that no two of them can be covered by a single angular rectangle in R.*

6.3.2 Covering Two Supermodular Bi-set Functions

If our bi-set functions are not only crossing supermodular, but supermodular, then it is possible to simultaneously cover two functions by a minimum number of directed edges. More precisely, the following has been proved in Bérczi and Frank (2008).

Theorem 3.10. *Let p_1 and p_2 be two supermodular bi-set functions on a ground set V. The minimum number of directed edges that cover both p_1 and p_2 equals*

$$\max\{p_1(X) + p_2(Y) : \ X \textit{ and } Y \textit{ are independent bi-sets}\}.$$

Note that the analogue of Theorem 2.57 is not true: the linear system

$$\{x \in \mathbb{R}^A : \ 0 \le x \le g, \ \varrho_x(Z) \ge \max\{p_1(Z), p_2(Z)\} \text{ for every bi-set } Z\}$$

is not necessarily TDI. Theorem 3.2 does not carry over to this case either: the in-degrees of covering digraphs do not necessarily form the integer vectors of a contra-polymatroid.

6.3.3 Bi-set Extension of Edmonds' Disjoint Arborescence Theorem

In this section we show an abstract generalization of Edmonds' disjoint arborescence theorem that involves families of bi-sets (a weaker generalization involving families of sets was proved by Szegő 2001). Let V be a ground set and let $\mathcal{F}_1, \dots, \mathcal{F}_k$ be intersecting bi-set families that satisfy the following *mixed intersection property*:

$$\text{If } X \in \mathcal{F}_i \text{ and } Y \in \mathcal{F}_j \text{ are intersecting bi-sets, then } X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j. \tag{30}$$

We say that a set A of directed edges *covers* a family of bi-sets if for every member of the family there is an edge that enters both the inner and outer sets of that member. The following theorem is proved in Bérczi and Frank (2008).

Theorem 3.11. *Let $D = (V, A)$ be a digraph, and let $\mathcal{F}_1, \dots, \mathcal{F}_k$ be intersecting bi-set families that satisfy* (30)*. The edge set A can be partitioned into k parts $F_1, \dots, F_k$ such that F_i covers $\mathcal{F}_i$ if and only if*

$$\varrho_D(X) \geq |\{i : X \in \mathcal{F}_i\}| \quad \textit{for every bi-set } X.$$

This implies a nice result of Kamiyama et al. (2008) where instead of spanning arborescences one is interested in arborescences that contain each node reachable from their prescribed root.

Theorem 3.12. *Let $D = (V, A)$ be a directed graph and let $R = \{r_1, \dots, r_k\}$ be a set of k distinct roots. Let V_i denote the set of nodes reachable from r_i in D. There exist edge-disjoint arborescences $A_i \subseteq A$ $(i = 1, \dots, k)$ such that A_i is an r_i-arborescence spanning V_i if and only if*

$$\varrho_D(X) \geq |\{i : r_i \notin X, V_i \cap X \neq \emptyset\}| \quad \textit{for every } X \subseteq V.$$

We give only an indication how this follows from Theorem 3.11. Call two nodes u and v in V *equivalent* if they are not separated by any V_i. A *subatom* is a subset of equivalent nodes. Let us define the bi-set families $\mathcal{F}_i$ $(i = 1, \dots, k)$ as follows. For each non-empty subatom $X \subseteq V_i - r_i$ and each subset $Y \subseteq V - V_i$, let the bi-set $(X \cup Y, X)$ be a member of $\mathcal{F}_i$. We can apply Theorem 3.11 to this bi-set family to obtain Theorem 3.12.

6.3.4 Covering Crossing Supermodular Functions by Directed Hypergraphs

We return to problems involving set functions, and consider their covering by hyperarcs. First we describe an abstract form of the splitting-off operation. Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing supermodular set function. Let furthermore $g_i : V \to \mathbb{Z}_+$ be an indegree-bound and $g_o : V \to \mathbb{Z}_+$ an outdegree-bound for which $g_i(V) \leq g_o(V)$. Suppose that $g_i(X) \geq p(X)$ and $g_o(V - X) \geq p(X)$ for every $X \subseteq V$. We define the *splitting-off operation* as follows. A hyperarc e can be *split off* from (p, g_i, g_o) if $g_i(h_e) > 0$ and $g_o(v) > 0$ for every $v \in e - h_e$. For such a hyperarc let

$$g_i^e(v) := \begin{cases} g_i(v) - 1 & \text{if } v = h_e, \\ g_i(v) & \text{otherwise,} \end{cases}$$

$$g_o^e(v) := \begin{cases} g_o(v) - 1 & \text{if } v \in e - h_e, \\ g_o(v) & \text{otherwise,} \end{cases}$$

$$p^e(X) := \begin{cases} p(X) - 1 & \text{if } e \text{ enters } X, \\ p(X) & \text{otherwise.} \end{cases}$$

The splitting-off operation is *feasible* if $g_i^e(X) \geq p^e(X)$ and $g_o^e(V - X) \geq p^e(X)$ for every $X \subseteq V$. The operation is called a *feasible r-splitting* if e is an r-hyperarc. It can be verified that p^e is crossing supermodular.

