Chapter 2
Derived Functors

Derived functors are Δ-functors out of derived categories, giving rise, upon application of homology, to functors such as Ext, Tor, and their sheaf-theoretic variants—in particular sheaf cohomology. Derived functors are characterized in §2.1 below by a universal property, and conditions for their existence are given in 2.2, leading up to the construction of right-derived functors via injective resolutions in 2.3 and, dually, of some left-derived functors via flat resolutions in 2.5. We use ideas of Spaltenstein [**Sp**] to deal throughout with unbounded complexes. The basic examples $\mathsf{R}\mathcal{H}om^\bullet$ and $\underline{\underline{\otimes}}$ are described in 2.4 and 2.5 respectively. Illustrating all that has gone before, their relation "adjoint associativity" is given in 2.6, which also includes an abbreviated discussion of what is, in all conscience, involved in constructing natural transformations of multivariate derived functors: a host of underlying category-theoretic trivialities, usually ignored, but of whose existence one should at least be aware. The last section 2.7 develops further refinements.

2.1 Definition of Derived Functors

Fix an abelian category $\mathcal{A}$, let $\mathbf{J}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A})$, let $\mathbf{D_J}$ be the corresponding derived category, and let

$$Q = Q_J \colon \mathbf{J} \to \mathbf{D_J}$$

be the canonical Δ-functor (see (1.7)). For any Δ-functors F and G from $\mathbf{J}$ to another Δ-category $\mathbf{E}$, or from $\mathbf{D_J}$ to $\mathbf{E}$, $\mathrm{Hom}(F, G)$ will denote the abelian group of Δ-functor morphisms from F to G.

Definition 2.1.1. A Δ-functor $F \colon \mathbf{J} \to \mathbf{E}$ is right-derivable if there exists a Δ-functor

$$\mathsf{R}F \colon \mathbf{D_J} \to \mathbf{E}$$

J. Lipman, M. Hashimoto, *Foundations of Grothendieck Duality for Diagrams of Schemes*, Lecture Notes in Mathematics 1960,

and a morphism of Δ-functors

$$\zeta\colon F \to \mathbf{R}F \circ Q$$

such that for every Δ-functor $G\colon \mathbf{D_J} \to \mathbf{E}$ the composed map

$$\operatorname{Hom}(\mathbf{R}F,\, G) \xrightarrow{\text{natural}} \operatorname{Hom}(\mathbf{R}F \circ Q,\, G \circ Q) \xrightarrow{\text{via } \zeta} \operatorname{Hom}(F,\, G \circ Q)$$

is an isomorphism (i.e., by (1.5.1), the map "via ζ" is an isomorphism).

The Δ-functor F is left-derivable if there exists a Δ-functor

$$\mathbf{L}F\colon \mathbf{D_J} \to \mathbf{E}$$

and a morphism of Δ-functors

$$\xi\colon \mathbf{L}F \circ Q \to F$$

such that for every Δ-functor $G\colon \mathbf{D_J} \to \mathbf{E}$ the composed map

$$\operatorname{Hom}(G,\, \mathbf{L}F) \xrightarrow{\text{natural}} \operatorname{Hom}(G \circ Q,\, \mathbf{L}F \circ Q) \xrightarrow{\text{via } \xi} \operatorname{Hom}(G \circ Q,\, F)$$

is an isomorphism (i.e., by (1.5.1), the map "via ξ" is an isomorphism).

Such a pair $(\mathbf{R}F, \zeta)$ $\big($respectively: $(\mathbf{L}F, \xi)\big)$ is called a right-derived (respectively: left-derived) *functor of* F.

As in (1.5.1), composition with Q gives an embedding of Δ-functor categories

$$\mathbf{Hom}_{\boldsymbol{\Delta}}(\mathbf{D_J},\, \mathbf{E}) \hookrightarrow \mathbf{Hom}_{\boldsymbol{\Delta}}(\mathbf{J},\, \mathbf{E}), \tag{2.1.1.1}$$

with image the full subcategory whose objects are the Δ-functors which transform quasi-isomorphisms into isomorphisms. Consequently we can regard a right-(left-)derived functor of F as an *initial* (*terminal*) *object* [**M**, p. 20] in the category of Δ-functor morphisms $F \to G'$ $(G' \to F)$ where G' ranges over all Δ-functors from $\mathbf{J}$ to $\mathbf{E}$ which transform quasi-isomorphisms into isomorphisms. As such, the pair $(\mathbf{R}F, \zeta)$ (or $(\mathbf{L}f, \xi)$)—if it exists—is unique up to canonical isomorphism.

Complement 2.1.2. Let $\mathcal{A}'$ be another abelian category. Any additive functor $F\colon \mathcal{A} \to \mathcal{A}'$ extends to a Δ-functor $\bar{F}\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}')$ (see (1.5.2)). $Q'\colon \mathbf{K}(\mathcal{A}') \to \mathbf{D}(\mathcal{A}')$ being the canonical map, we will refer to derived functors of $Q'\bar{F}$, or of the restriction of $Q'\bar{F}$ to some specified Δ-subcategory $\mathbf{J}$ of $\mathbf{K}(\mathcal{A})$, as being "derived functors of F" and denote them by $\mathbf{R}F$ or $\mathbf{L}F$.

Example 2.1.3. If $F\colon \mathbf{J} \to \mathbf{E}$ transforms quasi-isomorphisms into isomorphisms then $F = \widetilde{F} \circ Q$ for a unique $\widetilde{F}\colon \mathbf{D_J} \to \mathbf{E}$; and $(\widetilde{F}$, identity) is both a right-derived and a left-derived functor of F.

Remarks 2.1.4. Let $\mathcal{A}'$ be an abelian category, and in (2.1.1) suppose that $\mathbf{E}$ is a Δ-subcategory of $\mathbf{K}(\mathcal{A}')$ or of $\mathbf{D}(\mathcal{A}')$. If $\mathbf{R}F$ exists we can set

$$\mathbf{R}^iF(A) := H^i(\mathbf{R}F(A)) \qquad (A \in \mathbf{J},\ i \in \mathbb{Z}).$$

Since $\mathbf{R}F$ is a Δ-functor, any triangle $A \to B \to C \to A[1]$ in $\mathbf{J}$ is transformed by $\mathbf{R}F$ into a triangle in $\mathbf{E}$, and hence we have an exact homology sequence (see $(1.4.5)^{\mathrm{H}}$):

$$\cdots \to \mathbf{R}^{i-1}F(C) \to \mathbf{R}^iF(A) \to \mathbf{R}^iF(B) \to \mathbf{R}^iF(C) \to \mathbf{R}^{i+1}F(A) \to \cdots \tag{2.1.4$^{\mathrm{H}}$}$$

This applies in particular to the triangle $(1.4.4.2)^{\sim}$ associated to an exact sequence of $\mathcal{A}$-complexes

$$0 \to A \to B \to C \to 0 \qquad (A, B, C \in \mathbf{J}).$$

A similar remark can be made for $\mathbf{L}F$.

2.2 Existence of Derived Functors

Derivability of a given functor is often proved by reduction, via suitable Δ-equivalences of categories, to the trivial example (2.1.3), as we now explain—and summarize in (2.2.6).

We consider, as in (1.7), a diagram

$$\begin{tikzcd} \mathbf{J}' \arrow[r, "j"] \arrow[d, "Q'"'] & \mathbf{J}'' \arrow[d, "Q''"] \\ \mathbf{D}' \arrow[r, "\tilde{\jmath}"'] & \mathbf{D}'' \end{tikzcd}$$

where $\mathbf{J}' \subset \mathbf{J}''$ are Δ-subcategories of $\mathbf{K}(\mathcal{A})$, $\mathbf{D}'$ and $\mathbf{D}''$ are the corresponding derived categories, Q' and Q'' are the canonical Δ-functors, j is the inclusion, and $\tilde{\jmath}$ is the unique Δ-functor making the diagram commute; and we assume that the conditions of (1.7.2) or of $(1.7.2)^{\mathrm{op}}$ obtain. In other words we have a family of quasi-isomorphisms

$$\psi_X \colon A_X \to X, \qquad X \in \mathbf{J}'',\ A_X \in \mathbf{J}', \quad \text{(see (1.7.2))}, \tag{2.2.1}$$

or a family of quasi-isomorphisms

$$\varphi_X \colon X \to A_X, \qquad X \in \mathbf{J}'',\ A_X \in \mathbf{J}', \quad \text{(see (1.7.2)}^{\mathrm{op}}\text{)}. \tag{2.2.1$^{\mathrm{op}}$}$$

In either situation, $\tilde{\jmath}$ identifies $\mathbf{D}'$ with a Δ-subcategory of $\mathbf{D}''$; there is a Δ-functor $(\rho,\theta)\colon \mathbf{D}'' \to \mathbf{D}'$ with

$$\rho(X) = A_X \qquad (X \in \mathbf{J}'');$$

and there are isomorphisms of Δ-functors

$$\mathbf{1}_{\mathbf{D}''} \xrightarrow{\sim} \tilde{\jmath}\rho, \qquad \mathbf{1}_{\mathbf{D}'} \xrightarrow{\sim} \rho\tilde{\jmath} \tag{2.2.2}$$

induced by ψ or by φ.

Proposition 2.2.3. *With preceding notation, let* $\mathbf{E}$ *be a* Δ*-category, let* $F\colon \mathbf{J}'' \to \mathbf{E}$ *be a* Δ*-functor, and suppose that the restricted functor*

$$F' := F \circ j\colon \mathbf{J}' \to \mathbf{E}$$

has a right-derived functor

$$\mathsf{R}F'\colon \mathbf{D}' \to \mathbf{E}, \qquad \zeta'\colon F' \to \mathsf{R}F' \circ Q'.$$

If there exists a family $\varphi_X\colon X \to A_X$ *as in* $(2.2.1)^{\mathsf{op}}$, *whence a functor* ρ *as above, then* F *has the right-derived functor* $(\mathsf{R}F, \zeta)$ *where*

$$\mathsf{R}F = \mathsf{R}F' \circ \rho\colon \mathbf{D}'' \to \mathbf{E}$$

so that

$$\mathsf{R}F(X) = \mathsf{R}F'(A_X) \qquad (X \in \mathbf{J}''),$$

and where for each $X \in \mathbf{J}''$, $\zeta(X)$ *is the composition*

$$F(X) \xrightarrow{F(\varphi_X)} F(A_X) = F'(A_X) \xrightarrow{\zeta'(A_X)} \mathsf{R}F'(A_X) = \mathsf{R}F(X)\,.$$

A similar statement holds for left-derived functors when there exists a family ψ_X *as in* (2.2).

Proof. We check first that ζ is actually a morphism of Δ-functors. Consider a map $u\colon X \to Y$ in $\mathbf{J}''$. Since $Q''(\varphi_X)$ is an isomorphism, there is a unique map $\tilde{u}\colon A_X \to A_Y$ in $\mathbf{D}''$ (and hence in the full subcategory $\mathbf{D}'$) making the following $\mathbf{D}''$-diagram commute:

$$\begin{tikzcd} X \arrow[r, "Q''(\varphi_X)"] \arrow[d, "Q''(u)"'] & A_X \arrow[d, "\tilde{u}"] \\ Y \arrow[r, "Q''(\varphi_Y)"'] & A_Y \end{tikzcd}$$

By the definition of the functor ρ (see proof of (1.7.2)), that ζ is a morphism of functors means that the following diagram $\mathcal{D}(u)$ commutes for all u:

$$
\begin{tikzcd}
F(X) \arrow[r, "F(\varphi_X)"] \arrow[d, "F(u)"'] & F(A_X) \arrow[r, "\zeta'(A_X)"] \arrow[d, dashed, "?"'] & \mathbf{R}F'(A_X) \arrow[d, "\mathbf{R}F'(\tilde{u})"] \\
F(Y) \arrow[r, "F(\varphi_Y)"'] & F(A_Y) \arrow[r, "\zeta'(A_Y)"'] & \mathbf{R}F'(A_Y)
\end{tikzcd}
$$

If there were a $\mathbf{J}'$-map $u'\colon A_X \to A_Y$ such that $u'\varphi_X = \varphi_Y u$, whence $Q''(u')Q''(\varphi_X) = Q''(\varphi_Y)Q''(u)$ and $\tilde{u} = Q''(u') = Q'(u')$, then the broken arrow in $\mathcal{D}(u)$ could be replaced by the map $F(u')$, making both resulting subdiagrams of $\mathcal{D}(u)$, and hence $\mathcal{D}(u)$ itself, commute. We don't know that such a u' exists; but, I claim, *there exists a quasi-isomorphism* $v\colon Y \to Z$ *such that* (*with self-explanatory notation*) *both* v' *and* $(vu)'$ *exist.* This being so, both diagrams $\mathcal{D}(v)$ and $\mathcal{D}(vu)$ commute; and since $\tilde{v}$ is an isomorphism (because v is a quasi-isomorphism), therefore $\mathbf{R}F'(\tilde{v})$ is an isomorphism, and it follows easily that $\mathcal{D}(u)$ also commutes, as desired.

To verify the claim, use (1.6.3) to construct in $\mathbf{J}''$ a commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "\varphi_X"] \arrow[d, "u"'] \arrow[dr] & A_X \arrow[drr, "w"] & & & \\
Y \arrow[r, "\varphi_Y"'] & A_Y \arrow[rr, "\varphi"'] & & Z \arrow[r, "\varphi_Z"'] & A_Z
\end{tikzcd}
$$

with φ a quasi-isomorphism, and set

$$
\begin{aligned}
v &:= \varphi \circ \varphi_Y \\
v' &:= \varphi_Z \circ \varphi \\
(vu)' &:= \varphi_Z \circ w.
\end{aligned}
$$

Then $v'\varphi_Y = \varphi_Z v$ and $(vu)'\varphi_X = \varphi_Z(vu)$, as desired.

Thus ζ is a morphism of functors; and it is straightforward to check, via commutativity of (1.7.2.2), that ζ is in fact a morphism of Δ-functors.

Now we need to show (see (2.1.1)) that for every Δ-functor $G\colon \mathbf{D}'' \to \mathbf{E}$ the composed map

$$
\mathrm{Hom}(\mathbf{R}F,\, G) \xrightarrow{(1.5.1)} \mathrm{Hom}(\mathbf{R}F \circ Q'',\, G \circ Q'') \xrightarrow{\text{via } \zeta} \mathrm{Hom}(F,\, G \circ Q'')
$$

is *bijective*. For this it suffices to check that the following natural composition is an inverse map:

$$
\begin{aligned}
\mathrm{Hom}(F,\, G\circ Q'') &\longrightarrow \mathrm{Hom}(F\circ j,\, G\circ Q''\circ j)\\
&=\!=\!= \mathrm{Hom}(F',\, G\circ\tilde{\jmath}\circ Q')\\
&\xrightarrow{(2.1.1)} \mathrm{Hom}(\mathbf{R}F',\, G\circ\tilde{\jmath})\\
&\longrightarrow \mathrm{Hom}(\mathbf{R}F'\circ\rho,\, G\circ\tilde{\jmath}\circ\rho)\\
&\xrightarrow{(2.2.2)} \mathrm{Hom}(\mathbf{R}F'\circ\rho,\, G)\\
&=\!=\!= \mathrm{Hom}(\mathbf{R}F,\, G)\,.
\end{aligned}
$$

This checking is left to the reader, as is the proof for left-derived functors. Q.E.D.

Example 2.2.4 [**H**, p. 53, Thm. 5.1]. Let $j\colon \mathbf{J}' \hookrightarrow \mathbf{J}''$, $F\colon \mathbf{J}'' \to \mathbf{E}$, and $\varphi_X\colon X \to A_X$ be as above, and suppose that the restricted functor $F' := F\circ j$ transforms quasi-isomorphisms into isomorphisms (or, equivalently, $F(C)\cong 0$ for every exact complex $C\in\mathbf{J}'$, see (1.5.1)). Then by (2.1.3), F' has a right-derived functor $(\mathbf{R}F',\, \mathbf{1})$ where $F' = \mathbf{R}F'\circ Q'$ and $\mathbf{1}$ is the identity morphism of F'.

So by (2.2.3), F has a right-derived functor $(\mathbf{R}F, \zeta)$ with

$$\mathbf{R}F(X) = F(A_X)$$

and

$$\zeta(X) = F(\varphi_X)\colon F(X) \to F(A_X) = \mathbf{R}F(X)$$

for all $X\in\mathbf{J}''$. Note that if $X\in\mathbf{J}'$ then φ_X is a quasi-isomorphism in $\mathbf{J}'$, whence $\zeta(X)$ is an isomorphism.

