

---

## Contents

<b>Preface</b> .....	VII
<b>Introduction</b> .....	1
<b>I Preliminaries</b> .....	7
1 Lie Groups and Lie Algebras .....	7
1.1 Lie Groups and an Infinite-Dimensional Setting .....	7
1.2 The Lie Algebra of a Lie Group .....	9
1.3 The Exponential Map .....	12
1.4 Abstract Lie Algebras .....	15
2 Adjoint and Coadjoint Orbits .....	17
2.1 The Adjoint Representation .....	17
2.2 The Coadjoint Representation .....	19
3 Central Extensions .....	21
3.1 Lie Algebra Central Extensions .....	22
3.2 Central Extensions of Lie Groups .....	24
4 The Euler Equations for Lie Groups .....	26
4.1 Poisson Structures on Manifolds .....	26
4.2 Hamiltonian Equations on the Dual of a Lie Algebra ...	29
4.3 A Riemannian Approach to the Euler Equations .....	30
4.4 Poisson Pairs and Bi-Hamiltonian Structures .....	35
4.5 Integrable Systems and the Liouville–Arnold Theorem .	38
5 Symplectic Reduction .....	40
5.1 Hamiltonian Group Actions .....	41
5.2 Symplectic Quotients .....	42
6 Bibliographical Notes .....	44
<b>II Infinite-Dimensional Lie Groups: Their Geometry, Orbits, and Dynamical Systems</b> .....	47
1 Loop Groups and Affine Lie Algebras .....	47
1.1 The Central Extension of the Loop Lie algebra .....	47

1.2	Coadjoint Orbits of Affine Lie Groups . . . . .	52
1.3	Construction of the Central Extension of the Loop Group . . . . .	58
1.4	Bibliographical Notes . . . . .	65
2	Diffeomorphisms of the Circle and the Virasoro–Bott Group . .	67
2.1	Central Extensions . . . . .	67
2.2	Coadjoint Orbits of the Group of Circle Diffeomorphisms	70
2.3	The Virasoro Coadjoint Action and Hill’s Operators . . .	72
2.4	The Virasoro–Bott Group and the Korteweg–de Vries Equation . . . . .	80
2.5	The Bi-Hamiltonian Structure of the KdV Equation . . .	82
2.6	Bibliographical Notes . . . . .	86
3	Groups of Diffeomorphisms . . . . .	88
3.1	The Group of Volume-Preserving Diffeomorphisms and Its Coadjoint Representation . . . . .	88
3.2	The Euler Equation of an Ideal Incompressible Fluid . . .	90
3.3	The Hamiltonian Structure and First Integrals of the Euler Equations for an Incompressible Fluid . . . .	91
3.4	Semidirect Products: The Group Setting for an Ideal Magnetohydrodynamics and Compressible Fluids . . . . .	95
3.5	Symplectic Structure on the Space of Knots and the Landau–Lifschitz Equation . . . . .	99
3.6	Diffeomorphism Groups as Metric Spaces . . . . .	105
3.7	Bibliographical Notes . . . . .	109
4	The Group of Pseudodifferential Symbols . . . . .	111
4.1	The Lie Algebra of Pseudodifferential Symbols . . . . .	111
4.2	Outer Derivations and Central Extensions of $\psi$ DS . . . .	113
4.3	The Manin Triple of Pseudodifferential Symbols . . . . .	117
4.4	The Lie Group of $\alpha$ -Pseudodifferential Symbols . . . . .	119
4.5	The Exponential Map for Pseudodifferential Symbols . .	122
4.6	Poisson Structures on the Group of $\alpha$ -Pseudodifferential Symbols . . . . .	124
4.7	Integrable Hierarchies on the Poisson Lie Group $\tilde{G}_{\text{INT}}$ .	129
4.8	Bibliographical Notes . . . . .	132
5	Double Loop and Elliptic Lie Groups . . . . .	134
5.1	Central Extensions of Double Loop Groups and Their Lie Algebras . . . . .	134
5.2	Coadjoint Orbits . . . . .	136
5.3	Holomorphic Loop Groups and Monodromy . . . . .	138
5.4	Digression: Definition of the Calogero–Moser Systems . .	142
5.5	The Trigonometric Calogero–Moser System and Affine Lie Algebras . . . . .	146
5.6	The Elliptic Calogero–Moser System and Elliptic Lie Algebras . . . . .	149
5.7	Bibliographical Notes . . . . .	152

