Chapter 2
Generalized Convex Functions

Convexity is one of the most frequently used hypotheses in optimization theory. It is usually introduced to give global validity to propositions otherwise only locally true, for instance, a local minimum is also a global minimum for a convex function. Moreover, convexity is also used to obtain sufficiency for conditions that are only necessary, as with the classical Fermat theorem or with Kuhn-Tucker conditions in nonlinear programming. In microeconomics, convexity plays a fundamental role in general equilibrium theory and in duality theory. For more applications and historical reference, see, Arrow and Intriligator (1981), Guerraggio and Molho (2004), Islam and Craven (2005). The convexity of sets and the convexity and concavity of functions have been the object of many studies during the past one hundred years. Early contributions to convex analysis were made by Holder (1889), Jensen (1906), and Minkowski (1910, 1911). The importance of convex functions is well known in optimization problems. Convex functions come up in many mathematical models used in economics, engineering, etc. More often, convexity does not appear as a natural property of the various functions and domain encountered in such models. The property of convexity is invariant with respect to certain operations and transformations. However, for many problems encountered in economics and engineering the notion of convexity does no longer suffice. Hence, it is necessary to extend the notion of convexity to the notions of pseudo-convexity, quasi-convexity, etc. We should mention the early work by de de Finetti (1949), Fenchel (1953), Arrow and Enthoven (1961), Mangasarian (1965), Ponstein (1967), and Karamardian (1967). In the recent years, several extensions have been considered for the classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson (1981). Hanson's initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences.

In this chapter, we shall discuss about various concepts of generalized convex functions introduced in the literature in last thirty years for the purpose of weakening the limitations of convexity in mathematical programming. Hanson (1981) introduced the concept of invexity as a generalization of convexity for scalar constrained optimization problems, and he showed that weak duality and sufficiency of

S.K. Mishra et al., *Generalized Convexity and Vector Optimization,*
Nonconvex Optimization and Its Applications.

the Kuhn-Tucker optimality conditions hold when invexity is required instead of the usual requirement of convexity of the functions involved in the problem.

2.1 Convex and Generalized Convex Functions

Definition 2.1.1. *A subset X of R^n is convex if for every $x_1, x_2 \in X$ and $0 < \lambda < 1$, we have*

$$\lambda x_1 + (1-\lambda)x_2 \in X.$$

Definition 2.1.2. *A function $f : X \to R$ defined on a convex subset X of R^n is convex if for any $x_1, x_2 \in X$ and $0 < \lambda < 1$, we have*

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2).$$

If we have strict inequality for all $x_1 \neq x_2$ in the above definition, the function is said to be strictly convex.

Historically the first type of generalized convex function was considered by de Finetti (1949) who first introduced the quasiconvex functions (a name given by Fenchel (1953)) after 6 years.

Definition 2.1.3. *A function $f : X \to R$ is quasiconvex on X if*

$$f(x) \le f(y) \Rightarrow f(\lambda x + (1-\lambda)y) \le f(y), \quad \forall x,\ y \in X,\ \forall\, \lambda \in [0,\ 1]$$

or, equivalently, in non-Euclidean form

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x),\ f(y)\}, \quad \forall x,\ y \in X,\ \forall\, \lambda \in [0,\ 1].$$

For further study and characterization of quasiconvex functions, one can see Giorgi et al. (2004).

In the differentiable case, we have the following definition given in Avriel et al. (1988):

Definition 2.1.4. *A function $f : X \to R$ is said to be quasiconvex on X if*

$$f(x) \le f(y) \Rightarrow (x-y)\nabla f(y) \le 0, \quad \forall x,\ y \in X.$$

An important property of a differentiable convex function is that any stationary point is also a global minimum point; however, this useful property is not restricted to differentiable convex functions only. The family of pseudoconvex functions introduced by Mangasarian (1965) and under the name of semiconvex functions by Tuy (1964), strictly includes the family of differentiable convex functions and has the above mentioned property as well.

Definition 2.1.5. *Let $f : X \to R$ be differentiable on the open set $X \subset R^n$; then f is pseudoconvex on X if:*

$$f(x) < f(y) \Rightarrow (x-y)\nabla f(y) < 0, \quad \forall x,\ y \in X$$

or equivalently if

$$(x-y)\nabla f(y) \geq 0 \Rightarrow f(x) \geq f(y), \quad \forall x,\ y \in X.$$

From this definition it appears obvious that, if f is pseudoconvex and $\nabla f(y) = 0$, then y is a global minimum of f over X. Pseudoconvexity plays a key role in obtaining sufficient optimality conditions for a nonlinear programming problem as, if a differentiable objective function can be shown or assumed to be pseudoconvex, then the usual first-order stationary conditions are able to produce a global minimum.

The function $f : X \to R$ is called *pseudoconcave* if $-f$ is pseudoconvex.

Functions that are both pseudoconvex and pseudoconcave are called *pseudolinear*. Pseudolinear functions are particularly important in certain optimization problems, both in scalar and vector cases; see Chew and Choo (1984), Komlosi (1993), Rapcsak (1991), Kaul et al. (1993), and Mishra (1995).

The following result due to Chew and Choo (1984) characterizes the class of pseudolinear functions.

Theorem 2.1.1. *Let $f : X \to R$, where $X \subset R^n$ is an open convex set. Then the following statements are equivalent:*

(i) f is pseudolinear.
(ii) For any $x, y \in X$, it is $(x-y)\nabla f(y) = 0$ if and only if $f(x) = f(y)$.
(iii) There exists a function $p : X \times X \to R_+$ such that

$$f(x) = f(y) + p(x,y) \cdot (x-y)\nabla f(y).$$

The class of pseudolinear functions includes many classes of functions useful for applications, e.g., the class of linear fractional functions (see, e.g., Chew and Choo (1984)).

An example of a pseudolinear function is given by $f(x) = x + x^3,\ x \in R$. More generally, Kortanek and Evans (1967) observed that if f is pseudolinear on the convex set $X \subset R^n$, then the function $F = f(x) + [f(x)]^3$ is also pseudolinear on X.

For characterization of the solution set of a pseudolinear program, one can see Jeyakumar and Yang (1995).

Ponstein (1967) introduced the concept of strictly pseudoconvex functions for differentiable functions.

