2 Diffeomorphisms of the Circle and the Virasoro–Bott Group

This section deals with the Lie group $\mathrm{Diff}(S^1)$ of orientation-preserving diffeomorphisms of the circle and its Lie algebra $\mathrm{Vect}(S^1)$ of smooth vector fields on the circle, as well as with their central extensions. We start by showing that the Lie algebra of vector fields on the circle admits a unique nontrivial central extension, the so-called Virasoro algebra. This central extension gives rise to a central extension of the Lie group of circle diffeomorphisms, which is called the Virasoro–Bott group. Similar to the case of loop groups, the Virasoro–Bott group has a "nicer" coadjoint representation than the nonextended group of circle diffeomorphisms. Its coadjoint orbits can be classified in terms of conjugacy classes of the finite-dimensional group $\mathrm{SL}(2,\mathbb{R})$. Finally, we describe the Euler equations corresponding to right-invariant metrics on the Virasoro–Bott group and encounter the KdV and related partial differential equations among them.

2.1 Central Extensions

Let us consider the Lie algebra $\mathrm{Vect}(S^1)$ of smooth vector fields on the circle. After fixing a coordinate θ on the circle, any vector field can be written as $f(\theta)\partial_\theta$, where f is a smooth function on S^1 and ∂_θ stands for $\frac{\partial}{\partial\theta}$. Under this identification, the commutator of two elements in $\mathrm{Vect}(S^1)$ is given by

$$[f(\theta)\partial_\theta, g(\theta)\partial_\theta] = (f'(\theta)g(\theta) - g'(\theta)f(\theta))\partial_\theta\,,$$

where f' denotes the derivative in θ of the function f.[11]

Definition / Proposition 2.1 *The map* $\omega : \mathrm{Vect}(S^1)\times\mathrm{Vect}(S^1) \to \mathbb{R}$ *given by*

$$\omega(f(\theta)\partial_\theta, g(\theta)\partial_\theta) = \int_{S^1} f'(\theta)g''(\theta)d\theta \tag{2.5}$$

is a nontrivial 2*-cocycle on* $\mathrm{Vect}(S^1)$*, called the* Gelfand–Fuchs cocycle*. The corresponding central extension of* $\mathrm{Vect}(S^1)$ *is called the* Virasoro algebra *and is denoted by* $\mathfrak{vir}$.

Exercise 2.2 Prove the cocycle identity for ω.

The following proposition shows that the Virasoro algebra is the unique (up to isomorphism) nontrivial central extension of the Lie algebra $\mathrm{Vect}(S^1)$.

Proposition 2.3 *The second continuous cohomology group* $H^2(\mathrm{Vect}(S^1),\mathbb{R})$ *is one-dimensional and is generated by the Gelfand–Fuchs cocycle* ω.

[11] Note that this Lie bracket is the negative of the commonly assumed commutator of vector fields, as the calculations in Exercise 2.3 of Chapter I shows; see [24].

Proof. The proof of this proposition (see [322]) is similar to that of Proposition 1.6. Our goal is to show that, up to a coboundary, any continuous 2-cocycle ω on the Lie algebra $\mathrm{Vect}(S^1)$ is a multiple of the Gelfand–Fuchs cocycle. First, let us extend the cocycle ω from $\mathrm{Vect}(S^1)$ to a complex bilinear form on the complexification $\mathrm{Vect}(S^1)_{\mathbb{C}} = \mathrm{Vect}(S^1) \otimes \mathbb{C}$ of the Lie algebra $\mathrm{Vect}(S^1)$. An element $f(\theta)\partial_\theta \in \mathrm{Vect}(S^1)_{\mathbb{C}}$ can be expanded into a Fourier series

$$f(\theta) = \sum f_n e^{in\theta} .$$

By continuity, the cocycle ω is completely determined by its values on the basis fields $L_n = ie^{in\theta}\partial_\theta$. Note that the commutator of the fields L_n and L_m is given by

$$[L_n, L_m] = (m-n)L_{n+m} .$$

The cocycle identity for ω and the triple L_0, L_m, L_n gives

$$\omega([L_0, L_m], L_n) + \omega(L_m, [L_0, L_n]) = \omega(L_0, [L_m, L_n]) ,$$

which implies that the cohomology class of the cocycle ω is unchanged under rotations of S^1 that are generated by the vector field L_0. Indeed, the right-hand side of the equation above is an *exact cocycle* (i.e., coboundary) $d\alpha$, where α is the linear functional on $\mathrm{Vect}(S^1)$ defined by $\alpha(L_m) := \omega(L_0, L_m)$. (Here by definition $d\alpha(L_n, L_m) := \alpha([L_n, L_m])$.) In particular, the cocycle obtained from ω by averaging over S^1 belongs to the same cohomology class as ω. Therefore, we can assume ω to be *rotation invariant*, i.e.,

$$\omega([L_0, L_m], L_n) + \omega(L_m, [L_0, L_n]) = 0 . \tag{2.6}$$

Set $\omega_{n,m} := \omega(L_n, L_m)$. Then the commutator relation of the fields L_n and L_m and equation (2.6) imply

$$m\omega_{m,n} + n\omega_{m,n} = 0 .$$

This implies that $\omega_{m,n} = 0$ for $m + n \neq 0$. Antisymmetry of the cocycle ω implies $\omega_{n,-n} = \omega_{-n,n}$, so that it is enough to determine $\omega_{n,-n}$ for $n \in \mathbb{N}$.

The cocycle identity for ω evaluated on the triple L_m, L_n, L_{-m-n} implies

$$(m-n)\omega_{m+n,-n-m} + (2m+n)\omega_{n,-n} - (2n+m)\omega_{m,-m} = 0 .$$

In particular, for $m = 1$ the equation above reads as follows:

$$(-n+1)\omega_{n+1,-n-1} + (n+2)\omega_{n,-n} - (2n+1)\omega_{1,-1} = 0 .$$

Hence $\omega_{n,-n}$ is defined recursively once $\omega_{1,-1}$ and $\omega_{2,-2}$ are fixed. This shows that the space of the bilinear forms ω that satisfy the 2-cocycle condition is at most two-dimensional. Two linear independent elements of this space are given by $\omega_{n,-n} = n^3$ and $\omega_{n,-n} = n$. But the 2-cocycle defined by $\omega_{n,-n} = n$

is exact, since it coincides with $d\widetilde{\alpha}$, where $\widetilde{\alpha}$ is the linear functional defined by $\widetilde{\alpha}(L_n) = -\frac{1}{2}\delta_{n,0}$.

So up to a 2-coboundary, any 2-cocycle ω has the "cubic" form

$$\omega(L_n, L_m) = c\delta_{n,-m}n^3$$

for some $c \in \mathbb{C}$.

It remains to show that the "cubic" cocycle ω is nontrivial. Suppose that $\omega = d\beta$ for some 1-cocycle β. This means that β is a linear map and $\omega(L_n, L_m) = \beta([L_n, L_m])$. In particular, we have $\beta([L_n, L_{-n}]) = 2ni\beta(L_0)$, which shows that in this case $\omega(L_n, L_{-n})$ would have to depend linearly on n. This contradiction completes the proof. $\square$

Our next goal is to show that the central extension of the Lie algebra of vector fields $\mathrm{Vect}(S^1)$ defined by the Gelfand–Fuchs cocycle ω can be lifted to a central extension of the group of circle diffeomorphisms $\mathrm{Diff}(S^1)$. It turns out that the situation here is much simpler than that in the case of the loop groups. The central extension of the group $\mathrm{Diff}(S^1)$ corresponding to the Lie algebra $\mathfrak{vir}$ is topologically trivial and hence can be defined by a continuous group 2-cocycle.

