# Preface

A fundamental object of study in the theory of groups is the lower central series of groups whose terms are defined for a group $G$ inductively by setting

$$\gamma_1(G) = G, \quad \gamma_{n+1}(G) = [G,\ \gamma_n(G)] \qquad (n \geq 1),$$

where, for subsets $H,\ K$ of $G$, $[H,\ K]$ denotes the subgroup of $G$ generated by the commutators $[h,\ k] := h^{-1}k^{-1}hk$ for $h \in H$ and $k \in K$. The lower central series of free groups was first investigated by Magnus [Mag35]. To recall Magnus's work, let $F$ be a free group with basis $\{x_i\}_{i\in I}$ and $\mathcal{A} = \mathbb{Z}[[X_i \mid i \in I]]$ the ring of formal power series in the non-commuting variables $\{X_i\}_{i\in I}$ over the ring $\mathbb{Z}$ of integers. Let $\mathcal{U}(\mathcal{A})$ be the group of units of $\mathcal{A}$. The map $x_i \mapsto 1 + X_i,\ i \in I$, extends to a homomorphism

$$\theta : F \to \mathcal{U}(\mathcal{A}), \tag{1}$$

since $1 + X_i$ is invertible in $\mathcal{A}$ with $1 - X_i + X_i^2 - \cdots$ as its inverse. The homomorphism $\theta$ is, in fact, a monomorphism (Theorem 5.6 in [Mag66]). For $a \in \mathcal{A}$, let $a_n$ denote its homogeneous component of degree $n$, so that

$$a = a_0 + a_1 + \cdots + a_n + \cdots .$$

Define

$$\mathcal{D}_n(F) := \{f \in F \mid \theta(f)_i = 0,\ 1 \leq i < n\}, \quad n \geq 1.$$

It is easy to see that $\mathcal{D}_n(F)$ is a normal subgroup of $F$ and the series $\{\mathcal{D}_n(F)\}_{n\geq 1}$ is a central series in $F$, i.e., $[F,\ \mathcal{D}_n(F)] \subseteq \mathcal{D}_{n+1}(F)$ for all $n \geq 1$. Clearly, the intersection of the series $\{\mathcal{D}_n(F)\}_{n\geq 1}$ is trivial. Since $\{\mathcal{D}_n(F)\}_{n\geq 1}$

is a central series, we have $\gamma_n(F) \subseteq \mathcal{D}_n(F)$ for all $n \geq 1$. Thus, it follows that the intersection $\bigcap_{n\geq 1} \gamma_n(F)$ is trivial, i.e., $F$ is residually nilpotent.

Let $G$ be an arbitrary group and $R$ a commutative ring with identity. The *group ring* of $G$ over $R$, denoted by $R[G]$, is the $R$-algebra whose elements are the formal sums $\sum \alpha(g)g,\ g \in G,\ \alpha(g) \in R$, with only finitely many coefficients $\alpha(g)$ being non zero. The addition and multiplication in $R[G]$ are defined as follows:

$$\sum_{g\in G} \alpha(g)g + \sum_{g\in G} \beta(g)g = \sum_{g\in G} \big(\alpha(g) + \beta(g)\big)g,$$

$$\sum_{g\in G} \alpha(g)g \sum_{h\in G} \beta(h)h = \sum_{x\in G} \left(\sum_{gh=x} \alpha(g)\beta(h)\right)x.$$

The group $G$ can be identified with a subgroup of the group of units of $R[G]$, by identifying $g \in G$ with $1_R g$, where $1_R$ is the identity element of $R$, and it then constitutes an $R$-basis for $R[G]$. The map

$$\epsilon : R[G] \to R, \quad \sum \alpha(g)g \mapsto \sum \alpha(g),$$

is an algebra homomorphism and is called the *augmentation map*; its kernel is called the *augmentation ideal* of $R[G]$; we denote it by $\Delta_R(G)$. In the case when $R$ is the ring $\mathbb{Z}$ of integers, we refer to $\mathbb{Z}[G]$ as the integral group ring of $G$ and denote the augmentation ideal also by $\mathfrak{g}$, the corresponding Euler fraktur lowercase letter.