Definition 2.1.6. *A function $f : X \to R$, differentiable on the open set $X \subset R^n$, is strictly pseudoconvex on X if*

$$f(x) \leq f(y) \Rightarrow (x-y)\nabla f(y) < 0, \quad \forall\, x,\ y \in X,\ x \neq y,$$

or equivalently if

$$(x-y)\nabla f(y) \geq 0 \Rightarrow f(x) > f(y), \quad \forall\, x,\, y \in X,\, x \neq y.$$

The comparison of the definitions of pseudoconvexity and strict pseudoconvexity shows that strict pseudoconvexity implies pseudoconvexity. Ponstein (1967) showed that pseudoconvexity plus strict quasiconvexity implies strict pseudoconvexity and that strict pseudoconvexity implies strict quasiconvexity.

In a minimization problem, if the strict pseudoconvexity of the objective function can be shown or assumed, then the solution to the first-order optimality conditions is a unique global minimum. Many other characterization of strict pseudoconvex functions are given by Diewert et al. (1981).

Convex functions play an important role in optimization theory. The optimization problem:

$$\text{minimize}\, f(x) \;\text{ for } x \in X \subseteq R^n, \quad \text{subject to } \; g(x) \leqq 0,$$

is called a convex program if the functions involved are convex on some subset X of R^n. Convex programs have many useful properties:

1. The set of all feasible solutions is convex.
2. Any local minimum is a global minimum.
3. The Karush–Kuhn–Tucker optimality conditions are sufficient for a minimum.
4. Duality relations hold between the problem and its dual.
5. A minimum is unique if the objective function is strictly convex.

However, for many problems encountered in economics and engineering the notion of convexity does no longer suffice. To meet this demand and the convexity requirement to prove sufficient optimality conditions for a differentiable mathematical programming problem, the notion of invexity was introduced by Hanson (1981) by substituting the linear term $(x-y)$, appearing in the definition of differentiable convex, pseudoconvex and quasiconvex functions, with an arbitrary vector-valued function.

2.2 Invex and Generalized Invex Functions

Definition 2.2.1. *A function $f : X \to R$, X open subset of R^n, is said to be invex on X with respect to η if there exists vector-valued function $\eta : X \times X \to R^n$ such that*

$$f(x) - f(y) \geq \eta^T(x,y)\nabla f(y), \quad \forall x,\, y \in X.$$

The name "invex" was given by Craven (1981) and stands for "invariant convex."

Similarly f is said to be *pseudoinvex* on X with respect to η if there exists vector-valued function $\eta : X \times X \to R^n$ such that

$$\eta^T(x,y)\nabla f(y) \geq 0 \Rightarrow f(x) \geq f(y), \quad \forall x, y \in X.$$

The function $f : X \to R$, X open subset of R^n, is said to be *quasiinvex* on X with respect to η if there exists vector-valued function $\eta : X \times X \to R^n$ such that

$$f(x) \leq f(y) \Rightarrow \eta^T(x,y)\nabla f(y) \leq 0, \quad \forall x, y \in X.$$

Craven (1981) gave necessary and sufficient conditions for function f to be invex assuming that the functions f and η are twice continuously differentiable.

Ben-Israel and Mond (1986) and Kaul and Kaur (1985) also studied some relationships among the various classes of (generalized) invex functions and (generalized) convex functions. Let us list their results for the sake completion:

(1) A differentiable convex function is also invex, but not conversely, see example, Kaul and Kaur (1985).
(2) A differentiable pseudo-convex function is also pseudo-invex, but not conversely, see example, Kaul and Kaur (1985).
(3) A differentiable quasi-convex function is also quasi-invex, but not conversely, see example, Kaul and Kaur (1985).
(4) Any invex function is also pseudo-invex for the same function $\eta(x,\bar{x})$, but not conversely, see example, Kaul and Kaur (1985).
(5) Any pseudo-invex function is also quasi-invex, but not conversely.

Further insights on these relationships can be deduced by means of the following characterizations of invex functions:

Theorem 2.2.1. *(Ben-Israel and Mond (1986)) Let $f : X \to R$ be differentiable on the open set $X \subset R^n$; then f is invex if and only if every stationary point of f is a global minimum of f over X.*

It is adequate, in order to apply invexity to the study of optimality and duality conditions, to know that a function is invex without identifying an appropriate function $\eta(x,\bar{x})$. However, Theorem 2.2.1 allows us to find a function $\eta(x,\bar{x})$, when $f(x)$ is known to be invex; viz.

$$\eta(x,\bar{x}) = \begin{cases} \dfrac{[f(x) - f(\bar{x})]\nabla f(\bar{x})}{\nabla f(\bar{x})\nabla f(\bar{x})}, & \text{if} \quad \nabla f(\bar{x}) \neq 0 \\ 0, & \text{if} \quad \nabla f(\bar{x}) = 0. \end{cases}$$

Remark 2.2.1. If we consider an invex function f on set $X_0 \subseteq X$, with X_0 not open, it is not true that any local minimum of f on X_0 is also a global minimum. Let us consider the following example.

Example 2.2.1. Let $f(x,y) = y\left(x^2 - 1\right)^2$ and $X_0 = \{(x,y) : (x,y) \in R^2, x \geq -1/2, y \geq 1\}$. Every stationary point of f on X_0 is a global minimum of f on X_0, and therefore f is invex on X_0. The point $(-1/2,1)$ is a local minimum point of f on X_0, with $f(-1/2,1) = 9/16$, but the global minimum is $f(1,y) = f(-1,y) = 0$.

In order to consider some type of invexity for nondifferentiable functions, Ben-Israel and Mond (1986) and Weir and Mond (1988) introduced the following function:

Definition 2.2.2. *A function $f : X \to R$ is said to be pre-invex on X if there exists a vector function $\eta : X \times X \to R^n$ such that*

$$(y + \lambda \eta (x,\ y)) \in X, \quad \forall \lambda \in [0,\ 1], \quad \forall\ x,\ y \in X$$

and

$$f(y + \lambda \eta (x,\ y)) \leq \lambda f(x) + (1-\lambda) f(y), \quad \forall \lambda \in [0,\ 1], \quad \forall\ x,\ y \in X.$$

Weir and Mond (1988a) gave the following example of a pre-invex function which is not convex.