Let $\varphi : \theta \mapsto \varphi(\theta)$ be a diffeomorphism of the circle, and φ' stands for its derivative in θ.

Definition / Proposition 2.4 *The map $B : \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1) \to S^1$ given by*

$$(\varphi, \psi) \mapsto \frac{1}{2}\int_{S^1} \log(\varphi \circ \psi)' d \log \psi'$$

is a continuous 2-cocycle on the group $\mathrm{Diff}(S^1)$. The Lie algebra of the corresponding central extension $\widehat{\mathrm{Diff}}(S^1)$ is the Virasoro algebra $\mathfrak{vir}$. The 2-cocycle B is called the Bott cocycle, *and the corresponding central extension of the group $\mathrm{Diff}(S^1)$ is called the* Virasoro–Bott group.

PROOF. To show that the map B defines a group 2-cocycle, we have to check the identity

$$B(\varphi \circ \psi, \eta) + B(\varphi, \psi) = B(\varphi, \psi \circ \eta) + B(\psi, \eta)\,.$$

It is provided by the chain rule, which immediately gives

$$B(\varphi\circ\psi, \eta) = \frac{1}{2}\int_{S^1} \log(\varphi\circ\psi\circ\eta)'\, d\log\eta' = \frac{1}{2}\int_{S^1} \log(\varphi'\circ\psi\circ\eta)\, d\log\eta' + B(\psi, \eta)$$

and

$$\begin{aligned}B(\varphi, \psi \circ \eta) &= \frac{1}{2} \int_{S^1} \log(\varphi \circ \psi \circ \eta)' \, d \log(\psi \circ \eta)' \\ &= B(\varphi, \psi) + \frac{1}{2} \int_{S^1} \log(\varphi' \circ \psi \circ \eta) \, d \log \eta' \,.\end{aligned}$$

Now we verify that the infinitesimal version of the Bott group cocycle B coincides with the Gelfand–Fuchs Lie algebra 2-cocycle ω. Let $f\partial_\theta$ and $g\partial_\theta$ be two smooth vector fields on S^1 and consider the corresponding flows φ_s and ψ_t on S^1, starting at the identity diffeomorphism: $\varphi_0 = \psi_0 = \mathrm{id}$.

We have to check that

$$\omega(f\partial_\theta, g\partial_\theta) = \frac{d^2}{dt\, ds}\Big|_{t=0,s=0} B(\varphi_t, \psi_s) - \frac{d^2}{dt\, ds}\Big|_{t=0,s=0} B(\psi_s, \varphi_t)$$

(see Proposition 3.14 of Chapter I). The latter holds, since

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} B(\varphi_t, \psi_s) &= \frac{1}{2} \int_{S^1} \left(\log'(\varphi_0 \circ \psi_s)'\right) (f \circ \psi_s)' d \log \psi_s' \\ &= \frac{1}{2} \int_{S^1} (f' \circ \psi_s) \, d \log \psi_s'\end{aligned}$$

and

$$\frac{d}{ds}\Big|_{s=0} \frac{1}{2} \int_{S^1} (f' \circ \psi_s) \, d \log \psi_s' = \frac{1}{2} \int_{S^1} f' dg' \,.$$

Similarly, we obtain

$$\frac{d^2}{dt\, ds}\Big|_{t=0,s=0} B(\psi_s, \varphi_t) = \frac{1}{2} \int_{S^1} g' df' = -\frac{1}{2} \int_{S^1} f' dg' \,,$$

which, combined with the equation above, yields the assertion. □

2.2 Coadjoint Orbits of the Group of Circle Diffeomorphisms

Before we start classifying the coadjoint orbits of the Virasoro group, let us take a look at the coadjoint representation of the nonextended group of orientation-preserving diffeomorphisms of the circle. Observe that the dual spaces to the infinite-dimensional Lie algebras considered below are always understood as smooth duals, i.e., they are identified with appropriate spaces of smooth functions.

Let $\mathrm{Diff}(S^1)$ be the group of all orientation-preserving diffeomorphisms of S^1 and let $\mathrm{Vect}(S^1)$ be its Lie algebra.

Proposition 2.5 ([202]) *The (smooth) dual space* $\mathrm{Vect}(S^1)^*$ *is naturally identified with the space of quadratic differentials* $\Omega^{\otimes 2}(S^1) = \{u(\theta)(d\theta)^2\}$ *on the circle. The pairing is given by the formula*

$$\langle u(\theta)(d\theta)^2,\ v(\theta)\partial_\theta\rangle = \int_{S^1} u(\theta)v(\theta)\ d\theta$$

for any vector field $v(\theta)\partial_\theta \in \mathrm{Vect}(S^1)$. The coadjoint action coincides with the action of a diffeomorphism on the quadratic differential: for a diffeomorphism $\varphi \in \mathrm{Diff}(S^1)$ the action is

$$\mathrm{Ad}^*_{\varphi^{-1}}:\quad u\,(d\theta)^2 \mapsto u(\varphi)\cdot(\varphi')^2\,(d\theta)^2 = u(\varphi)\cdot(d\varphi)^2.$$

It follows from this proposition that the square root $\sqrt{u(\theta)(d\theta)^2}$ (when it makes sense) transforms under a diffeomorphism as a differential 1-form. In particular, if the function $u(\theta)$ does not have any zeros on the circle (say, $u(\theta) > 0$), then $\Phi(u(\theta)(d\theta)^2) := \int_{S^1}\sqrt{u(\theta)}\,d\theta$ is a Casimir function, i.e., an invariant of the coadjoint action. One can see that there is only one Casimir function in this case, since the corresponding orbit has codimension 1 in the dual space $\mathrm{Vect}(S^1)^*$. Indeed, there exists a diffeomorphism that sends the quadratic differential $u(\theta)(d\theta)^2$ without zeros to the constant quadratic differential $u_0(d\theta)^2$, where the constant u_0 is such that $\sqrt{u_0}$ is the average value of the 1-form $\sqrt{u(\theta)}\,d\theta$ on the circle:

$$2\pi\sqrt{u_0} = \int_{S^1}\sqrt{u(\theta)}\,d\theta\,.$$

The value u_0 parametrizes the orbits close to $u(\theta)(d\theta)^2$, and hence all these orbits have codimension 1 in $\Omega^{\otimes 2}(S^1)$. The stabilizer of a constant quadratic differential is the group S^1 of rigid rotations, so that the orbit through $u(\theta)(d\theta)^2$ is diffeomorphic to $\mathrm{Diff}(S^1)/S^1$.

On the other hand, if a differential $u(\theta)(d\theta)^2$ changes sign on the circle, then the integrals

$$\int_a^b \sqrt{|u(\theta)|}\,d\theta\,,$$

evaluated between any two consecutive zeros a and b of the function $u(\theta)$, are invariant. In particular, since $u(\theta)$ has at least two zeros, the coadjoint orbit of such a differential $u(\theta)(d\theta)^2$ necessarily has codimension higher than 1, and there exist coadjoint orbits of the group $\mathrm{Diff}(S^1)$ of arbitrarily high codimension. The classification of orbits in $\mathrm{Vect}(S^1)^*$ was described in [201, 203].

Remark 2.6 One can show that if the function $u(\theta)$ has two simple zeros, changing sign exactly twice on the circle, then the corresponding coadjoint orbit of the group $\mathrm{Diff}(S^1)$ has codimension 2; see [203]. (The corresponding two Casimirs are the integrals of $\sqrt{|u(\theta)|}\,d\theta$ over two different parts of the circle between these two zeros, while there are no extra local invariants at zeros themselves: quadratic differentials with simple zeros are all locally diffeomorphic to $\pm\theta\,(d\theta)^2$.)