The action of $\mathbf{R}F$ on maps can be described thus: if $u\colon X\to Y$ is a map in $\mathbf{J}''$ then with v' and $(vu)'$ as in the preceding proof,

$$\mathbf{R}F(u/1) = F(v')^{-1}\circ F((vu'))\,;$$

and for any map f/s in $\mathbf{D}''$ (see §1.2), we have

$$\mathbf{R}F(f/s) = \mathbf{R}F(f/1)\circ\mathbf{R}F(s/1)^{-1}.$$

As for the Δ-structure on $\mathbf{R}F$, one has for each X the isomorphism

$$\theta(X)\colon \mathbf{R}F(X[1]){=}F(A_{X[1]}) \xrightarrow[F(\eta_X)]{\sim} F(A_X[1]) \xrightarrow[\theta_F]{\sim} F(A_X)[1] = \mathbf{R}F(X)[1]$$

where

$$\eta_X := Q''(\varphi_X[1])\circ Q''(\varphi_{X[1]})^{-1}\colon A_{X[1]} \xrightarrow{\sim} A_X[1]\,,$$

and where the isomorphism θ_F comes from the Δ-functoriality of F.

(2.2.5). Let $\mathcal{A}$ be an abelian category, let $\mathbf{J}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A})$, and let F be a Δ-functor from $\mathbf{J}$ to a Δ-category $\mathbf{E}$. We say that a complex X in $\mathbf{J}$ is *right-F-acyclic* if for each quasi-isomorphism $u\colon X \to Y$ in $\mathbf{J}$ there is a quasi-isomorphism $v\colon Y \to Z$ in $\mathbf{J}$ such that the map $F(vu)\colon F(X) \to F(Z)$ is an isomorphism. *Left-F-acyclicity* is defined similarly, with arrows reversed.

For example, if $\mathbf{J} := \mathbf{J}''$ in (2.2.4), then every complex $X \in \mathbf{J}'$ is right-F-acyclic—just take $Z := A_Y$ and $v := \varphi_Y$. Conversely:

Lemma 2.2.5.1. *The right-F-acyclic complexes in $\mathbf{J}$ are the objects of a localizing subcategory* (§1.7). *Moreover, the restriction of F to this subcategory transforms quasi-isomorphisms into isomorphisms; in other words, if the complex X is both exact and right-F-acyclic, then $F(X) \cong 0$* (*see* (1.5.1)).

Proof. Since F commutes with translation—up to isomorphism—it is clear that X is right-F-acyclic iff so is $X[1]$.

Next, suppose we have a triangle $X \to X_1 \to X_2 \to X[1]$ in which X_1 and X_2 are right-F-acyclic. We will show that then X is right-F-acyclic. Any quasi-isomorphism $u\colon X \to Y$ can be embedded into a map of triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & X_1 \arrow[r] \arrow[d, "u_1"'] & X_2 \arrow[r] \arrow[d, "u_2"'] & X[1] \arrow[d, "u[1]"] \\
Y \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & Y[1]
\end{tikzcd}$$

where u_1 is a quasi-isomorphism whose existence is given by (1.6.3), and where u_2 is then given by $(\Delta3)'$ and $(\Delta3)''$ in §1.4. Such a u_2 is also a quasi-isomorphism, as one sees by applying the five-lemma to the natural map between the homology sequences of the two triangles (see $(1.4.5)^{\mathrm{H}}$). Similarly, from the definition of right-F-acyclic we deduce a triangle-map

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d, "v_1"'] & Y_2 \arrow[r] \arrow[d, "v_2"'] & Y[1] \arrow[r] \arrow[d, "v{[1]}"'] & Y_1[1] \arrow[d, "v_1{[1]}"] \\
Z_1 \arrow[r] & Z_2 \arrow[r] & Z[1] \arrow[r] & Z_1[1]
\end{tikzcd}$$

where v_1, v_2, and v are quasi-isomorphisms such that $F(v_1u_1)$ and $F(v_2u_2)$ are isomorphisms. (Here $(\Delta2)$ in §1.4 should be kept in mind.) We can then apply the Δ-functor F to the map of triangles

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "v_1u_1"'] & X_2 \arrow[r] \arrow[d, "v_2u_2"'] & X[1] \arrow[r] \arrow[d, "(vu){[1]}"'] & X_1[1] \arrow[d, "(v_1u_1){[1]}"] \\
Z_1 \arrow[r] & Z_2 \arrow[r] & Z[1] \arrow[r] & Z_1[1]
\end{tikzcd}$$

and deduce from $(\Delta3)^*$ that $F((vu)[1])$, and hence $F(vu)$, is also an isomorphism. Thus X is indeed right-F-acyclic.

In particular, the direct sum of two right-F-acyclic complexes is right-F-acyclic, because the direct sum is the summit of a triangle whose base is the zero-map from one to the other, see (1.4.2.1). Also, $0 \in \mathbf{J}$ is clearly right-F-acyclic. We see then that the right-F-acyclic complexes are the objects of a Δ-subcategory of $\mathbf{J}$.

For this subcategory to be localizing it suffices, by $(1.7.1)^{\mathsf{op}}$, that if $X \to Y \to Z$ is as in the definition of right-F-acyclic, then Z is right-F-acyclic; and this follows from:

Lemma 2.2.5.2. *If X is right-F-acyclic and if there exists a quasi-isomorphism $\alpha\colon X \to Z$ such that $F(\alpha)\colon F(X) \to F(Z)$ is an epimorphism, then Z is right-F-acyclic.*

Proof. Given a quasi-isomorphism $Z \to Y'$, there exists a quasi-isomorphism $Y' \to Z'$ such that $F(X) \to F(Z) \to F(Z')$ is an isomorphism (since X is right-F-acyclic); and since $F(X) \to F(Z)$ is an epimorphism, therefore $F(Z) \to F(Z')$ is an isomorphism. Q.E.D.

To justify the last assertion in (2.2.5.1), take $Y := 0$ in the definition of right-F-acyclicity. Q.E.D.

We leave it to the reader to establish a corresponding statement for left-F-acyclic complexes.

In summary:

Proposition 2.2.6. *Let $\mathcal{A}$ be an abelian category, let $\mathbf{J}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A})$, and let F be a Δ-functor from $\mathbf{J}$ to a Δ-category $\mathbf{E}$. Suppose $\mathbf{J}$ contains a family of quasi-isomorphisms $\varphi_X\colon X \to A_X$ $(X \in \mathbf{J})$ such that A_X is right-F-acyclic for all X, see* (2.2.5). *Then F has a right-derived functor $(\mathsf{R}F, \zeta)$ such that for all $X \in \mathbf{J}$,*

$$\mathsf{R}F(X) = F(A_X) \quad \text{and} \quad \zeta(X) = F(\varphi_X)\colon F(X) \to F(A_X) = \mathsf{R}F(X)\,.$$

Moreover, X is right-F-acyclic $\Leftrightarrow \zeta(X)$ is an isomorphism.

Proof. Everything is contained in (2.2.4) and (2.2.5), except for the fact that if $\zeta(X)$ is an isomorphism then X is right-F-acyclic, which is proved by taking, in (2.2.5), $Z := A_Y$, $v := \varphi_Y$, and noting that then $F(vu)$ is the composite isomorphism

$$F(X) \underset{\zeta(X)}{\xrightarrow{\;\sim\;}} \mathsf{R}F(X) \xrightarrow{\;\sim\;} \mathsf{R}F(Y) = F(Z).$$

Q.E.D.

Corollary 2.2.6.1. *With assumptions as in* (2.2.6), *if $G\colon \mathbf{E} \to \mathbf{E}'$ is any Δ-functor then $(G \circ \mathsf{R}F,\ G(\zeta))$ is a right-derived functor of GF.*

Proof. Clearly, right-F-acyclic complexes are right-(GF)-acyclic. It follows then from (2.2.4) and (2.2.5) that the assertion need only be proved for the restriction of F to the subcategory of right-F-acyclic complexes, in which case it follows from (2.1.3). Q.E.D.

Corollary 2.2.7. *Let $\mathcal{A}$, $\mathcal{A}'$ be abelian categories, let $\mathbf{J} \subset \mathbf{K}(\mathcal{A})$, $\mathbf{J}' \subset \mathbf{K}(\mathcal{A}')$ be Δ-subcategories with canonical functors $Q\colon \mathbf{J} \to \mathbf{D_J}$, $Q'\colon \mathbf{J}' \to \mathbf{D_{J'}}$ to their respective derived categories, and let $F\colon \mathbf{J} \to \mathbf{J}'$ and $G\colon \mathbf{J}' \to \mathbf{E}$ be Δ-functors. Assume that G has a right-derived functor $\mathbf{R}G$ and that every complex $X \in \mathbf{J}$ admits a quasi-isomorphism into a right-$(Q'F)$-acyclic complex A_X such that $F(A_X)$ is right-G-acyclic. Then $Q'F$ and GF have right-derived functors, denoted $\mathbf{R}F$ and $\mathbf{R}(GF)$, and there is a unique Δ-functorial isomorphism*

$$\alpha\colon \mathbf{R}(GF) \xrightarrow{\ \sim\ } \mathbf{R}G\,\mathbf{R}F$$

such that the following natural diagram commutes for all $X \in \mathbf{J}$:

$$\begin{tikzcd} GF(X) \arrow[r] \arrow[d] & \mathbf{R}(GF)(QX) \arrow[d, "\alpha(QX)", "\simeq"'] \\ \mathbf{R}GQ'F(X) \arrow[r] & \mathbf{R}G\mathbf{R}F(QX) \end{tikzcd} \tag{2.2.7.1}$$

Proof. Derivability of $Q'F$ results from (2.2.6). Derivability of GF results similarly once we show, as follows, that A_X is right-(GF)-acyclic: note for any quasi-isomorphism $A_X \to Y$ in $\mathbf{J}$ that, by (2.2.5.1), the resulting composed map $F(A_X) \to F(Y) \to F(A_Y)$ is a quasi-isomorphism and so $GF(A_X) \xrightarrow{\sim} GF(A_Y)$. The existence of a unique Δ-functorial α making (2.2.7.1) commute follows from the definition of right-derived functor. Since A_X is right-(GF)-acyclic and right-$(Q'F)$-acyclic, and $F(A_X)$ is right-G-acyclic, (2.2.6) implies that $\alpha(QX)$ is isomorphic to the identity map of $GF(A_X)$. Thus α is an isomorphism. Q.E.D.

We leave the corresponding statements for left-F-acyclic complexes and left-derived functors to the reader.

Incidentally, (2.2.6) generalizes in a simple way to triangulation-compatible multiplicative systems in any Δ-category (see [**H**, p. 31]). It is of course of little interest unless we can construct a family (φ_X). That matter is addressed in the following sections.

Exercises 2.2.8. (a) Verify that F transforms quasi-isomorphisms into isomorphisms iff every complex $X \in \mathbf{J}$ is right-F-acyclic.

(b) Verify that if $X \in \mathbf{J}$ is exact then X is right-F-acyclic iff $F(X) \cong 0$.

(c) Let F be a Δ-functor from $\mathbf{J}$ to a Δ-category $\mathbf{E}$. Let $\mathbf{J}'$ be the full subcategory of $\mathbf{J}$ whose objects are all the complexes in $\mathbf{J}$ admitting a quasi-isomorphism to a right-F-acyclic complex. Then $\mathbf{J}'$ is a Δ-subcategory of $\mathbf{J}$.

(d) X is right-F-acyclic iff every map $C \to X$ in $\mathbf{J}$ with C exact factors as $C \to C' \to X$ with C' exact and $F(C') \cong 0$.

(e) X is said to be "unfolded for F" if for every $Z \in \mathbf{E}$ the natural map

$$\mathrm{Hom}_{\mathbf{E}}(Z, F(X)) \to \varinjlim_{X \to Y} \mathrm{Hom}_{\mathbf{E}}(Z, F(Y))$$

is an isomorphism, where the $\varinjlim$ is taken over the category of all quasi-isomorphisms $X \to Y$ in $\mathbf{J}$ [**De**, p. 274, (iv)]. Check that any right-F-acyclic X is unfolded for F; and that the converse holds under the hypotheses of (2.2.6).

(f) Show: X is unfolded for F iff every map $C \to X$ in $\mathbf{J}$ with C exact factors as $C \to C' \to X$ with C' exact and $F(C) \to F(C')$ the zero map. (For this, the octahedral axiom in $\mathbf{E}$ may be needed, see §1.4.)

2.3 Right-Derived Functors via Injective Resolutions

The basic example of a family (φ_X) as in (2.2.6) arises when $\mathcal{A}$ has enough injectives, i.e., every object of $\mathcal{A}$ admits a monomorphism into an injective object. Then every complex $X \in \overline{\mathbf{K}}^{+}(\mathcal{A})$ admits a quasi-isomorphism $\varphi_X : X \to I_X$ into a bounded-below complex of injectives (see (1.8.2)); and by (2.3.4) and (2.3.2.1) below, this I_X is right-F-acyclic for *every* Δ-functor $F : \overline{\mathbf{K}}^{+}(\mathcal{A}) \to \mathbf{E}$, whence F is right-derivable.

Later on, however, it will become important for us to be able to deal with *unbounded* complexes; and for this purpose the following more general injectivity notion is, via (2.3.5), essential.

Definition 2.3.1. Let $\mathcal{A}$ be an abelian category, and let $\mathbf{J}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A})$. A complex $I \in \mathbf{J}$ is said to be q-injective in $\mathbf{J}$ (or $\mathbf{J}$-q-injective) if for every diagram $Y \xleftarrow{s} X \xrightarrow{f} I$ in $\mathbf{J}$ with s a quasi-isomorphism, there exists $g : Y \to I$ such that $gs = f$.[1]

Lemma 2.3.2. *$I \in \mathbf{J}$ is $\mathbf{J}$-q-injective iff every quasi-isomorphism $I \to Y$ in $\mathbf{J}$ has a left inverse.*

Proof. In (2.3.1) take $X := I$ and $f :=$ identity to see that if I is q-injective then the quasi-isomorphism s has a left inverse. Conversely, by (1.6.3) any diagram $Y \xleftarrow{s} X \xrightarrow{f} I$ is part of a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & I \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ Y & \xrightarrow[f']{} & Y' \end{array}$$

in which s' is a quasi-isomorphism; and then if t is a left inverse for s' and $g := tf'$, we have $gs = f$. Q.E.D.

[1] Here "q" stands for the class of quasi-isomorphisms. The equivalent term "K-injective" in [**Sp**, p. 127] seems to me less suggestive.

Corollary 2.3.2.1. *$I \in \mathbf{J}$ is $\mathbf{J}$-q-injective iff I is right-F-acyclic for every Δ-functor $F\colon \mathbf{J} \to \mathbf{E}$.*

Proof. If any quasi-isomorphism $I \to Y$ has a left inverse, then setting $X := I$ in (2.2.5) we see at once that I is right-F-acyclic. Conversely, if I is right-F-acyclic for the identity functor $\mathbf{J} \to \mathbf{J}$, then every quasi-isomorphism $I \to Y$ has a left inverse. Q.E.D.

Taking $F :=$ identity in (2.2.5.1), we deduce:

Corollary 2.3.2.2. *The $\mathbf{J}$-q-injective complexes are the objects of a localizing subcategory $\mathbf{I}$. Every quasi-isomorphism in $\mathbf{I}$ is an isomorphism, so the pair $(\mathbf{I}, \text{identity})$ has the universal property of the derived category $\mathbf{D_I}$ (§1.2), and therefore $\mathbf{I} \cong \mathbf{D_I}$ can be identified with a Δ-subcategory of $\mathbf{D_J}$.*

Corollary 2.3.2.3. *Suppose that there exists a family of* q-injective resolutions $\varphi_X\colon X \to I_X$ $(X \in \mathbf{J})$, *i.e., for each X, φ_X is a quasi-isomorphism and I_X is $\mathbf{J}$-q-injective. Then any Δ-functor $F\colon \mathbf{J} \to \mathbf{E}$ has a right-derived functor $(\mathsf{R}F, \zeta)$*[2] *with*

$$\mathsf{R}F(X) = F(I_X) \quad \text{and} \quad \zeta(X) = F(\varphi_X)\colon F(X) \to F(I_X) = \mathsf{R}F(X)\,,$$

and such that for any morphism $f/s\colon X_1 \xleftarrow{s} X \xrightarrow{f} X_2$ in $\mathbf{D_J}$,

$$\mathsf{R}F(f/s) = F(f') \circ F(s')^{-1}$$

where f' is the unique map in $\mathbf{I}$ making the following square in $\mathbf{J}$ commute

$$\begin{tikzcd} X \arrow[r, "\varphi_X"] \arrow[d, "f"'] & I_X \arrow[d, "f'"] \\ X_2 \arrow[r, "\varphi_{X_2}"'] & I_{X_2} \end{tikzcd}$$

and similarly for s'.