Example 2.2.2. $f(x) = -|x|,\ x \in R$. Then f is a pre-invex function with η given as follows:

$$\eta(x,\ y) = \begin{cases} x-y, & \text{if } y \leq 0 \quad \text{and } x \leq 0 \\ x-y, & \text{if } y \geq 0 \quad \text{and } x \geq 0 \\ y-x, & \text{if } y > 0 \quad \text{and } x < 0 \\ y-x, & \text{if } y < 0 \quad \text{and } x > 0. \end{cases}$$

As for convex functions, any local minimum of a pre-invex function is a global minimum and nonnegative linear combinations of pre-invex functions are pre-invex. Pre-invex functions were utilized by Weir and Mond (1988a) to establish proper efficiency results in multiple objective optimization problems.

2.3 Type I and Related Functions

Subsequently, Hanson and Mond (1982) introduced two new classes of functions which are not only sufficient but are also necessary for optimality in primal and dual problems, respectively. Let

$$P = \{x : x \in X, g(x) \leqq 0\} \quad \text{and} \quad D = \{x : (x,y) \in Y\},$$

where $Y = \{(x,y) : x \in X, y \in R^m, \nabla_x f(x) + y^T \nabla_x g(x) = 0; y \geqq 0\}$.

Hanson and Mond (1982) defined:

Definition 2.3.1. *$f(x)$ and $g(x)$ as Type I objective and constraint functions, respectively, with respect to $\eta(x)$ at $\bar{x}$ if there exists an n-dimensional vector function $\eta(x)$ defined for all $x \in P$ such that*

$$f(x) - f(\bar{x}) \geqq [\nabla_x f(\bar{x})]^T \eta(x,\bar{x})$$

and

$$-g(\bar{x}) \geqq [\nabla_x g(\bar{x})]^T \eta(x,\bar{x}),$$

the objective and constraint functions $f(x)$ and $g(x)$ are calledstrictly Type I if we have strict inequalities in the above definition.

Definition 2.3.2. *$f(x)$ and $g(x)$ as Type II objective and constraint functions, respectively, with respect to $\eta(x)$ at $\bar{x}$ if there exists an n-dimensional vector function $\eta(x)$ defined for all $x \in P$ such that*

$$f(\bar{x}) - f(x) \geqq [\nabla_x f(x)]^T \eta(x, \bar{x})$$

and

$$-g(x) \geqq [\nabla_x g(x)]^T \eta(x, \bar{x}).$$

the objective and constraint functions $f(x)$ and $g(x)$ are called strictly Type II if we have strict inequalities in the above definition.

Rueda and Hanson (1988) established the following relations:

1. If $f(x)$ and $g(x)$ are convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are Type I, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.1. The functions $f : \left(0, \frac{\pi}{2}\right) \to R$ and $g : \left(0, \frac{\pi}{2}\right) \to R$ defined by $f(x) = x + \sin x$ and $g(x) = -\sin x$ are Type I functions with respect to $\eta(x) = \left(\frac{2}{\sqrt{3}}\right) \left(\sin x - \frac{1}{2}\right)$ at $\bar{x} = \pi/6$, but $f(x)$ and $g(x)$ are not convex with respect to the same $\eta(x) = \left(\frac{2}{\sqrt{3}}\right)\left(\sin x - \frac{1}{2}\right)$ as can be seen by taking $x = \pi/4$ and $\bar{x} = \pi/6$.

2. If $f(x)$ and $g(x)$ are convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are Type II, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.2. The functions $f : \left(0, \frac{\pi}{2}\right) \to R$ and $g : \left(0, \frac{\pi}{2}\right) \to R$ defined by $f(x) = x + \sin x$ and $g(x) = -\sin x$ are Type II functions with respect to $\eta(x) = \frac{\left(\frac{1}{2} - \sin x\right)}{\cos x}$ at $\bar{x} = \pi/6$, but $f(x)$ and$g(x)$ are not convex with respect to the same $\eta(x) = \frac{\left(\frac{1}{2} - \sin x\right)}{\cos x}$ at $\bar{x} = \pi/6$.

3. If $f(x)$ and $g(x)$ are strictly convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are strictly Type I, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.3. The functions $f : \left(0, \frac{\pi}{2}\right) \to R$ and $g : \left(0, \frac{\pi}{2}\right) \to R$ defined by $f(x) = -x + \cos x$ and $g(x) = -\cos x$ are strictly Type I functions with respect to $\eta(x) = 1 - \left(\frac{2}{\sqrt{2}}\right) \cos x$ at $\bar{x} = \pi/4$, but $f(x)$ and$g(x)$ are not strictly convex with respect to the same $\eta(x) = 1 - \left(\frac{2}{\sqrt{2}}\right) \cos x$.

4. If $f(x)$ and $g(x)$ are strictly convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are strictly Type II, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.4. The functions $f : \left(0, \frac{\pi}{2}\right) \to R$ and $g : \left(0, \frac{\pi}{2}\right) \to R$ defined by $f(x) = -x + \cos x$ and $g(x) = -\cos x$ are strictly Type II functions with respect to $\eta(x) = \frac{\left(\cos x - \frac{\sqrt{2}}{2}\right)}{\sin x}$ at $\bar{x} = \pi/4$, but $f(x)$ and $g(x)$ are not strictly convex with respect to the same $\eta(x) = 1 - \left(\frac{2}{\sqrt{2}}\right)\cos x$.

Rueda and Hanson (1988) defined:

Definition 2.3.3. *$f(x)$ and $g(x)$ as pseudo-Type I objective and constraint functions, respectively, with respect to $\eta(x)$ at $\bar{x}$ if there exists an n-dimensional vector function $\eta(x)$ defined for all $x \in P$ such that*

$$[\nabla_x f(x)]^T \eta(x,\bar{x}) \geqq 0 \Rightarrow f(\bar{x}) - f(x) \geqq 0$$

and

$$[\nabla_x g(x)]^T \eta(x,\bar{x}) \geqq 0 \Rightarrow -g(x) \geqq 0.$$

Definition 2.3.4. *$f(x)$ and $g(x)$ as quasi-Type I objective and constraint functions, respectively, with respect to $\eta(x)$ at $\bar{x}$ if there exists an n-dimensional vector function $\eta(x)$ defined for all $x \in P$ such that*

$$f(x) - f(\bar{x}) \leqq 0 \Rightarrow [\nabla_x f(\bar{x})]^T \eta(x,\bar{x}) \leqq 0.$$

and

$$-g(x) \leqq 0 \Rightarrow [\nabla_x g(x)]^T \eta(x,\bar{x}) \leqq 0.$$

Pseudo-Type II and quasi-Type II objective and constraint functions are defined similarly.