The augmentation ideal $\Delta_R(G)$ leads to the following filtration of $R[G]$:

$$R[G] \supseteq \Delta_R(G) \supseteq \Delta_R^2(G) \supseteq \ldots \supseteq \Delta_R^n(G) \supseteq \ldots . \tag{2}$$

Note that the subset $G \cap \big(1 + \Delta_R^n(G)\big)$, $n \geq 1$, is a normal subgroup of $G$; this subgroup is called the $n$th *dimension subgroup* of $G$ over $R$ and is denoted by $D_{n,R}(G)$. It is easy to see that $\{D_{n,R}(G)\}_{n\geq 1}$ is a central series in $G$, and therefore $\gamma_n(G) \subseteq D_{n,R}(G)$ for all $n \geq 1$. In the case when $R$ is the ring $\mathbb{Z}$ of integers, we drop the suffix $\mathbb{Z}$ and write $D_n(G)$ for $D_{n,\mathbb{Z}}(G)$. The quotients $\Delta_R^n(G)/\Delta_R^{n+1}(G),\ n \geq 1$, are $R[G]$-modules with trivial $G$-action. There then naturally arise the following problems about dimension subgroups and augmentation powers.

**Problem 0.1** *Identify the subgroups* $D_{n,R}(G) = G \cap \big(1 + \Delta_R^n(G)\big),\ n \geq 1.$

**Problem 0.2** *Describe the structure of the quotients* $\Delta_R^n(G)/\Delta_R^{n+1}(G), n \geq 1.$

**Problem 0.3** *Describe the intersection* $\cap_{n\geq 1}\Delta_R^n(G)$*; in particular, characterize the case when this intersection is trivial.*

In the case when $F$ is a free group, then, for all $n \geq 1$, $\gamma_n(F) \subseteq D_n(F) \subseteq \mathcal{D}_n(F)$. The homomorphism $\theta : F \to \mathcal{A}$, defined in (1), extends by linearity to the integral group ring $\mathbb{Z}[F]$ of the free group $F$; we continue to denote the extended map by $\theta$:

$$\theta : \mathbb{Z}[F] \to \mathcal{A}.$$

Let $\mathfrak{f}$ be the augmentation ideal of $\mathbb{Z}[F]$; then, for $\alpha \in \mathfrak{f}^n$, $\theta(\alpha)_i = 0,\ i \leq n-1$. With the help of free differential calculus, it can be seen that the intersection of the ideals $\mathfrak{f}^n$, $n \geq 1$, is zero and the homomorphism $\theta : \mathbb{Z}[F] \to \mathcal{A}$ is a monomorphism (see Chap. 4 in [Gru70]). A fundamental result about free groups ([Mag37], [Gru36], [Wit37]; see also [Röh85]) is that the inclusions $\gamma_n(F) \subseteq D_n(F) \subseteq \mathcal{D}_n(F)$ are equalities:

$$\gamma_n(F) = D_n(F) = \mathcal{D}_n(F), \text{ for all } n \geq 1. \tag{3}$$

This result exhibits a close relationship among the lower central series, the dimension series, and the powers of the augmentation ideal of the integral group ring of a free group. Thus, for free groups, Problems 1 and 3 have a definitive answer in the integral case. Problem 2 also has a simple answer in this case: for every $n \geq 1$, the quotient $\mathfrak{f}^n/\mathfrak{f}^{n+1}$ is a free abelian group with the set of elements $(x_{i_1} - 1)\dots(x_{i_n} - 1) + \mathfrak{f}^{n+1}$ as basis, where $x_{i_j}$ range over a basis of $F$ (see p. 116 in [Pas79]).