Proof. Since φ_X becomes an isomorphism in $\mathbf{D_J}$, the map f' exists uniquely in $\mathbf{D_J}$, hence in $\mathbf{I}$ (2.3.2.2). For the rest see (2.2.4), with $\mathbf{J}' := \mathbf{I}$, $\mathbf{J}'' := \mathbf{J}$, and $v :=$ identity. Q.E.D.

Example 2.3.3. An object I in $\mathcal{A}$ is injective iff when considered as a complex vanishing in all nonzero degrees it is q-injective in $\mathbf{K}(\mathcal{A})$ (or in $\mathbf{K}^{\mathsf{b}}(\mathcal{A})$).

Sufficiency: for any $\mathcal{A}$-diagram $Y^0 \xleftarrow{s^0} X \xrightarrow{f} I$ with s^0 a monomorphism, take Y to be the complex which looks like the natural map $Y^0 \to \operatorname{coker}(s^0)$ in degrees 0 and 1, and vanishes elsewhere, and take $s\colon X \to Y$ to be the obvious quasi-isomorphism; then deduce from (2.3.1) that if I is q-injective there exists $g^0\colon Y^0 \to I$ such that $g^0 s^0 = f$—so that I is $\mathcal{A}$-injective.

[2] So the embedding functor (2.1.1.1) has a *left adjoint,* taking F to $\mathsf{R}F$.

For necessity, use (2.3.2): to find a left inverse in $\mathbf{K}(\mathcal{A})$ for a quasi-isomorphism $\beta\colon I \to Y$ we may replace Y by the complex $\tau_{\geq 0}Y$, to which Y maps quasi-isomorphically (§1.10), i.e., we may assume that Y vanishes in all negative degrees; then β induces a monomorphism (in $\mathcal{A}$) $\beta^0\colon I \to Y^0$, which has a left inverse if I is $\mathcal{A}$-injective, and that gives rise, obviously, to a left inverse for β. (One could also use (iv) in (2.3.8) below.)

Example 2.3.4. Any bounded-below complex I of $\mathcal{A}$-injectives is q-injective in $\mathbf{K}(\mathcal{A})$. Indeed, by [**H**, p. 41, Lemma 4.5], I satisfies the condition in (2.3.2). (One could also use (2.3.8)(iv).) Thus (2.3.2.3) applies to $\mathbf{J} := \overline{\mathbf{K}}^{+}(\mathcal{A})$ whenever $\mathcal{A}$ has enough injectives (see beginning of §2.3). In that case, further, every $\overline{\mathbf{K}}^{+}(\mathcal{A})$-q-injective complex admits a quasi-isomorphism, hence, by (2.3.2.2), an *isomorphism,* to a bounded-below complex of $\mathcal{A}$-injectives.

Example 2.3.5. Let U be a topological space, $\mathcal{O}$ a sheaf of rings on U, and $\mathcal{A}$ the abelian category of left $\mathcal{O}$-modules. Then a theorem of Spaltenstein [**Sp**, p. 138, Theorem 4.5] asserts that *every complex in* $\mathbf{K}(\mathcal{A})$ *admits a q-injective resolution.* Hence by (2.3.2.3), every Δ-functor out of $\mathbf{K}(\mathcal{A})$ is right-derivable.

More generally, a q-injective resolution exists for every complex in any Grothendieck category, i.e., an abelian category with exact direct limits and having a generator [**AJS**, p. 243, Theorem 5.4]. For example, injective Cartan-Eilenberg resolutions [**EGA**, III, Chap. 0, (11.4.2)] always exist in Grothendieck categories; and their totalizations—which generally require countable direct products—give q-injective resolutions when such products of epimorphisms are epimorphisms (a condition which holds in the category of modules over a fixed ring, but fails, for instance, in most categories of sheaves on topological spaces).

Example 2.3.6. Let $\mathcal{A}_1$, $\mathcal{A}_2$ be abelian categories, $\mathcal{A}_1$ having enough injectives. As in (1.5.2) any additive functor $F\colon \mathcal{A}_1 \to \mathcal{A}_2$ extends to a Δ-functor $\bar{F}\colon \mathbf{K}^{+}(\mathcal{A}_1) \to \mathbf{K}^{+}(\mathcal{A}_2)$ which has, by (2.3.4), a right-derived functor

$$\mathbf{R}^{+}\bar{F}\colon \mathbf{D}^{+}(\mathcal{A}_1) \to \mathbf{K}^{+}(\mathcal{A}_2)$$

satisfying, for a given family $\varphi_X\colon X \to I_X$ of injective resolutions,

$$\mathbf{R}^{+}\bar{F}(X) = \bar{F}(I_X)\,.$$

We can extend the domain of $\mathbf{R}^{+}\bar{F}$ to $\overline{\mathbf{D}}^{+}(\mathcal{A}_1)$ by composing with the equivalence τ^{+} defined in (1.8.1).

Moreover, if every $\mathcal{A}_1$-complex has a q-injective resolution, then there is a further extension to a derived functor $\mathbf{R}\bar{F}\colon \mathbf{D}(\mathcal{A}_1) \to \mathbf{K}(\mathcal{A}_2)$—whose composition with the canonical map $\mathbf{K}(\mathcal{A}_2) \to \mathbf{D}(\mathcal{A}_2)$ is $\mathbf{R}F$, see (2.1.2).

With H^i the usual homology functor, let $R^iF\colon \mathcal{A}_1 \to \mathcal{A}_2$ $(i \in \mathbb{Z})$ be the composition

$$\mathcal{A}_1 \xrightarrow{(1.2.2)} \mathbf{D}^+(\mathcal{A}_1) \xrightarrow{\mathbf{R}^+F} \mathbf{K}^+(\mathcal{A}_2) \xrightarrow{H^i} \mathcal{A}_2$$

(cf. (2.1.4)). Then $R^iF = 0$ for $i < 0$, and there is a natural map of functors $F \to R^0F$ which is an isomorphism if and only if F is left-exact.

Example 2.3.7. Let $f\colon U_1 \to U_2$ be a continuous map of topological spaces. Let $\mathcal{A}_i$ be the category of sheaves of abelian groups on U_i $(i = 1, 2)$. Then $\mathcal{A}_i$ is abelian, and has enough injectives. The direct image functor $f_*\colon \mathcal{A}_1 \to \mathcal{A}_2$ is left-exact, and has, as in (2.3.6), a derived functor

$$\mathbf{R}^+\overline{f}_*\colon \overline{\mathbf{D}}^+(\mathcal{A}_1) \to \mathbf{K}^+(\mathcal{A}_2)\,.$$

By (2.3.5), the composition $\mathbf{K}(\mathcal{A}_1) \xrightarrow{\overline{f}_*} \mathbf{K}(\mathcal{A}_2) \xrightarrow{Q} \mathbf{D}(\mathcal{A}_2)$ has a derived functor $\mathbf{R}f_*$, whose restriction to $\overline{\mathbf{D}}^+(\mathcal{A}_1)$ is isomorphic to $Q \circ \mathbf{R}^+\overline{f}_*$.

In particular, when U_2 is a single point then $\mathcal{A}_2 = \mathfrak{Ab}$, the category of abelian groups, and f_* is the global section functor $\Gamma = \Gamma(U_1, -)$. In this case one usually sets, for $i \in \mathbb{Z}$, see (2.1.4),

$$\mathbf{R}f_* = \mathbf{R}\Gamma, \qquad \mathbf{R}^if_* = \mathbf{R}^i\Gamma = \mathbf{H}^i, \qquad R^if_*(-) = H^i(U_1, -)\,.$$

Here are some other characterizations of q-injectivity, see [**Sp**, p. 129, Prop. 1.5], [**BN**, Def. 2.6 etc.].

Proposition 2.3.8. *Let $\mathcal{A}$ be an abelian category, and let $\mathbf{J}$ be a Δ-subcategory. of $\mathbf{K}(\mathcal{A})$. The following conditions on a complex $I \in \mathbf{J}$ are equivalent:*

(i) *I is q-injective in $\mathbf{J}$.*

(i)′ *For every diagram $Y \xleftarrow{s} X \xrightarrow{f} I$ in $\mathbf{J}$ with s a quasi-isomorphism there is a* unique *$g\colon Y \to I$ such that $gs = f$.*

(ii) *Every quasi-isomorphism $I \to Y$ in $\mathbf{J}$ has a left inverse.*

(ii)′ *Every quasi-isomorphism $I \to Y$ in $\mathbf{J}$ is a monomorphism.*

(iii) *I is right-F-acyclic for every Δ-functor $F\colon \mathbf{J} \to \mathbf{E}$.*

(iii)′ *I is right-F-acyclic for F the identity functor $\mathbf{J} \to \mathbf{J}$.*

(iv) *For every exact complex $X \in \mathbf{J}$, we have* $\mathrm{Hom}_{\mathbf{J}}(X, I) = 0$.

(iv)′ *The Δ-functor* $\mathrm{Hom}^\bullet(-, I)\colon \mathbf{J} \to \mathbf{K}(\mathfrak{Ab})$ *of* (1.5.3) *takes quasi-isomorphisms into quasi-isomorphisms.*

(v) *For every complex $X \in \mathbf{J}$, the natural map*

$$\mathrm{Hom}_{\mathbf{J}}(X, I) \to \mathrm{Hom}_{\mathbf{D}_{\mathbf{J}}}(X, I)$$

is bijective.

Proof. The equivalence of (i), (ii), (iii) and (iii)′ has already been shown (see (2.3.2) and the proof of (2.3.2.1)). For (ii) $\Leftrightarrow$ (ii)′ see (1.4.2.1). Taking $Y := 0$ in (2.3.1), we see that (i) $\Rightarrow$ (iv). The equivalence of (iv) and (iv)′ results from the footnote in (1.5.1) and the easily-checked relation

$$H^n\big(\mathrm{Hom}^\bullet(X,I)\big) \cong \mathrm{Hom}_{\mathbf{J}}(X[-n],I) \qquad (n\in\mathbb{Z},\ X\in\mathbf{J}). \tag{2.3.8.1}$$

The implications (v) ⇒ (i)′ ⇒ (i) are simple to verify.

We show next that (iv) ⇒ (ii). Let X be the summit of a triangle T in $\mathbf{J}$ whose base is a quasi-isomorphism $I \to Y$. By [**H**, p. 23, 1.1 b)], the resulting sequence

$$\mathrm{Hom}(X,I) \to \mathrm{Hom}(Y,I) \to \mathrm{Hom}(I,I) \to \mathrm{Hom}(X[-1],I)$$

is *exact.* Moreover, the exact homology sequence $(1.4.5)^{\mathrm{H}}$ of T shows that X is exact. So if (iv) holds, then $\mathrm{Hom}(Y,I) \to \mathrm{Hom}(I,I)$ is bijective, and (ii) follows.

Finally, we show (ii) ⇒ (v). For any map $f/s\colon X \to I$ in $\mathbf{D_J}$, (1.6.3) yields a commutative diagram in $\mathbf{J}$, with s' a quasi-isomorphism:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & I \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle s'} \\ X & \xrightarrow[\ f'\ ]{} & B \end{array}$$

If $ts' =$ identity, then $f/s = (s'/1)^{-1}(f'/1) = (tf')/1$, and so the map $\mathrm{Hom}_{\mathbf{J}}(X,I) \to \mathrm{Hom}_{\mathbf{D_J}}(X,I)$ is surjective. For injectivity, given $f\colon X \to I$ in $\mathbf{J}$, note that $f/1 = 0 \implies$ there exists a quasi-isomorphism $t\colon X' \to X$ such that $ft = 0$ (see §1.2) $\implies$ there exists a quasi-isomorphism $s\colon I \to Y$ such that $sf = 0$ [**H**, p. 37]; and if s has a left inverse, then $sf = 0 \implies f = 0$. Q.E.D.

Exercise 2.3.9. Show that if $\mathcal{A}$ is a Grothendieck category then $\mathbf{D}(\mathcal{A})$ is equivalent to the homotopy category of q-injective complexes. Hence if $\mathcal{A}$ has inverse limits then so does $\mathbf{D}(\mathcal{A})$.

2.4 Derived Homomorphism Functors

Let $\mathcal{A}$ be an abelian category, and let $\mathbf{L}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A})$ in which there exists a family of quasi-isomorphisms $\varphi_X\colon X \to I_X$ $(X \in \mathbf{L})$ such that $I_X \in \mathbf{L}$ is q-injective in $\mathbf{K}(\mathcal{A})$ for every X. Then for any quasi-isomorphism $s\colon X \to Y$ with Y in $\mathbf{K}(\mathcal{A})$ there exists, by (2.3.1), a map $g\colon Y \to I_X$, necessarily a quasi-isomorphism, such that $gs = \varphi_X$; and hence by $(1.7.1)^{\mathsf{op}}$, $\mathbf{L}$ is a localizing subcategory of $\mathbf{K}(\mathcal{A})$, i.e., the derived category $\mathbf{D_L}$ identifies naturally with a Δ-subcategory of $\mathbf{D}(\mathcal{A})$.

For example, if $\mathcal{A}$ has enough injectives we could take $\mathbf{L} := \overline{\mathbf{K}}^{+}(\mathcal{A})$, see (2.3.4). Or, if U is a topological space with a sheaf of rings $\mathcal{O}$ and $\mathcal{A}$ is the category of left $\mathcal{O}$-modules, we could take $\mathbf{L} := \mathbf{K}(\mathcal{A})$, see (2.3.5).

By (2.3.2.3), every Δ-functor $F\colon \mathbf{L} \to \mathbf{E}$ is right-derivable. So for any fixed object $A \in \mathbf{K}(\mathcal{A})$, the Δ-functor $F_A\colon \mathbf{L} \to \mathbf{K}(\mathfrak{Ab})$ given by

$$F_A(B) = \mathrm{Hom}^\bullet(A,\, B) \qquad (B \in \mathbf{L})$$

(see (1.5.3)) has a right-derived functor

$$\mathsf{R}F_A\colon \mathbf{D_L} \to \mathbf{K}(\mathfrak{Ab})$$

with

$$\mathsf{R}F_A(B) = \mathrm{Hom}^\bullet(A,\, I_B).$$

For fixed B and variable A, $\mathrm{Hom}^\bullet(A,\, I_B)$ is a contravariant Δ-functor from $\mathbf{K}(\mathcal{A})$ to $\mathbf{K}(\mathfrak{Ab})$ (see 1.5.3), which takes quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ to quasi-isomorphisms in $\mathbf{K}(\mathfrak{Ab})$ ((2.3.8)(iv)$'$) and hence—after composition with the natural functor $Q'\colon \mathbf{K}(\mathfrak{Ab}) \to \mathbf{D}(\mathfrak{Ab})$—to isomorphisms in $\mathbf{D}(\mathfrak{Ab})$. So by (1.5.1)—and the exercise preceding it—there results a Δ-functor $\mathbf{D}(\mathcal{A})^{\mathsf{op}} \to \mathbf{D}(\mathfrak{Ab})$. Thus we obtain a functor of two variables

$$\mathsf{R}\mathrm{Hom}^\bullet(A,\, B)\colon \mathbf{D}(\mathcal{A})^{\mathsf{op}} \times \mathbf{D_L} \to \mathbf{D}(\mathfrak{Ab})$$

which, together with appropriate θ (see (1.5.3)), is a Δ-functor in each variable separately:

$$\mathsf{R}\mathrm{Hom}^\bullet(A,\, B) = Q'\mathrm{Hom}^\bullet(A,\, I_B) \tag{2.4.1}$$

for all objects $A \in \mathbf{D}(\mathcal{A})^{\mathsf{op}}$, $B \in \mathbf{D_L}$; and we leave it to the reader to make explicit the effect of $\mathsf{R}\mathrm{Hom}^\bullet$ on morphisms in $\mathbf{D}(\mathcal{A})^{\mathsf{op}}$ and $\mathbf{D_L}$ respectively.

From (2.3.8)(v) and (2.3) (with $\mathbf{J} := \mathbf{K}(\mathcal{A})$), we deduce canonical isomorphisms (*Yoneda theorem*):

$$H^n(\mathsf{R}\mathrm{Hom}^\bullet(X,\, B)) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X,\, B[n]) \qquad (n \in \mathbb{Z}). \tag{2.4.2}$$

This leads, in particular, to an elementary interpretation of the exact sequence $(2.1.4)^{\mathrm{H}}$ when $F := F_X$, see [**H**, p. 23, Prop. 1.1, b)].