The following theorem from Király and Makai (2007) describes conditions when a feasible splitting-off is available (the special case corresponding to k-edge-connectivity augmentation was proved in Berg et al. (2003)).

Theorem 3.13. *Let p be a crossing supermodular function, $g_i : V \to \mathbb{Z}_+$ and $g_o : V \to \mathbb{Z}_+$ degree bounds such that $g_i(V) \le g_o(V) \le r g_i(V)$ for some integer r, and*

$$g_i(X) \ge p(X) \quad \text{for every } X \subseteq V, \tag{31}$$

$$g_o(V - X) \ge p(X) \quad \text{for every } X \subseteq V. \tag{32}$$

Let $u \in V$ be such that $g_i(u) > 0$. Then there is a hyperarc e with $h_e = u$ and $|e| \le r + 1$ that can be feasibly split off.

The splitting-off operation has the following nice property:

Lemma 3.14. *If for some $r_1 > r > r_2$ there is a feasible r_1-spitting and a feasible r_2-splitting with head u, then there is a feasible r-splitting with head u.*

Using this, we can prove a theorem on the existence of a degree-bounded directed hypergraph covering a crossing supermodular set function.

Theorem 3.15 (Király and Makai 2007). *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing supermodular set function, $g_i : V \to \mathbb{Z}_+$ and $g_o : V \to \mathbb{Z}_+$ degree bounds such that $g_i(V) \le g_o(V) \le r g_i(V)$ for some positive integer r, and*

$$g_i(X) \ge p(X) \quad \text{for every } X \subseteq V, \tag{33}$$

$$g_o(V - X) \ge p(X) \quad \text{for every } X \subseteq V. \tag{34}$$

Then there is a directed $(r+1)$-hypergraph D such that $\delta_D(v) \le g_o(v)$ and $\varrho_D(v) \le g_i(v)$ for every $v \in V$, and

$$\varrho_D(X) \ge p(X) \quad \text{for every } X \subseteq V.$$

Proof. We can assume that $r = \lceil g_o(V)/g_i(V) \rceil$. According to Theorem 3.13 we can obtain a directed hypergraph D^* by successive feasible splitting-off operations such that $\delta_{D^*}(v) \le g_o(v)$, $\varrho_{D^*}(v) \le g_i(v)$ for every $v \in V$, and $\varrho_{D^*}(X) \ge p(X)$ for every $X \subseteq V$. Suppose that D^* contains a hyperarc e of size $r_1 > r + 1$. This means that there is a feasible r_1-splitting with head h_e for some $r_1 > r + 1$. However, we know by Theorem 3.13 that there is also a feasible r_2-splitting with head h_e for some $r_2 \le r + 1$ since $g_o(V) \le r g_i(V)$.

So Lemma 3.14 implies that there is an $(r + 1)$-hyperarc e that can be feasibly split off. Since $g_i^e(V) \le g_o^e(V) \le r g_i^e(V)$, we can continue the splitting-off process, until we obtain a directed $(r + 1)$-hypergraph that covers p. □

Theorem 3.15 implies that both the feasible in-degree bounds and the feasible out-degree bounds can be described as the integer points of a contra-polymatroid. This means that we can efficiently find directed hypergraphs covering a crossing supermodular set function p with

- minimum total size,
- minimum number of hyperedges, all of bounded size,
- given total size and given number of hyperedges.

Theorem 3.16. *Let p be a crossing supermodular function, let $r \geq 2$ be an integer, and let γ denote the minimum number of r-hyperarcs that cover p. Let p^γ be the set function defined by*

$$p^\gamma(X) = \begin{cases} \gamma & \text{if } X = V, \\ p(X) & \text{otherwise.} \end{cases}$$

Then the in-degree vectors of directed r-hypergraphs that cover p are exactly the integer vectors in the contra-polymatroid $C(p^\gamma)$.

Since a feasible in-degree vector and a feasible out-degree vector can be chosen independently, we can obtain a linking result.

Corollary 3.17. *Let p be a crossing supermodular set function, and let $g_i : V \to \mathbb{Z}_+$ and $g_o : V \to \mathbb{Z}_+$ be upper bounds of the in-degrees and out-degrees, respectively. If p can be covered by at most γ r-hyperarcs, it can be covered by a directed r-hypergraph whose in-degrees are bounded by g_i, and it can be covered by a directed r-hypergraph whose out-degrees are bounded by g_o, then p can be covered by a directed r-hypergraph that satisfies all these conditions simultaneously.*

6.3.5 Combined Augmentation and Orientation

We have seen that for a crossing supermodular function both the problem of covering it with a minimum number of hyperarcs and the problem of orienting a hypergraph to cover it can be solved. In this section we present a combination of these two problems. Let p be a crossing supermodular function on the ground set V. Suppose we have a hypergraph that cannot be oriented to cover it; what is the minimum number (or total size) of hyperedges that have to be added in order to have a good orientation?

We first show a characterization for the degree-bounded version of the problem, taken from Frank and Király (2003) and Király and Makai (2007).

