

Chapter 17

Hom-Lie Admissible Hom-Coalgebras and Hom-Hopf Algebras

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Abstract The aim of this paper is to develop the coalgebra counterpart of the notions introduced by the authors in a previous paper, we introduce the notions of Hom-coalgebra, Hom-coassociative coalgebra and G -Hom-coalgebra for any subgroup G of permutation group \mathcal{S}_3 . Also we extend the concept of Lie-admissible coalgebra by Goze and Remm to Hom-coalgebras and show that G -Hom-coalgebras are Hom-Lie admissible Hom-coalgebras, and also establish duality correspondence between classes of G -Hom-coalgebras and G -Hom-algebras. In another hand, we provide relevant definitions and basic properties of Hom-Hopf algebras generalizing the classical Hopf algebras and define the module and comodule structure over Hom-associative algebra or Hom-coassociative coalgebra.

17.1 Introduction

In [4, 7, 8], the class of quasi-Lie algebras and subclasses of quasi-hom-Lie algebras and Hom-Lie algebras have been introduced. These classes of algebras are tailored in a way suitable for simultaneous treatment of the Lie algebras, Lie superalgebras, the color Lie algebras and the deformations arising in connection with twisted, discretized or deformed derivatives [5] and corresponding generalizations, discretizations and deformations of vector fields and differential calculus. It has been shown in [4, 7–9] that the class of quasi-Hom-Lie algebras contains as a subclass on the one hand the color Lie algebras and in particular Lie superalgebras and Lie algebras, and

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on the other hand various known and new single and multi-parameter families of algebras obtained using twisted derivations and constituting deformations and quasi-deformations of universal enveloping algebras of Lie and color Lie algebras and of algebras of vector-fields. The main feature of quasi-Lie algebras, quasi-Hom-Lie algebras and Hom-Lie algebras is that the skew-symmetry and the Jacobi identity are twisted by several deforming twisting maps and also in quasi-Lie and quasi-Hom-Lie algebras the Jacobi identity in general contains six twisted triple bracket terms.

In the paper [12], we provided a different way for constructing Hom-Lie algebras by extending the fundamental construction of Lie algebras from associative algebras via commutator bracket multiplication. To this end we defined the notion of Hom-associative algebras generalizing associative algebras to a situation where associativity law is twisted, and showed that the commutator product defined using the multiplication in a Hom-associative algebra leads naturally to Hom-Lie algebras. We introduced also Hom-Lie-admissible algebras and more general G -Hom-associative algebras with subclasses of Hom-Vinberg and pre-Hom-Lie algebras, generalizing to the twisted situation Lie-admissible algebras, G -associative algebras, Vinberg and pre-Lie algebras respectively, and show that for these classes of algebras the operation of taking commutator leads to Hom-Lie algebras as well. We constructed also all the twistings so that the brackets $[X_1, X_2] = 2X_2$, $[X_1, X_3] = -2X_3$, $[X_2, X_3] = X_1$ determine a three-dimensional Hom-Lie algebra. Finally, we provided for a subclass of twistings, the list of all three-dimensional Hom-Lie algebras. This list contains all three-dimensional Lie algebras for some values of structure constants. The families of Hom-Lie algebras in these list can be viewed as deformations of Lie algebras into a class of Hom-Lie algebras. The notion, constructions and properties of the enveloping algebras of Hom-Lie algebras are yet to be properly studied in full generality. An important progress in this direction has been made in the recent work by D. Yau [14].

In the present paper we develop the coalgebra counterpart of the notions and results of [12], extending in particular in the framework of Hom-associative and Hom-Lie algebras and Hom-coalgebras, the notions and results on associative and Lie admissible coalgebras obtained in [2]. In Sect. 17.2 we summarize the relevant definitions of Hom-associative algebra, Hom-Lie algebra, Hom-Leibniz algebra, and define the notions of Hom-coalgebras and Hom-coassociative coalgebras. In Sect. 17.3, we introduce the concept of Hom-Lie admissible Hom-coalgebra, describe some useful relations between coproduct, opposite coproduct, the cocommutator defined as their difference, and their β -twisted coassociators and β -twisted co-Jacobi sums. We also introduce the notion of G -Hom-coalgebra for any subgroup G of permutation group S_3 . We show that G -Hom-coalgebras are Hom-Lie admissible Hom-coalgebras, and also establish duality correspondence between classes of G -Hom-coalgebras and G -Hom-algebras. Section 17.4 is dedicated to relevant definitions and basic properties of the Hom-Hopf algebra which generalize the classical Hopf algebra structure. We also define the module and comodule structure over Hom-associative algebra or Hom-coassociative coalgebra.

17.2 Hom-Algebra and Hom-Coalgebra Structures

A Hom-algebra structure is a multiplication on a vector space where the structure is twisted by a homomorphism. The structure of Hom-Lie algebra was introduced by Hartwig et al. [4]. In the following we summarize the definitions of Hom-associative, Hom-Leibniz, and Hom-Lie-admissible algebraic structures introduced in [12] and generalizing the well known associative, Leibniz and Lie-admissible algebras. By dualization of Hom-associative algebra we define the Hom-coassociative coalgebra structure.

17.2.1 Hom-Algebra Structures

Let \mathbb{K} be an algebraically closed field of characteristic 0 and V be a linear space over \mathbb{K} .

Definition 17.1. A *Hom-associative algebra* is a triple (V, μ, α) consisting of a linear space V , a linear map $\mu : V \otimes V \rightarrow V$ and a homomorphism α satisfying

$$\mu(\alpha(x) \otimes \mu(y \otimes z)) = \mu(\mu(x \otimes y) \otimes \alpha(z)). \quad (17.1)$$

The Hom-associativity condition (17.1) may be expressed by the following commutative diagram.

$$\begin{array}{ccc} V \otimes V \otimes V & \xrightarrow{\mu \otimes \alpha} & V \otimes V \\ \downarrow \alpha \otimes \mu & & \downarrow \mu \\ V \otimes V & \xrightarrow{\mu} & V \end{array}$$

The Hom-associative algebra is unital if there exists a homomorphism $\eta : \mathbb{K} \rightarrow V$ such that the following diagrams are commutative

$$\begin{array}{ccccc} \mathbb{K} \otimes V & \xrightarrow{\eta \otimes id} & V \otimes V & \xleftarrow{id \otimes \eta} & V \otimes \mathbb{K} \\ & \searrow \cong & \downarrow \mu & \swarrow \cong & \\ & & V & & \end{array}$$

In the language of Hopf algebra, a Hom-associative algebra \mathcal{A} is a quadruple (V, μ, α, η) where V is the vector space, μ is the Hom-associative multiplication, α is the twisting homomorphism and η is the unit.