In other words, in a family of quadratic differentials $\bar{u}^\epsilon := u^\epsilon(\theta)\,(d\theta)^2$, where the function u^ϵ is everywhere positive for $\epsilon > 0$, has a double zero for $\epsilon = 0$, and has two simple zeros for $\epsilon < 0$ (e.g., $u^\epsilon = \cos\theta + 1 + \epsilon$) the codimension of the coadjoint orbit of $\bar{u}^\epsilon = u^\epsilon(d\theta)^2$ changes from 1 for $\epsilon > 0$ to 2 for $\epsilon \le 0$, since the number of Casimirs jumps from 1 to 2. (Note that for $\epsilon = 0$ the orbit codimension of $\bar{u}^0$ is also 2, since the existence of a double zero imposes an extra constraint on a quadratic differential.)

This change of "codimension parity" of the (infinite-dimensional) coadjoint orbits is rather surprising, since in finite dimensions the existence of a symplectic structure on each coadjoint orbit forces all of them to be even-dimensional, and hence codimensions of coadjoint orbits for a given (finite-dimensional) group are always of the same parity: either all even or all odd. However, for $\mathrm{Vect}(S^1)^* = \Omega^{\otimes 2}(S^1)$ we observe that there exist orbits of both codimensions 1 and 2!

In particular, this shows that the Weinstein theorem [384] on the existence of the transverse Poisson structure to symplectic leaves does not hold for infinite-dimensional Poisson manifolds; cf. Remark I.4.7. Indeed, on a two-dimensional transversal to $\bar{u}^0$ in $\mathrm{Vect}(S^1)^*$, neighboring coadjoint orbits of $\bar{u}^\epsilon$ have traces of both codimensions 1 and 2. One can consider the following example, clarifying how the change of parity can occur for infinite-dimensional symplectic leaves. Define the Poisson structure in an infinite-dimensional vector space $\{(x_0, x_1, x_2, \dots)\}$ by the bivector field

$$\Pi = x_0\, \partial_{x_1} \wedge \partial_{x_2} + \partial_{x_2} \wedge \partial_{x_3} + \partial_{x_3} \wedge \partial_{x_4} + \dots .$$

Its symplectic leaves are hyperplanes $\{x_0 = \text{const} \neq 0\}$, while for $x_0 = 0$ the symplectic leaves are planes $\{x_0 = 0,\, x_1 = \text{const}\}$ of codimension 2.

In the next section, however, we shall see that coadjoint orbits of the Virasoro group, the central extension of the diffeomorphism group $\mathrm{Diff}(S^1)$, do respect the codimension parity and behave much more like finite-dimensional coadjoint orbits.

2.3 The Virasoro Coadjoint Action and Hill's Operators

Let $\mathfrak{vir}$ be the Virasoro algebra, whose elements are pairs $(f(\theta)\partial_\theta, c)$, where $f(\theta)\partial_\theta$ is a vector field and c is a real number. We can think of its (smooth) dual space as the space of pairs $\mathfrak{vir}^* = \{(u(\theta)(d\theta)^2, a)\}$ consisting of a quadratic differential and a real number (the cocentral term). The pairing between $\mathfrak{vir}$ and $\mathfrak{vir}^*$ is given by

$$\langle (f(\theta)\partial_\theta, c), (u(\theta)(d\theta)^2, a)\rangle = \int_{S^1} f(\theta)u(\theta)d\theta + c \cdot a\,.$$

Our goal in this section is to derive a classification of the coadjoint orbits of the Virasoro–Bott group $\widehat{\mathrm{Diff}}(S^1)$. This classification turns out to be similar to that of the coadjoint orbits of the centrally extended loop groups in Section 1.2.

We begin by noticing that the center of the group $\widehat{\mathrm{Diff}}(S^1)$ acts trivially on the dual space $\mathfrak{vir}^*$. This is why to describe the coadjoint representation of the Virasoro–Bott group we need the action of (nonextended) diffeomorphisms only.

Definition / Proposition 2.7 *The coadjoint action of a diffeomorphism $\varphi \in \mathrm{Diff}(S^1)$ on the dual $\mathfrak{vir}^*$ of the Virasoro algebra is given by the following formula:*

$$\mathrm{Ad}^*_{\varphi^{-1}}: \quad (u\,(d\theta)^2,\, a) \mapsto \big(u(\varphi)\cdot(\varphi')^2\,(d\theta)^2 + aS(\varphi)\,(d\theta)^2,\, a\big)\,, \tag{2.7}$$

where

$$S(\varphi) = \frac{\varphi'\varphi''' - \frac{3}{2}(\varphi'')^2}{(\varphi')^2}$$

is the Schwarzian derivative *of the diffeomorphism φ.*

The same formula can be used to define the Schwarzian derivative $S(\phi)$ for a smooth map $\phi : \mathbb{R} \to \mathbb{R}$ or $\mathbb{R} \to \mathbb{R}P^1 \simeq S^1$.

Proof. The coadjoint action of the Virasoro algebra is defined by the identity

$$\langle \mathrm{ad}^*_{(v\partial_\theta, b)}(u(d\theta)^2,\, a), (w\partial_\theta, c)\rangle = -\langle (u(d\theta)^2,\, a), [(v\partial_\theta, b), (w\partial_\theta, c)]\rangle.$$

Using the definition of the Virasoro commutator and integrating by parts we obtain that the right-hand side is equal to

$$\int_{S^1} -w(2uv' + u'v + av''')\; d\theta\,.$$

Thus the coadjoint operator is

$$\mathrm{ad}^*_{(v\partial_\theta, b)}(u(d\theta)^2,\, a) = -((2uv' + u'v + av''')(d\theta)^2,\;\; 0). \tag{2.8}$$

It remains to check that equation (2.7) indeed defines a representation of the group $\mathrm{Diff}(S^1)$ on the space $\mathfrak{vir}^*$ and that the infinitesimal version of this action is given by equation (2.8). Both assertions can be checked by direct calculations, which we leave to the reader. □

Exercise 2.8 Prove the following transformation law for the Schwarzian derivative:

$$S(\varphi \circ \psi) = (S(\varphi)\circ\psi)\cdot(\psi')^2 + S(\psi)\,. \tag{2.9}$$

Check that the formula (2.7) defines a group representation by using this law.

It turns out to be more convenient to regard the dual Virasoro space $\mathfrak{vir}^*$ not as the space of pairs $\{(u(\theta)(d\theta)^2, a)\}$, but as the space of *Hill's operators*, i.e., differential operators $a\partial_\theta^2 + u(\theta)$, where ∂_θ^2 stands for the second derivative $d^2/d\theta^2$. Indeed, the group action on Hill's operators

$$\mathrm{Ad}^*_{\varphi^{-1}}: \quad a\partial_\theta^2 + u(\theta) \mapsto a\partial_\theta^2 + u(\varphi) \cdot (\varphi')^2 + aS(\varphi) \tag{2.10}$$

has the following nice geometric interpretation (see, e.g., [342, 202, 205, 304]).

Look at a hyperplane $a = \mathrm{const}$ corresponding to nonzero a in the dual space $\mathfrak{vir}^*$. For instance, we fix $a = 1$ and consider Hill's operators of the form $\partial_\theta^2 + u(\theta)$, where θ is a coordinate on S^1. Let f and g be two independent solutions of the corresponding Hill differential equation

$$(\partial_\theta^2 + u(\theta))y = 0 \tag{2.11}$$

for an unknown function y. Although this equation has periodic coefficients, the solutions need not necessarily be periodic, but instead are defined over $\mathbb{R}$. Consider the ratio $\eta := f/g : \mathbb{R} \to \mathbb{R}P^1$. (Below we use the same notation θ for the coordinate on the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and on its cover $\mathbb{R}$.)

