Chapter II

NONHOLONOMIC SYSTEMS

From the analog of Newton's law, Maggi's equations are deduced which are the most convenient equations of the nonholonomic mechanics. From Maggi's equations the most useful forms of equations of motion of nonholonomic systems are obtained. The connection between Maggi's equations and the Suslov–Jourdain principle is considered. The notion of ideal nonholonomic constraints is discussed. In studying nonholonomic systems the approach, applied in Chapter I to analysis of the motion of holonomic systems, is employed. The role of of Chetaev's type constraints for the development of nonholonomic mechanics is considered. For the solution of a number of nonholonomic problems, the different methods are applied.

§ 1. Nonholonomic constraint reaction

Consider the Cartesian coordinates $Ox_1x_2x_3$ with the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$. If on the motion of material point of the mass m it is imposed the nonholonomic constraint

$$\varphi(t, x, \dot{x}) = 0\,, \qquad x = (x_1, x_2, x_3)\,, \tag{1.1}$$

then the second Newton's law can be represented as

$$m\mathbf{w} = \mathbf{F} + \mathbf{R}'\,, \tag{1.2}$$

where $\mathbf{F} = (X_1, X_2, X_3)$ is an active force, acting on the point, and $\mathbf{R}' = (R'_1, R'_2, R'_3)$ is constraint reaction (1.1).

Consider the vector $\mathbf{R}'$. We differentiate equation of constraint (1.1) with respect to time:

$$\dot{\varphi} \equiv \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial x_k}\dot{x}_k + \frac{\partial\varphi}{\partial\dot{x}_k}\ddot{x}_k = 0\,, \qquad k = 1, 2, 3\,. \tag{1.3}$$

Together with the usual vector $\boldsymbol{\nabla}\varphi = \dfrac{\partial\varphi}{\partial x_k}\mathbf{i}_k$ we introduce the new vector $\boldsymbol{\nabla}'\varphi$ proposed by N. N. Polyakhov [185]:

$$\boldsymbol{\nabla}'\varphi = \frac{\partial\varphi}{\partial\dot{x}_k}\mathbf{i}_k\,.$$

Then equation (1.3) can be rewritten as

$$\frac{\partial\varphi}{\partial t} + \boldsymbol{\nabla}\varphi\cdot\mathbf{v} + \boldsymbol{\nabla}'\varphi\cdot\mathbf{w} = 0\,. \tag{1.4}$$

Multiplying scalarly equation (1.2) by $\boldsymbol{\nabla}'\varphi$ and equation (1.4) by m, we obtain

$$\mathbf{R}' \cdot \boldsymbol{\nabla}'\varphi = -m\Big(\frac{\partial\varphi}{\partial t} + \boldsymbol{\nabla}\varphi \cdot \mathbf{v}\Big) - \mathbf{F} \cdot \boldsymbol{\nabla}'\varphi\,.$$

This implies that the vector $\mathbf{R}'$ can be represented in the form

$$\mathbf{R}' = \Lambda\boldsymbol{\nabla}'\varphi + \mathbf{T}_0 = \mathbf{N} + \mathbf{T}_0\,,$$
$$\Lambda = -\frac{m\dfrac{\partial\varphi}{\partial t} + m\boldsymbol{\nabla}\varphi \cdot \mathbf{v} + \mathbf{F} \cdot \boldsymbol{\nabla}'\varphi}{|\boldsymbol{\nabla}'\varphi|^2}\,, \qquad \mathbf{T}_0 \cdot \mathbf{N} = 0\,. \tag{1.5}$$

Note that the only component $\mathbf{N}$ of constraint reaction depends on (1.1), in which case by formulas (1.5) it is defined as a certain function of $t, x, \dot{x}$. In particular, equations (1.1) and (1.2) are also valid for $\mathbf{T}_0 = 0$. The nonholonomic constraints of such type we shall called *ideal*. If $\mathbf{T}_0 \neq 0$, then the construction of the vector $\mathbf{T}_0$ should be described separately, based on the additional characteristics of the physical realization of constraint (1.1). As a rule, $\mathbf{T}_0$ essentially depends on the quantities $|\mathbf{N}|$ and, in lesser degree, on $t, x, \dot{x}$.