Let (V, μ, α, η) and $(V', \mu', \alpha', \eta')$ be two Hom-associative algebras. A linear map $f : V \rightarrow V'$ is a morphism of Hom-associative algebras if

$$\mu' \circ (f \otimes f) = f \circ \mu, \quad f \circ \eta = \eta' \quad \text{and} \quad f \circ \alpha = \alpha' \circ f.$$

In particular, (V, μ, α, η) and $(V', \mu', \alpha', \eta')$ are isomorphic if there exists a bijective linear map f such that

$$\mu = f^{-1} \circ \mu' \circ (f \otimes f), \quad \eta = f^{-1} \circ \eta' \quad \text{and} \quad \alpha = f^{-1} \circ \alpha' \circ f.$$

The tensor product of two Hom-associative algebras $(V_1, \mu_1, \alpha_1, \eta_1)$ and $(V_2, \mu_2, \alpha_2, \eta_2)$ is defined in an obvious way as the Hom-associative algebra $(V_1 \otimes V_2, \mu_1 \otimes \mu_2, \alpha_1 \otimes \alpha_2, \eta_1 \otimes \eta_2)$.

The Hom-Lie algebras were initially introduced in [4] motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields.

Definition 17.2. A *Hom-Lie algebra* is a triple $(V, [\cdot, \cdot], \alpha)$ consisting of a linear space V , bilinear map $[\cdot, \cdot] : V \times V \rightarrow V$ and a linear space homomorphism $\alpha : V \rightarrow V$ satisfying

$$\begin{aligned} [x, y] &= -[y, x] \quad (\text{skew-symmetry}) \\ \bigcirc_{x,y,z} [\alpha(x), [y, z]] &= 0 \quad (\text{Hom-Jacobi condition}) \end{aligned}$$

for all x, y, z from V , where $\bigcirc_{x,y,z}$ denotes summation over the cyclic permutation on x, y, z .

In a similar way we have the following definition of Hom-Leibniz algebra.

Definition 17.3. A *Hom-Leibniz algebra* is a triple $(V, [\cdot, \cdot], \alpha)$ consisting of a linear space V , bilinear map $[\cdot, \cdot] : V \times V \rightarrow V$ and a homomorphism $\alpha : V \rightarrow V$ satisfying

$$[[x, y], \alpha(z)] = [[x, z], \alpha(y)] + [\alpha(x), [y, z]]. \quad (17.2)$$

Note that if a Hom-Leibniz algebra is skewsymmetric then it is a Hom-Lie algebra.

17.2.2 Hom-Coalgebra Structures

Definition 17.4. A *Hom-coassociative coalgebra* is a quadruple $(V, \Delta, \beta, \varepsilon)$ where V is a \mathbb{K} -vector space and

$$\Delta : V \rightarrow V \otimes V, \quad \beta : V \rightarrow V \quad \text{and} \quad \varepsilon : V \rightarrow \mathbb{K}$$

are linear maps satisfying the following conditions:

$$\begin{aligned} \text{(C1)} \quad & (\beta \otimes \Delta) \circ \Delta = (\Delta \otimes \beta) \circ \Delta \\ \text{(C2)} \quad & (id \otimes \varepsilon) \circ \Delta = id \quad \text{and} \quad (\varepsilon \otimes id) \circ \Delta = id. \end{aligned}$$

The condition (C1) expresses the Hom-coassociativity of the comultiplication Δ . Also, it is equivalent to the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\Delta} & V \otimes V \\ \downarrow \Delta & & \downarrow \beta \otimes \Delta \\ V \otimes V & \xrightarrow{\Delta \otimes \beta} & V \otimes V \otimes V \end{array}$$

The condition (C2) expresses that ε is the counit which is also equivalent to the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{K} \otimes V & \xleftarrow{\varepsilon \otimes id_V} & V \otimes V & \xrightarrow{id \otimes \varepsilon} & V \otimes \mathbb{K} \\
& \nwarrow \cong & \uparrow \Delta & \nearrow \cong & \\
& & V & &
\end{array}$$

Let $(V, \Delta, \beta, \varepsilon)$ and $(V', \Delta', \beta', \varepsilon')$ be two Hom-coassociative coalgebras. A linear map $f : V \rightarrow V'$ is a morphism of Hom-coassociative coalgebras if

$$(f \otimes f) \circ \Delta = \Delta' \circ f, \quad \varepsilon = \varepsilon' \circ f \quad \text{and} \quad f \circ \beta = \beta' \circ f.$$

If $V = V'$, then the previous Hom-coassociative coalgebras are isomorphic if there exists a bijective linear map $f : V \rightarrow V$ such that

$$\Delta' = (f \otimes f) \circ \Delta \circ f^{-1}, \quad \varepsilon' = \varepsilon \circ f^{-1} \quad \text{and} \quad \beta = f^{-1} \circ \beta' \circ f.$$

In the sequel, we call *Hom-coalgebra* a triple (V, Δ, β) where V is a \mathbb{K} -vector space, Δ is a comultiplication not necessarily coassociative or Hom-coassociative, that is a linear map $\Delta : V \rightarrow V \otimes V$, and β is a linear map $\beta : V \rightarrow V$.

17.3 Hom-Lie Admissible Hom-Coalgebras

Let \mathbb{K} be an algebraically closed field of characteristic 0 and V be a vector space over \mathbb{K} . Let (V, Δ, β) be a Hom-coalgebra where $\Delta : V \rightarrow V \otimes V$ and $\beta : V \rightarrow V$ are linear maps and Δ is not necessarily coassociative or Hom-coassociative.

By a β -coassociator of Δ we call a linear map $\mathbf{c}_\beta(\Delta)$ defined by

$$\mathbf{c}_\beta(\Delta) := (\Delta \otimes \beta) \circ \Delta - (\beta \otimes \Delta) \circ \Delta.$$

Let \mathcal{S}_3 be the symmetric group of order 3. Given $\sigma \in \mathcal{S}_3$, we define a linear map

$$\Phi_\sigma : V^{\otimes 3} \longrightarrow V^{\otimes 3}$$

by

$$\Phi_\sigma(x_1 \otimes x_2 \otimes x_3) = x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}.$$

Recall that $\Delta^{op} = \tau \circ \Delta$ where τ is the usual flip that is $\tau(x \otimes y) = y \otimes x$.