Theorem 3.18. *Let $H_0 = (V, \mathcal{E}_0)$ be a hypergraph, $p : 2^V \to \mathbb{Z}_+$ a monotone decreasing or symmetric non-negative crossing supermodular set function, $g : V \to \mathbb{Z}_+$ a degree bound function and $0 \leq \gamma \leq g(V)/2$ an integer. There exists a hypergraph H with γ hyperedges satisfying the degree bounds such that $H_0 + H$ has an orientation covering p if and only if the following hold for every partition $\mathcal{P}$ of V:*

$$\gamma \geq \sum_{Z \in \mathcal{P}} p(Z) - e_{H_0}(\mathcal{P}), \tag{35}$$

$$\min_{X \in \mathcal{P}} g(V - X) \geq \sum_{Z \in \mathcal{P}} p(Z) - e_{H_0}(\mathcal{P}). \tag{36}$$

In addition, the rank of H can be bounded by $\lceil g(V)/\gamma \rceil$.

Proof. We only show the main ideas behind the proof. Suppose that (35) and (36) are satisfied. We extend the hypergraph H_0 by adding a new node z and for every $v \in V$ adding $g(v)$ parallel edges between v and z. It can be shown using Theorem 2.24 that this hypergraph has an orientation D^* so that $\min\{\varrho_{D^*}(X), \varrho_{D^*}(X+z)\} \geq p(X)$ for every $X \subseteq V$, and $\varrho_{D^*}(V) = \gamma$ (the property that p is monotone decreasing or symmetric implies that the conditions of Theorem 2.24 are satisfied).

For $v \in V$ let $g_i(v)$ be the number of new edges oriented towards v, and let $g_o(v)$ be the number of new edges oriented away from v. Let D_0 denote the orientation of H_0 induced by D^*. The construction implies that $g_i(v) + g_o(v) = g(v)$ for every $v \in V$, $g_i(V) = \gamma$, $g_i(X) \geq p(X) - \varrho_{D_0}(X)$ for every $X \subseteq V$ and $g_o(V - X) \geq p(X) - \varrho_{D_0}(X)$ for every $X \subseteq V$. Hence we can use Theorem 3.15 to obtain a directed hypergraph D of γ hyperarcs such that

- $\varrho_D(X) \geq p(X) - \varrho_{D_0}(X)$ for every $X \subseteq V$,
- $\varrho_D(v) \leq g_i(v)$ for every $v \in V$,
- $\delta_D(v) \leq g_o(v)$ for every $v \in V$.

Let H be the underlying undirected hypergraph of D. Then H has γ hyperedges, it satisfies the degree bound g, and $H_0 + H$ has an orientation that covers p. This means that H satisfies the conditions of the theorem. □

As in the previous sections, the characterization of the degree bounds that allow a good augmentation helps to deduce a characterization of the minimum number (or minimum total size) of hyperedges needed. Again, this follows from the fact that the feasible degree-specifications are the integer points of a contra-polymatroid (but in this case the proof is more involved). To formulate the theorem, we have to extend the definition of $e_H(\mathcal{F})$ to any family of sets: given a hypergraph $H = (V, \mathcal{E})$ and a family $\mathcal{F}$ of node sets let

$$e_H(\mathcal{F}) = \sum_{e \in \mathcal{E}} \max_{u \in e} |\{X \in \mathcal{F} : \ u \in X, e \not\subseteq X\}|.$$

Theorem 3.19. *Let $H_0 = (V, \mathcal{E}_0)$ be a hypergraph, $p : 2^V \to \mathbb{Z}_+$ a monotone decreasing or symmetric non-negative crossing supermodular set function, $\sigma \geq 0$ and $0 \leq \gamma \leq \sigma/2$ integers. There exists a hypergraph H with γ hyperedges of total size at most σ such that $H_0 + H$ has an orientation covering p if and only if the following hold:*

$$\gamma \geq \sum_{Z \in \mathcal{P}} p(Z) - e_{H_0}(\mathcal{P}) \quad \textit{for every partition } \mathcal{P},$$

$$\sigma \geq \sum_{Z \in \mathcal{F}} p(Z) - e_{H_0}(\mathcal{F})$$

whenever $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ where $\mathcal{F}_1$ is a partition of some $X \subseteq V$ and $\mathcal{F}_2$ is obtained by complementing the members of a partition of X that is coarser than $\mathcal{F}_1$. In addition, the rank of H can be bounded by $\lceil \sigma/\gamma \rceil$.

Application

A hypergraph H is *k-partition-connected* if for every t, one has to delete at least kt hyperedges to dismantle it into $t+1$ components. Equivalently, $e_H(\mathcal{P}) \geq k(|\mathcal{P}|-1)$ for every partition $\mathcal{P}$.

For non-negative integers k and l, we say that a hypergraph is (k,l)*-partition-connected* if $e_H(\mathcal{P}) \geq k(|\mathcal{P}|-1)+l$ for every nontrivial partition $\mathcal{P}$. Clearly, $(k,0)$-partition-connectivity is equivalent to k-partition-connectivity, and $(0,l)$-partition-connectivity is equivalent to l-edge-connectivity.

As an application of Theorem 3.19 we solve (k,l)-partition-connectivity augmentation problems if $k \geq l$. The following characterization, proved in Frank et al. (2003a), is at the heart of our approach. A directed hypergraph $D=(V,A)$ is called (k,l)*-edge-connected* from root $r \in V$ if $\varrho_D(X) \geq k$ and $\delta_D(X) \geq l$ for every $\emptyset \neq X \subseteq V-r$.

