2

The Skorohod Integral

The Wiener–Itô chaos expansion is a convenient starting point for the introduction of several important stochastic concepts. In this chapter we focus on the *Skorohod integral*. This stochastic integral, introduced for the first time by A. Skorohod in 1975 [216], may be regarded as an extension of the Itô integral to integrands that are not necessarily $\mathbb{F}$-adapted, see also, for example, [30, 31]. The Skorohod integral is also connected to the Malliavin derivative, which is introduced with full detail in Chap. 3.

As for other extensions of the Itô integral closely related to the Skorohod integral, we can mention the *noncausal integral* (also called *Ogawa integral*) and refer to [174, 175]; see also [85].

2.1 The Skorohod Integral

Let $u = u(t, \omega)$, $t \in [0, T], \omega \in \Omega$, be a measurable stochastic process such that, for all $t \in [0, T]$, $u(t)$ is a $\mathcal{F}_T$-measurable random variable and

$$E[u^2(t)] < \infty.$$

Then, for each $t \in [0, T]$, we can apply the Wiener–Itô chaos expansion to the random variable $u(t) = u(t, \omega)$, $\omega \in \Omega$, and thus there exist the symmetric functions $f_{n,t} = f_{n,t}(t_1, \ldots, t_n)$, $(t_1, \ldots, t_n) \in [0, T]^n$, in $\widetilde{L}^2([0, T]^n)$, $n = 1, 2, \ldots$, such that

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}).$$

Note that the functions $f_{n,t}$, $n = 1, 2, \ldots$, depend on the parameter $t \in [0, T]$, and so we can write

$$f_n(t_1, \ldots, t_n, t_{n+1}) = f_n(t_1, \ldots, t_n, t) := f_{n,t}(t_1, \ldots, t_n)$$

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and we may regard f_n as a function of $n+1$ variables. Since this function is symmetric with respect to its first n variables, its *symmetrization* $\widetilde{f}_n$ is given by

$$\widetilde{f}_n(t_1,\ldots,t_{n+1}) = \frac{1}{n+1}\Big[f_n(t_1,\ldots,t_{n+1}) \\ +f_n(t_2,\ldots,t_{n+1},t_1)+\cdots+f_n(t_1,\ldots,t_{n-1},t_{n+1},t_n)\Big], \quad (2.1)$$

see (1.5).

Example 2.1. Let us consider

$$f_{2,t}(t_1,t_2) = f_2(t_1,t_2,t) = \frac{1}{2}\big[\chi_{\{t_1<t<t_2\}} + \chi_{\{t_2<t<t_1\}}\big].$$

Then the symmetrization $\widetilde{f}_2$ of f_2 is given by

$$\widetilde{f}_2(t_1,t_2,t_3) = \frac{1}{3}\Big[\frac{1}{2}(\chi_{\{t_1<t_3<t_2\}} + \chi_{\{t_2<t_3<t_1\}}) \\ +\frac{1}{2}(\chi_{\{t_1<t_2<t_3\}} + \chi_{\{t_3<t_2<t_1\}}) + \frac{1}{2}(\chi_{\{t_3<t_1<t_2\}} + \chi_{\{t_2<t_1<t_3\}})\Big],$$

which gives

$$\widetilde{f}_2(t_1,t_2,t_3) = \frac{1}{6}. \quad (2.2)$$

Definition 2.2. *Let $u(t)$, $t \in [0,T]$, be a measurable stochastic process such that for all $t \in [0,T]$ the random variable $u(t)$ is $\mathcal{F}_T$-measurable and $E[u^2(t)] < \infty$. Let its Wiener–Itô chaos expansion be*

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)).$$

Then we define the Skorohod integral *of u by*

$$\delta(u) := \int_0^T u(t)\delta W(t) := \sum_{n=0}^{\infty} I_{n+1}(\widetilde{f}_n) \quad (2.3)$$

when convergent in $L^2(P)$. Here $\widetilde{f}_n$, $n = 1,2,...$, are the symmetric functions (2.1) *derived from $f_n(\cdot,t)$, $n = 1,2,....$ We say that u is* Skorohod integrable, *and we write $u \in Dom(\delta)$ if the series in* (2.3) *converges in $L^2(P)$ (see also Problem 2.2).*

Remark 2.3. By (1.17) a stochastic process u belongs to $Dom(\delta)$ if and only if

$$E[\delta(u)^2] = \sum_{n=0}^{\infty}(n+1)!\|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty\,. \quad (2.4)$$

Note that (2.4) naturally implies that

$$\|u\|^2_{L^2(P\times\lambda)} := E\Big[\int_0^T u^2(t)dt\Big] < \infty,$$

so $Dom(\delta) \subseteq L^2(P\times\lambda)$. See Problem 2.1.