Definition 17.5. A triple (V, Δ, β) is a *Hom-Lie admissible Hom-coalgebra* if the linear map

$$\Delta_L : V \longrightarrow V \otimes V$$

defined by $\Delta_L = \Delta - \Delta^{op}$, is a Hom-Lie coalgebra multiplication, that is the following condition is satisfied

$$\mathbf{c}_\beta(\Delta_L) + \Phi_{(213)} \circ \mathbf{c}_\beta(\Delta_L) + \Phi_{(231)} \circ \mathbf{c}_\beta(\Delta_L) = 0 \quad (17.3)$$

where (213) and (231) are the two cyclic permutations of order 3 in \mathcal{S}_3 .

Remark 17.1. Since $\Delta_L = \Delta - \Delta^{op}$, the equality $\Delta_L^{op} = -\Delta_L$ holds.

Lemma 17.1. *Let (V, Δ, β) be a Hom-coalgebra where $\Delta : V \rightarrow V \otimes V$ and $\beta : V \rightarrow V$ are linear maps and Δ is not necessarily coassociative or Hom-coassociative, then the following relations are true*

$$\mathbf{c}_\beta(\Delta^{op}) = -\Phi_{(13)} \circ \mathbf{c}_\beta(\Delta) \quad (17.4)$$

$$(\beta \otimes \Delta^{op}) \circ \Delta = \Phi_{(13)} \circ (\Delta \otimes \beta) \circ \Delta^{op} \quad (17.5)$$

$$(\beta \otimes \Delta) \circ \Delta^{op} = \Phi_{(13)} \circ (\Delta^{op} \otimes \beta) \circ \Delta \quad (17.6)$$

$$(\Delta \otimes \beta) \circ \Delta^{op} = \Phi_{(213)} \circ (\beta \otimes \Delta) \circ \Delta \quad (17.7)$$

$$(\Delta^{op} \otimes \beta) \circ \Delta = \Phi_{(12)} \circ (\Delta \otimes \beta) \circ \Delta. \quad (17.8)$$

Lemma 17.2. *The β -coassociator of Δ_L is expressed using Δ and Δ^{op} as follows:*

$$\mathbf{c}_\beta(\Delta_L) = \mathbf{c}_\beta(\Delta) + \mathbf{c}_\beta(\Delta^{op}) \quad (17.9)$$

$$\begin{aligned} & -(\Delta \otimes \beta) \circ \Delta^{op} - (\Delta^{op} \otimes \beta) \circ \Delta + \\ & \Phi_{(13)} \circ (\Delta \otimes \beta) \circ \Delta^{op} + \Phi_{(13)} \circ (\Delta^{op} \otimes \beta) \circ \Delta \\ & = \mathbf{c}_\beta(\Delta) - \Phi_{(13)} \circ \mathbf{c}_\beta(\Delta) \\ & - \Phi_{(213)} \circ (\beta \otimes \Delta) \circ \Delta - \Phi_{(12)} \circ (\Delta \otimes \beta) \circ \Delta \\ & + \Phi_{(23)} \circ (\beta \otimes \Delta) \circ \Delta + \Phi_{(231)} \circ (\Delta \otimes \beta) \circ \Delta. \end{aligned} \quad (17.10)$$

Proposition 17.1. *Let (V, Δ, β) be a Hom-coalgebra. Then one has*

$$\mathbf{c}_\beta(\Delta_L) + \Phi_{(213)} \circ \mathbf{c}_\beta(\Delta_L) + \Phi_{(231)} \circ \mathbf{c}_\beta(\Delta_L) = 2 \sum_{\sigma \in \mathcal{S}_3} (-1)^{\varepsilon(\sigma)} \Phi_\sigma \circ \mathbf{c}_\beta(\Delta) \quad (17.11)$$

where $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ .

Proof. By (17.10) and multiplication rules in the group \mathcal{S}_3 , it follows that

$$\begin{aligned} \Phi_{(213)} \circ \mathbf{c}_\beta(\Delta_L) &= \Phi_{(213)} \circ \mathbf{c}_\beta(\Delta) - \Phi_{(213)} \circ \Phi_{(13)} \circ \mathbf{c}_\beta(\Delta) \\ & - \Phi_{(213)} \circ \Phi_{(213)} \circ (\beta \otimes \Delta) \circ \Delta - \Phi_{(213)} \circ \Phi_{(12)} \circ (\Delta \otimes \beta) \circ \Delta \\ & + \Phi_{(213)} \circ \Phi_{(23)} \circ (\beta \otimes \Delta) \circ \Delta + \Phi_{(213)} \circ \Phi_{(231)} \circ (\Delta \otimes \beta) \circ \Delta \\ & = \Phi_{(213)} \circ \mathbf{c}_\beta(\Delta) - \Phi_{(12)} \circ \mathbf{c}_\beta(\Delta) \end{aligned} \quad (17.12)$$

$$\begin{aligned} & - \Phi_{(231)} \circ (\beta \otimes \Delta) \circ \Delta - \Phi_{(23)} \circ (\Delta \otimes \beta) \circ \Delta \\ & + \Phi_{(13)} \circ (\beta \otimes \Delta) \circ \Delta + (\Delta \otimes \beta) \circ \Delta, \\ \Phi_{(231)} \circ \mathbf{c}_\beta(\Delta_L) &= \Phi_{(231)} \circ \mathbf{c}_\beta(\Delta) - \Phi_{(231)} \circ \Phi_{(13)} \circ \mathbf{c}_\beta(\Delta) \\ & - \Phi_{(231)} \circ \Phi_{(213)} \circ (\beta \otimes \Delta) \circ \Delta - \Phi_{(231)} \circ \Phi_{(12)} \circ (\Delta \otimes \beta) \circ \Delta \\ & + \Phi_{(231)} \circ \Phi_{(23)} \circ (\beta \otimes \Delta) \circ \Delta + \Phi_{(231)} \circ \Phi_{(231)} \circ (\Delta \otimes \beta) \circ \Delta \\ & = \Phi_{(231)} \circ \mathbf{c}_\beta(\Delta) - \Phi_{(23)} \circ \mathbf{c}_\beta(\Delta) \\ & - (\beta \otimes \Delta) \circ \Delta - \Phi_{(13)} \circ (\Delta \otimes \beta) \circ \Delta \\ & + \Phi_{(12)} \circ (\beta \otimes \Delta) \circ \Delta + \Phi_{(213)} \circ (\Delta \otimes \beta) \circ \Delta. \end{aligned} \quad (17.13)$$

After summing up the equalities (17.10), (17.12) and (17.13) the terms on the right hand sides may be pairwise combined into the terms of the form $(-1)^{\varepsilon(\sigma)} \Phi_\sigma \circ \mathbf{c}_\beta(\Delta)$ with each one being present in the sum twice for all $\sigma \in \mathcal{S}_3$.

Definition 17.5 together with (17.11) yields the following corollary.