Proposition 2.9 *The potential u is (one-half) the Schwarzian derivative of the ratio η:*

$$u = \frac{S(\eta)}{2}.$$

Proof. First we note that the *Wronskian*

$$W(f,g) := \det \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = fg' - f'g$$

is constant, since it satisfies $W' = 0$. For two independent solutions the Wronskian does not vanish, and we normalize W by setting $W = -1$.

This additional condition allows one to find the potential u from the ratio η. Indeed, first one reconstructs the solutions f and g from the ratio η by differentiating:

$$\eta' = \frac{f'g - fg'}{g^2} = \frac{-W}{g^2} = \frac{1}{g^2}\,.$$

Therefore,

$$g = \frac{1}{\sqrt{\eta'}}, \qquad f = g \cdot \eta = \frac{\eta}{\sqrt{\eta'}}.$$

Given two solutions f and g, one immediately finds the corresponding differential equation they satisfy by writing out the following 3×3 determinant:

$$\det \begin{bmatrix} y & f & g \\ y' & f' & g' \\ y'' & f'' & g'' \end{bmatrix} = 0\,.$$

Since f and g satisfy the equation $y'' + u \cdot y = 0$, one obtains from the determinant above that

$$u = -\det \begin{bmatrix} f' & g' \\ f'' & g'' \end{bmatrix} .$$

The explicit formula for u expressed in terms of η turns out to be one-half the Schwarzian derivative of η. □

Corollary 2.10 *The Schwarzian derivative $S(\eta)$ is invariant with respect to a Möbius transformation $\eta \mapsto (a\eta+b)/(c\eta+d)$, where a, b, c, d are real numbers such that $ad - bc = 1$.*

In particular, if η itself is a Möbius transformation $\eta : \theta \mapsto (a\theta+b)/(c\theta+d)$, then $S(\eta) = S(\mathrm{id}) = 0$, where $\mathrm{id} : \theta \mapsto \theta$.

Proof. Indeed, for a given potential u the solutions f and g of the corresponding differential equation are not defined uniquely, but up to a transformation of the pair (f, g) by a matrix from $\mathrm{SL}(2, \mathbb{R})$. Then the ratio η changes by a Möbius transformation. Thus Möbius equivalent ratios η correspond to the same potential $u = S(\eta)/2$. For the identity diffeomorphism $\mathrm{id} : \theta \mapsto \theta$, the explicit formula for the Schwarzian derivative gives $S(\mathrm{id}) = 0$. □

Proposition 2.11 *The Virasoro coadjoint action of a diffeomorphism φ on the potential $u(\theta)$ gives rise to a diffeomorphism change of coordinate in the ratio η:*

$$\varphi : \quad \eta(\theta) \to \eta(\varphi(\theta)) .$$

Proof. We look at the corresponding infinitesimal action on the solutions of the differential equation $(\partial_\theta^2 + u(\theta))y = 0$. For a diffeomorphism $\varphi^{-1}(\theta) = \theta + \epsilon v(\theta)$ close to the identity, consider the infinitesimal Virasoro action of φ^{-1} on the potential $u(\theta)$:

$$u \mapsto u + \epsilon \cdot \delta u, \quad \text{where} \quad \delta u = 2uv' + u'v + \frac{1}{2}v'''$$

(cf. formula (2.8) for $a = \frac{1}{2}$ and note that we are considering the action of φ^{-1}). It is consistent with the following action on a solution y of the above differential equation:

$$y \mapsto y + \epsilon \cdot \delta y, \quad \text{where} \quad \delta y = -\frac{1}{2}yv' + y'v.$$

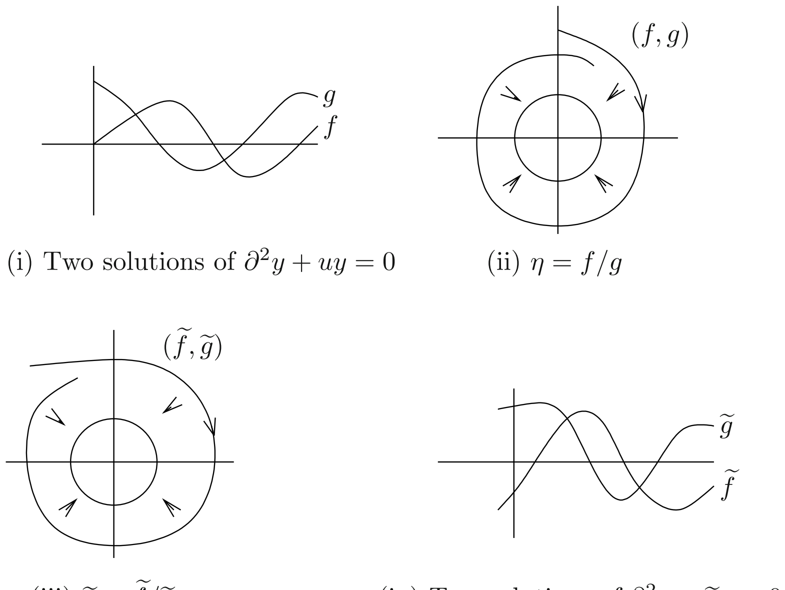
The consistency means that $(\partial_\theta^2 + u + \epsilon \cdot \delta u)(y + \epsilon \cdot \delta y) = 0 + \mathcal{O}(\epsilon^2)$.

Note that the action $\epsilon \cdot \delta y = \epsilon \cdot (-\frac{1}{2}yv' + y'v)$ is an infinitesimal version of the following action of the diffeomorphism $\varphi^{-1}(\theta) = \theta + \epsilon v(\theta)$ on y:

$$\varphi^{-1}: \quad y(\theta) \mapsto y(\varphi(\theta))(\varphi'(\theta))^{-1/2}.$$

Thus solutions to Hill's equation transform as densities of degree $-1/2$. Therefore the ratio η of two solutions transforms as a function under a diffeomorphism action. $\square$

In short, to calculate the coadjoint action on the potential u one can first pass from this potential to the ratio of two solutions of the corresponding Hill equation, then change the variable in the ratio, and finally take the Schwarzian derivative of the new ratio to reconstruct the new potential $\mathrm{Ad}^*_{\varphi^{-1}}u$ (see Figure 2.1).



(i) Two solutions of $\partial^2 y + uy = 0$ (ii) $\eta = f/g$

(iii) $\widetilde{\eta} = \widetilde{f}/\widetilde{g}$ (iv) Two solutions of $\partial^2 y + \widetilde{u}y = 0$

Fig. 2.1. Schematic picture of the action of a diffeomorphism of S^1 on Hill's operators and their solutions: To two solutions of the equation $(\partial^2 + u)y = 0$, one associates their ratio $\eta : \mathbb{R} \to \mathbb{R}P^1$. A diffeomorphism φ acts on the ratio η by reparametrization, and one reconstructs the corresponding solutions and Hill's operator $\partial^2 + \widetilde{u}$ from the new ratio $\widetilde{\eta}$.

Now we return to our goal, the classification of the Virasoro coadjoint orbits. Any (a = const)-hyperplane in the Virasoro dual $\mathfrak{vir}^*$ is invariant under the coadjoint action (see equation (2.10)), and identified with Hill's operators for $a \neq 0$. While all of the above considerations of Hill's operators were of local nature (local in θ), now we will make use of the fact that Hill's operators are periodic: $u(\theta)$ is defined on a circle.

Consider the universal covering $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of the group $\mathrm{SL}(2,\mathbb{R})$. The group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ admits an outer automorphism of order 2, taking the inverse of a matrix. One can see that the identity is its only fixed point on the universal covering group. Consider the set $(\widetilde{\mathrm{SL}}(2,\mathbb{R}) \setminus \{\mathrm{id}\})/\mathbb{Z}_2$, where we first dropped the identity element from $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ before taking the quotient. The main result of this section is the following theorem.