The foregoing results about free groups naturally raise the problem of investigation of the relationship among the lower central series $\{\gamma_n(G)\}_{n\geq 1}$, dimension series $\{D_{n,\,R}(G)\}_{n\geq 1}$, and augmentation quotients $\Delta_R^n(G)/\Delta_R^{n+1}(G)$, $n \geq 1$, of an arbitrary group $G$ over the commutative ring $R$. While these series have been extensively studied by various authors over the last several decades (see [Pas79], [Gup87c]), we are still far from a definitive theory. The most challenging case here is that when $R$ is the ring $\mathbb{Z}$ of integers, where a striking feature is that, unlike the case of free groups, the lower central series can differ from the dimension series, as first shown by Rips [Rip72].

Besides being purely of algebraic interest, lower central series and augmentation powers occur naturally in several other contexts, notably in algebraic K-theory, number theory, and topology. For example, the lower central series is the main ingredient of the theory of Milnor's $\bar{\mu}$-invariants of classical links [Mil57]; the lower central series and augmentation powers come naturally in [Cur71], [Gru80], [Qui69], and in the works of many other authors.

The main object of this monograph is to present an exposition of different methods related to the theory of the lower central series of groups, the dimension subgroups, and the augmentation powers. We will also be concerned with another important related series, namely, the derived series whose terms are defined, for a given group $G$, inductively by setting

$$\delta_0(G) = G, \quad \delta_{n+1}(G) = [\delta_n(G),\ \delta_n(G)] \text{ for } n \geq 0.$$

Our focus will be primarily on homological, homotopical, and combinatorial methods for the study of group rings. Simplicial methods, in fact, provide new possibilities for the theory of groups, Lie algebras, and group rings. For example, the derived functors of endofunctors on the category of groups come into play. Thus, working with simplicial objects and homotopy theory suggests new approaches for studying invariants of group presentations, a point of view which may be termed as "homotopical group theory." By homological group theory one normally means the study of properties of groups based on the properties of projective resolutions over their group rings. In contrast to this theory, by homotopical group theory we may understand the study of groups with the help of simplicial resolutions. From this point of view, homological group theory then appears as an abelianization of the homotopical one.

We now briefly describe the contents of this monograph.

Chapter 1: Lower central series We discuss examples and methods for investigating the lower central series of groups with a view to examining residual nilpotence, i.e., the property that this series intersects in the identity subgroup. We begin with Magnus's theorem [Mag35] on residual nilpotence of free groups and Gruenberg's [Gru57] result on free polynilpotent groups. Mal'cev's observation [Mal68] on the adjoint group of an algebra provides a method for constructing residually nilpotent groups. We consider next free products and describe Lichtman's characterization [Lic78] for the residual nilpotence of a free product of groups. If a group $G$ is such that the augmentation ideal $\mathfrak{g}$ of its integral group ring $\mathbb{Z}[G]$ is residually nilpotent, i.e., $\bigcap_{n\geq 0}\mathfrak{g}^n = 0$, then it is easily seen that $G$ is residually nilpotent. This property, namely, the residual nilpotence of $\mathfrak{g}$, has been characterized by Lichtman [Lic77]. Our next object is to discuss residual nilpotence of wreath products. We give a detailed account of Hartley's work, along with Shmelkin's theorem [Shm73] on verbal wreath products. For HNN-extensions we discuss a method introduced by Raptis and Varsos [Rap89]. Turning to linear groups, we give an exposition of recent work of Mikhailov and Bardakov [Bar07]. An interesting class of groups arising from geometric considerations is that of braid groups; we discuss the result of Falk and Randell [Fal88] on pure braid groups.