Corollary 17.1. *A triple (V, Δ, β) is a Hom-Lie admissible Hom-coalgebra if and only if*

$$\sum_{\sigma \in \mathcal{S}_3} (-1)^{\varepsilon(\sigma)} \Phi_\sigma \circ \mathbf{c}_\beta(\Delta) = 0$$

where $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ .

17.3.1 G -Hom-Coalgebra Structures

In this section we introduce, as in the multiplication case, the notion of G -Hom-coalgebra where G is a subgroup of the symmetric group \mathcal{S}_3 .

Definition 17.6. Let G be a subgroup of the symmetric group \mathcal{S}_3 , A Hom-coalgebra (V, Δ, β) is called G -Hom-coalgebra if

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \Phi_\sigma \circ \mathbf{c}_\beta(\Delta) = 0 \quad (17.14)$$

where $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ .

Proposition 17.2. *Let G be a subgroup of the permutations group \mathcal{S}_3 . Then any G -Hom-Coalgebra (V, Δ, β) is a Hom-Lie admissible Hom-coalgebra.*

Proof. The skew-symmetry follows straightaway from the definition. Take the set of conjugacy classes $\{gG\}_{g \in I}$ where $I \subseteq G$, and for any $\sigma_1, \sigma_2 \in I, \sigma_1 \neq \sigma_2 \Rightarrow \sigma_1 G \cap \sigma_2 G = \emptyset$. Then

$$\sum_{\sigma \in \mathcal{S}_3} (-1)^{\varepsilon(\sigma)} \Phi_\sigma \circ \mathbf{c}_\beta(\Delta) = \sum_{\sigma_1 \in I} \sum_{\sigma_2 \in \sigma_1 G} (-1)^{\varepsilon(\sigma)} \Phi_\sigma \circ \mathbf{c}_\beta(\Delta) = 0.$$

The subgroups of \mathcal{S}_3 are

$$G_1 = \{Id\}, G_2 = \{Id, \tau_{12}\}, G_3 = \{Id, \tau_{23}\},$$

$$G_4 = \{Id, \tau_{13}\}, G_5 = A_3, G_6 = \mathcal{S}_3,$$

where A_3 is the alternating group and where τ_{ij} is the transposition between i and j .

We obtain the following type of Hom-Lie-admissible Hom-coalgebras:

- The G_1 -Hom-coalgebras are the Hom-associative coalgebras defined above.
- The G_2 -Hom-coalgebras satisfy the condition

$$\mathbf{c}_\beta(\Delta) + \Phi_{(12)} \mathbf{c}_\beta(\Delta) = 0.$$

- The G_3 -Hom-coalgebras satisfy the condition

$$\mathbf{c}_\beta(\Delta) + \Phi_{(23)}\mathbf{c}_\beta(\Delta) = 0.$$

- The G_4 -Hom-coalgebras satisfy the condition

$$\mathbf{c}_\beta(\Delta) + \Phi_{(13)}\mathbf{c}_\beta(\Delta) = 0.$$

- The G_5 -Hom-coalgebras satisfy the condition

$$\mathbf{c}_\beta(\Delta) + \Phi_{(213)}\mathbf{c}_\beta(\Delta) + \Phi_{(231)}\mathbf{c}_\beta(\Delta) = 0.$$

If the product μ is skewsymmetric then the previous condition is exactly the Hom-Jacobi identity.

- The G_6 -Hom-coalgebras are the Hom-Lie-admissible coalgebras.

The G_2 -Hom-coalgebras may be called Vinberg-Hom-coalgebras and G_3 -Hom-coalgebras may be called preLie-Hom-coalgebras. The two classes define in fact the same class.

Definition 17.7. A *Vinberg-Hom-coalgebra* is a triple (V, Δ, β) consisting of a linear space V , a linear map $\mu : V \rightarrow V \times V$ and a homomorphism β satisfying

$$\mathbf{c}_\beta(\Delta) + \Phi_{(12)}\mathbf{c}_\beta(\Delta) = 0.$$

Definition 17.8. A *preLie-Hom-coalgebra* is a triple (V, Δ, β) consisting of a linear space V , a linear map $\mu : V \rightarrow V \times V$ and a homomorphism β satisfying

$$\mathbf{c}_\beta(\Delta) + \Phi_{(23)}\mathbf{c}_\beta(\Delta) = 0.$$

More generally, by dualization we have a correspondence between G -Hom-associative algebras introduced in [12] and G -Hom-coalgebras for a subgroup G of \mathcal{S}_3 .

Let G be a subgroup of \mathcal{S}_3 and (V, μ, α) be a G -Hom-associative algebra that is $\mu : V \otimes V \rightarrow V$ and $\alpha : V \rightarrow V$ are linear maps and the following condition is satisfied

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} a_{\alpha, \mu} \circ \Phi_\sigma = 0. \quad (17.15)$$

where $a_{\alpha, \mu}$ is the α -associator that is $a_{\alpha, \mu} = \mu \circ (\mu \otimes \alpha) - \mu \circ (\alpha \otimes \mu)$.

Setting

$$(\mu \otimes \alpha)_G = \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} (\mu \otimes \alpha) \circ \Phi_\sigma \quad \text{and} \quad (\alpha \otimes \mu)_G = \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} (\alpha \otimes \mu) \circ \Phi_\sigma$$

the condition (17.15) is equivalent to the following commutative diagram

$$\begin{array}{ccc}
V \otimes V \otimes V & \xrightarrow{(\mu \otimes \alpha)_G} & V \otimes V \\
\downarrow (\alpha \otimes \mu)_G & & \downarrow \mu \\
V \otimes V & \xrightarrow{\mu} & V
\end{array}$$

By the dualization of the square one may obtain the following commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{\Delta} & V \otimes V \\
\downarrow \Delta & & \downarrow (\beta \otimes \Delta)_G \\
V \otimes V & \xrightarrow{(\Delta \otimes \beta)_G} & V \otimes V \otimes V
\end{array}$$

where

$$(\beta \otimes \Delta)_G = \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \Phi_\sigma \circ (\beta \otimes \Delta) \quad \text{and} \quad (\Delta \otimes \beta)_G = \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \Phi_\sigma \circ (\Delta \otimes \beta).$$

The previous commutative diagram expresses that (V, Δ, β) is a G -Hom-coalgebra. More precisely we have the following connection between G -Hom-coalgebras and G -Hom-associative algebras.