It was shown by Rueda and Hanson (1988) that:

1. Type I objective and constraint functions $\Rightarrow$ pseudo-Type I objective and constraint functions, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.5. The functions $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ and $g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ defined by $f(x) = -\cos^2 x$ and $g(x) = -\cos x$ are pseudo-Type I functions with respect to $\eta(x) = -\frac{1}{2} + \left(\sqrt{2}/2\right)\cos x$ at $\bar{x} = -\pi/4$, but $f(x)$ and $g(x)$ are not Type I with respect to the same $\eta(x) = -\frac{1}{2} + \left(\sqrt{2}/2\right)\cos x$ as can be seen by taking $x = 0$.

2. Type II objective and constraint functions $\Rightarrow$ pseudo-Type II objective and constraint functions, but the converse is not necessarily true, as can be seen from the following.

Example 2.3.6. The functions $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ and $g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ defined by $f(x) = -\cos^2 x$ and $g(x) = -\cos x$ are pseudo-Type II functions with respect to $\eta(x) = \sin x\left(\cos x - \sqrt{2}/2\right)$ at $\bar{x} = -\pi/4$, but $f(x)$ and $g(x)$ are not Type II with respect to the same $\eta(x) = \sin x\left(\cos x - \sqrt{2}/2\right)$ as can be seen by taking $x = \pi/3$.

3. Type I objective and constraint functions $\Rightarrow$ quasi-Type I objective and constraint functions, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.7. The functions $f : (0,\pi) \to R$ and $g : (0,\pi) \to R$ defined by $f(x) = \sin^3 x$ and $g(x) = -\cos x$ are quasi-Type I functions with respect to $\eta(x) = -1$ at $\bar{x} = \pi/2$, but $f(x)$ and $g(x)$ are not Type I with respect to the same $\eta(x) = -1$ as can be seen by taking $\bar{x} = \pi/4$.

4. Type II objective and constraint functions $\Rightarrow$ quasi-Type II objective and constraint functions, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.8. The functions $f : (0,\infty) \to R$ and $g : (0,\infty) \to R$ defined by $f(x) = -\frac{1}{x}$ and $g(x) = 1-x$ are quasi-Type II functions with respect to $\eta(x) = 1-x$ at $\bar{x} = 1$, but $f(x)$ and $g(x)$ are not Type II with respect to the same $\eta(x) = 1-x$ as can be seen by taking $x = 2$.

5. Strictly Type I objective and constraint functions $\Rightarrow$ Type I objective and constraint functions, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.9. The functions $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ and $g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ defined by $f(x) = -\sin x$ and $g(x) = -\cos x$ are Type I functions with respect to $\eta(x) = \sin x$ at $\bar{x} = 0$, but $f(x)$ and $g(x)$ are not strictly Type I with respect to the same $\eta(x) = \sin x$ at $\bar{x} = 0$.

6. Strictly Type II objective and constraint functions $\Rightarrow$ Type II objective and constraint functions, but the converse is not necessarily true, as can be seen from the following example.

Example 2.3.10. The functions $f : \left(0, \frac{\pi}{2}\right) \to R$ and $g : \left(0, \frac{\pi}{2}\right) \to R$ defined by $f(x) = -\sin x$ and $g(x) = -e^{-x}$ are Type II functions with respect to $\eta(x) = 1$ at $\bar{x} = 0$, but $f(x)$ and $g(x)$ are not strictly Type I with respect to the same $\eta(x)$ at $\bar{x} = 0$.

Kaul et al. (1994) further extended the concepts of Rueda and Hanson (1988) to pseudo-quasi Type I, quasi-pseudo Type I objective and constraint functions as follows.

Definition 2.3.5. *$f(x)$ and $g(x)$ as quasi-pseudo-Type I objective and constraint functions, respectively, with respect to $\eta(x)$ at $\bar{x}$ if there exists an n-dimensional vector function $\eta(x)$ defined for all $x \in P$ such that*

$$f(x) - f(\bar{x}) \leqq 0 \Rightarrow [\nabla_x f(\bar{x})]^T \eta(x,\bar{x}) \leqq 0$$

and

$$[\nabla_x g(x)]^T \eta(x,\bar{x}) \geqq 0 \Rightarrow -g(x) \geqq 0.$$

Definition 2.3.6. *$f(x)$ and $g(x)$ as pseudo-quasi-Type I objective and constraint functions, respectively,with respect to $\eta(x)$ at $\bar{x}$ if there exists an n-dimensional vector function $\eta(x)$ defined for all $x \in P$ such that*

$$[\nabla_x f(\bar{x})]^T \eta(x,\bar{x}) \geqq 0 \Rightarrow f(x) - f(\bar{x}) \geqq 0$$

and

$$-g(x) \leqq 0 \Rightarrow [\nabla_x g(x)]^T \eta(x,\bar{x}) \leqq 0.$$

2.4 Univex and Related Functions

Let f be a differentiable function defined on a nonempty subset X of R^n and let $\phi : R \to R$ and $k : X \times X \to R_+$. For $x, \bar{x} \in X$, we write $k(x,\bar{x}) = \lim_{\lambda \to 0} b(x,\bar{x},\lambda) \geqq 0$. Bector et al. (1992) defined b-invex functions as follows.

Definition 2.4.1. *The function f is said to be B-invex with respect to η and k, at $\bar{x}$ if for all $x \in X$, we have*

$$k(x,\bar{x})[f(x) - f(\bar{x})] \geqq [\nabla_x f(\bar{x})]^T \eta(x,\bar{x}).$$

Bector et al. (1992) further extended this concept to univex functions as follows.