Consider the partial case of holonomic constraint, namely

$$f(t,x) = 0\,. \tag{1.6}$$

Represent it in the form of (1.1):

$$\varphi \equiv \dot{f} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_k}\dot{x}_k = 0\,.$$

Then we have

$$\frac{\partial\varphi}{\partial\dot{x}_k} = \frac{\partial f}{\partial x_k}\,,$$

and therefore for holonomic constraint (1.6) the vector $\boldsymbol{\nabla}'\varphi$, introduced above, coincides with the usual vector $\boldsymbol{\nabla}f$. Here, as is shown in Chapter I, the vector $\mathbf{N}$ is directed along a normal to the surface, given by equation (1.6), and the vector $\mathbf{T}_0$ lies in the plane tangential to this surface. In particular, if equation (1.6) gives a certain material surface, on which the point must move, then for the ideally burnished surface (for ideal holonomic constraint) we have $\mathbf{T}_0 = 0$. Otherwise we need to point out a rule for construction of the vector $\mathbf{T}_0$, for example, to give Coulomb's law (1.12) from Chapter I.

Assume now that on the motion of material point it is imposed two nonholonomic constraints

$$\varphi^{\varkappa}(t,x,\dot{x}) = 0\,, \qquad x = (x_1, x_2, x_3)\,, \qquad \varkappa = 1,2\,.$$

Arguing as above, we obtain

$$\frac{\partial\varphi^{\varkappa}}{\partial t} + \boldsymbol{\nabla}\varphi^{\varkappa} \cdot \mathbf{v} + \boldsymbol{\nabla}'\varphi^{\varkappa} \cdot \mathbf{w} = 0\,, \qquad \varkappa = 1,2\,.$$

The differential equation of motion has, as before, the form of (1.2). This law permits us to eliminate the vector $\mathbf{w}$ from the previous relations and to write them as

$$\mathbf{R}' \cdot \boldsymbol{\nabla}' \varphi^{\varkappa} \equiv R'^{\varkappa} = -\Big(m \frac{\partial \varphi^{\varkappa}}{\partial t} + m \boldsymbol{\nabla} \varphi^{\varkappa} \cdot \mathbf{v} + \mathbf{F} \cdot \boldsymbol{\nabla}' \varphi^{\varkappa} \Big) , \qquad \varkappa = 1, 2 .$$

This implies that if we represent the vector $\mathbf{R}'$ as the sum

$$\mathbf{R}' = \Lambda_{\varkappa} \boldsymbol{\nabla}' \varphi^{\varkappa} + \mathbf{T}_0 , \tag{1.7}$$

where $\mathbf{T}_0$ is a certain unknown vector orthogonal to the vectors $\boldsymbol{\nabla}' \varphi^{\varkappa}$, then the coefficients $\Lambda_{\varkappa}$ can be found from the following system of equations

$$\begin{aligned} \Lambda_1 |\boldsymbol{\nabla}' \varphi^1|^2 + \Lambda_2 \boldsymbol{\nabla}' \varphi^1 \cdot \boldsymbol{\nabla}' \varphi^2 &= R'^1 , \\ \Lambda_1 \boldsymbol{\nabla}' \varphi^1 \cdot \boldsymbol{\nabla}' \varphi^2 + \Lambda_2 |\boldsymbol{\nabla}' \varphi^2|^2 &= R'^2 . \end{aligned}$$

Thus, the components $\Lambda_{\varkappa} \boldsymbol{\nabla}' \varphi^{\varkappa}$ of the vector $\mathbf{R}'$ are uniquely defined by equations of constraints and the force $\mathbf{F}$.

We remark that a similar reasoning can also be used for two holonomic constraints since in this case we have $\boldsymbol{\nabla}' \varphi^{\varkappa} = \boldsymbol{\nabla} f^{\varkappa}$. Therefore if there are the two holonomic constraints, then the reaction can be represented as

$$\mathbf{R}' = \Lambda_{\varkappa} \boldsymbol{\nabla} f^{\varkappa} + \mathbf{T}_0 , \qquad \varkappa = 1, 2 .$$

We consider now the motion of representation point under the condition that there exist k nonholonomic constraints:

$$\varphi^{\varkappa}(t, y, \dot{y}) = 0 , \qquad \varkappa = \overline{1, k} . \tag{1.8}$$

Then like the motion of one material point we can write

$$M \mathbf{W} = \mathbf{Y} + \mathbf{R} , \tag{1.9}$$

which has the form of the second Newton's law in the vector form. In the sequel relation (1.9) is called the second Newton's law just as in Chapter I.

