2

The Electromagnetic Fields Created by Time-Sinusoidal Current

Abstract. This chapter is a recall of the properties of the long-range part of the electromagnetic fields created by time-periodic currents, as they may be observed in particular in the Zeeman effect. The aim of this part is also to place the vector frame of these observations, that is, one of the spherical coordinates, which is in the center of the presentation in the real formalism of the relativistic central potential problem. This frame is the one in which are expressed the Dirac probability current, associated with a state and with the transition between two states. But it is to notice that, as a specificity of the real formalism, the form given by Hestenes to the wave function of the electron, strictly equivalent to the Dirac spinor, may be presented, in the case of central potential, as a combination of the vectors of this frame.

2.1 Properties of the Electromagnetic Field Emitted by an Electron Bound in an Atom

The observation of the electromagnetic fields emitted by electrons bound in an atom, achieved when a magnetic field is present (Zeeman effect), shows that the field owns the following particularities:

1. The field is time-sinusoidal and polarized.
2. If the observation is orthogonal to the direction of the magnetic field, the polarization appears as being linear along this direction.
3. If the observation is parallel to this direction, the polarization appears as being circular and in a plane orthogonal to this direction.

Such data of the observations allow one to precise the general form of the electric currents, which are the source of the field.

The extension of these particularities to the transitions where no magnetic field is present, that is, spontaneous or stimulated emissions processes, is not directly observable. But it is confirmed not only by other experimental data, but also by the fact that the theoritical construction of the transition currents is deduced from the Darwin solutions of the Dirac equation, and that these solutions give exactly (if one excepts the small variation called the Lamb shift) the values of the levels of energy of an electron bound in an atom.

2.2 The Field at Large Distance of a Time-Periodic Current

The calculation of the field that is used here is based on the pure laws of Maxwell, without quantization. Indeed, using Quantum Field Theory is not a necessity in the domain studied here. It leads exactly to the same results (see [12]), with sometimes longer calculations.

We consider only the long-range part of the field by applying the following theorem [41]. If the source of the field is negligible outside a small neighbourhood of the origin O, the long-range part of the field is deduced from the integral formula of the retarded potential in such a way that

$$\mathbf{E}(x^0, \mathbf{r}) = -q \frac{\partial}{\partial x^0} \int \frac{\mathbf{j}^{\perp}(x^0 - R, \mathbf{r}\,')}{R}\, \mathrm{d}\tau', \tag{2.1}$$

$$\mathbf{H}(x^0, \mathbf{r}) = -q \frac{\partial}{\partial x^0} \int \frac{\mathbf{n} \times \mathbf{j}^{\perp}(x^0 - R, \mathbf{r}\,')}{R}\, \mathrm{d}\tau', \tag{2.2}$$

where the coordinates x^{μ} are in the form $(x^0 = ct, \boldsymbol{r})$ and q is the charge of the source.

The vector $\boldsymbol{j}^{\perp}$ is the component of the spatial part $\boldsymbol{j} = (j^1, j^2, j^3)$ of the space–time vector j^{μ}, orthogonal to the vector $\boldsymbol{n} = \boldsymbol{R}/R$, where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$, $R = |\boldsymbol{R}|$. Note that the time component j^0 of the current does not intervene.

In the theory of the electron, the vector j^{μ} has the meaning of a current of probability of the presence of the electron and $q = -e$ is the charge.

We can notice furthermore that if j^{μ} in time-independent, the long-range part of the field is null. As it is the case of the Dirac probability current j^{μ} associated with the state of a bound electron, this explains the reason why no electromagnetic field may be observed outside a passage from a state to one another.

If the field is time-sinusoidal, the source current is of the form

$$q\, \mathbf{j}(x^0, \mathbf{r}) = q\, [\cos\, \omega x^0\, \mathbf{j}_1(\mathbf{r}) + \sin\, \omega x^0\, \mathbf{j}_2(\mathbf{r})], \tag{2.3}$$

where the vectors $\boldsymbol{j}_k$ are to be precised.

At large distance r from the origin O, we may replace $\boldsymbol{r} - \boldsymbol{r}'$ by $\boldsymbol{r} = r\boldsymbol{n}$ in (2.1) and write

$$\boldsymbol{E}(x^0, \boldsymbol{r}) = -\frac{q}{r} \frac{\partial}{\partial x^0} \int \boldsymbol{j}^{\perp}(x^0 - r, \boldsymbol{r}')\, \mathrm{d}\tau' \tag{2.4}$$

and so we can write

$$\boldsymbol{E}(x^0,\boldsymbol{r}) = q\frac{\omega}{r}[\sin\omega(x^0-r)\,\boldsymbol{U}_1^{\perp} - \cos\omega(x^0-r)\,\boldsymbol{U}_2^{\perp}], \tag{2.5}$$

where

$$\boldsymbol{U}_k^{\perp} = \int \boldsymbol{j}_k^{\perp}(\boldsymbol{r}')\,\mathrm{d}\tau', \quad k=1,2\,. \tag{2.6}$$