Definition 2.4.2. *The function f is said to be univex with respect to η, ϕ and k, at $\bar{x}$ if for all $x \in X$, we have*

$$k(x,\bar{x})\,\phi[f(x) - f(\bar{x})] \geqq [\nabla_x f(\bar{x})]^T \eta(x,\bar{x}).$$

Definition 2.4.3. *The function f is said to be quasi-univex with respect to η, ϕ and k, at $\bar{x}$ if for all $x \in X$, we have*

$$\phi[f(x) - f(\bar{x})] \leqq 0 \Rightarrow k(x,\bar{x})\,\eta(x,\bar{x})^{\mathrm{T}} \nabla_x f(\bar{x}) \leqq 0.$$

Definition 2.4.4. *The function f is said to be pseudo-univex with respect to η, ϕ and k, at $\bar{x}$ if for all $x \in X$, we have*

$$\eta(x,\bar{x})^{\mathrm{T}} \nabla_x f(\bar{x}) \geqq 0 \Rightarrow k(x,\bar{x})\,\phi[f(x) - f(\bar{x})] \geqq 0.$$

Bector et al. (1992) gave the following relations with some other generalized convex functions existing in the literature.

1. Every B-invex function is univex function with $\phi : R \to R$ defined as $\phi(a) = a, \forall a \in R$, but not conversely.

Example 2.4.1. Let $f : R \to R$ be defined by $f(x) = x^3$, where,

$$\eta(x,\bar{x}) = \begin{cases} x^2 + \bar{x}^2 + x\bar{x}, & x > \bar{x} \\ x - \bar{x}, x & \leqq \bar{x} \end{cases}$$

and

$$k(x,\bar{x}) = \begin{cases} \bar{x}^2/(x-\bar{x}), & x > \bar{x} \\ 0, & x \leqq \bar{x}. \end{cases}$$

Let $\phi : R \to R$ be defined by $\phi(a) = 3a$. The function f is univex but not b-invex, because for $x = 1,\ \bar{x} = 1/2,\ k(x,\bar{x})\,\phi[f(x) - f(\bar{x})] < \eta(x,\bar{x})^{\mathrm{T}} \nabla_x f(\bar{x})$.

2. Every invex function is univex function with $\phi : R \to R$ defined as $\phi(a) = a, \forall a \in R$, and $k(x,\bar{x}) \equiv 1$, but not conversely.

Example 2.4.2. The function considered in above example is univex but not invex, because for $x = -3, \bar{x} = 1, f(x) - f(\bar{x}) < \eta(x,\bar{x})^{\mathrm{T}} \nabla_x f(\bar{x})$.

3. Every convex function is univex function with $\phi : R \to R$ defined as $\phi(a) = a, \forall a \in R, k(x,\bar{x}) \equiv 1$, and $\eta(x,\bar{x}) \equiv x - \bar{x}$, but not conversely.

Example 2.4.3. The function considered in above example is univex but not convex, because for $x = -2, \bar{x} = 1, f(x) - f(\bar{x}) < (x-\bar{x})^{\mathrm{T}} \nabla_x f(\bar{x})$.

4. Every b-vex function is univex function with $\phi : R \to R$ defined as $\phi(a) = a, \forall a \in R$, and $\eta(x,\bar{x}) \equiv x - \bar{x}$, but not conversely.

Example 2.4.4. The function considered in above example is univex but not b-vex, because for $x = \frac{1}{10}, \bar{x} = \frac{1}{100}, k(x,\bar{x})[f(x) - f(\bar{x})] < (x-\bar{x})^{\mathrm{T}} \nabla_x f(\bar{x})$.

Rueda et al. (1995) obtained optimality and duality results for several mathematical programs by combining the concepts of type I functions and univex functions. They combined the Type I and univex functions as follows.

Definition 2.4.5. *The differentiable functions $f(x)$ and $g(x)$ are called Type I univex objective and constraint functions, respectively with respect to $\eta, \phi_0, \phi_1, b_0, b_1$ at $\bar{x} \in X$, if for all $x \in X$, we have*

$$b_0(x,\bar{x})\,\phi_0[f(x) - f(\bar{x})] \geqq \eta(x,\bar{x})^{\mathrm{T}} \nabla_x f(\bar{x})$$

and

$$-b_1(x,\bar{x})\,\phi_1[g(\bar{x})] \geqq \eta(x,\bar{x})^{\mathrm{T}} \nabla_x g(\bar{x}).$$

Rueda et al. (1995) gave examples of functions that are univex but not Type I univex.

Example 2.4.5. The functions $f, g : [1,\infty) \to R$, defined by $f(x) = x^3$ and $g(x) = 1 - x$, are univex at $\bar{x} = 1$ with respect to $b_0 = b_1 = 1, \eta(x,\bar{x}) = x - \bar{x}, \phi_0(a) = 3a, \phi_1(a) = 1$, but g does not satisfy the second inequality of the above definition at $\bar{x} = 1$.

They also pointed out that there are functions which are Type I univex but not univex.

Example 2.4.6. The functions $f, g : [1,\infty) \to R$, defined by $f(x) = -1/x$ and $g(x) = 1 - x$, are Type I univex with respect to $b_0 = b_1 = 1, \eta(x,\bar{x}) = -1/(x-\bar{x})$, $\phi_0(a) = a$, $\phi_1(a) = -a$, at $\bar{x} = 1$, but g is not univex at $\bar{x} = 1$.

Following Rueda et al. (1995), Mishra (1998b) gave several sufficient optimality conditions and duality results for multiobjective programming problems by combining the concepts of Pseudo-quasi-Type I, quasi-pseudo-Type I functions and univex functions.

2.5 *V*-Invex and Related Functions

Jeyakumar and Mond (1992) introduced the notion of V-invexity for a vector function $f = (f_1, f_2, \ldots, f_p)$ and discussed its applications to a class of constrained multiobjective optimization problems. We now give the definitions of Jeyakumar and Mond (1992) as follows.

Definition 2.5.1. *A vector function $f : X \to R^p$ is said to be V-invex if there exist functions $\eta : X \times X \to R^n$ and $\alpha_i : X \times X \to R^+ - \{0\}$ such that for each $x, \bar{x} \in X$ and for $i = 1, 2, \ldots, p$,*

$$f_i(x) - f_i(\bar{x}) \geqq \alpha_i(x, \bar{x}) \nabla f_i(\bar{x}) \eta(x, \bar{x}).$$

for $p = 1$ and $\bar{\eta}(x, \bar{x}) = \alpha_i(x, \bar{x}) \eta(x, \bar{x})$ the above definition reduces to the usual definition of invexity given by Hanson (1981).