Using formula (1.7) in the case of representation point, we have

$$\mathbf{R} = \mathbf{N} + \mathbf{T}_0 , \quad \mathbf{N} = \Lambda_{\varkappa} \boldsymbol{\nabla}' \varphi^{\varkappa} , \quad \boldsymbol{\nabla}' \varphi^{\varkappa} = \frac{\partial \varphi^{\varkappa}}{\partial \dot{y}_{\mu}} \mathbf{j}_{\mu} , \quad \mathbf{T}_0 \cdot \mathbf{N} = 0 . \tag{1.10}$$

In Chapter IV the notion of a tangent space to the manifold of all possible configurations of a mechanical system will be introduced. The set of the Lagrange equations of the second kind describing a motion of the unconstrained mechanical system is written in this space as a single vector-valued equality which has a form of the second Newton law. This makes it possible to generalize formulas (1.5), (1.7) not only to mechanical systems, consisting of the finite number of particles, but also to arbitrary mechanical systems with the finite number of degrees of freedom.

Thus, the conclusion on the structure of the constraint reaction obtained for a single particle is of general nature. It is fundamental. Expressions (1.5), (1.7), (1.10) for the reaction force were obtained by N. N. Polyakhov in 1974 [185]. Later these results were included into the treatise (manual for universities) [189]. In 2001 O. M. O'Reilly and A. R. Srinivasa [416] devoted their work to deriving and discussion of expressions (1.5).

§ 2. Equations of motion of nonholonomic systems. Maggi's equations

Assume that the nonlinear nonholonomic constraints, imposed on a motion of system, in the curvilinear coordinates $q = (q^1, \dots, q^s)$ have the form

$$\varphi^{\varkappa}(t, q, \dot{q}) = 0\,, \qquad \varkappa = \overline{1, k}\,. \tag{2.1}$$

In the case of the motion of system of N material points $s = 3N$.

Now we pass from the variables $\dot{q} = (\dot{q}^1, \dots, \dot{q}^s)$ to the new nonholonomic variables $v_* = (v_*^1, \dots, v_*^s)$ by formulas

$$v_*^\rho = v_*^\rho(t, q, \dot{q})\,, \qquad \rho = \overline{1, s}\,. \tag{2.2}$$

If the solvability conditions are satisfied, then we can write the inverse transformation

$$\dot{q}^\sigma = \dot{q}^\sigma(t, q, v_*)\,, \qquad \sigma = \overline{1, s}\,. \tag{2.3}$$

Assuming that the derivatives of functions (2.2), (2.3) are continuous, we can introduce the two systems of linearly independent vectors:

$$\boldsymbol{\varepsilon}_\tau = \frac{\partial \dot{q}^\sigma}{\partial v_*^\tau}\mathbf{e}_\sigma\,, \quad \boldsymbol{\varepsilon}^\rho = \frac{\partial v_*^\rho}{\partial \dot{q}^\tau}\mathbf{e}^\tau\,, \qquad \rho, \tau = \overline{1, s}\,. \tag{2.4}$$

Since the product is as follows

$$\boldsymbol{\varepsilon}^\rho \cdot \boldsymbol{\varepsilon}_\tau = \frac{\partial v_*^\rho}{\partial \dot{q}^\sigma}\frac{\partial \dot{q}^\sigma}{\partial v_*^\tau} = \delta_\tau^\rho = \begin{cases} 0\,, & \rho \neq \tau\,, \\ 1\,, & \rho = \tau\,, \end{cases}$$

vectors (2.4) can be regarded as the vectors of the fundamental and reciprocal bases. We shall say that bases (2.4) are nonholonomic.

By assumption, the equations of constrains (2.1) are such that

$$|\nabla' \varphi^{\varkappa} \cdot \nabla' \varphi^{\varkappa^*}| \neq 0\,, \qquad \varkappa, \varkappa^* = \overline{1, k}\,.$$

In this case in transition formulas (2.2) the last functions can be given in the following way

$$v_*^{l+\varkappa} = \varphi^{\varkappa}(t, q, \dot{q})\,, \qquad l = s - k\,, \quad \varkappa = \overline{1, k}\,. \tag{2.5}$$