Theorem 3.20. *Let $k \geq l$ be non-negative integers. A hypergraph is (k,l)-partition-connected if and only if it has a (k,l)-edge-connected orientation (from any root).*

Theorems 3.19 and 3.20 imply the following on (k,l)-partition-connectivity augmentation if $k \geq l$:

Corollary 3.21. *Let $H_0=(V,\mathcal{E}_0)$ be a hypergraph, $\sigma \geq 0$, $0 \leq \gamma \leq \sigma/2$, and $k \geq l$ non-negative integers. There is a hypergraph H with γ hyperedges of total size at most σ such that H_0+H is (k,l)-partition-connected if and only if the following two conditions are met*:

(i) $\gamma \geq (|\mathcal{P}|-1)k+l-e_{H_0}(\mathcal{P})$ *for every nontrivial partition* $\mathcal{P}$,
(ii) $\sigma \geq |\mathcal{F}_1|k+|\mathcal{F}_2|l-e_{H_0}(\mathcal{F}_1+\mathcal{F}_2)$ *whenever $\mathcal{F}_1$ is a partition of some $X \subseteq V$ and $\mathcal{F}_2$ is obtained by complementing the members of a partition of X that is coarser than $\mathcal{F}_1$.*

In addition, the rank of H can be bounded by $\lceil \sigma/\gamma \rceil$.

6.3.6 Covering Symmetric Crossing Supermodular Functions by Uniform Hypergraphs

In the previous section we showed how to solve (k,l)-partition-connectivity augmentation when $k \geq l$. In this section we deal with the case when $k=0$, i.e. l-edge-connectivity augmentation. Very little is known about (k,l)-partition-connectivity augmentation of hypergraphs when $0<k<l$ (note that this case is irrelevant for graphs, since there it is equivalent to $(k+l)$-edge-connectivity).

The generalization considered in this section is the covering of symmetric crossing supermodular functions. Let $p: 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric crossing supermodular set function. Let $r \geq 2$ be an integer, and $g: V \to \mathbb{Z}_+$ a degree bound function such that r divides $g(V)$. First we consider the problem of finding an r-hypergraph of $g(V)/r$ hyperedges satisfying the degree bound that covers the set function p.

We call a partition $\mathcal{P} = \{V_1, \dots, V_l\}$ *p-full* if $l > r$ and $p(\bigcup_{i\in I} V_i) > 0$ for every $\emptyset \neq I \subsetneq \{1, \dots, l\}$. Suppose that we have an r-hypergraph H of $g(V)/r$ hyperedges that covers p, and let $\mathcal{P}$ be a p-full partition. Then the hypergraph obtained from H by contracting the members of $\mathcal{P}$ must be connected. Since a connected hypergraph of rank r on a ground set of size $|\mathcal{P}|$ must have at least $(|\mathcal{P}| - 1)/(r - 1)$ hyperedges, we have $(|\mathcal{P}| - 1)/(r - 1) \leq g(V)/r$. This motivates the following definition.

A p-full partition $\mathcal{P}$ is called a *deficient partition* if $(|\mathcal{P}| - 1)/(r - 1) > g(V)/r$.

Theorem 3.22 (Király 2004). *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric crossing supermodular set function, $r \geq 2$ an integer, and $g : V \to \mathbb{Z}_+$ a degree bound function such that $r \mid g(V)$. There is an r-hypergraph H of $g(V)/r$ hyperedges covering p such that $d_H(v) \leq g(v)$ for every $v \in V$ if and only if the following hold*:

$$g(X) \geq p(X) \quad \textit{for every } X \subseteq V,$$
$$\frac{g(V)}{r} \geq p(X) \quad \textit{for every } X \subseteq V,$$

there are no deficient partitions.

Since the last two conditions do not depend on the individual values of g, but only on $g(V)$, the characterization of the degree-bounded problem can be used in the usual way to prove a min-max theorem on the corresponding minimum cardinality problem.

Theorem 3.23. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric crossing supermodular set function, and $r \geq 2$ an integer. There is an r-hypergraph with γ hyperedges that covers p if and only if the following hold*:

$$r\gamma \geq \sum_{X\in\mathcal{P}} p(X) \quad \textit{for every partition } \mathcal{P},$$
$$\gamma \geq p(X) \quad \textit{for every } X \subseteq V,$$
$$\gamma \geq \frac{l-1}{r-1} \quad \textit{if there is a } p\textit{-full partition with } l \textit{ members}.$$

We also obtain a contra-polymatroid containing feasible degree vectors, which gives rise to a kind of linking property.

Theorem 3.24. *Let γ be the minimum number of r-hyperedges that cover p. Let p^γ be the set function defined by*

$$p^\gamma(X) = \begin{cases} r\gamma & \textit{if } X = V, \\ p(X) & \textit{otherwise}. \end{cases}$$

Then the integer vectors in the contra-polymatroid $C(p^\gamma)$ are degree vectors of r-hypergraphs that cover p.