Example 2.4. Let us verify that

$$\int_0^T W(T)\delta W(t) = W^2(T) - T.$$

The Wiener–Itô chaos expansion of the integrand $u(t) = W(T) = \int_0^T 1\, dW(s)$, $t \in [0,T]$, is given by $f_0 = 0$, $f_1 = 1$, and $f_n = 0$ for $n \geq 2$. Hence

$$\delta(u) = I_2(\widetilde{f}_1) = I_2(1) = 2\int_0^T\int_0^{t_2} 1\, dW(t_1)dW(t_2) = W^2(T) - T.$$

Note that, even if the integrand does not depend on t, we have

$$\int_0^T W(T)\delta W(t) \neq W(T)\int_0^T \delta W(t).$$

This last remark illustrates that, *for $u \in Dom(\delta)$, even if G is an $\mathcal{F}_T$-measurable random variable such that $Gu \in Dom(\delta)$, we have in general that*

$$\int_0^T Gu(t)\delta W(t) \neq G\int_0^T u(t)\delta W(t). \tag{2.5}$$

Example 2.5. What is $\int_0^T W(t)\big[W(T) - W(t)\big]\delta W(t)$? Note that

$$\begin{aligned}\int_0^T\int_0^{t_2} \chi_{\{t_1<t<t_2\}}(t_1,t_2)dW(t_1)dW(t_2) &= \int_0^T W(t)\chi_{\{t<t_2\}}(t_2)dW(t_2)\\ &= W(t)\big[W(T) - W(t)\big].\end{aligned}$$

Hence

$$u(t) = W(t)\big[W(T) - W(t)\big] = I_2(f_2(\cdot,t)),$$

where

$$f_{2,t}(t_1,t_2) = f_2(t_1,t_2,t) = \frac{1}{2}\Big(\chi_{\{t_1<t<t_2\}} + \chi_{\{t_2<t<t_1\}}\Big).$$

Hence by Example 2.1 we have

$$\delta(u) = I_3(\widetilde{f}_2) = I_3(\frac{1}{6}) = \frac{1}{6}I_3(1) = \frac{1}{6}\Big[W^3(T) - 3T\,W(T)\Big].$$

2.2 Some Basic Properties of the Skorohod Integral

The reader accustomed with classical analysis and Itô stochastic integration may find (2.3) to be just a formal definition for an operator, which can hardly be matched with the general meaning of integral. The purpose of the two following sections is to motivate Definition 2.2, showing that the operator (2.3) is a meaningful stochastic integral having strong links with the Itô stochastic integral itself. In the forthcoming Chaps. 3 and 5, more will be said about the properties of the Skorohod integral.

First of all we recognize that, just like any integral in classical analysis, the Skorohod integral (2.3) is a *linear* operator:

$$L^2(P \times \lambda) \supseteq Dom(\delta) \ni u \quad \Longrightarrow \quad \delta(u) \in L^2(P).$$

See Problem 2.3.

Another typical property of integrals is the additivity on adjacent intervals of integration. This also holds for the Skorohod integral.

Proposition 2.6. *For any fixed* $t \in [0,T]$ *and* $u \in Dom(\delta)$ *we have* $\chi_{(0,t]}u \in Dom(\delta)$ *and* $\chi_{(t,T]}u \in Dom(\delta)$ *and*

$$\int_0^t u(s)\delta W(s) = \int_0^T \chi_{(0,t]}(s)u(s)\delta W(s) \text{ and } \int_t^T u(s)\delta W(s) = \int_0^T \chi_{(t,T]}(s)u(s)\delta W(s),$$

with

$$\int_0^T u(s)\delta W(s) = \int_0^t u(s)\delta W(s) + \int_t^T u(s)\delta W(s).$$

Proof The proof, based on the Wiener-Itô chaos expansions and (2.4), is left as an exercise. See Problem 2.4. □

Proposition 2.7. *For any* $u \in Dom(\delta)$ *the Skorohod integral has zero expectation, that is,*

$$E\big[\delta(u)\big] = 0. \tag{2.6}$$

Proof This is a trivial consequence of the fact that Itô integrals and thus also iterated Itô integrals have zero expectation. □

Here we address all those who associate the name of "integral" to the operators resulting from the classical construction, which defines the integral as some limit of certain finite sums derived from simple functions (e.g., Rieman integral, Lebesgue integral, and Itô integral). In some sense also the Skorohod integral can be regarded as such (see, e.g., [168]). A full characterization in this sense can be given in the white noise framework, see Theorem 5.20 and Corollary 5.21.