Proposition 17.3. *Let (V, Δ, β) be a G -Hom-coalgebra where G is a subgroup of \mathcal{S}_3 . Its dual vector space V^* is provided with a G -Hom-associative algebra (V^*, Δ^*, β^*) where Δ^*, β^* are the transpose maps.*

Proof. Let (V, Δ, β) be a G -Hom-coalgebra. Let V^* be the dual space of V ($V^* = \text{Hom}(V, \mathbb{K})$).

Consider the map

$$\begin{aligned}
\lambda_n : (V^*)^{\otimes n} &\longrightarrow (V^*)^{\otimes n} \\
f_1 \otimes \cdots \otimes f_n &\longrightarrow \lambda_n(f_1 \otimes \cdots \otimes f_n)
\end{aligned}$$

such that for $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$

$$\lambda_n(f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) = f_1(v_1) \otimes \cdots \otimes f_n(v_n)$$

and set

$$\mu := \Delta^* \circ \lambda_2 \quad \alpha := \beta^*$$

where the star \star denotes the transpose linear map. Then, the quadruple (V^*, μ, η, α) is a G -Hom-associative algebra. Indeed, $\mu(f_1, f_2) = \mu_{\mathbb{K}} \circ \lambda_2(f_1 \otimes f_2) \circ \Delta$ where $\mu_{\mathbb{K}}$ is the multiplication of \mathbb{K} and $f_1, f_2 \in V^*$. One has

$$\begin{aligned}
\mu \circ (\mu \otimes \alpha)(f_1 \otimes f_2 \otimes f_3) &= \mu(\mu(f_1 \otimes f_2) \otimes \alpha(f_3)) \\
&= \mu_{\mathbb{K}} \circ \lambda_2(\mu(f_1 \otimes f_2) \otimes \alpha(f_3)) \circ \Delta \\
&= \mu_{\mathbb{K}} \circ \lambda_2(\lambda_2((f_1 \otimes f_2) \circ \Delta) \otimes \alpha(f_3)) \circ \Delta \\
&= \mu_{\mathbb{K}} \circ (\mu_{\mathbb{K}} \otimes id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes \beta) \circ \Delta.
\end{aligned}$$

Similarly

$$\mu \circ (\alpha \otimes \mu)(f_1 \otimes f_2 \otimes f_3) = \mu_{\mathbb{K}} \circ (id \otimes \mu_{\mathbb{K}}) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\beta \otimes \Delta) \circ \Delta.$$

Using the associativity and the commutativity of $\mu_{\mathbb{K}}$, the α -associator may be written as

$$a_{\alpha, \mu} = \mu_{\mathbb{K}} \circ (id \otimes \mu_{\mathbb{K}}) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ ((\Delta \otimes \beta) \circ \Delta - (\beta \otimes \Delta) \circ \Delta).$$

Then we have the following connection between the α -associator and β -coassociator

$$a_{\alpha, \mu} = \mu_{\mathbb{K}} \circ (id \otimes \mu_{\mathbb{K}}) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ \mathbf{c}_{\beta}(\Delta).$$

Therefore if (V, Δ, β) is a G -Hom-coalgebra, then the (V^*, Δ^*, β^*) is a G -Hom-associative algebra.

Proposition 17.4. *Let (V, μ, α) be a finite-dimensional G -Hom-associative algebra where G is a subgroup of \mathcal{S}_3 . Its dual vector space V^* is provided with a G -Hom-coalgebra (V^*, μ^*, α^*) , where μ^*, α^* are the transpose maps.*

Proof. Let $\mathcal{A} = (V, \mu, \alpha)$ be a n -dimensional Hom-associative algebra (n finite). Let $\{e_1, \dots, e_n\}$ be a basis of V and $\{e_1^*, \dots, e_n^*\}$ be the dual basis. Then $\{e_i^* \otimes e_j^*\}_{i,j}$ is a basis of $\mathcal{A}^* \otimes \mathcal{A}^*$. The comultiplication $\Delta = \mu^*$ on \mathcal{A}^* is defined for $f \in \mathcal{A}^*$ by

$$\Delta(f) = \sum_{i,j=1}^n f(\mu(e_i \otimes e_j)) e_i^* \otimes e_j^*.$$

Set $\mu(e_i \otimes e_j) = \sum_{k=1}^n C_{ij}^k e_k$ and $\alpha(e_i) = \sum_{k=1}^n \alpha_i^k e_k$. Then $\Delta(e_k^*) = \sum_{i,j=1}^n C_{ij}^k e_i^* \otimes e_j^*$ and $\beta(e_i) = \alpha^*(e_i) = \sum_{k=1}^n \alpha_i^k e_k$.

The condition (17.14) of G -Hom-coassociativity of Δ , applied to any element e_k^* of the basis, is equivalent to

$$\sum_{p,q,s=1}^n \sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \left(\sum_{i,j=1}^n \alpha_s^j C_{ij}^k C_{pq}^i - \alpha_p^i C_{ij}^k C_{qs}^j \right) e_{\sigma^{-1}(p)}^* \otimes e_{\sigma^{-1}(q)}^* \otimes e_{\sigma^{-1}(s)}^* = 0.$$

Therefore Δ is G -Hom-coassociative if for any $p, q, s, k \in \{1, \dots, n\}$ one has

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \left(\sum_{i,j=1}^n \alpha_s^j C_{ij}^k C_{pq}^i - \alpha_p^i C_{ij}^k C_{qs}^j \right) = 0.$$

The previous system is exactly the condition (17.15) of G -Hom-associativity of μ , written on $e_{p'} \otimes e_{q'} \otimes e_{s'}$ and setting $p = \sigma(p')$, $q = \sigma(q')$, $s = \sigma(s')$.

Corollary 17.2. *The dual vector space of a Hom-coassociative coalgebra $(V, \Delta, \beta, \varepsilon)$ is a Hom-associative algebra $(V^*, \Delta^*, \beta^*, \varepsilon^*)$, where V^* is the dual vector space and the star for the linear maps denotes the transpose map. The dual vector space of finite-dimensional Hom-associative algebra is a Hom-coassociative coalgebra.*

Proof. It is a particular case of the previous Propositions ($G = G_1$).

17.4 Hom-Hopf Algebras

In this section, we introduce a generalization of Hopf algebras and show some relevant properties of the new structure. We also define the module and comodule structure over Hom-associative algebra or Hom-coassociative coalgebra. For classical Hopf algebras theory, we refer to [1, 3, 6, 10, 11, 13]. Let \mathbb{K} be an algebraically closed field of characteristic 0 and V be a vector space over \mathbb{K} .

Definition 17.9. A Hom-bialgebra is a quintuple $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ where

- (B1) (V, μ, α, η) is a Hom-associative algebra
- (B2) $(V, \Delta, \beta, \varepsilon)$ is a Hom-coassociative coalgebra
- (B3) The linear maps Δ and ε are morphisms of algebras (V, μ, α, η) .