(2.4.3). The variables A, B are treated quite differently in the above definition of $\mathsf{R}\mathrm{Hom}^\bullet$. But there is a more symmetric characterization of this derived functor, analogous to the one in (2.1.1). This is given in (2.4.4), after the necessary preparation.

Let $\mathbf{K_1}$, $\mathbf{K_2}$, $\mathbf{E}$ be Δ-categories, with respective translation functors T_1, T_2, T. A Δ-functor from $\mathbf{K_1} \times \mathbf{K_2}$ to $\mathbf{E}$ is defined to be a triple (F, θ_1, θ_2) with

$$F\colon \mathbf{K_1} \times \mathbf{K_2} \to \mathbf{E}$$

a functor and

$$\theta_1\colon F \circ (T_1 \times 1) \xrightarrow{\ \sim\ } T \circ F, \qquad \theta_2\colon F \circ (1 \times T_2) \xrightarrow{\ \sim\ } T \circ F$$

isomorphisms of functors, such that for each $B \in \mathbf{K_2}$ the functor

$$F_B(A) := F(A, B)$$

together with θ_1 is a Δ-functor from $\mathbf{K_1}$ to $\mathbf{E}$, and for each $A \in \mathbf{K_1}$ the functor

$$F_A(B) := F(A, B)$$

together with θ_2 is a Δ-functor from $\mathbf{K_2}$ to $\mathbf{E}$; and such that furthermore the composed functorial isomorphisms

$$F(T_1 \times T_2) = F(T_1 \times 1)(1 \times T_2) \xrightarrow{\text{via } \theta_1} TF(1 \times T_2) \xrightarrow{\text{via } \theta_2} TTF$$
$$F(T_1 \times T_2) = F(1 \times T_2)(T_1 \times 1) \xrightarrow{\text{via } \theta_2} TF(T_1 \times 1) \xrightarrow{\text{via } \theta_1} TTF$$

are *negatives* of each other. Similarly, we can define Δ-functors of three or more variables—with a condition indicated by the equation

$$(\text{via } \theta_i) \circ (\text{via } \theta_j) = -(\text{via } \theta_j) \circ (\text{via } \theta_i) \qquad (i \neq j).$$

Morphisms of Δ-functors are defined in the obvious way, see (1.5).

For example, let $\mathbf{L} \subset \mathbf{K} := \mathbf{K}(\mathcal{A})$ be as above, with respective derived categories $\mathbf{D_L} \subset \mathbf{D}$, and consider the functor

$$\mathrm{Hom}^\bullet \colon \mathbf{K}^{\mathsf{op}} \times \mathbf{L} \to \mathbf{K}(\mathfrak{Ab}).$$

As in the exercise preceding (1.5.1), we can consider the opposite category $\mathbf{K}^{\mathsf{op}}$ to be triangulated, with translation inverse to that in $\mathbf{K}$, in such a way that the canonical contravariant functor $\mathbf{K} \to \mathbf{K}^{\mathsf{op}}$ and its inverse, together with $\theta =$ identity, are both Δ-functors. This being so, one checks then that $\mathrm{Hom}^\bullet$ is a Δ-functor (see (1.5.3)).

Similarly

$$\mathsf{R}\mathrm{Hom}^\bullet \colon \mathbf{D}^{\mathsf{op}} \times \mathbf{D_L} \to \mathbf{D}(\mathfrak{Ab})$$

is a Δ-functor. Furthermore, the q-injective resolution maps $\varphi_B \colon B \to I_B$ induce a natural morphism of Δ-functors

$$\eta \colon Q'\mathrm{Hom}^\bullet(A,\, B) \to Q'\mathrm{Hom}^\bullet(A,\, I_B) \overset{(2.4.1)}{=} \mathsf{R}\mathrm{Hom}^\bullet(QA,\, QB)$$

where $Q \colon \mathbf{K} \to \mathbf{D}$ is the canonical functor. This η is, in the following sense, *universal* (hence unique up to isomorphism):

Lemma 2.4.4. *Let*

$$G \colon \mathbf{D}^{\mathsf{op}} \times \mathbf{D_L} \to \mathbf{D}(\mathfrak{Ab})$$

be a Δ-functor, and let

$$\mu \colon Q'\mathrm{Hom}^\bullet(A,\, B) \to G(QA, QB) \qquad (A \in \mathbf{K}^{\mathsf{op}},\ B \in \mathbf{L})$$

be a morphism of Δ-functors. Then there exists a unique morphism of Δ-functors

$$\overline{\mu}\colon \mathbf{R}\mathrm{Hom}^\bullet \to G$$

such that $\mu = \overline{\mu}\eta$.

Proof. $\overline{\mu}$ is the composition

$$\mathbf{R}\mathrm{Hom}^\bullet(QA,\, QB) = Q'\mathrm{Hom}^\bullet(A,\, I_B) \xrightarrow{\ \mu\ } G(QA, QI_B) \xrightarrow{\ \sim\ } G(QA, QB)\,.$$

The rest is left to the reader. (See also (2.6.5) below.)

(2.4.5). Next we discuss the *sheafified version* of the above. Let U be a topological space, $\mathcal{O}$ a sheaf of commutative rings, and $\mathcal{A}$ the abelian category of (sheaves of) $\mathcal{O}$-modules. The "sheaf-hom" functor

$$\mathcal{H}om\colon \mathcal{A}^{\mathsf{op}} \times \mathcal{A} \to \mathcal{A}$$

extends naturally to a Δ-functor

$$\mathcal{H}om^\bullet\colon \mathbf{K}(\mathcal{A})^{\mathsf{op}} \times \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$$

(essentially because everything in (1.5.3) is compatible with restriction to open subsets—details left to the reader).

Taking note of the following Lemma, we can proceed as above to derive a Δ-functor

$$\mathbf{R}\mathcal{H}om^\bullet\colon \mathbf{D}(\mathcal{A})^{\mathsf{op}} \times \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})\,.$$

Lemma 2.4.5.1. *If I is a q-injective complex in $\mathbf{K}(\mathcal{A})$ then the functor $\mathcal{H}om^\bullet(-,\, I)$ takes quasi-isomorphisms to quasi-isomorphisms.*

Proof. For $A \in \mathbf{K}(\mathcal{A})$ and $i \in \mathbb{Z}$, the homology $H^i(\mathcal{H}om^\bullet(A, I))$ is the sheaf associated to the presheaf

$$V \mapsto H^i\big(\Gamma(V, \mathcal{H}om^\bullet(A,\, I)\big) = H^i\big(\mathrm{Hom}^\bullet(A|V,\, I|V)\big) \qquad (V \text{ open in } U).$$

We can then apply (2.3.8)(iv)$'$ to the category $\mathcal{A}_V$ of $(\mathcal{O}|V)$-modules, as soon as we know:

Lemma 2.4.5.2. *Let V be an open subset of U, with inclusion map $i\colon V \hookrightarrow U$. Then for any q-injective complex $I \in \mathbf{K}(\mathcal{A})$, the restriction $i^*I = I|_V$ is q-injective in $\mathbf{K}(\mathcal{A}_V)$.*

Proof. The *extension by zero* of an $\mathcal{O}_V$-module M is the sheaf $i_!M$ associated to the presheaf on U which assigns $M(W)$ to any open $W \subset V$ and 0 to any open $W \not\subset V$. The restriction $i^*i_!M$ can be identified with M; and the stalk of $i_!M$ at any point $w \notin V$ is 0. So $i_!$ is an exact functor.

Now from any diagram $Y \xleftarrow{s} X \xrightarrow{f} i^*I$ of maps of $\mathcal{A}_V$-complexes with s a quasi-isomorphism, we get the diagram

$$i_!Y \xleftarrow{i_!s} i_!X \xrightarrow{i_!f} i_!i^*I \overset{\alpha}{\hookrightarrow} I$$

where $i_!s$ is a quasi-isomorphism (since $i_!$ is exact) and α is the natural map. By (2.3.1), there exists a map $g\colon i_!X \to I$ such that $g\circ i_!s = \alpha\circ i_!f$ in $\mathbf{K}(\mathcal{A})$; and then we have, in $\mathbf{K}(\mathcal{A}_V)$,

$$i^*g\circ s = i^*g\circ i^*i_!s = i^*\alpha\circ i^*i_!f = 1\circ f = f\,.$$

Thus i^*I is indeed q-injective. Q.E.D.

(2.4.5.3). Similarly, any functor having an exact left adjoint preserves q-injectivity.

2.5 Derived Tensor Product

Let U be a topological space, $\mathcal{O}$ a sheaf of commutative rings, and $\mathcal{A}$ the abelian category of (sheaves of) $\mathcal{O}$-modules. Recall from (1.5.4) the definition of the tensor product (over $\mathcal{O}$) of two complexes in $\mathbf{K}(\mathcal{A})$, and its Δ-functorial properties. The standard theory of the derived tensor product, via resolutions by complexes of flat modules, applies to complexes in $\overline{\mathbf{D}}^-(\mathcal{A})$, see e.g., [**H**, p. 93]. Following Spaltenstein [**Sp**] we can use direct limits to extend the theory to *arbitrary* complexes in $\mathbf{D}(\mathcal{A})$. Before defining, in (2.5.7), the derived tensor product, we need to develop an appropriate acyclicity notion, "q-flatness."

Definition 2.5.1. A complex $P \in \mathbf{K}(\mathcal{A})$ is q-flat if for every quasi-isomorphism $Q_1 \to Q_2$ in $\mathbf{K}(\mathcal{A})$, the resulting map $P\otimes Q_1 \to P\otimes Q_2$ is also a quasi-isomorphism; or equivalently (see footnote under (1.5.1)), if for every exact complex $Q \in \mathbf{K}(\mathcal{A})$, the complex $P\otimes Q$ is also exact.

Example 2.5.2. $P \in \mathbf{K}(\mathcal{A})$ is q-flat iff for each point $x \in U$, the stalk P_x is q-flat in $\mathbf{K}(\mathcal{A}_x)$, where $\mathcal{A}_x$ is the category of modules over the ring $\mathcal{O}_x$. (In verifying this statement, note that an exact $\mathcal{O}_x$-complex Q_x is the stalk at x of the exact $\mathcal{O}$-complex Q which associates Q_x to those open subsets of U which contain x, and 0 to those which don't.)

For instance, a complex P which vanishes in all degrees but one (say n) is q-flat if and only if tensoring with the degree n component P^n is an exact functor in the category of $\mathcal{O}$-modules, i.e., P^n is a flat $\mathcal{O}$-module, i.e., for each $x \in U$, P^n_x is a flat $\mathcal{O}_x$-module.

Example 2.5.3. Tensoring with a fixed complex Q is a Δ-functor, and so the exact homology sequence $(1.4.5)^{\mathrm{H}}$ of a triangle yields that the q-flat complexes are the objects of a Δ-subcategory of $\mathbf{K}(\mathcal{A})$.

A bounded complex

$$P: \quad \cdots \to 0 \to 0 \to P^m \to \cdots \to P^n \to 0 \to 0 \to \cdots$$

fits into a triangle $P' \to P \to P'' \to P'[1]$ where P' is P^n in degree n and 0 elsewhere, and where P'' is the cokernel of the obvious map $P' \to P$. So starting with (2.5.2) we see by induction on $n - m$ that any bounded complex of flat $\mathcal{O}$-modules is q-flat.

Example 2.5.4. Since (filtered) direct limits commute with both tensor product and homology, therefore any such limit of q-flat complexes is again q-flat.

A bounded-above complex

$$P: \quad \cdots \to P^m \to \cdots \to P^n \to 0 \to 0 \to \cdots$$

is the limit of the direct system $P_0 \to P_1 \to \cdots \to P_i \to \cdots$ where P_i is obtained from P by replacing all the components P^j with $j < n - i$ by 0, and the maps are the obvious ones. Hence, any bounded-above complex of flat $\mathcal{O}$-modules is q-flat.

A *q-flat resolution* of an $\mathcal{A}$-complex C is a quasi-isomorphism $P \to C$ where P is q-flat. The totality of such resolutions (with variable P and C) is the class of objects of a category, whose morphisms are the obvious ones.

Proposition 2.5.5. *Every $\mathcal{A}$-complex C is the target of a quasi-isomorphism ψ_C from a q-flat complex P_C, which can be constructed to depend functorially on C, and so that $P_{C[1]} = P_C[1]$ and $\psi_{C[1]} = \psi_C[1]$.*

Proof. Every $\mathcal{O}$-module is a quotient of a flat one; in fact there exists a functor P_0 from $\mathcal{A}$ to its full subcategory of flat $\mathcal{O}$-modules, together with a functorial epimorphism $P_0(\mathcal{F}) \twoheadrightarrow \mathcal{F}$ $(\mathcal{F} \in \mathcal{A})$. Indeed, for any open $V \subset U$ let $\mathcal{O}_V$ be the extension of $\mathcal{O}|V$ by zero, (i.e., the sheaf associated to the presheaf taking an open W to $\mathcal{O}(W)$ if $W \subset V$ and to 0 otherwise), so that $\mathcal{O}_V$ is flat, its stalk at $x \in U$ being $\mathcal{O}_x$ if $x \in V$ and 0 otherwise. There is a canonical isomorphism

$$\psi\colon \mathcal{F}(V) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{O}_V, \mathcal{F}) \qquad (\mathcal{F} \in \mathcal{A})$$

such that $\psi(\lambda)$ takes $1 \in \mathcal{O}_V(V)$ to λ. With $\mathcal{O}_\lambda := \mathcal{O}_V$ for each $\lambda \in \mathcal{F}(V)$, the maps $\psi(\lambda)$ define an epimorphism, with flat source,

$$P_0(\mathcal{F}) := \Big(\bigoplus_{V \text{ open}} \ \bigoplus_{\lambda \in \mathcal{F}(V)} \mathcal{O}_\lambda \Big) \twoheadrightarrow \mathcal{F},$$

and this epimorphism depends functorially on $\mathcal{F}$.

We deduce then, for each $\mathcal{F}$, a functorial flat resolution

$$\cdots \to P_2(\mathcal{F}) \to P_1(\mathcal{F}) \to P_0(\mathcal{F}) \twoheadrightarrow \mathcal{F}$$

with $P_1(\mathcal{F}) := P_0\big(\ker(P_0(\mathcal{F}) \twoheadrightarrow \mathcal{F})\big)$, etc. Set $P_n(\mathcal{F}) = 0$ if $n < 0$. Then to a complex C we associate the *flat* complex $P = P_C$ such that $P^r := \oplus_{m-n=r} P_n(C^m)$ and the restriction of the differential $P^r \to P^{r+1}$ to $P_n(C^m)$ is $P_n(C^m \to C^{m+1}) \oplus (-1)^m\big(P_n(C^m) \to P_{n-1}(C^m)\big)$, together with the natural map of complexes $P \to C$ induced by the epimorphisms $P_0(C^m) \twoheadrightarrow C^m$ $(m \in \mathbb{Z})$. Elementary arguments, with or without spectral sequences, show that for the truncations $\tau_{\le m}C$ of §1.10, the maps $P_{\tau_{\le m}C} \to \tau_{\le m}C$ are *quasi-isomorphisms.* Since homology commutes with direct limits, the resulting map

$$\psi_C\colon P_C = \varinjlim_m P_{\tau_{\le m}C} \to \varinjlim_m \tau_{\le m}\, C = C,$$

(which depends functorially on C) is a quasi-isomorphism; and by (2.5.4), P_C is q-flat. That $P_{C[1]} = P_C[1]$ and $\psi_{C[1]} = \psi_C[1]$ is immediate. Q.E.D.

Exercises 2.5.6. (a) Let P and Q be complexes of $\mathcal{O}$-modules, and suppose that for all integers s, t, u, v the complex $\tau_{\le s}\tau_{\ge t}P \otimes_{\mathcal{O}} \tau_{\le u}\tau_{\ge v}Q$ is exact. Then

$$P \otimes Q = \varinjlim_{s,u} \tau_{\le s}P \otimes \tau_{\le u}Q$$

is exact.

(b) If for all $n \in \mathbb{Z}$ the homology $H^n(P)$ is a flat $\mathcal{O}$-module and furthermore, for all n the kernel of $P^n \to P^{n+1}$ is a direct summand of P^n (or, for all n the image of $P^n \to P^{n+1}$ is a direct summand of P^{n+1}), then P is q-flat. (Use (a) to reduce to where P is bounded; then apply induction to the number of n such that $P^n \ne 0$.)