2.3 Source Currents of Time-Sinusoidal Polarized Field

Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be an orthogonal frame of the three-space of the laboratory galilean frame. The most convenient coordinates system for defining the current is the (r,θ,φ) spherical coordinate system, in which the vector $\boldsymbol{e}_3$ defines a privileged direction, the one of the magnetic field in the case of the presence of this field,

$$\begin{aligned}
&\boldsymbol{u} = \cos\varphi\;\boldsymbol{e}_1 + \sin\varphi\;\boldsymbol{e}_2, \quad \boldsymbol{v} = -\sin\varphi\;\boldsymbol{e}_1 + \cos\varphi\;\boldsymbol{e}_2,\\
&\boldsymbol{n} = \cos\theta\;\boldsymbol{e}_3 + \sin\theta\;\boldsymbol{u}, \quad \boldsymbol{w} = -\sin\theta\;\boldsymbol{e}_3 + \cos\theta\;\boldsymbol{u}, \quad \boldsymbol{r} = r\boldsymbol{n}.
\end{aligned} \tag{2.7}$$

For taking into account the polarizations, the components $\boldsymbol{j}_1, \boldsymbol{j}_2$ of the current may be then defined in the following way:

$$\boldsymbol{j}_1 = \cos\epsilon\varphi\;\boldsymbol{j}_I + \sin\epsilon\varphi\;\boldsymbol{j}_{II}, \quad \boldsymbol{j}_2 = -\sin\epsilon\varphi\;\boldsymbol{j}_I + \cos\epsilon\varphi\;\boldsymbol{j}_{II}, \tag{2.8}$$

where

$$\boldsymbol{j}_I = b(r,\theta)\;\boldsymbol{v}, \quad \boldsymbol{j}_{II} = a(r,\theta)\;\boldsymbol{u} + c(r,\theta)\;\boldsymbol{e}_3, \tag{2.9}$$

and where ϵ may be taken equal to 0 or ±1. We consider the vector

$$\boldsymbol{U} = \cos\omega x^0\;\boldsymbol{U}_1 + \sin\omega x^0\;\boldsymbol{U}_2, \quad \boldsymbol{U}_k = \int \boldsymbol{j}_k(\boldsymbol{r})\,\mathrm{d}\tau. \tag{2.10}$$

2.3.1 Linear Polarization: $\epsilon = 0$

In this case we have $\boldsymbol{j}_1 = \boldsymbol{j}_I$ and $\boldsymbol{j}_2 = \boldsymbol{j}_{II}$. The relations $\mathrm{d}\tau = (r\sin\theta\mathrm{d}\varphi)(r\mathrm{d}\theta)\mathrm{d}r$ and $\int_0^{2\pi}\boldsymbol{u}\;\mathrm{d}\varphi = 0 = \int_0^{2\pi}\boldsymbol{v}\;\mathrm{d}\varphi$ give

$$\boldsymbol{U}_1 = 0,\; \boldsymbol{U}_2 = C\boldsymbol{e}_3,\; C = 2\pi\int_0^{\infty}\int_0^{\pi} c(r,\theta)r^2\sin\theta\;\mathrm{d}r\mathrm{d}\theta \quad \boldsymbol{U} = \sin\omega x^0\;C\boldsymbol{e}_3. \tag{2.11}$$

2.3.2 Circular Polarizations: $\epsilon = \pm1$

In this case we deduce immediately

$$\begin{aligned}
\boldsymbol{j}_1^{\pm} &= (\pm a - b)\cos\varphi\sin\varphi\;\boldsymbol{e}_1 + (b\,\cos^2\varphi \pm a\,\sin^2\varphi)\;\boldsymbol{e}_2 \pm c\,\sin\varphi\;\boldsymbol{e}_3,\\
\boldsymbol{j}_2^{\pm} &= (\pm b\,\sin^2\varphi + a\,\cos^2\varphi)\;\boldsymbol{e}_1 + (a \mp b)\cos\varphi\sin\varphi\;\boldsymbol{e}_2 + c\,\cos\varphi\;\boldsymbol{e}_3
\end{aligned}$$

and after integration

$$
\begin{aligned}
&\boldsymbol{U}_1^{\pm} = (J_I \pm J_{II})\,\boldsymbol{e}_2, \quad \boldsymbol{U}_2^{\pm} = (\pm J_I + J_{II})\,\boldsymbol{e}_1, \\
&\boldsymbol{U}^{+} = A(\sin\omega x^0\ \boldsymbol{e}_1 + \cos\omega x^0\ \boldsymbol{e}_2), \ \ A = J_{II} + J_I, \\
&\boldsymbol{U}^{-} = B(\sin\omega x^0\ \boldsymbol{e}_1 - \cos\omega x^0\ \boldsymbol{e}_2), \ \ B = J_{II} - J_I, \\
&\ \ J_I = \pi \int_0^{\infty}\int_0^{\pi} b(r,\theta) r^2 \sin\theta\ \mathrm{d}r\mathrm{d}\theta, \quad J_{II} = \pi \int_0^{\infty}\int_0^{\pi} a(r,\theta) r^2\ \sin\theta\ \mathrm{d}r\mathrm{d}\theta\,.
\end{aligned}
\tag{2.12}
$$

2.4 Flux of the Poynting Vector Through a Sphere of Large Radius

Let us consider the flux F, per unit of time, through a sphere S of large radius, of the Poynting vector of the field, created by the transition current between two states, of an electron bound in an atom. If we consider the energy E released at each transition, the ratio F/E gives the number of transitions per second.