Definition 2.5.2. *A vector function $f : X \to R^p$ is said to be V-pseudoinvex if there exist functions $\eta : X \times X \to R^n$ and $\beta_i : X \times X \to R^+ - \{0\}$ such that for each $x, \bar{x} \in X$ and for $i = 1, 2, \ldots, p$,*

$$\sum_{i=1}^{p} \nabla f_i(\bar{x}) \eta(x, \bar{x}) \geqq 0 \Rightarrow \sum_{i=1}^{p} \beta_i(x, \bar{x}) f_i(x) \geqq \sum_{i=1}^{p} \beta_i(x, \bar{x}) f_i(\bar{x}).$$

Definition 2.5.3. *A vector function $f : X \to R^p$ is said to be V-quasiinvex if there exist functions $\eta : X \times X \to R^n$ and $\delta_i : X \times X \to R^+ - \{0\}$ such that for each $x, \bar{x} \in X$ and for $i = 1, 2, \ldots, p$,*

$$\sum_{i=1}^{p} \delta_i(x, \bar{x}) f_i(x) \leqq \sum_{i=1}^{p} \delta_i(x, \bar{x}) f_i(\bar{x}) \Rightarrow \sum_{i=1}^{p} \nabla f_i(\bar{x}) \eta(x, \bar{x}) \leqq 0.$$

It is evident that every V-invex function is both V-pseudo-invex (with $\beta_i(x, \bar{x}) = \frac{1}{\alpha_i(x, \bar{x})}$) and V-quasi-invex (with $\delta_i(x, \bar{x}) = \frac{1}{\alpha_i(x, \bar{x})}$). Also if we set $p = 1$, $\alpha_i(x, \bar{x}) = 1$, $\beta_i(x, \bar{x}) = 1$, $\delta_i(x, \bar{x}) = 1$ and $\eta(x, \bar{x}) = x - \bar{x}$, then the above definitions reduce to those of convexity, pseudo-convexity and quasi-convexity, respectively.

Definition 2.5.4. *A vector optimization problem:*

$$(\text{VP}) \qquad V - \min(f_1, f_2, \ldots, f_p) \quad \textit{subject to } g(x) \leqq 0,$$

where $f_i : X \to R, i = 1, 2, \ldots, p$ and $g : X \to R^m$ are differentiable functions on X, is said to be V-invex vector optimization problem if each $f_1, f_2, \ldots, f_p$ and $g_1, g_2, \ldots, g_m$ is a V-invex function.

Note that, invex vector optimization problems are necessarily V-invex, but not conversely. As a simple example, we consider following example from Jeyakumar and Mond (1992).

Example 2.5.1. Consider

$$\min_{x_1, x_2 \in R} \left(\frac{x_1^2}{x_2}, \frac{x_1}{x_2} \right)$$
$$\text{subject to } 1 - x_1 \leq 1, 1 - x_2 \leq 1.$$

Then it is easy to see that this problem is a V-invex vector optimization problem with $\alpha_1 = \frac{\bar{x}_2}{x_2}$, $\alpha_2 = \frac{\bar{x}_1}{x_1}$, $\beta_1 = 1 = \beta_2$, and $\eta(x, \bar{x}) = x - \bar{x}$; but clearly, the problem does not satisfy the invexity conditions with the same η.

It is also worth noticing that the functions involved in the above problem are invex, but the problem is not necessarily invex.

It is known (see Craven (1981)) that invex problems can be constructed from convex problems by certain nonlinear coordinate transformations. In the following, we see that V-invex functions can be formed from certain nonconvex functions (in particular from convex-concave or linear fractional functions) by coordinate transformations.

Example 2.5.2. Consider function, $h : R^n \to R^p$ defined by $h(x) = (f_1(\phi(x)), \ldots, f_p(\phi(x)))$, where $f_i : R^n \to R, i = 1, 2, \ldots, p$, are strongly pseudo-convex functions with real positive functions $\alpha_i, \phi : R^n \to R^n$ is surjective with $\phi'(\bar{x})$ onto for each $\bar{x} \in R^n$. Then, the function h is V-invex.

Example 2.5.3. Consider the composite vector function $h(x) = (f_1(F_1(x)), \ldots, f_p(F_p(x)))$, where for each $i = 1, 2, \ldots, p$, $F_i : X_0 \to R$ is continuously differentiable and pseudolinear with the positive proportional function $\alpha_i(\cdot, \cdot)$, and $f_i : R \to R$ is convex. Then, $h(x)$ is $V-$invex with $\eta(x, y) = x - y$. This follows from the following convex inequality and pseudolinearity conditions:

$$\begin{aligned} f_i(F_i(x)) - f_i(F_i(y)) &\geq f_i'(F_i(y))(F_i(x) - F_i(y)) \\ &= f_i'(F_i(y))\,\alpha_i(x, y)\,F_i'(y)(x - y) \\ &= \alpha_i(x, y)(f_i \circ F_i)'(y)(x - y). \end{aligned}$$

For a simple example of a composite vector function, we consider

$$h(x_1, x_2) = \left(e^{x_1/x_2}, \frac{x_1 - x_2}{x_1 + x_2} \right), \quad \text{where} \quad X_0 = \left\{ (x_1, x_2) \in R^2 : x_1 \geq 1, x_2 \geq 1 \right\}.$$

Example 2.5.4. Consider the function $H(x) = (f_1((g_1 \circ \psi)(x)), \ldots, f_p((g_p \circ \psi)(x)))$, where each f_i is pseudolinear on R^n with proportional functions $\alpha_i(x, y)$, ψ is a differentiable mapping from R^n onto R^n such that $\psi'(y)$ is surjective for each $y \in R^n$, and $f_i : R \to R$ is convex for each i. Then H is V-invex.

Jeyakumar and Mond (1992) have shown that the V-invexity is preserved under a smooth convex transformation.

Proposition 2.5.1. *Let $\psi : R \to R$ be differentiable and convex with positive derivative everywhere; let $h : X_0 \to R^p$ be V-invex. Then, the function*

$$h_\psi(x) = (\psi(h_1(x)), \ldots, \psi(h_p(x))), \quad x \in X_0$$

is V-invex.

The following very important property of V-invex functions was also established by Jeyakumar and Mond (1992).

Proposition 2.5.2. *Let $f : R^n \to R^p$ be V-invex. Then $y \in R^n$ is a (global) weak minimum of f if and only if there exists $0 \neq \tau \in R^p$, $\tau \geq 0$, $\sum\limits_{i=1}^{p} \tau_i f_i'(y) = 0$.*

By Proposition 2.5.2, one can conclude that for a V-invex vector function every critical point (i.e., $f_i'(y) = 0,\ i = 1, \ldots, p$) is a global weak minimum.