(2.5.7). Let $\mathcal{A}$ be, as above, the category of $\mathcal{O}$-modules, and let

$$\mathbf{J}' \subset \mathbf{K} := \mathbf{K}(\mathcal{A})$$

be the Δ-subcategory of $\mathbf{K}$ whose objects are all the q-flat complexes, see (2.5.3). Fix $B \in \mathbf{K}$ and consider the Δ-functor

$$F_B\colon \mathbf{K} \to \mathbf{D} := \mathbf{D}(\mathcal{A})$$

such that

$$F_B(A) = A \otimes B \qquad \text{(see (1.5.4))}.$$

If A is both q-flat and exact, then $A \otimes B$ is exact: to see this, we may replace B by any quasi-isomorphic complex B' (since A is q-flat), and by (2.5.5) we may assume that B' is q-flat, whence, by (2.5.1), $A \otimes B'$ is exact. Hence the restriction of F_B to $\mathbf{J}'$ transforms quasi-isomorphisms into isomorphisms.

There exists, by (2.5.5), a functorial family of quasi-isomorphisms

$$\psi_A \colon P_A \to A \qquad (A \in \mathbf{K},\ P_A \in \mathbf{J}').$$

with $P_{A[1]} = P_A[1]$. An argument dual to that in (2.2.4) (with $\mathbf{J}'' := \mathbf{K}$) shows then that F_B has a left-derived Δ-functor

$$(\mathbf{L}F_B, \text{identity}) \colon \mathbf{D} \to \mathbf{D} \tag{2.5.7.1}$$

with

$$\mathbf{L}F_B(A) = P_A \otimes B \cong P_A \otimes P_B \cong A \otimes P_B,$$

the isomorphisms being the ones induced by ψ_A and ψ_B. Alternatively, P_A is left-F_B-acyclic for all A, B (see 2.5.10(d)), so one can apply (2.2.6).

For fixed A and variable B, $P_A \otimes B$ is a Δ-functor from $\mathbf{K}$ to $\mathbf{D}$ which takes quasi-isomorphisms to isomorphisms, so by (1.5.1) there results a Δ-functor from $\mathbf{D}$ to $\mathbf{D}$. Hence there is a functor of two variables, called a *derived tensor product,*

$$\underline{\underline{\otimes}} \colon \mathbf{D} \times \mathbf{D} \longrightarrow \mathbf{D}$$

which together with appropriate θ (see (1.5.4)) is a Δ-functor in each variable separately (i.e., it is a Δ-functor as defined in (2.4.3)).

Though the variables A and B have been treated differently in the foregoing, their roles are essentially equivalent. Indeed, there is a universal property analogous to (the dual of) that in (2.4.4), characterizing the natural composite map of Δ-functors from $\mathbf{K} \times \mathbf{K}$ to $\mathbf{D}$:

$$QA \underline{\underline{\otimes}} QB \xrightarrow{\ \sim\ } Q(P_A \otimes P_B) \longrightarrow Q(A \otimes B)\,.$$

Hence, in view of (1.5.4.1), there is a canonical Δ-bifunctorial isomorphism

$$B \underline{\underline{\otimes}} A \xrightarrow{\ \sim\ } A \underline{\underline{\otimes}} B.$$

This arises, in fact, from the natural isomorphism $P_B \otimes P_A \xrightarrow{\ \sim\ } P_A \otimes P_B$.

(2.5.8). The *local hypertor* sheaves are defined by

$$\mathcal{T}\mathit{or}_n(A, B) = H^{-n}(A \underline{\underline{\otimes}} B) \qquad (n \in \mathbb{Z};\ A, B \in \mathbf{D}).$$

As in (2.1.4), short exact sequences in either the A or B variable give rise to long exact hypertor sequences.

We remark that when U is a *scheme* and $\mathcal{O} = \mathcal{O}_U$, if the homology sheaves of the complexes A and B are all *quasi-coherent* then so are the sheaves $\mathcal{T}\mathit{or}_n(A, B)$. This is clear, by reduction to the affine case, if A and B are quasi-coherent $\mathcal{O}_X$-modules (i.e., complexes vanishing except in degree 0). In the general case, since

$$A \otimes B = \varinjlim_{s,u} \tau_{\leq s} A \otimes \tau_{\leq u} B\,,$$

we may assume A and B lie in $\mathbf{D}^-$, and then argue as in [**H**, p. 98, Prop. 4.3], or alternatively, use the Künneth spectral sequence

$$E^2_{pq} = \bigoplus_{i+j=q} \mathcal{T}\!or_p(H^{-i}(A),\, H^{-j}(B)) \Rightarrow \mathcal{T}\!or_\bullet(A,B)$$

(as described e.g., in [**B**, p. 186, Exercise 9(b)], with *flat* resolutions replacing projective ones). Thus, with notation as in (1.9), denoting by $\mathbf{D}_{\mathsf{qc}}$ the Δ-subcategory $\mathbf{D}_{\#} \subset \mathbf{D}$ with $\mathcal{A}^{\#} \subset \mathcal{A}$ the subcategory of *quasi-coherent* $\mathcal{O}_U$-modules (which is plump, see [**GD**, p. 217, (2.2.2) (iii)]), we have a Δ-functor

$$\underline{\underline{\otimes}}\colon \mathbf{D}_{\mathsf{qc}} \times \mathbf{D}_{\mathsf{qc}} \longrightarrow \mathbf{D}_{\mathsf{qc}}\ . \tag{2.5.8.1}$$

(2.5.9). The definitions in (1.5.4) can be extended to three (or more) variables, to give a Δ-functor $A \otimes B \otimes C$ from $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ to $\mathbf{K}$.

There exists a Δ-functor $T_3\colon \mathbf{D} \times \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ together with a Δ-functorial map

$$\eta\colon T_3(A,B,C) \longrightarrow A \otimes B \otimes C \qquad (A,B,C \in \mathbf{K})$$

such that for any Δ-functor $H\colon \mathbf{D} \times \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ and any Δ-functorial map $\mu\colon H(A,B,C) \longrightarrow A \otimes B \otimes C$ there is a unique Δ-functor map $\bar{\mu}\colon H \to T_3$ such that $\mu = \eta \circ \bar{\mu}$. (The reader can fill in the missing Q's.) In fact there is such a T_3 with

$$T_3(A,B,C) = P_A \otimes P_B \otimes P_C\ .$$

We usually write

$$T_3(A,B,C) = A \underline{\underline{\otimes}} B \underline{\underline{\otimes}} C\,.$$

There are canonical Δ-functorial isomorphisms

$$(A \underline{\underline{\otimes}} B) \underline{\underline{\otimes}} C \xrightarrow{\ \sim\ } A \underline{\underline{\otimes}} B \underline{\underline{\otimes}} C \xleftarrow{\ \sim\ } A \underline{\underline{\otimes}} (B \underline{\underline{\otimes}} C)\,.$$

Similar considerations hold for $n > 3$ variables. Details are left to the reader. (See, for example, (2.6.5) below.)

Exercises 2.5.10. (a) Show that if $A \in \mathbf{K}(\mathcal{A})$ is q-flat and $B \in \mathbf{K}(\mathcal{A})$ is q-injective then $\mathcal{H}om^\bullet(A,B)$ is q-injective.

(b) Let $\Gamma\colon \mathcal{A} \to \mathfrak{Ab}$ be the global section functor. Show that there is a natural isomorphism of Δ-functors (of two variables, see (2.4.3))

$$\mathbf{R}\mathrm{Hom}^\bullet(A,B) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma\mathbf{R}\mathcal{H}om^\bullet(A,B).$$

(Use (a) and (2.2.7), or [**Sp**, 5.14, 5.12, 5.17].)

(c) Let (A_α) be a (small, directed) inductive system of $\mathcal{A}$-complexes. Show that for any complex $B \in \mathbf{D}(\mathcal{A})$ there are natural isomorphisms

$$\varinjlim_{\alpha} \mathcal{T}\!or_n\big(A_\alpha,\, B\big) \xrightarrow{\ \sim\ } \mathcal{T}\!or_n\big((\varinjlim_{\alpha} A_\alpha),\, B\big) \qquad (n \in \mathbb{Z}).$$

(d) Show that for P to be q-flat it is necessary that P be left-F_B-acyclic for all B (F_B as in (2.5.7)), and sufficient that P be left-F_B-acyclic for all *exact* B. (For the last part, (2.2.6) could prove helpful.) Formulate and prove an analogous statement involving q-injectivity and $\mathrm{Hom}^\bullet$. (See (2.3.8).)

2.6 Adjoint Associativity

Again let U be a topological space, $\mathcal{O}$ a sheaf of commutative rings, and $\mathcal{A}$ the abelian category of $\mathcal{O}$-modules. Set $\mathbf{K} := \mathbf{K}(\mathcal{A})$, $\mathbf{D} := \mathbf{D}(\mathcal{A})$. This section is devoted to (2.6.1)—or better, (2.6.1)* at the end—which expresses the basic adjointness relation between the Δ-functors $\mathbf{R}\mathcal{H}om^\bullet \colon \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{D}$ and $\underline{\underline{\otimes}} \colon \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ defined in (2.4.5) and (2.5.7) respectively.

Proposition 2.6.1. *There is a natural isomorphism of Δ-functors*

$$\mathbf{R}\mathcal{H}om^\bullet(A \underline{\underline{\otimes}} B,\, C) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^\bullet(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,\, C))\,.$$

Remarks. (i) In fact, the Δ-functors $\mathbf{R}\mathcal{H}om^\bullet$ and $\underline{\underline{\otimes}}$ are defined only up to canonical isomorphism, by universal properties, as in (2.5.9). We leave it to the reader to verify that the map in (2.6.1) (to be constructed below) is compatible, in the obvious sense, with such canonical isomorphisms.

(ii) A proof similar to the following one[3] yields a natural isomorphism

$$\mathbf{R}\mathrm{Hom}^\bullet(A \underline{\underline{\otimes}} B,\, C) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,\, C))\,.$$

Applying homology H^0 we have, by (2.4), the *adjunction isomorphism*

$$\mathrm{Hom}_{\mathbf{D}}(A \underline{\underline{\otimes}} B,\, C) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}}(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,\, C))\,. \tag{2.6.1$'$}$$

(iii) Prop. (2.6.1) gives a derived-category upgrade of the standard sheaf isomorphism

$$\mathcal{H}om(F \otimes G,\, H) \xrightarrow{\ \sim\ } \mathcal{H}om(F,\, \mathcal{H}om(G,\, H)) \qquad (F, G, H \in \mathcal{A}). \tag{2.6.2}$$

Proof of (2.6.1). We discuss the proof at several levels of pedantry, beginning with the argument, in full, given in [**I**, p. 151, Lemme 7.4] (see also [**Sp**, p. 147, Prop. 6.6]): "Resolve C injectively and B flatly."

This argument can be expanded as follows. Choose quasi-isomorphisms

$$C \to I_C\,, \qquad P_B \to B$$

[3] Or application of the functor $\mathbf{R}\Gamma$ to (2.6.1), see (2.5.10),

where I_C is q-injective and P_B is q-flat. It follows from (2.3.8)(iv) that the complex of sheaves $\mathcal{H}om^\bullet(P_B,\, I_C)$ is *q-injective,* since for any exact complex $X \in \mathbf{K}$, the isomorphism of complexes

$$\mathrm{Hom}^\bullet(X \otimes P_B,\, I_C) \xrightarrow{\ \sim\ } \mathrm{Hom}^\bullet(X,\, \mathcal{H}om^\bullet(P_B,\, I_C))$$

coming out of (2.6) yields, upon application of homology H^0,

$$0 = \mathrm{Hom}_{\mathbf{K}}(X \otimes P_B,\, I_C) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{K}}(X,\, \mathcal{H}om^\bullet(P_B,\, I_C)).$$

Now consider the natural sequence of **D**-maps

$$\begin{tikzcd}
\mathbf{R}\mathcal{H}om^\bullet(A \underset{=}{\otimes} B,\, C) \arrow[d] & \mathbf{R}\mathcal{H}om^\bullet(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,\, C)) \arrow[d] \\
\mathbf{R}\mathcal{H}om^\bullet(A \underset{=}{\otimes} B,\, I_C) \arrow[d] & \mathbf{R}\mathcal{H}om^\bullet(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,\, I_C)) \arrow[d] \\
\mathbf{R}\mathcal{H}om^\bullet(A \underset{=}{\otimes} P_B,\, I_C) & \mathbf{R}\mathcal{H}om^\bullet(A,\, \mathbf{R}\mathcal{H}om^\bullet(P_B,\, I_C)) \\
\mathbf{R}\mathcal{H}om^\bullet(A \otimes P_B,\, I_C) \arrow[u] & \mathbf{R}\mathcal{H}om^\bullet(A,\, \mathcal{H}om^\bullet(P_B,\, I_C)) \arrow[u] \\
\mathcal{H}om^\bullet(A \otimes P_B,\, I_C) \arrow[u] \arrow[r, "\text{from (2.6.2)}"'] & \mathcal{H}om^\bullet(A,\, \mathcal{H}om^\bullet(P_B,\, I_C)) \arrow[u]
\end{tikzcd}$$

Since P_B is q-flat, and I_C and $\mathcal{H}om^\bullet(P_B,\, I_C)$ are q-injective, all these maps are isomorphisms (as follows, e.g., from the last assertion of (2.2.6)); so we can compose to get the isomorphism (2.6.1).

But we really should check that this isomorphism does not depend on the chosen quasi-isomorphisms, and that it is in fact Δ-functorial. This can be quite tedious. The following remarks outline a method for managing such verifications. The basic point is (2.6.4) below.

Let M be a set. An *M-category* is an additive category $\mathbf{C}$ plus a map $t\colon M \to \mathrm{Hom}(\mathbf{C}, \mathbf{C})$ from M into the set of additive functors from $\mathbf{C}$ to $\mathbf{C}$, such that with $T_m := t(m)$ it holds that $T_i \circ T_j = T_j \circ T_i$ for all $i, j \in M$. Such an M-category will be denoted $\mathbf{C}_M$, the map f—or equivalently, the commuting family $(T_m)_{m \in M}$—understood to have been specified; and when the context renders it superfluous, the subscript "M" may be omitted.

An *M-functor* $F\colon \mathbf{C}_M \to \mathbf{C}'_M$ is an additive functor $F\colon \mathbf{C} \to \mathbf{C}'$ together with isomorphisms of functors

$$\theta_i\colon F \circ T_i \xrightarrow{\ \sim\ } T'_i \circ F \qquad (i \in M)$$

(with $(T'_m)_{m\in M}$ the commuting family of functors defining the M-structure on $\mathbf{C}'$) such that for all $i \neq j$, the following diagram commutes:

$$\begin{tikzcd}
F\circ T_i\circ T_j \arrow[r, "\text{via }\theta_i"] \arrow[d, equal] & T'_i\circ F\circ T_j \arrow[r, "T'_i(\theta_j)"] & T'_i\circ T'_j\circ F \arrow[d, equal] \\
F\circ T_j\circ T_i \arrow[r, "\text{via }\theta_j"'] & T'_j\circ F\circ T_i \arrow[r, "-T'_j(\theta_i)"'] & T'_j\circ T'_i\circ F
\end{tikzcd}$$

where, for instance, $T'_j(\theta_i)$ is the isomorphism of functors such that for each object $X \in \mathbf{C}$, $[T'_j(\theta_i)](X)$ is the $\mathbf{C}'$-isomorphism

$$T'_j\big(\theta_i(X)\big)\colon T'_j\big(FT_i(X)\big) \xrightarrow{\sim} T'_j\big(T'_iF(X)\big).$$

A *morphism* $\eta\colon (F,\{\theta_i\}) \to (G,\{\psi_i\})$ of M-functors is a morphism of functors $\eta\colon F \to G$ such that for every $i \in M$ and every object X in $\mathbf{C}$, the following diagram commutes:

$$\begin{tikzcd}
FT_i(X) \arrow[r, "\theta_i(X)"] \arrow[d, "\eta(T_i(X))"'] & T'_iF(X) \arrow[d, "T'_i(\eta(X))"] \\
GT_i(X) \arrow[r, "\psi_i(X)"'] & T'_iG(X)
\end{tikzcd}$$

Composition of such η being defined in the obvious way, the M-functors from $\mathbf{C}$ to $\mathbf{C}'$, and their morphisms, form a category $\mathbf{H}:= \mathbf{Hom}_M(\mathbf{C},\,\mathbf{C}')$. If $M' \supset M$ and $\mathbf{C}'_{M'}$ is viewed as an M-category via "restriction of scalars" then $\mathbf{H}$ is itself an M'-category, with $j \in M'$ being sent to the functor $T^{\#}_j : \mathbf{H} \to \mathbf{H}$ such that on objects of $\mathbf{H}$,

$$T^{\#}_j\big(F,\,\{\theta_i\}\big) = \big(T'_j\circ F,\,\{-T'_j(\theta_i)\}\big)\,,$$

where the isomorphism of functors

$$T'_j(\theta_i)\colon (T'_j\circ F)\circ T_i \xrightarrow{\sim} T'_j\circ T'_i\circ F = T'_i\circ(T'_j\circ F)$$

is as above.[4] The definition of $T^{\#}_j\eta$ (η as above), and the verification that $\mathbf{H}$ is thus an M'-category, are straightforward.