Remark 17.2. The condition (B3) could be expressed by the following system:

$$\begin{cases} \Delta(e_1) = e_1 \otimes e_1 & \text{where } e_1 = \eta(1) \\ \Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} \mu(x^{(1)} \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes y^{(2)}) \\ \varepsilon(e_1) = 1 \\ \varepsilon(\mu(x \otimes y)) = \varepsilon(x) \varepsilon(y) \end{cases}$$

where the bullet \bullet denotes the multiplication on tensor product and by using the Sweedler's notation $\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}$. If there is no ambiguity we denote the multiplication by a dot.

Remark 17.3. One can consider a more restrictive definition where linear maps Δ and ε are morphisms of Hom-associative algebras that is the condition (B3) becomes equivalent to

$$\begin{cases} \Delta(e_1) = e_1 \otimes e_1 & \text{where } e_1 = \eta(1) \\ \Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} \mu(x^{(1)} \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes y^{(2)}) \\ \varepsilon(e_1) = 1 \\ \varepsilon(\mu(x \otimes y)) = \varepsilon(x) \varepsilon(y) \\ \Delta(\alpha(x)) = \sum_{(x)} \alpha(x^{(1)}) \otimes \alpha(x^{(2)}) \\ \varepsilon \circ \alpha(x) = \varepsilon(x) \end{cases}$$

Given a Hom-bialgebra $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$, we show that the vector space $\text{Hom}(V, V)$ with the multiplication given by the convolution product carries a structure of Hom-algebra.

Proposition 17.5. Let $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra. Then the algebra $\text{Hom}(V, V)$ with the multiplication given by the convolution product defined by

$$f \star g = \mu \circ (f \otimes g) \circ \Delta$$

and the unit being $\eta \circ \varepsilon$ is a Hom-associative algebra with the homomorphism map defined by $\gamma(f) = \alpha \circ f \circ \beta$.

Proof. Let $f, g, h \in \text{Hom}(V, V)$. Then

$$\begin{aligned}\gamma(f) * (g * h) &= \mu \circ (\gamma(f) \otimes (g * h)) \Delta \\ &= \mu \circ (\gamma(f) \otimes (\mu \circ (g \otimes h) \circ \Delta)) \Delta \\ &= \mu \circ (\alpha \otimes \mu) \circ (f \otimes g \otimes h) \circ (\beta \otimes \Delta) \Delta.\end{aligned}$$

Similarly

$$(f * g) * \gamma(h) = \mu \circ (\mu \otimes \alpha) \circ (f \otimes g \otimes h) \circ (\Delta \otimes \beta) \Delta.$$

Then, the Hom-associativity of μ and the Hom-coassociativity of Δ lead to the Hom-associativity of the convolution product. The unitality is as usual.

Definition 17.10. An endomorphism S of V is said to be an *antipode* if it is the inverse of the identity over V for the Hom-algebra $\text{Hom}(V, V)$ with the multiplication given by the convolution product defined by

$$f \star g = \mu \circ (f \otimes g) \Delta$$

and the unit being $\eta \circ \varepsilon$.

The condition being antipode may be expressed by the condition:

$$\mu \circ S \otimes \text{Id} \circ \Delta = \mu \circ \text{Id} \otimes S \circ \Delta = \eta \circ \varepsilon.$$

Definition 17.11. A *Hom-Hopf algebra* is a Hom-bialgebra with an antipode.

Then, a Hom-Hopf algebra over a \mathbb{K} -vector space V is given by

$$\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon, S)$$

where the following homomorphisms

$$\mu : V \otimes V \rightarrow V, \quad \eta : \mathbb{K} \rightarrow V, \quad \alpha : V \rightarrow V$$

$$\Delta : V \rightarrow V \otimes V, \quad \varepsilon : V \rightarrow \mathbb{K}, \quad \beta : V \rightarrow V$$

$$S : V \rightarrow \mathbb{K}$$

satisfy the following conditions:

1. (V, μ, α, η) is a unital Hom-associative algebra.
2. $(V, \Delta, \beta, \varepsilon)$ is a counital Hom-coalgebra.
3. Δ and ε are morphisms of algebras, which translate to

$$\begin{cases} \Delta(e_1) = e_1 \otimes e_1 & \text{where } e_1 = \eta(1) \\ \Delta(x \cdot y) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} x^{(1)} \cdot y^{(1)} \otimes x^{(2)} \cdot y^{(2)} \\ \varepsilon(e_1) = 1 \\ \varepsilon(x \cdot y) = \varepsilon(x) \varepsilon(y) \end{cases}$$

4. S is the antipode, so

$$\mu \circ S \otimes Id \circ \Delta = \mu \circ Id \otimes S \circ \Delta = \eta \circ \varepsilon.$$

Remark 17.4. Let V be a finite-dimensional \mathbb{K} -vector space.

If $\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon, S)$ is a Hom-Hopf algebra, then

$$\mathcal{H}^* = (V^*, \Delta^*, \beta^*, \varepsilon^*, \mu^*, \alpha^*, \eta^*, S^*)$$

is also a Hom-Hopf algebra.

17.4.1 Primitive Elements and Generalized Primitive Elements

In the following, we discuss the properties of primitive elements in a Hom-bialgebra.

Let $\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra and $e_1 = \eta(1)$ be the unit.

Definition 17.12. An element $x \in \mathcal{H}$ is called primitive if $\Delta(x) = e_1 \otimes x + x \otimes e_1$.

Let $x \in \mathcal{H}$ be a primitive element. The coassociativity of Δ implies

$$(\beta \otimes \Delta) \circ \Delta(x) = \tau_{13} \circ (\Delta \otimes \beta) \circ \Delta(x)$$

where τ_{13} is a permutation in the symmetric group \mathcal{S}_3 .

Lemma 17.3. Let x be a primitive element in \mathcal{H} , then $\varepsilon(x) = 0$.

Proof. By counity property, we have $x = (id \otimes \varepsilon) \circ \Delta(x)$. If $\Delta(x) = e_1 \otimes x + x \otimes e_1$, then $x = \varepsilon(x)e_1 + \varepsilon(e_1)x$, and since $\varepsilon(e_1) = 1$ it implies $\varepsilon(x) = 0$.

Proposition 17.6. Let $\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra and $e_1 = \eta(1)$ be the unit. If x and y are two primitive elements in \mathcal{H} . Then we have $\varepsilon(x) = 0$ and the commutator $[x, y] = \mu(x \otimes y) - \mu(y \otimes x)$ is also a primitive element.