If $1 \to R \to F \to G \to 1$ is a free presentation of a group $G$, then the quotient group $R/[R, R]$ of $R$ can be viewed as a $G$-module, called a "relation module." The relationship between the properties of $F/[R, R]$ and those of $F/R$ has been investigated by many authors [Gru70], [Gru], [Gup87c]. We discuss a generalization of this notion. Let $R$ and $S$ be normal subgroups of the free group $F$. Then the quotient group $(R\cap S)/[R, S]$ is abelian and it can be viewed, in a natural way, as a module over $F/RS$. Clearly the relation modules, and more generally, the higher relation modules $\gamma_n(R)/\gamma_{n+1}(R),\ n \geq 2$, are all special cases of this construction. Such modules are related to the second homotopy module $\pi_2(X)$ of the standard complex $X$ associated to the free presentation $G \simeq F/R$. We discuss here the work in [Mik06b]. The main point of investigation here is the faithfulness of the $F/RS$-module $(R\cap S)/[R, S]$.

We next turn to $k$-central extensions, namely, the extensions

$$1 \to N \to \tilde{G} \to G \to 1,$$

where $N$ is contained in the $k$th central subgroup $\zeta_k(\tilde{G})$ of $\tilde{G}$. We examine the connection between residual nilpotence of $G$ and that of $\tilde{G}$. In general, neither implies the other.

The construction of the lower central series $\{\gamma_n(G)\}_{n\geq 1}$ of a group $G$ can be extended in an obvious way to define the transfinite terms $\gamma_\tau(G)$ of the lower central series for infinite ordinals $\tau$. Let $\omega$ denote the first non-finite ordinal. The groups $G$ whose lower central series has the property that $\gamma_\omega(G) \neq \gamma_{\omega+1}(G)$, called groups with long lower central series, are of topological interest. We discuss methods for constructing such groups.

It has long been known that the Schur multiplicator is related to the study of the lower central series. We explore a similar relationship with generalized multiplicators, better known as Baer invariants of free presentations of groups. In particular, we discuss generalized Dwyer filtration of Baer invariants and its relation with the residual nilpotence of groups. Using a generalization of the Magnus embedding, we shall see that every 2-central extension of a one-relator residually nilpotent group is itself residually nilpotent [Mik07a].

The residually nilpotent groups with the same lower central quotients as some free group are known as para-free groups. Non-free para-free groups were first discovered by Baumslag [Bau67]. We make some remarks related to the para-free conjecture, namely, the statement that a finitely generated para-free group has trivial Schur multiplicator.

Next we study the nilpotent completion $Z_\infty(G)$, which is the inverse limit of the system of epimorphisms $G \to G/\gamma_n(G),\ n \geq 1$, and certain subgroups of this completion. We study the Bousfield–Kan completion $R_\infty X$ of a simplicial set $X$ over a commutative ring $R$, and homological localization (called HZ-localization) functor $L : G \mapsto L(G)$ on the category of groups, due to Bousfield, in particular, the uncountability of the Schur multiplicator of the free nilpotent completion of a non abelian free group. After discussing the homology of the nilpotent completion, we go on to study transfinite para-free groups. Given an ordinal number $\tau$, a group $G$ is defined to be $\tau$-para-free if there exists a homomorphism $F \to G$, with $F$ free, such that

$$L(F)/\gamma_\tau\big(L(F)\big) \simeq L(G)/\gamma_\tau\big(L(G)\big).$$

Our final topic of discussion in this chapter is the study of the lower central series and the homology of crossed modules. These modules were first defined by Whitehead [Whi41]. With added structure to take into account, the computations naturally become rather more complicated. It has recently been shown that the cokernel of a non aspherical projective crossed module with a free base group acts faithfully on its kernel [Mik06a]; we give an exposition of this and some other related results.

**Chapter 2: Dimension subgroups** In this chapter we study various problems concerning the dimension subgroups. The relationship between the lower central and the dimension series of groups is highly intriguing.