Hanson et al. (2001) extended the (scalarized) generalized type-I invexity into a vector (V-type-I) invexity.

Definition 2.5.5. *The vector problem (VP) is said to be V-type-I at $\bar{x} \in X$ if there exist positive real-valued functions α_i and β_j defined on $X \times X$ and an n-dimensional vector-valued function $\eta : X \times X \to R^n$ such that*

$$f_i(x) - f_i(\bar{x}) \geqq \alpha_i(x, \bar{x}) \nabla f_i(\bar{x}) \eta(x, \bar{x})$$

and

$$-g_j(\bar{x}) \geqq \beta_j(x, \bar{x}) \nabla g_j(\bar{x}) \eta(x, \bar{x}),$$

for every $x \in X$ and for all $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, m$.

Definition 2.5.6. *The vector problem (VP) is said to be quasi-V-type-I at $\bar{x} \in X$ if there exist positive real-valued functions α_i and β_j defined on $X \times X$ and an n-dimensional vector-valued function $\eta : X \times X \to R^n$ such that*

$$\sum_{i=1}^{p} \tau_i \alpha_i(x, \bar{x}) [f_i(x) - f_i(\bar{x})] \leqq 0 \Rightarrow \sum_{\mathrm{i}=1}^{\mathrm{p}} \tau_i \eta(x, \bar{x}) \nabla f_i(\bar{x}) \leqq 0$$

and

$$-\sum_{j=1}^{m} \lambda_j \beta_j(x, \bar{x}) g_j(\bar{x}) \leqq 0 \Rightarrow \sum_{j=1}^{m} \lambda_j \eta(x, \bar{x}) \nabla g_j(\bar{x}) \leqq 0,$$

for every $x \in X$.

Definition 2.5.7. *The vector problem (VP) is said to be pseudo-V-type-I at $\bar{x} \in X$ if there exist positive real-valued functions $\alpha_i$and β_j defined on $X \times X$ and an n-dimensional vector-valued function $\eta : X \times X \to R^n$ such that*

$$\sum_{i=1}^{p} \tau_i \eta(x,\bar{x}) \nabla f_i(\bar{x}) \geqq 0 \Rightarrow \sum_{i=1}^{p} \tau_i \alpha_i(x,\bar{x}) [f_i(x) - f_i(\bar{x})] \geqq 0$$

and

$$\sum_{j=1}^{m} \lambda_j \eta(x,\bar{x}) \nabla g_j(\bar{x}) \geqq 0 \Rightarrow -\sum_{j=1}^{m} \lambda_j \beta_j(x,\bar{x}) g_j(\bar{x}) \geqq 0,$$

for every $x \in X$.

Definition 2.5.8. *The vector problem (VP) is said to be quasi-pseudo-V-type-I at $\bar{x} \in X$ if there exist positive real-valued functions α_i and β_j defined on $X \times X$ and an n-dimensional vector-valued function $\eta : X \times X \to R^n$ such that*

$$\sum_{i=1}^{p} \tau_i \alpha_i(x,\bar{x}) [f_i(x) - f_i(\bar{x})] \leqq 0 \Rightarrow \sum_{i=1}^{p} \tau_i \eta(x,\bar{x}) \nabla f_i(\bar{x}) \leqq 0$$

and

$$\sum_{j=1}^{m} \lambda_j \eta(x,\bar{x}) \nabla g_j(\bar{x}) \geqq 0 \Rightarrow -\sum_{j=1}^{m} \lambda_j \beta_j(x,\bar{x}) g_j(\bar{x}) \geqq 0,$$

for every $x \in X$.

Definition 2.5.9. *The vector problem (VP) is said to be pseudo-quasi-V-type-I at $\bar{x} \in X$ if there exist positive real-valued functions α_i and β_j defined on $X \times X$ and an n-dimensional vector-valued function $\eta : X \times X \to R^n$ such that*

$$\sum_{i=1}^{p} \tau_i \eta(x,\bar{x}) \nabla f_i(\bar{x}) \geqq 0 \Rightarrow \sum_{i=1}^{p} \tau_i \alpha_i(x,\bar{x}) [f_i(x) - f_i(\bar{x})] \geqq 0$$

and

$$-\sum_{j=1}^{m} \lambda_j \beta_j(x,\bar{x}) g_j(\bar{x}) \leqq 0 \Rightarrow \sum_{j=1}^{m} \lambda_j \eta(x,\bar{x}) \nabla g_j(\bar{x}) \leqq 0,$$

for every $x \in X$.

Nevertheless the study of generalized convexity of a vector function is not yet sufficiently explored and some classes of generalized convexity have been introduced recently. Several attempts have been made by many authors to introduce possibly a most wide class of generalized convex function, which can meet the demand of a real life situation to formulate a nonlinear programming problem and therefore get a best possible solution for the same. Recently, Aghezzaf and Hachimi (2001) introduced a new class of functions, which we shall give in next section.

2.6 Further Generalized Convex Functions

Definition 2.6.1. *f is said to be weak strictly pseudoinvex with respect to η at $\bar{x} \in X$ if there exists a vector function $\eta(x,\bar{x})$ defined on $X \times X$ such that, for all $x \in X$,*

$$f(x) \leq f(\bar{x}) \Rightarrow \nabla f(\bar{x})\eta(x,\bar{x}) < 0.$$

This definition is a slight extension of that of the pseudoinvex functions. This class of functions does not contain the class of invex functions, but does contain the class of strictly pseudoinvex functions.

Every strictly pseudoinvex function is weak strictly pseudoinvex with respect to the same η. However, the converse is not necessarily true, as can be seen from the following example.