Theorem 2.12 ([342, 202]) *Given $a \neq 0$, there is a one-to-one correspondence between the set of coadjoint orbits of the Virasoro–Bott group in the hyperplane $\{a\partial_\theta^2 + u(\theta)\} \subset \mathfrak{vir}^*$ and the set of conjugacy classes in the quotient $(\widetilde{\mathrm{SL}}(2,\mathbb{R}) \setminus \{\mathrm{id}\})/\mathbb{Z}_2$*

Proof. Consider a pair (f, g) of linearly independent solutions of the Hill equation $(a\partial_\theta^2 + u(\theta))y = 0$. For a periodic potential $u(\theta)$ these solutions are quasiperiodic, i.e., the values $(f(\theta), g(\theta))$ and $(f(\theta+2\pi), g(\theta+2\pi))$ are related by a monodromy matrix $M \in \mathrm{SL}(2,\mathbb{R})$:

$$(f(\theta + 2\pi), g(\theta + 2\pi)) = (f(\theta), g(\theta))\, M\,. \tag{2.12}$$

Recall that we use the same notation θ for the coordinate on the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and on its cover $\mathbb{R}$. Note that the monodromy matrix M can be viewed as an element in the universal cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, where the lift to the cover is provided by the fundamental solution $(f(\theta), g(\theta))$, starting at the identity: $\begin{pmatrix} f & g \\ f' & g' \end{pmatrix}|_{\theta=0} = \mathrm{id}$.

Similarly, the values of the "projective solution," the ratios $\eta(\theta) := f(\theta)/g(\theta)$ and $\eta(\theta + 2\pi) := f(\theta + 2\pi)/g(\theta + 2\pi)$, are related by a Möbius transformation $\mathcal{M} \in \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm\,\mathrm{id}\}$. The monodromy matrix M (respectively $\mathcal{M}$) changes to a conjugate matrix if we pick a different pair of solutions (f, g) for the same differential equation.

Now regard the ratio $\eta = f/g$ for $\theta \in [0, 2\pi]$ as a map $\eta :\ [0, 2\pi] \to \mathbb{R}P^1$ describing a motion ("rotation") along the circle $\mathbb{R}P^1 \simeq S^1$. One can see that the condition $W \neq 0$ on the Wronskian is equivalent to the condition $\eta' = -W/g^2 \neq 0$, i.e., that the rotation "does not stop." Choosing the positive sign of the Wronskian, $W > 0$, we can assume that the rotation always goes in the negative direction: $\eta' < 0$.

Recall that the Virasoro action on η is, in fact, a circle reparametrization for the coordinate θ. By a diffeomorphism change of the coordinate $\theta \mapsto \varphi(\theta)$, one can always turn the map $\eta :\ [0, 2\pi] \to \mathbb{R}P^1$ into a *uniform* rotation along $\mathbb{R}P^1$, while keeping the boundary values of $\eta(\theta)$ on the segment $[0, 2\pi]$ satisfying the monodromy relation $\eta(\theta + 2\pi) = \eta(\theta)\mathcal{M}$. Furthermore, the number of rotations (the "winding number") for the map $\eta: [0, 2\pi] \to \mathbb{R}P^1$ does not change under a reparametrization by a circle diffeomorphism φ. In other words, the orbits of the maps η (or, equivalently, of the potentials $\{u(x)\}$) are described by one continuous parameter (the conjugacy class of M) and one discrete parameter (the winding number). One can see that these two parameters together encode nothing else but the conjugacy class of the monodromy matrices M in the universal covering of $\mathrm{SL}(2, \mathbb{R})$.

Note that the choice in the sign of the Wronskian reflects the $\mathbb{Z}_2$-action on the universal covering $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Indeed, with this choice ($W > 0$) the path η always goes in the negative direction ($\eta' < 0$), so one can reach only the "negative half" of the conjugacy classes in the universal cover $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

Finally, note that the identity matrix in the universal covering $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ (or in its projectivization $\widetilde{\mathrm{SL}}(2, \mathbb{R})/\mathbb{Z}_2$) cannot be obtained as a monodromy matrix for the maps $\eta: [0, 2\pi] \to \mathbb{R}P^1$. Indeed, any map η starting at the identity has to move out from it, since $\eta'(0) \neq 0$. □

Corollary 2.13 *The Virasoro orbits in the hyperplane* $\{a\partial_\theta^2 + u(\theta) \mid a = a_0\} \subset \mathfrak{vir}^*$ *with fixed* $a_0 \neq 0$ *are classified by the Jordan normal form of matrices in* $\mathrm{SL}(2, \mathbb{R})$ *and by a positive integer parameter, the winding number. In this hyperplane* $\{a = a_0\}$ *of the dual* $\mathfrak{vir}^*$ *the orbit containing Hill's operator* $a\partial_\theta^2 + u(\theta)$ *has codimension equal to the codimension in* $\mathrm{SL}(2, \mathbb{R})$ *of the conjugacy class of the monodromy matrix* M *corresponding to this Hill's operator.*

Matrices in the group $\mathrm{SL}(2, \mathbb{R})$ split into three classes, whose normal forms are the exponentials of the following three classes in the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, the complexification of $\mathfrak{sl}(2, \mathbb{R})$:

$$(i) \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \quad (ii) \begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix}, \text{ and } \quad (iii) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \tag{2.13}$$

see Figure 2.2. The codimensions of the corresponding conjugacy classes in $\mathrm{SL}(2, \mathbb{R})$ are 1 in cases (i) and (ii), and 3 in case (iii). Note that the set of real matrices that are exponentials of (i)-type matrices consists of the elliptic and hyperbolic parts: rotation matrices (for $\mu \in i\mathbb{R}$) and hyperbolic rotations (for $\mu \in \mathbb{R}$). Furthermore, hyperbolic rotations correspond to one-sheeted hyperboloids. Rotations in the clockwise and counterclockwise directions correspond to different sheets of two-sheeted hyperboloids, and they belong to different conjugacy classes in $\mathrm{SL}(2, \mathbb{R})$. (The rotation by 180° has a three-dimensional stabilizer and corresponds to a one-point conjugacy class.) The group $\mathrm{SL}(2, \mathbb{R})$ is topologically a solid torus, and the adjacency of conjugacy classes described in Figure 2.2 is observed near both id and $-$id in this group.

The equality of the codimensions of the Virasoro coadjoint orbits in $\mathfrak{vir}^*$ and the codimensions of (the conjugacy classes of) the corresponding

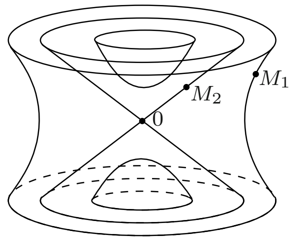


Fig. 2.2. The points M_1, M_2, and 0 in $\mathfrak{sl}(2,\mathbb{R})$ (which is a local picture of the group $\mathrm{SL}(2,\mathbb{R})$) correspond to Virasoro orbits of types (i), (ii), and (iii) respectively.

monodromy matrices in $\mathrm{SL}(2,\mathbb{R})$ follows from the smooth dependence on a parameter in the above classification. (The versal deformations of the Virasoro orbits can be defined in terms of the Jordan–Arnold normal forms of the monodromy matrices depending on a parameter; cf. [15, 233, 306].) Alternatively, one can describe the dimensions of the corresponding stabilizers; see [202, 342]. To visualize (the three-dimensional transversal to) the set of the Virasoro orbits, one can imagine the universal covering of $\mathrm{SL}(2,\mathbb{R})$ as a cylinder filled with an infinite number of copies of Figure 2.2, stacked one on top of another, while the $\mathbb{Z}_2$-quotient keeps only "half" of this infinite cylinder.