Suppose given such categories $\mathbf{A}_M$, $\mathbf{B}_N$, and $\mathbf{C}_{M\cup N}$, where the sets M and N are disjoint. $\mathbf{A}\times\mathbf{B}$ is considered to be an $(M\cup N)$-category, with $i \in M$ going to the functor $T_i \times 1$ and $j \in N$ to the functor $1 \times T_j$. Also, $\mathbf{Hom}_N(\mathbf{B},\,\mathbf{C})$ is considered, as above, to be an $(M\cup N)$-category.

[4] The reason for the minus sign in the definition of $T^{\#}_j$ is hidden in the details of the proof of Lemma (2.6.3) below.

Lemma 2.6.3. *With preceding notation, there is a natural isomorphism of $M \cup N$-categories*

$$\mathbf{Hom}_{M\cup N}\big(\mathbf{A}\times\mathbf{B},\,\mathbf{C}\big)\ \xrightarrow{\ \sim\ }\ \mathbf{Hom}_{M}\big(\mathbf{A},\mathbf{Hom}_{N}(\mathbf{B},\mathbf{C})\big)$$

The *proof,* left to the reader, requires very little imagination, but a good deal of patience.

For any positive integer n, let $\triangle_n$ be the set $\{1, 2, \dots, n\}$. From now on, we deal with Δ-categories, always considered to be $\triangle_1$-categories via their translation functors. If $\mathbf{C_1}, \dots, \mathbf{C_n}$ are Δ-categories, then the product category $\mathbf{C} = \mathbf{C_1} \times \mathbf{C_2} \times \cdots \times \mathbf{C_n}$ becomes a $\triangle_n$-category by the product construction used in (2.6.3). A Δ-category $\mathbf{E}$ can also be made into an $\triangle_n$-category by sending each $i \in \triangle_n$ to the translation functor of $\mathbf{E}$. With these understandings, we see that the $\triangle_n$-functors from $\mathbf{C_1} \times \mathbf{C_2} \times \cdots \times \mathbf{C_n}$ to $\mathbf{E}$ are just the Δ-functors of (2.4.3) (categories of which we denote by $\mathbf{Hom}_{\mathbf{\Delta}}$). For example, one checks that the source and target of the isomorphism in (2.6.1) are both $\triangle_3$-functors.

Now for $1 \le i \le n$ fix abelian categories $\mathcal{A}_i$, and let $\mathbf{L_i}$ be a Δ-subcategory of $\mathbf{K}(\mathcal{A}_i)$, with corresponding derived category $\mathbf{D_i}$ and canonical functor $Q_i\colon \mathbf{L_i} \to \mathbf{D_i}$. Let $\mathbf{E}$ be any Δ-category. We can generalize (1.5.1) as follows:

Proposition 2.6.4. *The canonical functor*

$$\mathbf{L_1}\times\cdots\times\mathbf{L_n}\ \xrightarrow[Q_1\times\ldots\times Q_n]{}\ \mathbf{D_1}\times\cdots\times\mathbf{D_n}$$

induces an isomorphism from the category $\mathbf{Hom}_{\mathbf{\Delta}}(\mathbf{D_1}\times\mathbf{D_2}\times\cdots\times\mathbf{D_n},\,\mathbf{E})$ *to the full subcategory of* $\mathbf{Hom}_{\mathbf{\Delta}}(\mathbf{L_1}\times\mathbf{L_2}\times\cdots\times\mathbf{L_n},\,\mathbf{E})$ *whose objects are the Δ-functors F such that for any quasi-isomorphisms $\alpha_1, \dots, \alpha_n$ in* $\mathbf{L_1}, \dots, \mathbf{L_n}$ *respectively, $F(\alpha_1, \dots, \alpha_n)$ is an isomorphism in* $\mathbf{E}$.

Proof. The case $n = 1$ is contained in (1.5.1). We can then proceed by induction on n, using the natural isomorphism

$$\begin{aligned}&\mathbf{Hom}_{\triangle_\mathbf{n}}\big(\mathbf{C_1}\times\mathbf{C_2}\times\cdots\times\mathbf{C_n},\,\mathbf{E}\big)\\ &\qquad\qquad\xrightarrow{\sim}\mathbf{Hom}_{\triangle_1}\big(\mathbf{C_1},\,\mathbf{Hom}_{\triangle_{\mathbf{n}-1}}(\mathbf{C_2}\times\ldots\times\mathbf{C_n},\,\mathbf{E})\big)\end{aligned}$$

provided by (2.6.3) (with $\mathbf{C_i} := \mathbf{D_i}$ or $\mathbf{L_i}$). Q.E.D.

Suppose next that we have pairs of Δ-subcategories $\mathbf{L'_i} \subset \mathbf{L''_i}$ in $\mathbf{K}(\mathcal{A}_i)$, with respective derived categories $\mathbf{D'_i}, \mathbf{D''_i}$, and canonical functors $Q'_i\colon \mathbf{L'_i} \to \mathbf{D'_i}$, $Q''_i\colon \mathbf{L''_i} \to \mathbf{D''_i}$ $(1 \le i \le n)$. Suppose further that every complex $A \in \mathbf{L''_i}$ admits a quasi-isomorphism into a complex $I_A \in \mathbf{L'_i}$. Then as in (1.7.2) the natural Δ-functors $\tilde{\jmath}_i\colon \mathbf{D'_i} \to \mathbf{D''_i}$ are Δ-equivalences, having quasi-inverses ρ_i satisfying $\rho_i(A) = I_A$ $(A \in \mathbf{L''_i})$. There result functors

$$\tilde{\jmath}^* : \mathbf{Hom_\Delta}(\mathbf{D_1''} \times \cdots \times \mathbf{D_n''}, \mathbf{E}) \longrightarrow \mathbf{Hom_\Delta}(\mathbf{D_1'} \times \cdots \times \mathbf{D_n'}, \mathbf{E})$$
$$\rho^* : \mathbf{Hom_\Delta}(\mathbf{D_1'} \times \cdots \times \mathbf{D_n'}, \mathbf{E}) \longrightarrow \mathbf{Hom_\Delta}(\mathbf{D_1''} \times \cdots \times \mathbf{D_n''}, \mathbf{E})$$

together with functorial isomorphisms

$$\tilde{\jmath}^*\rho^* \xrightarrow{\ \sim\ } \text{identity}, \qquad \rho^*\tilde{\jmath}^* \xrightarrow{\ \sim\ } \text{identity},$$

i.e., $\tilde{\jmath}^*$ and ρ^* are quasi-inverse equivalences of categories.

We deduce the following variation on the theme of (2.2.3), thereby arriving at a general method for specifying maps between Δ-functors on products of derived categories:[5]

Corollary 2.6.5. *With above notation let* $H\colon \mathbf{L_1'} \times \cdots \times \mathbf{L_n'} \to \mathbf{E}$, $F\colon \mathbf{D_1''} \times \cdots \times \mathbf{D_n''} \to \mathbf{E}$, *and* $G\colon \mathbf{D_1''} \times \cdots \times \mathbf{D_n''} \to \mathbf{E}$ *be* Δ*-functors. Let*

$$\zeta\colon H \xrightarrow{\ \sim\ } F\circ(\tilde{\jmath}_1 Q_1' \times \ldots \times \tilde{\jmath}_n Q_n'),$$
$$\beta\colon H \longrightarrow G\circ(\tilde{\jmath}_1 Q_1' \times \ldots \times \tilde{\jmath}_n Q_n')$$

be Δ*-functorial maps, with* ζ *an isomorphism. Then*:

(i) *There exists a unique* Δ*-functorial map* $\bar{\beta}\colon F \to G$ *such that for all* $A_1 \in \mathbf{L_1'}, \ldots, A_n \in \mathbf{L_n'}$, $\beta(A_1, \ldots, A_n)$ *factors as*

$$H(A_1,\ldots,A_n) \xrightarrow{\ \ \zeta\ \ } F(A_1,\ldots,A_n) \xrightarrow{\ \ \bar{\beta}\ \ } G(A_1,\ldots,A_n). \tag{2.6.5.1}$$

Moreover, if β *is an isomorphism then so is* $\bar{\beta}$.

(ii) *If* H *in* (i) *extends to a* Δ*-functor* $H\colon \mathbf{L_1''} \times \ldots \times \mathbf{L_n''} \to \mathbf{E}$, *and* ζ (*respectively* β) *to a* Δ*-functorial map* $\zeta\colon H \to F\circ(\tilde{\jmath}_1 Q_1'' \times \ldots \times \tilde{\jmath}_n Q_n'')$ (*respectively* $\beta\colon H \to G\circ(\tilde{\jmath}_1 Q_1'' \times \ldots \times \tilde{\jmath}_n Q_n'')$), *then the factorization* (2.6.5.1) *of* $\beta(A_1, \ldots, A_n)$ *holds for all* $A_1 \in \mathbf{L_1''}, \ldots, A_n \in \mathbf{L_n''}$.

Proof. (i) The assertion just means that $\bar{\beta}$ is the unique map (resp. isomorphism) $F \to G$ in the category $\mathbf{Hom_\Delta}(\mathbf{D_1''} \times \ldots \times \mathbf{D_n''}, \mathbf{E})$ corresponding via the above equivalence $\tilde{\jmath}^*$ and (2.6.4) to the map (resp. isomorphism) $\beta\zeta^{-1}$ in the category $\mathbf{Hom_\Delta}(\mathbf{L_1'} \times \ldots \times \mathbf{L_n'}, \mathbf{E})$.

(ii) Use quasi-isomorphisms $A_i \to I_{A_i}$ to map (2.6.5.1) into the corresponding diagram with $I_{A_i} \in \mathbf{L_i'}$ in place of A_i. To this latter diagram (i) applies; and as the resulting map $G(A_1, \ldots, A_n) \to G(I_{A_1}, \ldots, I_{A_n})$ is an isomorphism, the rest is clear. Q.E.D.

[5] This is no more (or less) than a careful formulation of the method used, e.g., throughout [**H**, Chapter II].

We can now derive (2.6.1) as follows. Take $n=3$, and set

$$\mathbf{L_1'} := \mathbf{K}$$

$$\mathbf{L_2'} := \begin{cases} \Delta\text{-subcategory of } \mathbf{K} \text{ whose objects are} \\ \text{the q-flat complexes (2.5.3).} \end{cases}$$

$$\mathbf{L_3'} := \begin{cases} \Delta\text{-subcategory of } \mathbf{K} \text{ whose objects are} \\ \text{the q-injective complexes (2.3.2.2).} \end{cases}$$

Let $\mathbf{D_1'}$, $\mathbf{D_2'}$, $\mathbf{D_3'}$ be the corresponding derived categories, and set

$$\mathbf{L_i''} := \mathbf{K}, \quad \mathbf{D_i''} := \mathbf{D} \qquad (i = 1, 2, 3),$$

so that the natural maps $j_i \colon \mathbf{D_i}' \to \mathbf{D_i''}$ are Δ-equivalences, with quasi-inverses obtained for $i=2$ and $i=3$ from q-flat (resp. q-injective) resolutions, i.e., from families of quasi-isomorphisms

$$P_B \to B \qquad (B \in \mathbf{K},\ P_B \in \mathbf{L_2'}),$$

$$C \to I_C \qquad (C \in \mathbf{K},\ I_C \in \mathbf{L_3'}).$$

In Corollary (2.6.5)(ii), let $H \colon \mathbf{L_1''} \times \mathbf{L_2''} \times \mathbf{L_3''} \to \mathbf{D}$ be the Δ-functor

$$H(A, B, C) := \mathcal{H}om^\bullet(A \otimes B,\, C),$$

let ζ be the natural composed Δ-functorial map

$$\mathcal{H}om^\bullet(A \otimes B,\, C) \to \mathbf{R}\mathcal{H}om^\bullet(A \otimes B, C) \to \mathbf{R}\mathcal{H}om^\bullet(A \underset{=}{\otimes} B, C),$$

and let β be the natural composed Δ-functorial map

$$\begin{aligned} \mathcal{H}om^\bullet(A \otimes B,\, C) &\underset{(2.6.2)}{\xrightarrow{\sim}} \mathcal{H}om^\bullet(A,\, \mathcal{H}om^\bullet(B,\, C)) \\ &\longrightarrow \mathbf{R}\mathcal{H}om^\bullet(A,\, \mathcal{H}om^\bullet(B,\, C)) \\ &\longrightarrow \mathbf{R}\mathcal{H}om^\bullet(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,\, C)). \end{aligned}$$

(Meticulous readers may wish to insert the missing Q's).

We saw near the beginning of the proof of (2.6.1), that for $(B, C) \in \mathbf{L_2'} \times \mathbf{L_3'}$, the complex $\mathcal{H}om^\bullet(B,\, C)$ is q-injective, and hence for such (B, C), *ζ and β are isomorphisms.* Modifying (2.6.5) in the obvious way to take contravariance into account, we deduce the following elaboration of (2.6.1):

Proposition (2.6.1)*. *There is a unique Δ-functorial isomorphism*

$$\alpha \colon \mathbf{R}\mathcal{H}om^\bullet(A \underset{=}{\otimes} B,\, C) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet(A,\, \mathbf{R}\mathcal{H}om^\bullet(B,\, C))$$

such that for all $A, B, C \in \mathbf{D}$, the following natural diagram (in which $\mathcal{H}^\bullet$ stands for $\mathcal{H}\mathrm{om}^\bullet$) commutes:

$$\begin{array}{ccccc}
\mathcal{H}^\bullet(A\otimes B, C) & \longrightarrow & \mathbf{R}\mathcal{H}^\bullet(A\otimes B, C) & \longrightarrow & \mathbf{R}\mathcal{H}^\bullet(A\underset{=}{\otimes} B, C) \\
{\scriptstyle \text{via}}\big\downarrow{\scriptstyle (2.6.2)} & & & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \alpha} \\
\mathcal{H}^\bullet(A,\, \mathcal{H}^\bullet(B, C)) & \longrightarrow & \mathbf{R}\mathcal{H}^\bullet(A,\, \mathcal{H}^\bullet(B, C)) & \longrightarrow & \mathbf{R}\mathcal{H}^\bullet(A,\, \mathbf{R}\mathcal{H}^\bullet(B, C))
\end{array}$$

This Δ-functorial isomorphism is the same as the one described—non-canonically, via P_B and I_C—near the beginning of this section. See also exercise (3.5.3)(e) below.

From (2.5.7.1) and (3.3.8) below (dualized), we deduce:

Corollary 2.6.6. *For fixed A the Δ-functor $F_A(-) := \mathrm{Hom}^\bullet(A, -)$ of §2.4 has a right-derived Δ-functor of the form $(\mathbf{R}F_A, \mathrm{identity})$.*

Exercise 2.6.7 (see [**De**, §1.2]). Define derived functors of several variables, and generalize the relevant results from §§2.2–2.3.

2.7 Acyclic Objects; Finite-Dimensional Derived Functors

This section contains additional results about acyclicity, used to get some more ways to construct derived functors, further illustrating (2.2.6). It can be skipped on first reading.

Let $\mathcal{A}$, $\mathcal{A}'$ be abelian categories, and let $\phi\colon \mathcal{A} \to \mathcal{A}'$ be an additive functor. We also denote by ϕ the composed Δ-functor

$$\mathbf{K}(\mathcal{A}) \xrightarrow{\mathbf{K}(\phi)} \mathbf{K}(\mathcal{A}') \xrightarrow{\;Q\;} \mathbf{D}(\mathcal{A}')$$

where $\mathbf{K}(\phi)$ is the natural extension of the original ϕ to a Δ-functor. We say then that an object in $\mathcal{A}$ is right-(or left-)ϕ-acyclic if it is so when viewed as a complex vanishing outside degree zero (see (2.2.5) with $\mathbf{J} := \mathbf{K}(\mathcal{A})$). In this section we deal mainly with the "left" context, and so we abbreviate "left-ϕ-acyclic" to "ϕ-acyclic." (The corresponding—dual—results in the "right" context are left to the reader. They are perhaps marginally less important because of the abundance of injectives in situations that we will deal with.)