Corollary 3.25. *Let p be a symmetric crossing supermodular set function, let $g : V \to \mathbb{Z}_+$ be an upper degree bound. If p can be covered by at most $g(V)/r$ r-hyperedges, and it can be covered by an r-hypergraph whose degrees are bounded by g, then p can be covered by an r-hypergraph that satisfies both conditions simultaneously.*

The k-edge-connectivity augmentation problem of an initial hypergraph by uniform hyperedges is a special case of Theorem 3.23. For a hypergraph $H = (V, \mathcal{E})$ let $c(H)$ denote the number of components of H.

Corollary 3.26. *Let $H_0 = (V_0, \mathcal{E}_0)$ be a hypergraph, and $r \geq 2$ an integer. There is an r-uniform hypergraph H with γ hyperedges such that $H_0 + H$ is k-edge-connected if and only if the following hold*:

$$\begin{aligned} r\gamma &\geq |\mathcal{F}|k - \sum_{X \in \mathcal{F}} d_{H_0}(X) \quad \textit{for every subpartition } \mathcal{F}, \\ \gamma &\geq k - d_{H_0}(X) \quad \textit{for every } X \subseteq V, \\ (r-1)\gamma &\geq c(H_0 - \mathcal{E}_0') - 1 \quad \textit{for every } \mathcal{E}_0' \subseteq \mathcal{E}_0 \textit{ for which } |\mathcal{E}_0'| = k - 1. \end{aligned}$$

6.3.7 Parity-Constrained Orientations Covering Non-negative Intersecting Supermodular Functions

If parity constraints are added to problems studied in this survey, most of them become difficult. For example, we do not know how to decide whether a graph has a strongly connected orientation where all in-degrees are even. There are cases though when the parity-constrained problems are tractable, and the present section describes a fairly general such case.

A set function $p : 2^V \to \mathbb{Z}$ is *monotone decreasing* if $p(X) \geq p(Y)$ whenever $\emptyset \neq X \subseteq Y$. It can be proved that an intersecting supermodular function p with $p(V) = 0$ is monotone decreasing if and only if it is non-negative.

Let $H = (V, \mathcal{E})$ be a hypergraph, $T \subseteq V$ a fixed set, and $p : 2^V \to \mathbb{Z}$ a set function such that $p(V) = 0$. An orientation of H is called (p, T)*-feasible* if it covers p and the in-degree of $v \in V$ is odd if and only if $v \in T$. A set $X \subseteq V$ is called *even* if $|X \cap T| + i_H(X) + p(X)$ is even; X is called *odd* if $|X \cap T| + i_H(X) + p(X)$ is odd. Clearly, $\varrho_D(X) \geq p(X) + 1$ must hold for an odd set X in a (p, T)-feasible orientation of H. This motivates the definition of the following set function:

$$p^T(X) := \begin{cases} p(X) & \text{if } X \text{ is even,} \\ p(X) + 1 & \text{if } X \text{ is odd.} \end{cases} \tag{37}$$

Note that p^T depends on H too. The definition implies that

$$p^T(X) \equiv |X \cap T| + i_H(X) \mod 2 \tag{38}$$

for every $X \subseteq V$. Given a partition $\mathcal{P}$, the value

$$\mu_T(\mathcal{P}) := \sum_{Z \in \mathcal{P}} p^T(Z) - e_H(\mathcal{P})$$

is called the *deficiency* of $\mathcal{P}$, which depends also on H and p. It can be checked that the deficiency of every partition has the same parity.

It is easy to see that if an orientation D of H is (p, U)-feasible for some $U \subseteq V$, then

$$|T \Delta U| \geq \max\{\mu_T(\mathcal{P}) : \ \mathcal{P} \text{ is a partition}\}.$$

It turns out that if p is non-negative intersecting supermodular, and there exists an orientation covering p, then equality can be attained, i.e. we can characterize the minimum number of nodes with wrong in-degree parity in an orientation covering p.

Theorem 3.27 (Király and Szabó to appear). *Let $H = (V, \mathcal{E})$ be a hypergraph, $T \subseteq V$ a fixed set, and $p : 2^V \to \mathbb{Z}_+$ an intersecting supermodular and non-negative set function for which $p(V) = 0$. Suppose that H has an orientation covering p. Then there exists a set $U \subseteq V$ such that*

$$|T \Delta U| = \max\{\mu_T(\mathcal{P}) : \ \mathcal{P} \textit{ is a partition}\} \tag{39}$$

and H has a (p, U)-feasible orientation.

Recently Makai et al. (2007) proved the above theorem using a general result on matroid parity and gave a polynomial time combinatorial algorithm.

Applications

One can deduce from Theorem 3.27 the Berge–Tutte formula on the size of a maximum matching of a graph. Let $G = (V, E)$ be an undirected graph. Let $\nu(G)$ denote the maximum size of a matching in G; then the Berge–Tutte formula can be written in the following form.

Theorem 3.28.

$$|V| - 2\nu(G) = \max\{odd_G(W) - |W| : W \subseteq V\},$$

where $odd_G(W)$ denotes the number of components of $G - W$ with an odd number of nodes.

To obtain this from Theorem 3.27, we subdivide each edge $e \in E$ by a new node u_e, resulting in the graph $G' = (V', E')$. For $v \in V$ let $p(\{v\}) = \deg_G(v) - 1$, and let $p(X) = 0$ for all other sets $X \subseteq V'$. Define $T \subseteq V'$ to consist of those nodes $v \in V$ for which $\deg_G(v) - 1$ is odd.