Remark 2.14 Regarded as homogeneous spaces, the orbits of type (i) are often denoted by $\mathrm{Diff}(S^1)/S^1$, the notation $\mathrm{Diff}(S^1)/\mathbb{R}^1$ stands for (ii) (and sometimes for the case $\mu \in \mathbb{R}$ in (i)), and $\mathrm{Diff}(S^1)/\,\mathrm{SL}(2,\mathbb{R})$ corresponds to (iii).

To see the reasoning for this, we describe the stabilizers for coadjoint orbits containing constant elements, i.e., Hill's operators $\partial_\theta^2 + u(\theta)$ with constant potentials $u(\theta) \equiv p = \mathrm{const}$. For such an operator, the corresponding monodromy matrix $M_p \in \mathrm{SL}(2,\mathbb{R})$ is given explicitly:

$$M_p = \begin{pmatrix} \cos(2\pi\sqrt{p}) & \frac{1}{\sqrt{p}}\sin(2\pi\sqrt{p}) \\ -\sqrt{p}\sin(2\pi\sqrt{p}) & \cos(2\pi\sqrt{p})\,, \end{pmatrix} \qquad \text{for } \; p > 0\,,$$

$$M_p = \begin{pmatrix} \cosh(2\pi\sqrt{-p}) & \frac{1}{\sqrt{-p}}\sinh(2\pi\sqrt{-p}) \\ \sqrt{-p}\sinh(2\pi\sqrt{-p}) & \cosh(2\pi\sqrt{-p}) \end{pmatrix} \qquad \text{for } \; p < 0\,,$$

and

$$M_0 = \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix} \qquad \text{for } \; p = 0\,.$$

One can see that for Hill's differential operators with potentials $p < 0$ or $p = 0$ the stabilizer is $\mathbb{R}$. In the case $p > 0$ it is the group S^1 of rigid rotations, provided that $M_p \neq \pm\,\mathrm{id} \in \mathrm{SL}(2,\mathbb{R})$. Finally, the stabilizer is three-dimensional, once the monodromy M_p is plus or minus the identity matrix, i.e., the exponential of type (iii) in the list (2.13). This can be the case only if $p = m^2/4$

for some $m \in \mathbb{N}$. In the latter case, the stabilizer of the corresponding Hill's equation is the m-fold covering of $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{Diff}(S^1)$.

Finally, we note that the trace function $\mathrm{tr} : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}$ is invariant on conjugacy classes in $\mathrm{SL}(2,\mathbb{R})$ and it gives rise to a Casimir function on the hyperplane $\{a = a_0\} \subset \mathfrak{vir}^*$ for the Lie–Poisson bracket: $\mathrm{tr}\{a_0\partial_\theta^2 + u(\theta)\} := \mathrm{tr}(M)$, where M is a monodromy matrix of the given Hill operator. Along with the value of a, this is the only Casimir for generic Virasoro orbits, since their codimension in the hyperplane $\{a = a_0\}$ is equal to 1.

2.4 The Virasoro–Bott Group and the Korteweg–de Vries Equation

The *Korteweg–de Vries* (or KdV) equation is the nonlinear evolution equation

$$u_t = -3uu' - au''' , \tag{2.14}$$

which describes traveling waves in a shallow canal. Here, u is a function of the time variable t and one space variable θ, u_t and u' denote the corresponding partial derivatives in t and θ, and a is a nonzero constant. A brief history of this equation can be found in [294].

In this section we show how the Korteweg–de Vries equation appears as the Euler equation with respect to a certain right-invariant metric on the Virasoro–Bott group. Recall that the Euler equation with respect to a right-invariant metric on a Lie group G is a dynamical system on the corresponding Lie algebra $\mathfrak{g}$ describing the evolution of the tangent vector along a geodesic on G, where this vector is pulled back to the Lie algebra of G by right translation; see Section I.4.

Consider the L^2-inner product on the Virasoro algebra $\mathfrak{vir} = \mathrm{Vect}(S^1) \oplus \mathbb{R}$ defined by

$$\langle (v(\theta)\partial_\theta, a), (w(\theta)\partial_\theta, c)\rangle = \int_{S^1} v(\theta)w(\theta)d\theta + a \cdot c\,. \tag{2.15}$$

Extend this quadratic form to every tangent space on the Virasoro–Bott group by right translations to define a (weak) right-invariant L^2-metric on the group.

Theorem 2.15 ([305]) *The Euler equation for the right-invariant L^2-metric on the Virasoro group is (the family of) the KdV equation:*

$$u_t = -3uu' - au''' , \tag{2.16}$$

$$a_t = 0\,. \tag{2.17}$$

PROOF. According to Arnold's Theorem I.4.14, the Euler equation on $\mathfrak{g}^*$ for the *right-invariant* metric on the group G has the form

$$\frac{d}{dt}m(t) = \mathrm{ad}^*_{A^{-1}m(t)}\, m(t)\,, \tag{2.18}$$

where $m(t)$ is a point in $\mathfrak{g}^*$, and $A : \mathfrak{g} \to \mathfrak{g}^*$ is the inertia operator defined by the metric (using the left-invariant metric would give the minus sign in the equation; cf. Remark I.4.16).

In the Virasoro coadjoint action, an element $(v\partial_\theta, c) \in \mathfrak{vir}$ of the Virasoro algebra acts on an element $(u(d\theta)^2, a) \in \mathfrak{vir}^*$ of the dual space via

$$\mathrm{ad}^*_{(v\partial_\theta,c)}(u(d\theta)^2, a) = ((-2v'u - vu' - av''')(d\theta)^2, 0)\,;$$

see equation (2.8). Furthermore, the L^2-inner product gives rise to the "identity" inertia operator $A : \mathfrak{vir} \to \mathfrak{vir}^*$:

$$(u\partial_\theta, a) \mapsto (u(d\theta)^2, a)\,,$$

mapping a vector field $u(\theta)\partial_\theta$ to the quadratic differential $u(\theta)(d\theta)^2$ with the same function $u(\theta)$.

Then, substituting $(u(d\theta)^2, a)$ for m, we see that the Euler equation (2.18) becomes

$$\frac{d}{dt}(u(d\theta)^2, a) = ((-3uu' - au''')(d\theta)^2, 0)\,,$$

from which one immediately reads off the KdV equation (2.16), (2.17). □

The component a does not change with time (see equation (2.17))and plays the role of a constant parameter in the KdV equation. It has the physical meaning of the characteristic thickness of the shallow-water approximation (see, e.g., [228], p. 169).

Remark 2.16 One can study more general metrics on the Virasoro algebra, some of which are of particular interest in mathematical physics. Consider, for instance, the following two-parameter family of weighted $H^1_{\alpha,\beta}$-inner products on $\mathfrak{vir}$:

$$\langle (v\partial_\theta, b), (w\partial_\theta, c)\rangle_{H^1_{\alpha,\beta}} = \int_{S^1} (\alpha\, vw + \beta\, v'w')\, d\theta + bc\,.$$

The case $\alpha = 1, \quad \beta = 0$ corresponds to the L^2 inner product above, while $\alpha = \beta = 1$ corresponds to the H^1 product.