<b>III Applications of Groups: Topological and Holomorphic Gauge Theories</b>	155
1 Holomorphic Bundles and Hitchin Systems	155
1.1 Basics on Holomorphic Bundles	155
1.2 Hitchin Systems	159
1.3 Bibliographical Notes	162
2 Poisson Structures on Moduli Spaces	163
2.1 Moduli Spaces of Flat Connections on Riemann Surfaces	163
2.2 Poincaré Residue and the Cauchy–Stokes Formula	170
2.3 Moduli Spaces of Holomorphic Bundles	173
2.4 Bibliographical Notes	179
3 Around the Chern–Simons Functional	180
3.1 A Reminder on the Lagrangian Formalism	180
3.2 The Topological Chern–Simons Action Functional	184
3.3 The Holomorphic Chern–Simons Action Functional	187
3.4 A Reminder on Linking Numbers	189
3.5 The Abelian Chern–Simons Path Integral and Linking Numbers	192
3.6 Bibliographical Notes	196
4 Polar Homology	197
4.1 Introduction to Polar Homology	197
4.2 Polar Homology of Projective Varieties	202
4.3 Polar Intersections and Linkings	206
4.4 Polar Homology for Affine Curves	209
4.5 Bibliographical Notes	211
<b>Appendices</b>	213
A.1 Root Systems	213
1.1 Finite Root Systems	213
1.2 Semisimple Complex Lie Algebras	215
1.3 Affine and Elliptic Root Systems	216
1.4 Root Systems and Calogero–Moser Hamiltonians	218
A.2 Compact Lie Groups	221
2.1 The Structure of Compact Groups	221
2.2 A Cohomology Generator for a Simple Compact Group	224
A.3 Krichever–Novikov Algebras	225
3.1 Holomorphic Vector Fields on $\mathbb{C}^*$ and the Virasoro Algebra	225
3.2 Definition of the Krichever–Novikov Algebras and Almost Grading	226
3.3 Central Extensions	228
3.4 Affine Krichever–Novikov Algebras, Coadjoint Orbits, and Holomorphic Bundles	231
A.4 Kähler Structures on the Virasoro and Loop Group Coadjoint Orbits	234

4.1	The Kähler Geometry of the Homogeneous Space $\text{Diff}(S^1)/S^1$ . . . . .	234
4.2	The Action of $\text{Diff}(S^1)$ and Kähler Geometry on the Based Loop Spaces . . . . .	237
A.5	Diffeomorphism Groups and Optimal Mass Transport . . . . .	240
5.1	The Inviscid Burgers Equation as a Geodesic Equation on the Diffeomorphism Group . . . . .	240
5.2	Metric on the Space of Densities and the Otto Calculus . . . . .	244
5.3	The Hamiltonian Framework of the Riemannian Submersion . . . . .	247
A.6	Metrics and Diameters of the Group of Hamiltonian Diffeomorphisms . . . . .	250
6.1	The Hofer Metric and Bi-invariant Pseudometrics on the Group of Hamiltonian Diffeomorphisms . . . . .	250
6.2	The Infinite $L^2$ -Diameter of the Group of Hamiltonian Diffeomorphisms . . . . .	252
A.7	Semidirect Extensions of the Diffeomorphism Group and Gas Dynamics . . . . .	256
A.8	The Drinfeld–Sokolov Reduction . . . . .	260
8.1	The Drinfeld–Sokolov Construction . . . . .	260
8.2	The Kupershmidt–Wilson Theorem and the Proofs . . . . .	263
A.9	The Lie Algebra $\mathfrak{gl}_\infty$ . . . . .	267
9.1	The Lie Algebra $\mathfrak{gl}_\infty$ and Its Subalgebras . . . . .	267
9.2	The Central Extension of $\mathfrak{gl}_\infty$ . . . . .	268
9.3	$q$ -Difference Operators and $\mathfrak{gl}_\infty$ . . . . .	269
A.10	Torus Actions on the Moduli Space of Flat Connections . . . . .	272
10.1	Commuting Functions on the Moduli Space . . . . .	272
10.2	The Case of $\text{SU}(2)$ . . . . .	274
10.3	$\text{SL}(n, \mathbb{C})$ and the Rational Ruijsenaars–Schneider System . . . . .	277
	<b>References</b> . . . . .	281
	<b>Index</b> . . . . .	301

The Geometry of Infinite-Dimensional Groups

Khesin, B.; Wendt, R.

2009, XII, 304 p., Softcover

ISBN: 978-3-540-85205-6