If we have an orientation of G' covering p that has a minimal number of nodes with wrong in-degree parity, then we may assume that $\varrho(u_e) \neq 1$ for every $e \in E$, and so the edges for which $\varrho(u_e) = 2$ form a matching. It can be shown that the size of this matching is exactly the value given by the Berge–Tutte formula.

Another application of Theorem 3.27 comes from the fact that it characterizes which graphs have a rooted k-edge-connected orientation where every in-degree is odd. This characterization was proved earlier in Frank et al. (2001). The special case when $k = 1$, proved by Nebeský (1981), surprisingly leads to the characterization of upper embeddable graphs.

6.3.8 Eulerian Splitting-off

In the previous section we saw that the addition of parity constraints can make the problem significantly more difficult. In this section we show a way in which parity can actually make things easier.

In his early seminal paper on splitting-off in Eulerian graphs (Lovász 1976), Lovász proved three theorems showing how the splitting operation may preserve certain connectivity properties of the graph. Here we look at some other problems where a similar approach may work, based partly on earlier results by Bertsimas and Teo (1997).

In this section we consider only graphs, and we allow parallel edges and loops. Let $p : 2^V \to \mathbb{Z}_+$ be a non-negative set function on the ground set V and let $m : V \to \mathbb{Z}_+$ be a function on V. We consider the problem of finding a graph $G = (V, E)$ for which $d_G(v) = m(v)$ for every $v \in V$ and $d_G(X) \geq p(X)$ for every $X \subseteq V$. We assume in the rest of the section that $p(\emptyset) = 0$ and the set function p is symmetric.

An obvious necessary condition for the existence of G is $m(X) \geq p(X)$ for every $X \subseteq V$; this motivates the introduction of the *excess function* m_p:

$$m_p(X) := m(X) - p(X) \quad (X \subseteq V).$$

In addition to the requirement that $m_p(X) \geq 0$ for every $X \subseteq V$, it is also necessary for $m(V)$ (or equivalently $m_p(V)$) to be even. Our strategy is to make an additional, seemingly very strong requirement on m_p: it should also be even for every $X \subseteq V$ for which $p(X) > 0$. In exchange, we can relax the requirements on the set function p.

A set function p is called *semi-skew-supermodular* if for any 3 sets X_1, X_2, X_3 with $p(X_i) > 0$ $(i = 1, 2, 3)$ at least one of the following four possibilities holds:

- $p(X_i) + p(X_j) \leq p(X_i \cap X_j) + p(X_i \cup X_j)$ for some $i \neq j$,
- $p(X_i) + p(X_j) \leq p(X_i - X_j) + p(X_j - X_i)$ for some $i \neq j$,
- $p(X_1) + p(X_2) + p(X_3) \leq p(X_1 \cap X_2 \cap X_3) + p(X_1 - (X_2 \cup X_3)) + p(X_2 - (X_1 \cup X_3)) + p(X_3 - (X_1 \cup X_2))$,
- $p(X_1) + p(X_2) + p(X_3) \leq p(X_1 \cup X_2 \cup X_3) + p((X_2 \cap X_3) - X_1) + p((X_1 \cap X_3) - X_2) + p((X_1 \cap X_2) - X_3)$.

Obviously every skew supermodular set function is semi-skew-supermodular. In addition, the following is implied by the definition.

- If p_1 and p_2 are skew supermodular set functions, then $\max\{p_1, p_2, 0\}$ is a semi-skew-supermodular set function.

- If p is semi-skew-supermodular and $G = (V, E)$ is a graph, then $\max\{p - d_G, 0\}$ is semi-skew-supermodular.
- If p is a non-symmetric semi-skew-supermodular set function, then $p'(X) := \max\{p(X), p(V - X)\}$ is also semi-skew-supermodular, hence the assumption of symmetry is not restrictive.

The next theorem (proved in Király 2007b) states that if p is semi-skew-supermodular and m_p is even-valued then the non-negativity of m_p is sufficient for the existence of a degree-specified graph that covers p. A similar result for a slightly more restricted class of set functions appeared in Bertsimas and Teo (1997).

Theorem 3.29. *Let $p : 2^V \to \mathbb{Z}_+$ be a symmetric and semi-skew-supermodular set function. Let $m : V \to \mathbb{Z}_+$ be a degree specification with the properties that $m(V)$ is even and $m_p(X)$ is non-negative and even-valued if $p(X) > 0$. Then there exists a graph G such that $d_G(v) = m(v)$ for every $v \in V$ and $d_G(X) \geq p(X)$ for every $X \subseteq V$.*

We present four applications of Theorem 3.29. We start with the parsimonious property that was the original motivation for the result of Bertsimas and Teo. The second application concerns the covering of graphs by edge-disjoint forests, and it is a slight extension of a result in Frank and Király (2003). The third one offers a simple proof for some known results on edge-disjoint paths. The fourth application is a proof of a theorem of Karzanov and Lomonosov (1978) on multiflows.