Theorem 2.17 ([192]) *The Euler equations for the right-invariant $H^1_{\alpha,\beta}$-metric (with $\alpha \neq 0$) on the Virasoro group are given by the following system:*

$$\alpha(u_t + 3uu') - \beta((u'')_t + 2u'u'' + uu''') + au''' = 0\,, \tag{2.19}$$

$$a_t = 0\,. \tag{2.20}$$

Exercise 2.18 Give a proof of the latter theorem along the lines of the proof of the L^2-case above. (Hint: The inertia operator for the weighted H^1 metric is

$$A : (v\partial_\theta, a) \mapsto ((\Lambda v)(d\theta)^2,\, a),$$

where $\Lambda := \alpha - \beta\partial_\theta^2$ is a second-order differential operator. Verify that in terms of $v = \Lambda^{-1}u$ the Euler equation has the form

$$\frac{d}{dt}(\Lambda v) = -2(\Lambda v)v' - (\Lambda v')v + av''' \,,$$

which is equivalent to equation (2.19).)

Remark 2.19 For $\alpha = 1$, $\beta = 0$, equation (2.19) is the KdV equation (2.16). For $\alpha = \beta = 1$, one recovers the *Camassa–Holm equation* (see [268]). For $\alpha = 0$, $\beta = 1$, equation (2.19) becomes the *Hunter–Saxton equation.* We note that in the case of $\alpha = 0$, the $H^1_{\alpha,\beta}$-metric becomes the homogeneous $\dot{H}^1$-metric, which is degenerate. Therefore, to define the Euler equations one has to pass to the homogeneous space $\widehat{\mathrm{Diff}}(S^1)/S^1$ (or $\mathrm{Diff}(S^1)/S^1$) and define the geodesic flow on it; see details in [192]. It turns out that the space $\mathrm{Diff}(S^1)/S^1$ equipped with the $\dot{H}^1$-metric is isometric to an open subset of an L^2-sphere; see [237, 238]. This isometry, in particular, allows one to extend solutions of the Hunter–Saxton equation beyond breaking time and interpret them after wave-breaking in an appropriate weak sense.

We also note that the case $a = 0$ corresponds to the nonextended Lie algebra $\mathrm{Vect}(S^1)$ of vector fields on the circle, rather than to the Virasoro algebra $\mathfrak{vir}$. In the nonextended case, depending on the values of α and β, one obtains the Hopf (or inviscid Burgers) equation $u_t + 3uu' = 0$ or the nonextended Camassa–Holm equation [69, 127]

$$u_t + 3uu' + 2u'u'' + uu''' + (u'')_t = 0\,.$$

2.5 The Bi-Hamiltonian Structure of the KdV Equation

The KdV equation is not only a Hamiltonian system; it also exhibits strong integrability properties. As we discussed before, there are various definitions of what an integrable infinite-dimensional system is: one can require from the system either the existence of action-angle coordinates, or the existence of "sufficiently many" independent integrals of motion, or some other properties, which may differ substantially in infinite dimensions. In this section we show that the KdV equation is not only Hamiltonian, but in fact bi-Hamiltonian, thus exhibiting one of the "strongest forms" of integrability. More precisely, in addition to being Hamiltonian with respect to the Lie–Poisson bracket on the dual space $\mathfrak{vir}^*$, this equation turns out to be Hamiltonian with respect to another compatible Poisson structure on the same space.

Recall that for any Lie algebra $\mathfrak{g}$, every point $m_0 \in \mathfrak{g}^*$ gives rise to a "constant" Poisson bracket $\{\ ,\ \}_0$ on $\mathfrak{g}^*$ by "freezing" the usual Lie–Poisson bracket $\{\ ,\ \}_{LP}$ at the point m_0. The constant Poisson bracket is defined for smooth functions f, g on $\mathfrak{g}^*$ by

$$\{f, g\}_0(m) := \langle [df_m, dg_m], m_0 \rangle\,;$$

see Section I.4.4. Furthermore, the Poisson brackets $\{\ ,\ \}_{LP}$ and $\{\ ,\ \}_0$ are compatible for all choices of the point m_0 (see Lemma I.4.21). The main goal of this section is to show that for a certain choice of the "freezing point" $m_0 \in \mathfrak{vir}^*$, the KdV equation is Hamiltonian with respect to the constant Poisson structure $\{\ ,\ \}_0$. Note that other equations discussed above (Camassa–Holm, Hunter–Saxton) have a similar bi-Hamiltonian structure, but related to different choices of the "freezing point"; see [192].

Theorem 2.20 *The KdV equation (2.14) is Hamiltonian with respect to the constant Poisson bracket on $\mathfrak{vir}^*$ with the "freezing point" $m_0 = (\frac{1}{2}(d\theta)^2, 0) \in \mathfrak{vir}^*$.*

Proof. Let $F(u, a)$ be a function on $\mathfrak{vir}^*$ and let $(v\partial_\theta, b) := (\delta F/\delta u, \delta F/\delta a) \in \mathfrak{vir}$ be the (variational) derivative $dF_{(u(d\theta)^2, a)}$ of F at $(u(d\theta)^2, a)$. Then the Hamiltonian equation with the Hamiltonian function F, computed with respect to the constant Poisson structure "frozen" at $(u_0(dx)^2, a_0)$, has the form

$$\frac{d}{dt}(u(d\theta)^2, a) = \mathrm{ad}^*_{(v\partial_\theta, b)}(u_0(d\theta)^2, a_0) = -\big((2u_0v' + (u_0)'v + a_0v''')(d\theta)^2, 0\big)\,.$$

(Here we use Remark I.4.22 and the explicit form (2.8) of the coadjoint action ad^* for the Virasoro algebra.)

Now specifying the "freezing" point to $(u_0(d\theta)^2, a_0) = (\frac{1}{2}(d\theta)^2, 0) \in \mathfrak{vir}^*$, we come to

$$\frac{d}{dt}(u(d\theta)^2, a) = -(v'(d\theta)^2, 0)\,, \tag{2.21}$$

where v is defined as the "partial derivative" of the functional F, i.e., $v\partial_\theta = \delta F/\delta u$.

Next, consider the functional F of the form

$$F(u, a) = \int_{S^1} \left(\frac{1}{2}u^3 - \frac{a}{2}(u')^2\right) d\theta\,.$$

By definition, the *variational derivative* $(\delta F/\delta u, \delta F/\delta a) \in \mathfrak{vir}$ of the functional F is determined by the following identity satisfied for any $(\xi(d\theta)^2, c) \in \mathfrak{vir}^*$:

$$\left\langle (\xi(d\theta)^2, c)\,, \left(\frac{\delta F}{\delta u}, \frac{\delta F}{\delta a}\right)\right\rangle = \frac{d}{d\epsilon}\Big|_{\epsilon=0} F(u + \epsilon\xi, a + \epsilon c).$$

For equation (2.21) we need only the partial derivative $\delta F/\delta u$, and we find it as follows:

$$\begin{aligned}\frac{d}{d\epsilon}\Big|_{\epsilon=0} F(u+\epsilon\xi, a) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{S^1} \left(\frac{1}{2}(u+\epsilon\xi)^3 - \frac{a}{2}(u'+\epsilon\xi')^2\right) d\theta \\ &= \int_{S^1} \left(\frac{3}{2}u^2\xi - au'\xi'\right) d\theta = \int_{S^1} \left(\frac{3}{2}u^2\xi + au''\xi\right) d\theta \\ &= \left\langle \left(\left(\frac{3}{2}u^2 + au''\right)\partial_\theta, a\right), (\xi(d\theta)^2, 0)\right\rangle .\end{aligned}$$

Hence we obtain the derivative $\delta F/\delta u = (\frac{3}{2}u^2 + au'')\partial_\theta$. Since $v\partial_\theta = \delta F/\delta u$, now we substitute $v = \frac{3}{2}u^2 + au''$ into the equation (2.21), which gives

$$u_t = -3uu' - au''' ,$$

the KdV equation. □

Corollary 2.21 *The KdV equation is bi-Hamiltonian with respect to the compatible Poisson structures $\{\ ,\ \}_{LP}$ and $\{\ ,\ \}_0$ on $\mathfrak{vir}^*$, where $\{\ ,\ \}_0$ denotes the constant Poisson structure, frozen in the point $(\frac{1}{2}(d\theta)^2, 0) \in \mathfrak{vir}^*$.*

The constant bracket for the KdV is usually called the *first KdV structure*, or the Gardner–Faddeev–Zakharov bracket [140, 392], while the linear Lie–Poisson structure is called the *second KdV*, or the Magri *bracket* [246]. The analogues of these structures for higher-order differential operators are called the first and second Adler–Gelfand–Dickey structures; see Section 4.