2.3 The Skorohod Integral as an Extension of the Itô Integral

As mentioned earlier, the Skorohod integral is an extension of the Itô integral. More precisely, if the integrand u is $\mathbb{F}$-adapted, then the two integrals coincide as elements of $L^2(P)$. To prove this, we need a characterization of adaptedness with respect to $\mathbb{F}$ in terms of the functions $f_n(\cdot,t)$, $n = 1, 2, ...$, in the chaos expansion.

Lemma 2.8. *Let $u = u(t)$, $t \in [0,T]$, be a measurable stochastic process such that, for all $t \in [0,T]$, the random variable $u(t)$ is $\mathcal{F}_T$-measurable and $E[u^2(t)] < \infty$. Let*

$$u(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$$

be its Wiener-Itô chaos expansion. Then u is $\mathbb{F}$-adapted if and only if

$$f_n(t_1,\dots,t_n,t) = 0 \qquad \text{if} \quad t < \max_{1\le i\le n} t_i\,. \tag{2.7}$$

The above equality is meant a.e. in $[0,T]^n$ with respect to Lebesgue measure.

Proof First note that for any $g \in \widetilde{L}^2([0,T]^n)$ we have

$$\begin{aligned} E[I_n(g)|\mathcal{F}_t] &= n!E[J_n(g)|\mathcal{F}_t] \\ &= n!E\Big[\int_0^T\int_0^{t_n}\cdots\int_0^{t_2} g(t_1,\dots,t_n)dW(t_1)\cdots dW(t_n)\Big|\mathcal{F}_t\Big] \\ &= n!\int_0^t\int_0^{t_n}\cdots\int_0^{t_2} g(t_1,\dots,t_n)dW(t_1)\cdots dW(t_n) \\ &= n!J_n(g(t_1,\dots,t_n)\cdot\chi_{\{\max t_i<t\}}) \\ &= I_n(g(t_1,\dots,t_n)\cdot\chi_{\{\max t_i<t\}}). \end{aligned} \tag{2.8}$$

Now, u is $\mathbb{F}$-adapted if and only if $E[u(t)|\mathcal{F}_t] = u(t)$. Namely, if and only if $\sum_{n=0}^{\infty} I_n(f_n(\cdot,t)) = \sum_{n=0}^{\infty} E[I_n(f_n(\cdot,t))|\mathcal{F}_t] = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)\cdot\chi_{\{\max t_i<t\}})$. And thus if and only if $f_n(t_1,\dots,t_n,t)\cdot\chi_{\{\max t_i<t\}} = f_n(t_1,\dots,t_n,t)$ a.e. in $[0,T]^n$ with respect to Lebesgue measure. By uniqueness of the sequence of deterministic functions in the Wiener-Itô chaos expansion and since the last identity is equivalent to (2.7), the lemma is proved. □

Theorem 2.9. *Let $u = u(t)$, $t \in [0,T]$, be a measurable $\mathbb{F}$-adapted stochastic process such that*

$$E\Big[\int_0^T u^2(t)dt\Big] < \infty.$$

Then $u \in Dom(\delta)$ and its Skorohod integral coincides with the Itô integral

$$\int_0^T u(t)\delta W(t) = \int_0^T u(t)dW(t). \tag{2.9}$$

Proof Let $u(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t))$ be the chaos expansion of $u(t)$. First note that by (2.1) and Lemma 2.8 we have

$$\widetilde{f}_n(t_1, \dots, t_n, t_{n+1}) = \frac{1}{n+1} f_n(t_{1,} \cdots, t_{j-1}, t_{j+1}, \dots, t_{n+1,} t_j),$$