For every group $G$ and integer $n \geq 1$, we have

$$\gamma_n(G) \subseteq D_n(G) := G \cap (1 + \mathfrak{g}^n).$$

As first shown by Rips [Rip72], equality does not hold in general. We pursue the counter example of Rips and the subsequent counterexamples constructed by N. Gupta, and construct several further examples of groups without the dimension property, i.e., groups where the lower central series does not coincide with the integral dimension series. We construct a 4-generator and 3-relator example of a group $G$ with $D_4(G) \neq \gamma_4(G)$ and show further that, in a sense, this is a minimal counter example by proving that every 2-relator group $G$ has the property that $D_4(G) = \gamma_4(G)$. At the moment it seems to be an intractable problem to compute the length of the dimension series of a finite 2-group of class 3. However, we show that for the group without the dimension property considered by Gupta and Passi (see p. 76 in [Gup87c]), the fifth dimension subgroup is trivial. Examples of groups with $D_n(G) \neq \gamma_n(G)$ with $n \geq 5$ were first constructed by Gupta [Gup90]. In this direction, for each $n \geq 5$, we construct a 5-generator 5-relator group $\mathfrak{G}_n$ such that $D_n(\mathfrak{G}_n) \neq \gamma_n(\mathfrak{G}_n)$. We also construct a nilpotent group of class 4 with non trivial sixth dimension subgroup. We hope that these examples will lead to a closer understanding of groups without the dimension property.

For each $n \geq 4$, in view of the existence of groups with $\gamma_n(G) \neq D_n(G)$, the class $\mathcal{D}_n$ of groups with trivial $n$th dimension subgroup is not a variety. This class is, however, a quasi-variety [Plo71]. We present an account of our work [Mik06c] showing that the quasi-variety $\mathcal{D}_4$ is not finitely based, thus answering a problem of Plotkin.

We next review the progress on the identification of integral dimension subgroups and on Plotkin's problems about the length of the dimension series of nilpotent groups.

Related to the dimension subgroups are the Lie dimension subgroups $D_{[n]}(G)$ and $D_{(n)}(G)$, $n \geq 1$, with $D_{[n]}(G) \subseteq D_{(n)}(G) \subseteq D_n(G)$ (see Sect. 2.10 for definitions). We show that, for every natural number $s$, there exists a group of class $n$ such that $D_{[n+s]}(G) \neq \gamma_{n+s}(G) \neq 1$. In contrast with the integral case, many more definitive results are known about the dimension subgroups and the Lie dimension subgroups over fields. We review these results and their applications, in particular, to the study of Lie nilpotency indices of the augmentation ideals.

**Chapter 3: Derived series** We study the derived series of free nilpotent groups. Combinatorial methods developed for the study of the dimension subgroups can be employed for this purpose. We give an exposition of the work of Gupta and Passi [Gup07].

Homological methods were first effectively applied by Strebel [Str74] for the investigation of the derived series of groups. We give an exposition of Strebel's work and then apply it to study properties analogous to those studied in Chap. 1 for the lower central series of groups; in particular, we study the behaviour of the transfinite terms of the derived series. We show that this study has an impact on Whitehead's asphericity question which asks whether every sub complex of an aspherical two-dimensional complex is itself aspherical.

Chapter 4: Augmentation powers The structure of the augmentation powers $\mathfrak{g}^n$ and the quotients $\mathfrak{g}^n/\mathfrak{g}^{n+1}$ for the group $G$ is of algebraic and number-theoretic interest. While the augmentation powers have been investigated by several authors, a solution to this problem, when $G$ is torsion free or torsion abelian group, has recently been given by Bak and Tang [Bak04]. We give an account of the main features of this work. We then discuss transfinite augmentation powers $\mathfrak{g}^\tau$, where $\tau$ is any ordinal number. We next give an exposition of some of Hartley's work on the augmentation powers.

A filtration $\{P_nH^2(G, \mathbb{T})\}_{n\geq 0}$ of the Schur multiplicator $H^2(G, \mathbb{T})$ arising from the notion of the polynomial 2-cocycles is a useful tool for the investigation of the dimension subgroups. This approach to dimension subgroups naturally leads to relative dimension subgroups $D_n(E, N) := E\cap(1+\mathfrak{e}^n+\mathfrak{n}\mathfrak{e})$, where $N$ is a normal subgroup of the group $E$. The relative dimension subgroups provide a generalization of dimension subgroups, and have been extensively studied by Hartl [Har08]. Using the above-mentioned filtration of the Schur multiplicator, Passi and Stammbach [Pas74] have given a characterization of para-free groups. These ideas were further developed in [Mik04, Mik05b]. We give here an account of this (co)homological approach for the study of subgroups determined by two-sided ideals in group rings.