If no external field is present, the transition is called spontaneous emission. The number of these transitions may be experimentally observed, and, for comparison, the theoretical calculation presents an interest (see Chap. 7).

We consider that F is averaged on a period $T = 2\pi\omega$ of the source current and denoted by

$$\langle X \rangle = \frac{1}{T}\int_0^T X\ \mathrm{d}x^0,$$

the average of X.

Because $\boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{n} \times \boldsymbol{E}$ are orthogonal to $\boldsymbol{n}$, we can write for a sphere S of center 0 of radius R

$$F = \frac{c}{4\pi}\int_{S_0} \langle \boldsymbol{n} \cdot (\boldsymbol{E} \times \boldsymbol{H} \rangle\ R^2 \mathrm{d}\sigma,$$

then

$$F = \frac{c}{4\pi}\int_{S_0} \langle \boldsymbol{E}^2 \rangle\ R^2 \mathrm{d}\sigma, \tag{2.13}$$

where S_0 is the sphere unity. Now

$$
\begin{aligned}
\langle \cos^2\omega(x^0 - R) \rangle = \langle \sin^2\omega(x^0 - R) \rangle = \frac{1}{2}, \\
\langle \cos 2\omega(x^0 - R)\sin\omega(x^0 - R) \rangle = 0\,.
\end{aligned}
$$

In other respect, let (θ_0, φ_0) be the system of spherical coordinates of S_0, such that the axis of the poles is colinear with one of the vectors $\boldsymbol{U}_k$.

We can write $(\boldsymbol{U}_k^{\perp})^2 = \boldsymbol{U}_k^2 \sin^2\theta_0$ and

$$\int_{S_0} (\boldsymbol{U}_k^{\perp})^2 \mathrm{d}\sigma = \int_0^{2\pi}\int_0^{\pi} [\boldsymbol{U}_k^2 \sin^2\theta_0] \sin\,\theta_0\, \mathrm{d}\varphi \mathrm{d}\theta_0 = \frac{8\pi}{3}\boldsymbol{U}_k^2, \tag{2.14}$$

and taking into account the presence of $1/R^2$ in $\boldsymbol{E}^2$, we can replace in all the cases of polarization (2.13) by the equation

$$F = \frac{c\omega^2 e^2}{3}(\boldsymbol{U}_1^2 + \boldsymbol{U}_2^2). \tag{2.15}$$

2.5 Units

The only constants we use are the three fundamental constants (revised in 1989 by B.N. Taylor):

1. The light speed $c = 2.99\,792\,458 \times 10^{10}\,\mathrm{cm\,s^{-1}}$.
2. The electron charge magnitude $e = 4.803\,206 \times\ 10^{-10}$ (e.s.u.).
3. The reduced Planck constant $\hbar = h/2\pi = 1.054\,572 \times 10^{-27}\,\mathrm{erg\,s}$. In addition we use
4. the electron mass $m = 9.109\,389 \times 10^{-28}$ g. All the other constants used will be derived from these four ones, in particular,
5. the fine structure constant

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.035\,989} \quad (e \text{ in e.s.u.}) \tag{2.16}$$

   and as unit of length:
6. the "radius of first Bohr orbit"

$$a = \hbar^2/(me^2) = \hbar/(mc\alpha) = 5.291\,772 \times 10^{-9}\,\mathrm{cm}\,. \tag{2.17}$$

Note

In other respects, one introduces in the expression of the electromagnetic potentials the factor $1/(4\pi\epsilon_0)$ (the presence of 4π is due to the writing $4\pi j^\mu$ instead of j^μ in the current term of the Maxwell equations), where ϵ_0 is the permittivity of free space, and e is expressed in e.m.u:

$$\epsilon_0 = 8.854187 \times 10^{-12}\,\mathrm{F\ m^{-1}}, \quad e = 1.602\,1777 \times 10^{-19}\ \text{(e.m.u.)}$$

That gives (with c expressed in metres) the same value of α with the expression

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}, \quad (e \text{ in e.m.u.}) \tag{2.18}$$

For simplicity and to be in agreement with the largest part of the reference articles and treatises mentioned here, we use the former expressions of the potentials and the constant α, in preference to these last ones.

http://www.springer.com/978-3-540-85549-1

Relativistic Transitions in the Hydrogenic Atoms
Elementary Theory
Boudet, R.
2009, XII, 136 p., Hardcover
ISBN: 978-3-540-85549-1