The Parsimonious Property

Let $G = (V, E)$ be an undirected graph with a cost function $c : E \to \mathbb{Z}_+$ on the edges, and let $p : 2^V \to \mathbb{Z}_+$ be a symmetric set function. Consider the following linear program:

$$\begin{aligned} &\min cx \\ &x(e) \geq 0 \quad \text{for every } e \in E, \\ &d_x(Z) \geq p(Z) \quad \text{for every } Z \subseteq V. \end{aligned} \tag{40}$$

We say that a node $v \in V$ has the *parsimonious property* if the linear system (40) has an optimal solution x^* with $d_{x^*}(v) = p(v)$. This property has several structural consequences, and it is useful in the analysis of approximation algorithms (see e.g. Goemans and Bertsimas 1993). The following theorem is a generalization of the result in (Bertsimas and Teo 1997) where a property stronger than subadditivity was required. A node $v \in V$ is called *subadditive* if $p(X) + p(v) \geq p(X + v)$ for every $X \subseteq V - v$.

Theorem 3.30. *If G is the complete graph, c satisfies the triangle inequality, and p is semi-skew-supermodular, then the linear system (40) has an optimal solution x^* with the following property*:

$$d_{x^*}(v) = p(v) \quad \textit{for every subadditive node } v.$$

Augmentation of k-Forest-Coverable Graphs

The following theorem of Nash-Williams (1964) characterizes graphs that can be covered by k forests.

Theorem 3.31. *The edge-set of a graph $G = (V, E)$ can be covered by k forests if and only if $i_G(X) \leq k(|X| - 1)$ for every non-empty subset X of V.*

Given a graph $G = (V, E)$ that can be covered by k forests and weights on the edges of the complete graph on V, we may want to find an edge set F of maximum weight for which the graph $G' = (V, E + F)$ can still be covered by k forests. This is an easy problem in the sense that it can be solved by finding the maximum weight independent set in a matroid. In fact, the following, more general problem can be solved using the weighted matroid intersection algorithm:

Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ that can both be covered by k forests and weights on the edges of the complete graph on V, find an edge set F of maximum weight such that the graphs $G'_1 = (V, E_1 + F)$ and $G'_2 = (V, E_2 + F)$ can still be covered by k forests.

What happens if we also want to prescribe the number of new edges incident to each node? The weighted problem cited above becomes NP-complete, even for $E_1 = E_2 = \emptyset$ and $k = 1$. However, the non-weighted degree-prescribed problem can be solved, and there is a simple necessary and sufficient condition for the existence of F. This extends the result in Frank and Király (2003) that dealt with the case $E_1 = E_2$. An interesting point about this application of Theorem 3.29 is that parity is not part of the definition of the problem; an even-valued excess function appears only implicitly.

Theorem 3.32. *Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs that can be covered by k forests, and let $m : V \to \mathbb{Z}_+$ be a degree specification with $m(V)$ even. There exists an edge set F for which $d_F(v) = m(v)$ for every $v \in V$ and both $G'_1 = (V, E_1 + F)$ and $G'_2 = (V, E_2 + F)$ can be covered by k forests if and only if*

$$\max\left\{m(X) - \frac{m(V)}{2}, 0\right\} \leq k(|X| - 1) - \max\{i_{G_1}(X), i_{G_2}(X)\}$$

for every $\emptyset \neq X \subseteq V$.

Edge-Disjoint Paths: The Eulerian Case

Let $G = (V, E)$ and $H = (V, F)$ be undirected graphs on the same ground set. Our goal is to find a family $\{P_f : f \in F\}$ of edge-disjoint paths in G such that for every $f \in F$ the end-nodes of the path P_f are the end-nodes of f.

We say that *the cut condition holds* if $d_G(X) \geq d_H(X)$ for every $X \subseteq V$. The cut condition is clearly necessary for the existence of the required edge-disjoint paths. It is known that if H is a double star, a K_4 or a C_5, possibly with multiple parallel edges, and $G + H$ is Eulerian, then the cut condition is also sufficient (see e.g. Schrijver 1991).

Theorem 3.33. *Let $G = (V, E)$ and $H = (V, F)$ be undirected graphs such that*

- *H is a double star, a K_4 or a C_5, possibly with multiple parallel edges,*
- *$G + H$ is Eulerian,*
- *$d_G(X) \geq d_H(X)$ for every $X \subseteq V$.*

Then there exists a family $\{P_f : f \in F\}$ of edge-disjoint paths in G such that for every $f \in F$ the end-nodes of the path P_f are the end-nodes of f.

This theorem can be proved quite easily using Theorem 3.29. The main observation is that in the cases mentioned in the theorem the function d_H is semi-skew-supermodular. This essentially means that we can either split off edge-pairs or remove edges that appear both in G and in H until we get an obviously solvable configuration.

Multiflows in Inner Eulerian Graphs

Let $G = (V, E)$ be an undirected graph and $T \subseteq V$ a set of terminal nodes. We say that the pair (G, T) is *inner Eulerian* if $d_G(v)$ is even for every $v \in V - T$. A *T-path* is a path with both end-nodes in T. If $\mathcal{P}$ is a family of T-paths and $Z \subseteq T$, then $d_{\mathcal{P}}(Z)$ denotes the number of paths in $\mathcal{P}$ that have exactly one end-node in Z. Let

$$\lambda_G(Z) := \min\{d_G(X) : X \subseteq V, X \cap T = Z\}.$$

If $\mathcal{P}$ is a family of edge-disjoint T-paths, then obviously $d_{\mathcal{P}}(Z) \leq \lambda_G(Z)$ for every $Z \subseteq T$. Generalizing a result of Lovász (1976), Karzanov and Lomonosov (1978) proved that equality can be attained on any given family $\mathcal{L}$ of subsets of T that is *3-cross-free*: it has no three members that are pairwise crossing on the ground set T.