Example 2.6.1. The function $f = (f_1, f_2)$ defined on $X = R$, by $f_1(x) = x(x+2)$ and $f_2(x) = x(x+2)^2$ is weak strictly pseudoinvex function with respect to $\eta(x,\bar{x}) = x+2$ at $\bar{x} = 0$, but it s not strictly pseudoinvex with respect to the same $\eta(x,\bar{x})$ at $\bar{x}$ because for $\bar{x} = -2$, we have

$$f(x) \leqq f(\bar{x}) \text{ but } \nabla f(\bar{x})\,\eta(x,\bar{x}) = 0 \not< 0.$$

Definition 2.6.2. *f is said to be strong pseudoinvex with respect to η at $\bar{x} \in X$ if there exists a vector function $\eta(x,\bar{x})$ defined on $X \times X$ such that, for all $x \in X$,*

$$f(x) \leq f(\bar{x}) \Rightarrow \nabla f(\bar{x})\,\eta(x,\bar{x}) \leq 0.$$

Instead of the class of weak strictly pseudoinvex, the class of strong pseudoinvex functions does contain the class of invex functions. Also, every weak strictly pseudoinvex function is strong pseudoinvex with respect to the same η. However, the converse is not necessarily true, as can be seen from the following example.

Example 2.6.2. The function $f = (f_1, f_2)$ defined on $X = R$, by $f_1(x) = x^3$ and $f_2(x) = x(x+2)^2$ is strongly pseudoinvex function with respect to $\eta(x,\bar{x}) = x$ at $\bar{x} = 0$, but it is not weak strictly pseudoinvex with respect to the same $\eta(x,\bar{x})$ at $\bar{x}$ because for $\bar{x} = -1$

$$f(x) \leq f(\bar{x}) \text{ but } \nabla f(\bar{x})\,\eta(x,\bar{x}) = (0,-4)^T \not< 0,$$

also f is not invex with respect to the same η at $\bar{x}$, as can be seen by taking $\bar{x} = -2$.

There exist functions f that are pseudoinvex but not strong pseudoinvex with respect to the same η. Conversely, we can find functions that are strong pseudoinvex, but they are not pseudoinvex with respect to the same η.

Example 2.6.3. The function $f : R \to R^2$, defined by $f_1(x) = x(x-2)^2$ and $f_2(x) = x(x-3)$, is pseudoinvex with respect to $\eta(x,\bar{x}) = x - \bar{x}$ at $\bar{x} = 0$, but it is not weak strictly pseudoinvex with respect to the same $\eta(x,\bar{x})$ when $x = 2$.

Example 2.6.4. The function $f : R \to R^2$, defined by $f_1(x) = x(x-2)$ and $f_2(x) = x^2(x-1)$, is strong pseudoinvex with respect to $\eta(x,\bar{x}) = x - \bar{x}$ at $\bar{x} = 0$, but it is not pseudoinvex with respect to the same $\eta(x,\bar{x})$ at that point.

Remark 2.6.1. If f is both pseudoinvex and quasiinvex with respect to η at $\bar{x} \in X$, then it is strong pseudoinvex function with respect to the same η at $\bar{x}$.

Definition 2.6.3. *f is said to be weak quasiinvex with respect to η at $\bar{x} \in X$ if there exists a vector function $\eta(x,\bar{x})$ defined on $X \times X$ such that, for all $x \in X$,*

$$f(x) \leq f(\bar{x}) \Rightarrow \nabla f(\bar{x})\,\eta(x,\bar{x}) \leqq 0.$$

Every quasiinvex function is weak quasiinvex with respect to the same η. However, the converse is not necessarily true.

Example 2.6.5. Define a function $f : R \to R^2$, by $f_1(x) = x(x-2)^2$ and $f_2(x) = x^2(x-2)$, then the function is weak quasiinvex with respect to $\eta(x,\bar{x}) = x - \bar{x}$ at $\bar{x} = 0$, but it is not quasiinvex with respect to the same $\eta(x,\bar{x})$ at $\bar{x} = 0$, because $f(x) \leqq f(\bar{x})$ but $\nabla f(\bar{x})\,\eta(x,\bar{x}) \nleqq 0$, for $x = 2$.

Definition 2.6.4. *f is said to be weak pseudoinvex with respect to η at $\bar{x} \in X$ if there exists a vector function $\eta(x,\bar{x})$ defined on $X \times X$ such that, for all $x \in X$,*

$$f(x) < f(\bar{x}) \Rightarrow \nabla f(\bar{x})\,\eta(x,\bar{x}) \leq 0.$$

The class of weak pseudoinvex functions does contain the class of invex functions, pseudoinvex functions, strong pseudoinvex functions and strong quasiinvex functions.

Remark 2.6.2. Notice from Examples 2.6.1–2.6.4, that the concepts of weak strictly pseudoinvex, strong pseudoinvex, weak pseudoinvex, and pseudoinvex vector-valued functions are different, in general. However, they coincide in the scalar-valued case.

Definition 2.6.5. *f is said to be strong quasiinvex with respect to η at $\bar{x} \in X$ if there exists a vector function $\eta(x,\bar{x})$ defined on $X \times X$ such that, for all $x \in X$,*

$$f(x) \leqq f(\bar{x}) \Rightarrow \nabla f(\bar{x})\,\eta(x,\bar{x}) \leq 0.$$

Every strong quasiinvex function s both quasiivex and strong pseudoinvex with respect to the same η.

Aghezzaf and Hachimi (2001) introduced the class of weak prequasiinex functions by generalzing the class of preinvex (Ben-Israel and Mond 1986) and the class prequasiinvex functions (Suneja et al. 1993).

Definition 2.6.6. *We say that f is weak prequasiinvex at $\bar{x} \in X$ with respect to η if X is invex at $\bar{x}$ with respect to η and, for each $x \in X$,*

$$f(x) \leq f(\bar{x}) \Rightarrow f(\bar{x} + \lambda\eta(x,\bar{x})) \leqq f(\bar{x}), \quad 0 < \lambda \leqq 0.$$

Every prequasiinvex function is weak prequasiinvex with respect to the same η. But the converse is not true.

Example 2.6.6. The function $f : R \to R^2$, defined by $f_1(x) = x(x-2)^2$ and $f_2(x) = x(x-2)$, is weak prequasiivex at $\bar{x} = 0$ with respect to $\eta(x,\bar{x}) = x - \bar{x}$, but it is not prequasiinvex with respect to the same $\eta(x,\bar{x})$ at $\bar{x} = 0$, because $f_1(x) = x(x-2)^2$ is not prequasiinvex at $\bar{x} = 0$ with respect to same $\eta(x,\bar{x})$.

Springer

http://www.springer.com/978-3-540-85670-2

Generalized Convexity and Vector Optimization
Mishra, S.K.; Wang, S.; Lai, K.K.
2009, X, 294 p., Hardcover
ISBN: 978-3-540-85670-2