If $X \in \mathcal{A}$ and $Z \to X$ is a quasi-isomorphism in $\mathbf{K}(\mathcal{A})$, then the natural map $\tau_{\le 0}Z \to Z$ of §1.10 is a quasi-isomorphism. If furthermore the induced map $\phi(Z) \to \phi(X)$ is a quasi-isomorphism and the functor ϕ is either right exact or left exact, then, one checks, the natural composition $\phi(\tau_{\le 0}Z) \to \phi(Z) \to \phi(X)$ is also a quasi-isomorphism.

One deduces the following characterization of ϕ-acyclicity:

Lemma 2.7.1. *If $X \in \mathcal{A}$ is such that every exact sequence*

$$\cdots \longrightarrow Y_2 \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow X \longrightarrow 0$$

embeds into a commutative diagram in $\mathcal{A}$

$$\begin{tikzcd}
\cdots \arrow[r] & Z_2 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & Z_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & Y_2 \arrow[r] & Y_1 \arrow[r] & Y_0 \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

with the top row and its image under ϕ both exact, then X is ϕ-acyclic; and the converse holds whenever ϕ is either right exact or left exact.

Proposition 2.7.2. *With preceding notation, let $\mathbf{P}$ be a class of objects in $\mathcal{A}$ such that*

(i) *every object in $\mathcal{A}$ is a quotient of (i.e., target of an epimorphism from) one in $\mathbf{P}$;*
(ii) *if A and B are in $\mathbf{P}$ then so is $A \oplus B$; and*
(iii) *for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, if B and C are in $\mathbf{P}$, then $A \in \mathbf{P}$ and moreover the corresponding sequence $0 \to \phi A \to \phi B \to \phi C \to 0$ in $\mathcal{A}'$ is also exact.*

Then every bounded-above $\mathbf{P}$-complex (i.e., complex with all components in $\mathbf{P}$)—in particular every object in $\mathbf{P}$—is ϕ-acyclic; the restriction ϕ_- of ϕ to $\overline{\mathbf{K}}^-(\mathcal{A})$ has a left-derived functor $\mathbf{L}\phi_- : \overline{\mathbf{D}}^-(\mathcal{A}) \to \mathbf{D}(\mathcal{A}')$; and if $\phi \not\cong 0$ then $\dim^+ \mathbf{L}\phi_- = 0$ (*see* (1.11.1)).

Proof. Since $\mathbf{P}$ is nonempty—by (i)—therefore (iii) with $B = C \in \mathbf{P}$ shows that $0 \in \mathbf{P}$. Then (ii) implies that the $\mathbf{P}$-complexes in $\mathbf{K}^-(\mathcal{A})$ are the objects of a Δ-subcategory, see (1.6). Starting from (i), an inductive argument ([**H**, p. 42, 4.6, 1)], dualized—and with assistance, if desired, from [**Iv**, p. 34, Prop. 5.2]) shows that every complex in $\mathbf{K}^-(\mathcal{A})$—and so, via $(1.8.1)^-$, in $\overline{\mathbf{K}}^-(\mathcal{A})$—is the target of a quasi-isomorphism from a bounded-above $\mathbf{P}$-complex. Hence, for the first assertion it suffices to show that ϕ transforms quasi-isomorphisms between bounded-above $\mathbf{P}$-complexes into isomorphisms, i.e., that *for any bounded-above exact $\mathbf{P}$-complex $X^\bullet$, $\phi(X^\bullet) \cong 0$* (see (1.5.1)).

Using (iii), we find by descending induction (starting with i_0 such that $X^j = 0$ for all $j > i_0$) that for every i, the kernel K^i of $X^i \to X^{i+1}$ lies in $\mathbf{P}$ and the obvious sequence

$$0 \to \phi(K^i) \to \phi(X^i) \to \phi(K^{i+1}) \to 0$$

is exact. Consequently, the complex obtained by applying ϕ to $X^\bullet$ is exact, i.e., $\phi(X^\bullet) \cong 0$ in $\mathbf{D}(\mathcal{A}')$.

Now by (2.2.4) (dualized) we see that $\mathbf{L}\phi_-$ exists and $\dim^+\mathbf{L}\phi_- \le 0$, with equality if $\phi(A) \ncong 0$ for some $A \in \mathcal{A}$, because there is a natural epimorphism $H^0\mathbf{L}\phi_- A \twoheadrightarrow \phi(A)$. Q.E.D.

Exercise 2.7.2.1. Let $\phi\colon \mathcal{A} \to \mathcal{A}'$ be as above. Let $(\Lambda_i)_{0\le i<\infty}$ be a "homological functor" [**Gr**, p. 140], with $\Lambda_0 = \phi$. Let $\mathbf{P}$ consist of all objects B in $\mathcal{A}$ such that $\Lambda_i(B) = 0$ for all $i > 0$, and suppose that every object $A \in \mathcal{A}$ is a quotient of one in $\mathbf{P}$. Then $\mathbf{L}\phi_-$ exists, and the homological functors (Λ_i) and $(\Lambda'_i) := (H^{-i}\mathbf{L}\phi_-)$ are coeffaceable, hence universal [**Gr**, p. 141, Prop. 2.2], hence isomorphic to each other.

Example 2.7.3. A *ringed space* is a pair $(X, \mathcal{O}_X)$ with X a topological space and $\mathcal{O}_X$ a sheaf of commutative rings on X; and a *morphism* of ringed spaces $(f,\theta)\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ is a continuous map $f\colon X \to Y$ together with a map $\theta\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings. Any such (f,θ) gives rise to a (left-exact) *direct image* functor

$$f_*\colon \{\mathcal{O}_X\text{-modules}\} \to \{\mathcal{O}_Y\text{-modules}\}$$

such that $[f_*M](U) = M(f^{-1}U)$ for any $\mathcal{O}_X$-module M and any open set $U \subset Y$, the $\mathcal{O}_Y$-module structure on f_*M arising via θ; and also to a (right-exact) *inverse image* functor

$$f^*\colon \{\mathcal{O}_Y\text{-modules}\} \to \{\mathcal{O}_X\text{-modules}\}$$

defined up to isomorphism as being left-adjoint to f_* [**GD**, Chap. 0, §4]. For every $\mathcal{O}_Y$-module N, the stalk $(f^*N)_x$ at $x \in X$ is $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} N_{f(x)}$.

An $\mathcal{O}_Y$-module F is *flat* if the stalk F_y is a flat $\mathcal{O}_{Y,y}$-module for all $y \in Y$. *The class* $\mathbf{P}$ *of flat* $\mathcal{O}_Y$*-modules satisfies the hypotheses of* (2.7.2) *when* $\phi = f^*$: (i) is given by [**H**, p. 86, Prop. 1.2], (ii) is easy, and for (iii) see [**B′**, Chap. 1, §2, no. 5]. Thus the restriction f^*_- of f^* to $\overline{\mathbf{K}}^-(Y)$ has a left-derived functor

$$\mathbf{L}f^*_-\colon \overline{\mathbf{D}}^-(Y) \to \mathbf{D}(X)$$

(where $\mathbf{D}(X)$ is the derived category of the category of $\mathcal{O}_X$-modules, etc.), defined via resolutions (on the left) by complexes of flat $\mathcal{O}_Y$-modules.

Using the family of quasi-isomorphisms $\psi_A\colon P_A \to A$ $(A \in \mathbf{D}(Y))$ with P_A q-flat (see (2.5.5)), we can, in view of (2.5.2) and (2.5.3), show as in (2.5.7) that $\mathbf{L}f^*_-$ extends to a derived Δ-functor

$$(\mathbf{L}f^*, \text{identity})\colon \mathbf{D}(Y) \to \mathbf{D}(X) \tag{2.7.3.1}$$

satisfying $\mathbf{L}f^*(A) = f^*(P_A)$.

For any $\mathcal{O}_Y$-module N, the stalk of the homology

$$L_i f^*(N) := H^{-i}\mathbf{L}f^*(N) \qquad (i \ge 0)$$

at any $x \in X$ is $\operatorname{Tor}_i^{\mathcal{O}_{Y,f(x)}}(\mathcal{O}_{X,x},\, N_{f(x)})$. So by the last assertion in (2.2.6) (dualized), or in (2.7.4), N is f^*-acyclic iff $\operatorname{Tor}_i^{\mathcal{O}_{Y,f(x)}}(\mathcal{O}_{X,x},\, N_{f(x)}) = 0$

for all $x \in X$ and $i > 0$. (Note here that since f^* is right exact, the natural map is an isomorphism $L_0 f^*(N) \xrightarrow{\sim} f^*(N)$.) Thus—or by (2.7.2)—any flat $\mathcal{O}_Y$-module is f^*-acyclic.

Recall that an $\mathcal{O}_X$-module M is *flasque* (or *flabby*) if the restriction map $M(X) \to M(U)$ is surjective for every open subset U of X. For example, injective $\mathcal{O}_X$-modules are flasque [**G**, p. 264, 7.3.2] (with $\mathcal{L} = \mathcal{O}_X$). *The class of flasque $\mathcal{O}_X$-modules satisfies the hypotheses of* (2.7.2) (*dual version*) *when* $\phi = f_*$: for (i) see [**G**, p. 147], (ii) is easy, and (iii) follows from the fact that if

$$0 \to F \to G \to H \to 0$$

is an exact sequence of $\mathcal{O}_X$-modules, with F flasque, then for all open sets $V \subset X$ the sequence

$$0 \to F(V) \to G(V) \to H(V) \to 0$$

is still exact [**G**, p. 148, Thm. 3.1.2]. So the restriction f_*^+ of f^* to $\overline{\mathbf{K}}^{+}(X)$ has a right-derived functor

$$\mathbf{R}f_*^+ : \overline{\mathbf{D}}^{+}(X) \to \mathbf{D}(Y)$$

defined via resolutions (on the right) by complexes of flasque $\mathcal{O}_X$-modules.

Of course we already know from (2.3.4), via (somewhat less elementary) *injective* resolutions, that $\mathbf{R}f_*^+$ exists, and by (2.3.5) it extends to a derived functor $\mathbf{R}f_* : \mathbf{D}(X) \to \mathbf{D}(Y)$. (See also (2.3.7).) In fact, in view of (2.7.3.1), it follows from (3.2.1) and (3.3.8) (dualized) that:

(2.7.3.2). *The* Δ*-functor* (f_*, identity) *has a derived* Δ*-functor of the form* ($\mathbf{R}f_*$, identity).

An $\mathcal{O}_X$-module M is f_*-acyclic iff the "higher direct image" sheaves

$$R^i f_*(M) := H^i \mathbf{R}f_*(M) \qquad (i \geq 0)$$

vanish for all $i > 0$, see last assertion in (2.2.6) or in (2.7.4) (dualized). (Since f_* is left-exact, the natural map is an isomorphism $f_* \xrightarrow{\sim} R^0 f_*$.) Flasque sheaves are f_*-acyclic.

For more examples involving flasque sheaves see [**H**, p. 225, Variations 6 and 7] ("cohomology with supports").

Proposition 2.7.4. *Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories, and let $\phi\colon \mathcal{A} \to \mathcal{A}'$ be a* right-exact *additive functor. If C is ϕ-acyclic, then for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ the corresponding sequence $0 \to \phi A \to \phi B \to \phi C \to 0$ is also exact, and A is ϕ-acyclic iff B is. So if every object in $\mathcal{A}$ is a quotient of a ϕ-acyclic one, then the conclusions of* (2.7.2) *hold with* $\mathbf{P}$ *the class of ϕ-acyclic objects; and then $D \in \mathcal{A}$ is ϕ-acyclic iff the natural map $\mathbf{L}\phi_-(D) \to \phi(D)$ is an isomorphism in $\mathbf{D}(\mathcal{A}')$, i.e., iff $H^{-i}\mathbf{L}\phi_-(D) = 0$ for all $i > 0$.*

Proof. For the first assertion, note that by (2.7.1) there exists a commutative diagram

$$\begin{tikzcd}
C_2 \arrow[r, "\delta"] \arrow[d] & C_1 \arrow[r, "\gamma"] \arrow[d] & C_0 \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r, "\alpha"'] & C \arrow[r] & 0
\end{tikzcd}$$

such that the top row is exact and remains so after application of ϕ. There results a commutative diagram

$$\begin{tikzcd}
 & & C_2 \arrow[r, equal] \arrow[d, "\delta"'] & C_2 \arrow[d, "\delta"] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & C_1 \arrow[r, equal] \arrow[d, "\gamma'"'] & C_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & C_0 \times_C B \arrow[r, "\pi"'] \arrow[d] & C_0 \arrow[r] \arrow[d] & 0 \\
 & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact columns, in which the middle row is *split exact,* a right inverse for the projection π being given by the graph of the map β.[6] (The coordinates of γ' are γ and 0.) Applying ϕ preserves split-exactness; and then, since ϕ is right-exact, so that e.g., $\phi C = \operatorname{coker}(\phi\gamma)$, the "snake lemma" yields an exact sequence

$$0 \to \ker(\phi\gamma') \to \ker(\phi\gamma) \to \phi A \to \phi B \to \phi C \to 0\,.$$

Since

$$\ker(\phi\gamma) = \operatorname{im}(\phi\delta) \subset \ker(\phi\gamma')$$

we conclude that $0 \to \phi A \to \phi B \to \phi C \to 0$ is exact, as asserted in (2.7.4).

In other words, if Z is the complex which looks like $A \to B$ in degrees -1 and 0 and which vanishes elsewhere, then the quasi-isomorphism $Z \to C$ given by the exact sequence $0 \to A \to B \to C \to 0$ becomes, upon application of ϕ, an isomorphism in $\mathbf{D}(\mathcal{A}')$; and hence, by (2.2.5.2) (dualized), Z is a ϕ-acyclic complex.

---

[6] Recall that $C_0 \times_C B$ is the kernel of the map $C_0 \oplus B \to C$ whose restriction to C_0 is $\alpha\beta$ and to B is $-\alpha$.

The natural semi-split sequence $0 \to B \to Z \to A[1] \to 0$ leads, as in (1.4.3), to a triangle

$$B \longrightarrow Z \longrightarrow A[1] \longrightarrow B[1]\,;$$

and since the ϕ-acyclic complexes are the objects of a Δ-subcategory, see (2.2.5.1), it follows that A is ϕ-acyclic iff B is.

Since Δ-subcategories are closed under direct sum, it is clear now that (ii) and (iii) in (2.7.2) hold when $\mathbf{P}$ is the class of ϕ-acyclic objects, whence the second-last assertion in (2.7.4). In view of (2.7.2) and its proof, the last assertion of (2.7.4) is contained in (2.2.6). Q.E.D.

The derived functor $\mathbf{L}\phi_{\_}$ of (2.7.4) satisfies $\dim^{+}\mathbf{L}\phi_{\_} = 0$ (unless $\phi \cong 0$, see (2.7.2)). When its lower dimension satisfies $\dim^{-}\mathbf{L}\phi_{\_} < \infty$, more can be said.

Proposition 2.7.5. *Let $\phi\colon \mathcal{A} \to \mathcal{A}'$ be a right-exact functor such that every object in $\mathcal{A}$ is a quotient of a ϕ-acyclic one, and let $\mathbf{L}\phi_{\_}$ be a left-derived functor of $\phi|\overline{\mathbf{K}}^{-}(\mathcal{A})$, see* (2.7.4). *Then the following conditions on an integer $d \ge 0$ are equivalent:*

(i) $\dim^{-}\mathbf{L}\phi_{\_} \le d$.

(ii) *For any $F \in \mathcal{A}$ we have*

$$L_j\phi(F) := H^{-j}\mathbf{L}\phi_{\_}(F) = 0 \quad \text{for all } j > d.$$

(iii) *In any exact sequence in $\mathcal{A}$*

$$0 \to 0 \to B_d \to B_{d-1} \to \dots \to B_0\,,$$

if $B_0, B_1, \dots, B_{d-1}$ are all ϕ-acyclic then so is B_d.[7]

(iv) *For any $F \in \mathcal{A}$ there is an exact sequence*

$$0 \to B_d \to B_{d-1} \to \dots \to B_0 \to F \to 0$$

in which every B_i is ϕ-acyclic.