Remark 2.22 Similarly, one can show that both the Camassa–Holm and Hunter–Saxton equations are bi-Hamiltonian with respect to the Lie–Poisson structure and a constant structure on $\mathfrak{vir}^*$. The respective "freezing" points of the constant Poisson bracket on $\mathfrak{vir}^*$ are $m_1 = (\frac{1}{2}(d\theta)^2, 1)$ for the Camassa–Holm equation and $m_2 = (0, 1)$ for the Hunter–Saxton equation (see Figure 2.3).

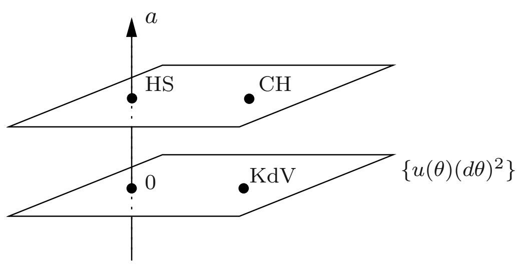


Fig. 2.3. The freezing points of the constant Poisson bracket on $\mathfrak{vir}^*$ that give rise to the bi-Hamiltonian structures for the KdV, the Camassa–Holm, and the Hunter–Saxton equations respectively.

Remark 2.23 The bi-Hamiltonian nature of the KdV equation allows one to obtain the whole hierarchy of the KdV Hamiltonians via the Lenard–Magri scheme discussed in the introduction.

Namely, consider the linear combination of the Poisson brackets for the KdV: the Virasoro Lie–Poisson and the constant ones:

$$\{\ ,\ \}_\lambda := \{\ ,\ \}_{LP} + \lambda^2\{\ ,\ \}_0\,.$$

Since the corresponding brackets are compatible, $\{\ ,\ \}_\lambda$ defines a Poisson bracket for all $\lambda \in \mathbb{R}$, which is the usual Lie–Poisson bracket on $\mathfrak{vir}^*$ shifted in the direction of $m_0 = (\frac{1}{2}(d\theta)^2, 0) \in \mathfrak{vir}^*$. (Note that one uses here the parameter λ^2 rather than λ for the expansion below to have a simpler form.)

Recall that we think of the dual space $\mathfrak{vir}^* = \{a\partial_\theta^2 + u(\theta)\}$ as the space of Hill's operators. The monodromy of a differential operator $\partial_\theta^2 + u(\theta)$, which is a matrix in $\mathrm{SL}(2,\mathbb{R})$, changes to a conjugate matrix under the Virasoro coadjoint action (see Theorem 2.12). Therefore, the trace of the monodromy is a Casimir function for the Poisson bracket $\{\ ,\ \}_{LP}$ on (the hyperplane $a = 1$ in) the dual space $\mathfrak{vir}^*$. The same reasoning allows one to obtain the following result.

Lemma 2.24 *Let M_λ denote the monodromy of the differential operator $\frac{d^2}{d\theta^2} + u(\theta) - \lambda^2$. Then the function*

$$h_\lambda := \log(\mathrm{tr}(M_\lambda))$$

is a Casimir function for the Poisson bracket $\{\ ,\ \}_\lambda$ on the space $\mathfrak{vir}^*$.

Indeed, $\{\ ,\ \}_\lambda$ is the usual Lie–Poisson bracket on $\mathfrak{vir}^*$ shifted in the direction of $m_0 = (0\cdot\partial_\theta^2 + 1/2) \in \mathfrak{vir}^*$, and so, instead of the Lie–Poisson Casimir $\mathrm{tr}(M)$, we can use the shifted Casimir $\mathrm{tr}(M_\lambda)$ or any function of it, in particular, $\log(\mathrm{tr}(M_\lambda))$.

Finally, by expanding h_λ into a power series in λ^{-1} one produces first integrals of the KdV equation:

$$h_\lambda = 2\pi\lambda - \sum_{n=1}^{\infty} h_{2n-1}\lambda^{1-2n}\,,$$

where

$$h_1 = \frac{1}{2}\int_{S^1} u\,d\theta, \quad h_3 = \frac{1}{8}\int_{S^1} u^2\,d\theta, \quad h_5 = \frac{1}{16}\int_{S^1} (u^3 - \frac{1}{2}(u')^2)\,d\theta, \quad \ldots;$$

see, e.g., [31, 37].

2.6 Bibliographical Notes

The cohomology of the Lie algebra of vector fields on the circle was computed by Gelfand and Fuchs in [143], where the term Virasoro algebra was coined after the paper [376] (see also [118, 138]). The Bott cocycle on the group $\mathrm{Diff}(S^1)$ first appeared in [53].

The group $\mathrm{Diff}(S^1)$ (and hence the Virasoro group) does not admit a natural complexification, i.e., a group corresponding to the complexified Lie algebra $\mathrm{Vect}(S^1)_{\mathbb{C}}$; see [205, 322] and Example I.1.25. However, for a cone in this complex Lie algebra consisting of those vector fields that "point outside of the circle," there exists a semigroup. This *annulus semigroup* appeared in the papers of Neretin [291] and Segal [343]. For more details on representations and applications of the Virasoro group and algebra see [292, 322, 153].

The classification of the coadjoint orbits of the Virasoro group can be found in the literature under different guises: as a classification of projective structures on the circle by Kuiper [223], as a classification of Hill's operators by Lazutkin and Pankratova [233], and in the present form, as Virasoro orbits, in the papers by Kirillov [202, 205] and Segal [342]; see also [164, 304, 388, 32]. The adjoint orbits of the diffeomorphism group of the circle were described in [164].

The Virasoro coadjoint orbit $\mathrm{Diff}(S^1)/S^1$ can also be understood as the universal Teichmüller space [326]; cf. [284, 285]. Curvatures of a Kähler metric on this orbit were described in [208], while its complex Hilbert manifold structure is discussed in [338, 363]. The Virasoro group itself admits a complex structure and can be viewed as a holomorphic $\mathbb{C}^*$-bundle over its orbit $\mathrm{Diff}(S^1)/S^1$ [236].

There is a vast literature related to the geometry and Hamiltonian properties of the KdV equation, which is one of the key examples in any book on soliton theory. The description of the KdV equation, as well as its superanalogue, as an Euler–Arnold equation on the Virasoro group with respect to the L^2-metric can be found in [305, 344, 346]. More general, H^1-type, metrics on this group were considered in [268, 189, 192], and we followed the latter paper in our exposition. Regularity properties of the Riemannian exponential maps for these and other Sobolev metrics on the Virasoro and the diffeomorphism groups are described in [73, 74].

The Adler–Gelfand–Dickey structures [3, 141, 142] are the generalizations of the KdV Poisson structures from Hill's operators to linear differential operators of higher order, and we discuss them in Section 4.

For the algebro-geometric approach to constructing solutions of the KdV equation we address the reader to [216]; the description of the corresponding infinite-dimensional Grassmann manifolds can be found in [347] and the references therein. Various analytical aspects of the KdV theory, its spectral theory, and the angle-action variables are discussed in the book [182]. The geometry related to the KAM theory for near-integrable Hamiltonian systems, applied in the infinite-dimensional context, e.g., to the KdV-type equations, is discussed in [224, 182].

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