The set of all primitive elements of \mathcal{H} , denoted by $\text{Prim}(\mathcal{H})$, has a structure of Hom-Lie algebra.

Proof. By a direct calculation one has

$$\begin{aligned} \Delta([x, y]) &= \Delta(\mu(x \otimes y) - \mu(y \otimes x)) \\ &= \Delta(x) \bullet \Delta(y) - \Delta(y) \bullet \Delta(x) \\ &= (e_1 \otimes x + x \otimes e_1) \bullet (e_1 \otimes y + y \otimes e_1) - (e_1 \otimes y + y \otimes e_1) \bullet (e_1 \otimes x + x \otimes e_1) \\ &= e_1 \otimes \mu(x \otimes y) + y \otimes x + x \otimes y + \mu(x \otimes y) \otimes e_1 \\ &\quad - e_1 \otimes \mu(y \otimes x) - x \otimes y - y \otimes x - \mu(y \otimes x) \otimes e_1 \\ &= e_1 \otimes (\mu(x \otimes y) - \mu(y \otimes x)) + (\mu(x \otimes y) - \mu(y \otimes x)) \otimes e_1 \\ &= e_1 \otimes [x, y] + [x, y] \otimes e_1 \end{aligned}$$

which means that $\text{Prim}(\mathcal{H})$ is closed under the bracket multiplication $[\cdot, \cdot]$.

We have seen in [12] that there is a natural map from the Hom-associative algebras to Hom-Lie algebras. The bracket $[x, y] = \mu(x \otimes y) - \mu(y \otimes x)$ is obviously skewsymmetric and one checks that the Hom-Jacobi condition is satisfied:

$$\begin{aligned} & [\alpha(x), [y, z]] - [[x, y], \alpha(z)] - [\alpha(y), [x, z]] = \\ & \mu(\alpha(x) \otimes \mu(y \otimes z)) - \mu(\alpha(x) \otimes \mu(z \otimes y)) - \mu(\mu(y \otimes z) \otimes \alpha(x)) \\ & + \mu(\mu(z \otimes y) \otimes \alpha(x)) - \mu(\mu(x \otimes y) \otimes \alpha(z)) + \mu(\mu(y \otimes x) \otimes \alpha(z)) \\ & + \mu(\alpha(z) \otimes \mu(x \otimes y)) - \mu(\alpha(z) \otimes \mu(y \otimes x)) - \mu(\alpha(y) \otimes \mu(x \otimes z)) \\ & + \mu(\alpha(y) \otimes \mu(z \otimes x)) + \mu(\mu(x \otimes z) \otimes \alpha(y)) - \mu(\mu(z \otimes x) \otimes \alpha(y)) = 0 \end{aligned}$$

We introduce now a notion of generalized primitive element.

Definition 17.13. An element $x \in \mathcal{H}$ is called generalized primitive element if it satisfies the conditions

$$(\beta \otimes \Delta) \circ \Delta(x) = \tau_{13} \circ (\Delta \otimes \beta) \circ \Delta(x) \quad (17.16)$$

$$\Delta^{op}(x) = \Delta(x) \quad (17.17)$$

where τ_{13} is a permutation in the symmetric group \mathcal{S}_3 .

Remark 17.5. 1. In particular, a primitive element in \mathcal{H} is a generalized primitive element.

2. The condition (17.16) may be written

$$(\Delta \otimes \beta) \circ \Delta(x) = \tau_{13} \circ (\beta \otimes \Delta) \circ \Delta(x).$$

Proposition 17.7. Let $\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra and $e_1 = \eta(1)$ be the unit. If x and y are two generalized primitive elements in \mathcal{H} . Then, we have $\varepsilon(x) = 0$ and the commutator $[x, y] = \mu(x \otimes y) - \mu(y \otimes x)$ is also a generalized primitive element.

The set of all generalized primitive elements of \mathcal{H} , denoted by $GPrim(\mathcal{H})$, has a structure of Hom-Lie algebra.

Proof. Let x and y be two generalized primitive elements in \mathcal{H} . In the following the multiplication μ is denoted by a dot. The following equalities hold:

$$\begin{aligned} (\Delta \otimes \beta) \circ \Delta(x \cdot y - y \cdot x) &= (\Delta \otimes \beta) \circ \Delta(x \cdot y) - (\Delta \otimes \beta) \circ \Delta(y \cdot x) \\ &= (\Delta \otimes \beta)(\Delta(x) \bullet \Delta(y)) - (\Delta \otimes \beta)(\Delta(y) \bullet \Delta(x)) \\ &= \Delta(x^{(1)} \cdot y^{(1)}) \otimes \beta(x^{(2)} \cdot y^{(2)}) - \Delta(y^{(1)} \cdot x^{(1)}) \otimes \beta(y^{(2)} \cdot x^{(2)}) \\ &= (x^{(1)(1)} \cdot y^{(1)(1)}) \otimes (x^{(1)(2)} \cdot y^{(1)(2)}) \otimes \beta(x^{(2)} \cdot y^{(2)}) \\ &\quad - (y^{(1)(1)} \cdot x^{(1)(1)}) \otimes (y^{(1)(2)} \cdot x^{(1)(2)}) \otimes \beta(y^{(2)} \cdot x^{(2)}). \end{aligned}$$

Then, using the fact that $\Delta^{op} = \Delta$ for generalized primitive elements one has:

$$\begin{aligned}
\tau_{13} \circ (\Delta \otimes \beta) \circ \Delta(x \cdot y - y \cdot x) &= \beta(x^{(2)} \cdot y^{(2)}) \otimes (x^{(1)(2)} \cdot y^{(1)(2)}) \otimes (x^{(1)(1)} \cdot y^{(1)(1)}) \\
&\quad - \beta(y^{(2)} \cdot x^{(2)}) \otimes (y^{(1)(2)} \cdot x^{(1)(2)}) \otimes (y^{(1)(1)} \cdot x^{(1)(1)}) \\
&= (\beta \otimes \Delta) \circ \Delta(x \cdot y - y \cdot x).
\end{aligned}$$

The structure of Hom-Lie algebra follows from the same argument as in the primitive elements case.

17.4.2 Antipode's Properties

Let $\mathcal{H} = (V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon, S)$ be a Hom-Hopf algebra.