Theorem 3.34 (Karzanov and Lomonosov 1978). *Let (G, T) be inner Eulerian and let $\mathcal{L}$ be a* 3*-cross-free family of subsets of T. Then there is a family $\mathcal{P}$ of edge-disjoint T-paths for which $d_{\mathcal{P}}(Z) = \lambda_G(Z)$ for every $Z \in \mathcal{L}$.*

This theorem also follows easily from Theorem 3.29, by splitting off edge-pairs from nodes of $V - T$.

6.3.9 Merging Hyperedges

This final section is related to the connectivity augmentation of hypergraphs, but the problem is formulated somewhat differently. The starting point is the same: we have a hypergraph $H = (V, \mathcal{E})$ that does not cover a given set function p, and we want to modify it to cover p. But instead of adding new hyperedges, we are allowed only to merge existing hyperedges.

By *merging* two disjoint hyperedges of H we mean the operation of replacing them in H by their union. "Merging some hyperedges of H" means repeating this operation a few times. Let us define the set function

$$b_H(X) := |\{e \in \mathcal{E} : e \cap X \neq \emptyset\}|.$$

Clearly, if there is a node-set for which $b_H(X) < p(X)$, then it is impossible to obtain a hypergraph covering p by merging hyperedges. It is easy to see that b_H is fully submodular and

$$b_H(X) + b_H(Y) \geq b_H(X - Y) + b_H(Y - X) + |\{e \in \mathcal{E} : \emptyset \neq e \cap Y \subseteq X \cap Y\}|.$$

The following theorem from Király (2007a) states that for symmetric skew supermodular functions the condition $b_H(X) \geq p(X)$ is sufficient. We include the whole proof since it is fairly straightforward.

Theorem 3.35 (Király 2007a). *Let $H = (V, \mathcal{E})$ be a hypergraph, and let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew supermodular set function for which*

$$b_H(X) \geq p(X) \quad \textit{for every } X \subseteq V. \tag{41}$$

Then by merging some hyperedges of H we can obtain a hypergraph $H_* = (V, \mathcal{E}_*)$ such that

$$d_{H_*}(X) \geq p(X) \quad \textit{for every } X \subseteq V. \tag{42}$$

Proof. We use induction on the number of hyperedges of H (it is clearly true if $\mathcal{E} = \emptyset$). A set $X \subseteq V$ is called *tight* if $b_H(X) = p(X)$. By the properties of b_H and p, if X and Y are tight, then either $X \cap Y$ and $X \cup Y$ are tight, or $X - Y$ and $Y - X$ are tight. Furthermore, if X and Y are tight and there is a hyperedge e such that $\emptyset \neq e \cap Y \subseteq X \cap Y$, then $X \cap Y$ and $X \cup Y$ are tight.

Let e_0 be an arbitrary hyperedge of H. If there is no tight set X such that $e_0 \subseteq X$, then let $H' := H - e_0$ and

$$p'(X) := \begin{cases} p(X) - 1 & \text{if } e_0 \cap X \neq \emptyset \text{ and } e_0 \cap (V - X) \neq \emptyset, \\ p(X) & \text{otherwise.} \end{cases}$$

The set function p' is symmetric and skew supermodular, and $b_{H'}(X) \geq p'(X)$ for every $X \subseteq V$, so by induction there is a hypergraph H'_*, obtained by merging some hyperedges of H', such that $d_{H'_*}(X) \geq p'(X)$ for every $X \subseteq V$. It follows that $H_* := H'_* + e_0$ covers p. We can thus assume that there is a tight set X_0 such that $e_0 \subseteq X_0$; we also assume that X_0 is a maximal tight set with that property.

Suppose that there is no hyperedge $e \in \mathcal{E}$ such that $e \cap X_0 = \emptyset$. Then $p(V - X_0) = p(X_0) = b_H(X_0) > b_H(V - X_0)$ since $e_0 \subseteq X_0$, contradicting (41). Thus there is a hyperedge $e_1 \in \mathcal{E}$ such that $e_1 \cap X_0 = \emptyset$. Consider the hypergraph $H' := (V, \mathcal{E} - \{e_0, e_1\} + (e_0 \cup e_1))$, i.e. the hypergraph obtained by merging e_0 and e_1. If $b_{H'}(X) < p(X)$ for some $X \subseteq V$, then $e_0 \cap X \neq \emptyset$, $e_1 \cap X \neq \emptyset$, and X was tight. Since $\emptyset \neq e_0 \cap X \subseteq X_0 \cap X$, $X_0 \cup X$ is also tight, which contradicts the maximality of X_0 since $X - X_0 \supseteq e_1 \cap X \neq \emptyset$.

We proved that H' and p satisfy (41), so by induction there is a hypergraph H_* obtained by merging some hyperedges of H' (hence obtained by merging some hyperedges of H) that satisfies (42). $\square$

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