(v) *For any complex $F^\bullet \in \mathbf{K}(\mathcal{A})$ and integers $m \le n$, if $F^j = 0$ for all $j \notin [m,n]$ then there exists a quasi-isomorphism $B^\bullet \to F^\bullet$ where B^j is ϕ-acyclic for all j and $B^j = 0$ for $j \notin [m-d, n]$.*

(vi) *For any complex $F^\bullet \in \mathbf{K}(\mathcal{A})$ and integer m, if $F^j = 0$ for all $j < m$ then there exists a quasi-isomorphism $B^\bullet \to F^\bullet$ where B^j is ϕ-acyclic for all j and $B^j = 0$ for all $j < m - d$.*

---

[7] For $d = 0$ this means that every $B \in \mathcal{A}$ is ϕ-acyclic, i.e., ϕ is an *exact* functor, see (2.7.4) (and then every $F^\bullet \in \mathbf{K}(\mathcal{A})$ is ϕ-acyclic, see (2.2.8(a))).

When there exists an integer $d \geq 0$ for which these conditions hold, then:

(a) *Every complex of ϕ-acyclic objects is a ϕ-acyclic complex.*

(b) *Every complex in $\mathcal{A}$ is the target of a quasi-isomorphism from a ϕ-acyclic complex.*

(c) *A left-derived functor* $\mathbf{L}\phi\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A}')$ *exists,* $\dim^{+}\mathbf{L}\phi = 0$ *(unless $\phi \cong 0$) and* $\dim^{-}\mathbf{L}\phi \leq d$.

(d) *The restriction* $\mathbf{L}\phi|_{\overline{\mathbf{D}}^{*}(\mathcal{A})}$ *is a left-derived functor of* $\phi|_{\overline{\mathbf{K}}^{*}(\mathcal{A})}$, *and*

$$\mathbf{L}\phi(\overline{\mathbf{D}}^{*}(\mathcal{A})) \subset \overline{\mathbf{D}}^{*}(\mathcal{A}') \qquad (* = \mathsf{+},\ \mathsf{-},\ \text{or } \mathsf{b}).$$

Proof. (i)⇔(ii). This is given by (iii) and (iv) in (1.11.2).

(iii)⇒(v)⇒(iv). Let $F^{\bullet}$ and $m \leq n$ be as in (v). As in the proof of (2.7.2), there is a quasi-isomorphism $P^{\bullet} \to F^{\bullet}$ with P^{j} ϕ-acyclic for all j and $P^{j} = 0$ for $j > n$. Let B^{m-d} be the cokernel of $P^{m-d-1} \to P^{m-d}$. If (iii) holds, then B^{m-d} is ϕ-acyclic: this is trivial if $d = 0$, and otherwise follows from the exact sequence

$$0 \to B^{m-d} \to P^{m-d+1} \to \cdots \to P^{m-1} \to P^{m}.$$

So all components of the complex $B^{\bullet} = \tau_{\geq m-d}P^{\bullet}$ (see (1.10)) are ϕ-acyclic, and clearly $P^{\bullet} \to F^{\bullet}$ factors naturally as $P^{\bullet} \to B^{\bullet} \to F^{\bullet} = \tau_{\geq m-d}F^{\bullet}$ where both arrows represent quasi-isomorphisms. Thus (iii)⇒(v); and (v)⇒(iv) is obvious.

Recalling from (2.7.4) that $B \in \mathcal{A}$ is ϕ-acyclic iff $L_i\phi(B) = 0$ for all $i > 0$, we easily deduce the implications (iv)⇒(ii)⇒(iii) from:

Lemma 2.7.5.1. *Let*

$$0 = B_{d+1} \to B_d \to B_{d-1} \to \cdots \to B_0 \to F \to 0$$

be an exact sequence in $\mathcal{A}$ with $B_0, B_1, \dots, B_{d-1}$ all ϕ-acyclic, and let K_j be the cokernel of $B_{j+1} \to B_j$ $(0 \leq j \leq d)$. Then for any $i > 0$, there results a natural sequence of isomorphisms

$$\begin{aligned} &L_{i+d}\phi(F) = L_{i+d}\phi(K_0) \xrightarrow{\sim} L_{i+d-1}\phi(K_1) \xrightarrow{\sim} \cdots \\ &\qquad \cdots \xrightarrow{\sim} L_{i+2}\phi(K_{d-2}) \xrightarrow{\sim} L_{i+1}\phi(K_{d-1}) \xrightarrow{\sim} L_i\phi(K_d) = L_i\phi(B_d)\,. \end{aligned}$$

Proof. When $d = 0$, it's obvious. If $d > 0$, apply $(2.1.4)^{\mathrm{H}}$ (dualized) to the natural exact sequences

$$0 \to K_j \to B_{j-1} \to K_{j-1} \to 0 \qquad (0 < j \leq d)$$

to obtain exact sequences

$$
\begin{aligned}
0 = L_{i+d-j+1}\phi(B_{j-1}) &\to L_{i+d-j+1}\phi(K_{j-1}) \\
&\to L_{i+d-j}\phi(K_j) \to L_{i+d-j}\phi(B_{j-1}) = 0. \qquad \text{Q.E.D.}
\end{aligned}
$$

(iii)⇒(vi). Condition (iii) coincides with condition (iii) of [**H**, p. 42, Lemma 4.6, 2)] (dualized, and with P the set of ϕ-acyclics in $\mathcal{A}$). Condition (i) of *loc. cit.* holds by assumption, and condition (ii) of *loc. cit.* is contained in (2.7.4). So if (iii) holds, *loc. cit.* gives the existence of a quasi-isomorphism $B^\bullet \to F^\bullet$ with B^j ϕ-acyclic for all j; and the recipe at the bottom of [**H**, p. 43] for constructing $B^\bullet$ allows us, when $F^j = 0$ for all $j < m$, to do so in such a way that $B^j = 0$ for all $j < m - d$.

(vi)⇒(ii). Assuming (vi), we can find for each object $F \in \mathcal{A}$ a quasi-isomorphism $B^\bullet \to F$ with all B^j ϕ-acyclic and $B^j = 0$ for $j < -d$. If K is the cokernel of $B^{-1} \to B^0$ then the natural composition

$$H^0(B^\bullet) \longrightarrow K \longrightarrow F$$

is an isomorphism, whence so are the functorially induced compositions

$$L_j\phi(H^0(B^\bullet)) \longrightarrow L_j\phi(K) \longrightarrow L_j\phi(F) \qquad (j \in \mathbb{Z}). \qquad (2.7.5.2)$$

But for every $j > d$, (2.7.5.1) with K in place of F yields $L_j\phi(K) = 0$, so that the isomorphism (2.7.5.2) is the zero-map. Thus (ii) holds.

Now suppose that (i)–(vi) hold for some $d \geq 0$. We have just seen, in proving that (iii)⇒(vi), that then every complex in $\mathcal{A}$ receives a quasi-isomorphism from a complex $B^\bullet$ of ϕ-acyclics; and so, as in the proof of (2.7.2), assertion (2.7.5)(a)—and hence (b)—will result if we can show that whenever such a $B^\bullet$ is exact, then so is $\phi(B^\bullet)$. But condition (iii) guarantees that when $B^\bullet$ is exact, the kernel K^i of $B^i \to B^{i+1}$ is ϕ-acyclic for all i, whence by (2.7.4) we have exact sequences

$$0 \to \phi(K^{i-1}) \to \phi(B^{i-1}) \to \phi(K^i) \to 0 \qquad (i \in \mathbb{Z})$$

which together show that $\phi(B^\bullet)$ is indeed exact.

The existence of $\mathbf{L}\phi$, via resolutions by complexes of ϕ-acyclic objects, follows now from (2.2.6); and the dimension statements follow, after application of $(1.8.1)^+$ or $(1.8.1)^-$, from (v) with $m = -\infty$ (obvious interpretation, see beginning of above proof that (iii)⇒(v)) and from (vi). Similar considerations yield (d). Q.E.D.

Example 2.7.6. The *dimension* $\dim f$ of a map $f\colon X \to Y$ of ringed spaces is the upper dimension (1.11) of the functor $\mathbf{R}f_*^+\colon \overline{\mathbf{D}}^{+}(X) \to \mathbf{D}(Y)$ of (2.7.3):

$$\dim f := \dim^+ \mathbf{R}f_*^+ \,,$$

a nonnegative integer unless $f_*\mathcal{O}_X \cong 0$, in which case $\dim f = -\infty$. When f has *finite* dimension, (2.7.5)(c) (dualized) gives the existence of a derived functor $\mathbf{R}f_*\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ via resolutions (on the right) by complexes of f_*-acyclic objects, and we have $\infty > \dim f = \dim^+\mathbf{R}f_*$.

The *tor-dimension* (or *flat dimension*) $\text{tor-dim}\, f$ of a map $f\colon X \to Y$ of ringed spaces is defined to be the lower dimension (see (1.11)) of the functor $\mathbf{L}f^*_-\colon \overline{\mathbf{D}}^-(Y) \to \mathbf{D}(X)$ of (2.7.3):

$$\text{tor-dim}\, f := \dim^-\mathbf{L}f^*_-\,,$$

a nonnegative integer unless $\mathcal{O}_X \cong 0$, in which case $\text{tor-dim}\, f = -\infty$. When f has *finite* tor-dimension, (2.7.5)(c) gives the existence of a derived functor $\mathbf{L}f^*\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ via resolutions (on the left) by complexes of f^*-acyclic objects, and we have $\infty > \text{tor-dim}\, f = \dim^-\mathbf{L}f^*$.

Following [**I**, p. 241, Définition 3.1] one says that an $\mathcal{O}_X$-complex E has *flat f-amplitude in* $[m,n]$ if for any $\mathcal{O}_Y$-module F,

$$H^i(E \underset{=}{\otimes} \mathbf{L}f^*F) = 0 \ \text{ for all } i \notin [m,n],$$

or equivalently, for the functor $L_E(F) := E \underset{=}{\otimes} \mathbf{L}f^*F$ of $\mathcal{O}_Y$-module F,

$$\dim^+ L \le m \ \text{ and } \ \dim^- L \le -n.$$

This means that the stalk E_x at each $x \in X$ is $\mathbf{D}(\mathcal{O}_{Y,f(x)})$-isomorphic to a flat complex vanishing in degrees outside $[m,n]$, see [**I**, p. 242, 3.3], or argue as in (2.7.6.4) below. E has *finite flat f-amplitude* if such m and n exist.

It follows from (2.7.6.4) below and [**I**, p. 131, 5.1] that f *has finite tor-dimension* $\iff$ $\mathcal{O}_X$ *has finite flat f-amplitude.*

(2.7.6.1). If X is a compact Hausdorff space of dimension $\le d$ (in the sense that each point has a neighborhood homeomorphic to a locally closed subspace of the Euclidean space $\mathbb{R}^d$), and $\mathcal{O}_X$ is the constant sheaf $\mathbb{Z}$, then $\dim f \le d$.

Indeed, if $I^\bullet$ is a flasque resolution of the abelian sheaf F, then for any open $U \subset Y$ the restriction $I^\bullet|f^{-1}(U)$ is a flasque resolution of $F|f^{-1}(U)$, and $R^jf_*(F)$ is, up to isomorphism, the sheaf associated to the presheaf taking any such U to the group $H^j(\Gamma(f^{-1}(U),\, I^\bullet|f^{-1}(U))$, a group isomorphic to $H^j(f^{-1}(U),\, F|f^{-1}(U))$ [**G**, p. 181, Thm. 4.7.1(a)], and hence vanishing for $j > d$, see [**Iv**, Chap. III, §9].

More generally, if X is locally compact and we assume only that the fibers $f^{-1}y$ ($y \in Y$) are compact and have dimension $\le d$, then $\dim f \le d$ (because the stalk $(R^jf_*F)_y$ is the cohomology $H^j(f^{-1}y,\, F|f^{-1}y)$, see [**Iv**, p. 315, Thm. 1.4], whose proof does not require any assumption on Y).

(2.7.6.2). (Grothendieck, see [**H**, p. 87]). If $(X, \mathcal{O}_X)$ is a noetherian scheme of finite Krull dimension d, then $\dim f \le d$.

(2.7.6.3). For a ringed-space map $f\colon X \to Y$ with $\mathcal{O}_X \ncong 0$, the following conditions are equivalent:
(i) tor-dim $f = 0$.
(i)$'$ Every $\mathcal{O}_Y$-module is f^*-acyclic.
(i)$''$ The functor f^* of $\mathcal{O}_Y$-modules is exact.
(ii) f is flat (i.e., $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module for all $x \in X$).

Proof. Since every $\mathcal{O}_X$-module is a quotient of a flat one, which is f^*-acyclic (see (2.7.3)), the equivalence of (i), (i)$'$, and (i)$''$ is given, e.g., by that of (i) and (iii) in (2.7.5) (for $d = 0$). The equivalence of (i) and (ii) is the case $d = 0$ of:

(2.7.6.4) Let $f\colon X \to Y$ be a ringed-space map and $d \geq 0$ an integer. Then tor-dim $f \leq d \iff$ for each $x \in X$ there exists an exact sequence of $\mathcal{O}_{Y,f(x)}$-modules

$$0 \to P_d \to P_{d-1} \to \ldots \to P_1 \to P_0 \to \mathcal{O}_{X,x} \to 0 \qquad (*)$$

with P_i flat over $\mathcal{O}_{Y,f(x)}$ $(0 \leq i \leq d)$.

Proof. ("*if*") Let F be an $\mathcal{O}_Y$-module and let $Q^\bullet \to F$ be a quasi-isomorphism with $Q^\bullet$ a flat complex (1.8.3). Then for $j \geq 0$, the homology

$$L_j f^*(F) \cong H^{-j}(f^*Q^\bullet) \qquad \text{(see (2.7.3))}$$

vanishes iff for each $x \in X$, with $y = f(x)$, $R = \mathcal{O}_{Y,y}$, and $S = \mathcal{O}_{X,x}$ we have

$$0 = H^{-j}\big((f^*Q^\bullet)_x\big) = H^{-j}\big(S \otimes_R Q^\bullet_y\big) = \mathrm{Tor}^R_j(S, F_y)$$

(where the last equality holds since $Q^\bullet_y \to F_y$ is an R-flat resolution of F_y), whence the assertion.

("*only if*") Suppose only that $L_{d+1}f^*(F) = 0$ for all F, so that (see above) $\mathrm{Tor}^R_{d+1}(S, F_y) = 0$; and let

$$\cdots \to P_2' \to P_1' \to P_0' \to S \to 0$$

be an R-flat resolution of S. Then, I claim, the module

$$P_d := \operatorname{coker}(P_{d+1}' \to P_d')$$

is *R-flat,* whence we have $(*)$ with $P_i = P_i'$ for $0 \leq i < d$.

Indeed, the flatness of P_d is equivalent to the vanishing of $\mathrm{Tor}^R_1(P_d, R/I)$ for all R-ideals I [**B$'$**, §4, Prop. 1]. But any such I is $\mathcal{I}_y$ where $\mathcal{I} \subset \mathcal{O}_Y$ is the $\mathcal{O}_Y$-ideal such that for any open $U \subset Y$,

$$\begin{aligned} \mathcal{I}(U) &= \{\, r \in \mathcal{O}_Y(U) \mid r_y \in I \,\} && \text{if } y \in U \\ &= 0 && \text{if } y \notin U; \end{aligned}$$

so that if $F = \mathcal{O}_Y/\mathcal{I}$, then $R/I = F_y$; and from the flat resolution

$$\cdots \to P_{d+2}' \to P_{d+1}' \to P_d' \to P_d \to 0$$

of P_d, we get the desired vanishing:

$$\mathrm{Tor}^R_1(P_d, R/I) = \mathrm{Tor}^R_1(P_d, F_y) = \mathrm{Tor}^R_{d+1}(S, F_y) = 0.$$

Exercise 2.7.6.5. (For amusement only.) If Y is a quasi-separated scheme, then $f\colon X \to Y$ satisfies tor-dim $f \le d$ if (and only if) for every quasi-coherent $\mathcal{O}_Y$-ideal $\mathcal{I}$, we have

$$L_{d+1}f^*(\mathcal{O}_Y/\mathcal{I}) = 0.$$

If in addition Y is quasi-compact or locally noetherian, then we need only consider *finite-type* quasi-coherent $\mathcal{O}_Y$-ideals.

[The following facts in [**GD**] can be of use here: p. 111, (5.2.8); p. 313, (6.7.1); p. 294, (6.1.9) (i); p. 295, (6.1.10)(iii); p. 318, (6.9.7).]

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