where

$$j := \operatorname{argmax}_{1 \le i \le n+1} t_i.$$

Hence

$$\begin{aligned}
\|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} &= (n+1)! \int_{S_{n+1}} \widetilde{f}_n^2(t_1, \dots, t_{n+1}) dt_1 \cdots dt_{n+1} \\
&= \frac{(n+1)!}{(n+1)^2} \int_{S_{n+1}} f_n^2(t_1, \dots, t_{n+1}) dt_1 \cdots dt_{n+1} \\
&= \frac{n!}{n+1} \int_0^T \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} f_n^2(t_1, \dots, t_n, t) dt_1 \cdots dt_n dt \\
&= \frac{n!}{n+1} \int_0^T \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n^2(t_1, \dots, t_n, t) dt_1 \cdots dt_n dt \\
&= \frac{1}{n+1} \int_0^T \|f_n(\cdot, t)\|^2_{L^2([0,T]^n)} dt,
\end{aligned}$$

again by using Lemma 2.8. Hence, by (1.17),

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+1)! \|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} &= \sum_{n=0}^{\infty} n! \int_0^T \|f_n(\cdot, t)\|^2_{L^2([0,T]^n)} dt \\
&= \int_0^T \sum_{n=0}^{\infty} n! \|f_n(\cdot, t)\|^2_{L^2([0,T]^n)} dt \\
&= E\Big[\int_0^T u^2(t) dt \Big] < \infty.
\end{aligned}$$

This proves that $u \in Dom(\delta)$, see (2.4). Finally, we prove the relationship (2.9):

$$\begin{aligned}
\int_0^T u(t)dW(t) &= \sum_{n=0}^{\infty} \int_0^T I_n(f_n(\cdot,t))dW(t) \\
&= \sum_{n=0}^{\infty} \int_0^T n! \int_{0\le t_1\le\cdots\le t_n\le t} f_n(t_1,\ldots,t_n,t)dW(t_1)\cdots dW(t_n)dW(t) \\
&= \sum_{n=0}^{\infty} \int_0^T n!(n+1) \int_{0\le t_1\le\cdots\le t_n\le t_{n+1}} \widetilde{f}_n(t_1,\ldots,t_n,t_{n+1})dW(t_1) \\
&\qquad\qquad \cdots dW(t_n)dW(t_{n+1}) \\
&= \sum_{n=0}^{\infty}(n+1)!J_{n+1}(\widetilde{f}_n) = \sum_{n=0}^{\infty} I_{n+1}(\widetilde{f}_n) = \int_0^T u(t)\delta W(t).
\end{aligned}$$

By this the proof is complete. □

2.4 Exercises

Problem 2.1. Prove that $Dom(\delta) \subseteq L^2(P \times \lambda)$. [*Hint.* Use (2.4)].

Problem 2.2. Let $u(t), 0 \le t \le T$, be a measurable stochastic process such that

$$E\left[\int_0^T u^2(t)dt\right] < \infty.$$

Show that there exists a sequence of deterministic measurable kernels $f_n(t_1,...,t_n,t)$ on $[0,T]^{n+1}$ $(n \ge 0)$, with

$$\int_{[0,T]^{n+1}} f_n^2(t_1,...,t_n,t)dt_1...dt_n dt < \infty$$

such that all f_n are symmetric with respect to the variables $t_1,...,t_n$ and such that

$$u(t) = u(\omega,t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))(\omega), \qquad \omega \in \Omega, t \in [0,T],$$

with convergence in $L^2(P \times \lambda)$. [*Hint.* Consider approximations of $u(t)$, $t \in [0,T]$, in $L^2(P\times\lambda)$ of the form $\sum_{i=1}^m a_i(\omega)b_i(t)$, $m = 1,2,...$, where $a_i \in L^2(P)$ and $b_i \in L^2([0,T])$.]

Problem 2.3. Prove the linearity of the Skorohod integral. [*Hint.* See Problem 1.2.]

Problem 2.4. Prove Proposition 2.6.

Problem 2.5. (*) Compute the following Skorohod integrals:

(a) $\int_0^T W(t)\delta W(t)$,

(b) $\int_0^T \Big(\int_0^T g(s)dW(s)\Big)\delta W(t)$, for a given function $g \in L^2([0,T])$,

(c) $\int_0^T W^2(t_0)\delta W(t)$, where $t_0 \in [0,T]$ is fixed,

(d) $\int_0^T \exp\{W(T)\}\delta W(t)$ [*Hint*. Use Problem 1.3.],

(e) $\int_0^T F\delta W(t)$, where $F = \int_0^T g(s)W(s)ds$, with $g \in L^2([0,T])$ [*Hint*. Use Problem 1.3].

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