In analogy with the notion of HZ-localization of groups, we study the Bousfield HZ localization of modules. Given a group $G$, a $G$-module homomorphism $f : M \to N$ is said to be an HZ-map if the maps $f_0 : H_0(G, M) \to H_0(G, N)$ and $f_1 : H_1(G, M) \to H_1(G, N)$ induced on the homology groups are such that $f_0$ is an epimorphism and $f_1$ is an isomorphism. In the category of $G$-modules, we examine the localization of $G$-modules with respect to the class $\mathcal{HZ}$ of HZ maps. We discuss the work of Brown and Dror [Bro75] and of Dwyer [Dwy75] where the relation between the HZ localization of a module $M$ and its $\mathfrak{g}$-adic completion $\varprojlim M/\mathfrak{g}^nM$ has been investigated.

Chapter 5: Homotopical aspects After recalling the construction of certain functors, we give an exposition of the work of Curtis [Cur63] on the lower central series of simplicial groups. Of particular interest to us are his two spectral sequences of homotopy exact couples arising from the application of the lower central and the augmentation power functors, $\gamma_n$ and $\Delta^n$, respectively, to a simplicial group. Our study is motivated by the work Stallings [Sta75] where he suggested a program and pointed out the main problem in its pursuit:

> Finally, Rips has shown that there is a difference between the "dimension subgroups" and the terms of the lower central series ... . The problem would be, how to compute with the Curtis spectral sequence, at least to the point of going through the Rips example in detail?

To some extent, this program was realized by Sjögren [Sjo79]. However, in general, it is an open question whether there exists any homotopical role of the groups without the dimension property. The simplicial approach for the investigation of the relationship between the lower central series and the augmentation powers of a group has been studied by Gruenenfelder [Gru80]. We compute certain initial terms of the Curtis spectral sequences and, following [Har08], discuss applications to the identification of dimension subgroups. We show that it is possible to place some of the known results on the dimension subgroups in a categorical setting, and thus hope that this point of view might lead to a deeper insight. One of the interesting features of the simplicial approach is the connection that it provides to the derived functors of non-additive functors in the sense of Dold and Puppe [Dol61]. Finally, we present a number of homotopical applications; in particular, we give an algebraic proof of the computation of the low-dimensional homotopy groups of the 2-sphere. Another application that we give is an algebraic proof of the well-known theorem, due to Serre [Ser51], about the $p$-torsion of the homotopy groups of the 2-sphere.

Chapter 6: Miscellanea In this concluding chapter we present some applications of the group ring construction in different, rather unexpected, contexts. As examples, we may mention here (1) the solution, due to Lam and Leung [Lam00], of a problem in number theory which asks, for a given natural number $m$, the computation of the set $W(m)$ of all possible integers $n$ for which there exist $m$th roots of unity $\alpha_1, \ldots, \alpha_n$ in the field $\mathbb{C}$ of complex numbers such that $\alpha_1 + \cdots + \alpha_n = 0$, and (2) application of dimension subgroups by Massuyeau [Mas07] in low-dimensional topology.

Appendix At several places in the text we need results about simplicial objects. Rather than interrupt the discussion at each such point, we have preferred to collect the results needed in an appendix to which the reader may refer as and when necessary. The material in the appendix is mainly that which is needed for the group-theoretic problems in hand. For a more detailed exposition of simplicial methods, the reader is referred to the books of Goerss and Jardine [Goe99] and May [May67].

To conclude, we may mention that a crucial point that emerges from the study of the various series carried out in this monograph is that the least transfinite step in all cases is the one which at the moment deserves most to be understood from the point of view of applications.

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