For any element $x \in V$, using the counity and Sweedler notation, one may write

$$x = \sum_{(x)} x^{(1)} \otimes \varepsilon(x^{(2)}) = \sum_{(x)} \varepsilon(x^{(1)}) \otimes x^{(2)}. \quad (17.18)$$

Then, for any $f \in \text{End}_{\mathbb{K}}(V)$, we have

$$f(x) = \sum_{(x)} f(x^{(1)}) \varepsilon(x^{(2)}) = \sum_{(x)} \varepsilon(x^{(1)}) \otimes f(x^{(2)}). \quad (17.19)$$

Let $f \star g = \mu \circ (f \otimes g) \Delta$ be the convolution product of $f, g \in \text{End}_{\mathbb{K}}(V)$. One may write

$$(f \star g)(x) = \sum_{(x)} \mu(f(x^{(1)}) \otimes g(x^{(2)})). \quad (17.20)$$

Since the antipode S is the inverse of the identity for the convolution product then S satisfies

$$\varepsilon(x) \eta(1) = \sum_{(x)} \mu(S(x^{(1)}) \otimes x^{(2)}) = \sum_{(x)} \mu(x^{(1)} \otimes S(x^{(2)})). \quad (17.21)$$

Proposition 17.8. *The antipode S is unique and we have*

- $S(\eta(1)) = \eta(1)$.
- $\varepsilon \circ S = \varepsilon$.

Proof. (1) We have $S \star id = id \star S = \eta \circ \varepsilon$. Thus, $(S \star id) \star S = S \star (id \star S) = S$. If S' is another antipode of \mathcal{H} then

$$S' = S' \star id \star S' = S' \star id \star S = S \star id \star S = S.$$

Therefore the antipode when it exists is unique.

(2) Setting $e_1 = \eta(1)$ and since $\Delta(e_1) = e_1 \otimes e_1$ one has

$$(S \star id)(e_1) = \mu(S(e_1) \otimes e_1) = S(e_1) = \eta(\varepsilon(e_1)) = e_1.$$

(3) Applying (17.19) to S , we obtain $S(x) = \sum_{(x)} S(x^{(1)}) \varepsilon(x^{(2)})$.

Applying ε to (17.21), we obtain

$$\varepsilon(x) = \varepsilon\left(\sum_{(x)} \mu(S(x^{(1)}) \otimes x^{(2)})\right).$$

Since ε is a Hom-algebra morphism, one has

$$\varepsilon(x) = \sum_{(x)} \varepsilon(S(x^{(1)}))\varepsilon(x^{(2)}) = \varepsilon\left(\sum_{(x)} S(x^{(1)})\varepsilon(x^{(2)})\right) = \varepsilon(S(x)).$$

Thus $\varepsilon \circ S = \varepsilon$.

17.4.3 Modules and Comodules

We introduce in the following the structure of module and comodule over Hom-associative algebras.

Let $\mathcal{A} = (V, \mu, \alpha)$ be a Hom-associative \mathbb{K} -algebra, an \mathcal{A} -module (left) is a triple (M, f, γ) where M is \mathbb{K} -vector space and f, γ are \mathbb{K} -linear maps, $f : M \rightarrow M$ and $\gamma : V \otimes M \rightarrow M$, such that the following diagram commutes:

$$\begin{array}{ccc} V \otimes V \otimes M & \xrightarrow{\mu \otimes f} & V \otimes M \\ \downarrow \alpha \otimes \gamma & & \downarrow \gamma \\ V \otimes M & \xrightarrow{\gamma} & M \end{array}$$

The dualization leads to comodule definition over a Hom-coassociative coalgebra.

Let $C = (V, \Delta, \beta)$ be a Hom-coassociative coalgebra. A C -comodule (right) is a triple (M, g, ρ) where M is a \mathbb{K} -vector space and g, ρ are \mathbb{K} -linear maps, $g : M \rightarrow M$ and $\rho : M \rightarrow M \otimes V$, such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes V \\ \downarrow \rho & & \downarrow g \otimes \Delta \\ M \otimes V & \xrightarrow{\rho \otimes \beta} & M \otimes V \otimes V \end{array}$$

Remark 17.6. A Hom-associative \mathbb{K} -algebra $\mathcal{A} = (V, \mu, \alpha)$ is a left \mathcal{A} -module with $M = V$, $f = \alpha$ and $\gamma = \mu$. Also, a Hom-coassociative coalgebra $C = (V, \Delta, \beta)$ is a right C -comodule with $M = V$, $g = \beta$ and $\rho = \Delta$. The properties of modules and comodules over Hom-associative algebras or Hom-coassociative algebras will be discussed in a forthcoming paper.

17.4.4 Examples

The classification of two-dimensional Hom-associative algebras, up to isomorphism, yields the following two classes. Let $B = \{e_1, e_2\}$ be a basis where $\eta(1) = e_1$ is the unit.

1. The multiplication μ_1 is defined by $\mu_1(e_1 \otimes e_i) = \mu_1(e_i \otimes e_1) = e_i$ for $i = 1, 2$ and $\mu_1(e_2 \otimes e_2) = e_2$, and the homomorphism α_1 is defined, with respect to the basis B , by $\begin{pmatrix} a_1 & 0 \\ a_2 - a_1 & a_2 \end{pmatrix}$.
2. The multiplication μ_2 is defined by $\mu_2(e_1 \otimes e_i) = \mu_2(e_i \otimes e_1) = e_i$ for $i = 1, 2$ and $\mu_2(e_2 \otimes e_2) = 0$, and the homomorphism α_2 is defined, with respect to the basis B , by $\begin{pmatrix} a_1 & 0 \\ a_2 & a_1 \end{pmatrix}$.

The Hom-bialgebras corresponding to the Hom-associative algebra defined by μ_1 and α_1 are given in the following table

	Comultiplication	Co-unit	homomorphism
1	$\Delta(e_1) = e_1 \otimes e_1$ $\Delta(e_2) = e_2 \otimes e_2$	$\varepsilon(e_1) = 1$ $\varepsilon(e_2) = 1$	$\begin{pmatrix} b_1 & 0 \\ b_3 & b_2 \end{pmatrix}$
2	$\Delta(e_1) = e_1 \otimes e_1$ $\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - 2e_2 \otimes e_2$	$\varepsilon(e_1) = 1$ $\varepsilon(e_2) = 0$	$\begin{pmatrix} b_1 & \frac{b_1 - b_3}{2} \\ b_2 & b_3 \end{pmatrix}$
3	$\Delta(e_1) = e_1 \otimes e_1$ $\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_2 \otimes e_2$	$\varepsilon(e_1) = 1$ $\varepsilon(e_2) = 0$	$\begin{pmatrix} b_1 & b_1 - b_3 \\ b_2 & b_3 \end{pmatrix}$

Only Hom-bialgebra (2) carries a structure of Hom-Hopf algebra with an antipode defined, with respect to a basis B , by the identity matrix.

Remark 17.7. There is no Hom-bialgebra associated to the Hom-associative algebra defined by the multiplication μ_2 and any homomorphism α_2 .

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