Therefore if constraint (2.1) is satisfied, then we have $v_*^{l+\varkappa} = 0$. Then by formulas (2.4) we have

$$\boldsymbol{\varepsilon}^{l+\varkappa} = \frac{\partial \varphi^\varkappa}{\partial \dot{q}^\tau} \mathbf{e}^\tau \equiv \boldsymbol{\nabla}' \varphi^\varkappa \,, \qquad \varkappa = \overline{1, k} \,.$$

We introduce two subspaces orthogonal to each other with the nonholonomic bases $\{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_l\}$ and $\{\boldsymbol{\varepsilon}^{l+1}, \ldots, \boldsymbol{\varepsilon}^s\}$ and call them L-space and K-space, respectively. Decompose the acceleration vector into the following two components

$$\mathbf{W} = \mathbf{W}_L + \mathbf{W}^K \,, \quad \mathbf{W}_L = \widetilde{W}^\lambda \boldsymbol{\varepsilon}_\lambda \,, \quad \mathbf{W}^K = \widetilde{W}_{l+\varkappa} \boldsymbol{\varepsilon}^{l+\varkappa} \,, \quad \mathbf{W}_L \cdot \mathbf{W}^K = 0 \,.$$

Here the wavy sign denotes that the components of acceleration vector are taken for nonholonomic bases (2.4) but not for the usual fundamental and reciprocal bases.

The second Newton's law (1.9) is replaced then by the following two equations:

$$M\mathbf{W}_L = \mathbf{Y}_L + \mathbf{R}_L \,, \tag{2.6}$$

$$M\mathbf{W}^K = \mathbf{Y}^K + \mathbf{R}^K \,. \tag{2.7}$$

Differentiating the equations of constraints (2.1) with respect to time and taking into account that the vector $\mathbf{W}$ can be represented as

$$\mathbf{W} = (\ddot{q}^\sigma + \Gamma^\sigma_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta) \mathbf{e}_\sigma \,, \qquad \sigma = \overline{1, s} \,, \quad \alpha, \beta = \overline{0, s} \,,$$

we obtain

$$\boldsymbol{\varepsilon}^{l+\varkappa} \cdot \mathbf{W} = \chi_1^\varkappa(t, q, \dot{q}) \,,$$
$$\chi_1^\varkappa(t, q, \dot{q}) = -\frac{\partial \varphi^\varkappa}{\partial t} - \frac{\partial \varphi^\varkappa}{\partial q^\sigma} \dot{q}^\sigma + \frac{\partial \varphi^\varkappa}{\partial \dot{q}^\sigma} \Gamma^\sigma_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta \,,$$
$$\varkappa = \overline{1, k} \,, \qquad \alpha, \beta = \overline{0, s} \,.$$

These equations are similar to equations (2.14) of Chapter I. This implies that the vector $\mathbf{W}^K$ as the function of $t, q^\sigma, \dot{q}^\sigma$, $\sigma = \overline{1, s}$ is uniquely determined by constraints equations. By equation (2.7) for the given force $\mathbf{Y}^K$ the vector $\mathbf{W}^K$ can be obtained by means of the constraint reaction $\mathbf{R}^K = \mathbf{N} = \Lambda_\varkappa \boldsymbol{\nabla}' \varphi^\varkappa$. Unlike the above the component $\mathbf{W}_L$ is independent of the mathematical definition of the equations of constraints. It can be determined from equation (2.6) for any vector $\mathbf{R}_L$, in particular, for $\mathbf{R}_L \equiv \mathbf{T}_0 = 0$ if in L-space the equation of properly motion has the form

$$M\mathbf{W}_L = \mathbf{Y}_L \,.$$

It is naturally to call nonholonomic constraints (2.1), which do not influence the vector $\mathbf{W}_L$, ideal. For these constraints the vector of reaction is as follows

$$\mathbf{R} = \mathbf{R}^K = \mathbf{N} = \Lambda_\varkappa \boldsymbol{\nabla}' \varphi^\varkappa \,. \tag{2.8}$$

By formulas (1.9) and (2.8) the second Newton's law for ideal nonholonomic constraints has the form

$$M\mathbf{W} = \mathbf{Y} + \Lambda_{\varkappa} \boldsymbol{\nabla}' \varphi^{\varkappa} \,. \tag{2.9}$$

Multiplying this equation by the vectors $\boldsymbol{\varepsilon}_\lambda$, $\lambda = \overline{1, l}$, we obtain *Maggi's equations*

$$\big(MW_\sigma - Q_\sigma\big) \frac{\partial \dot{q}^\sigma}{\partial v_*^\lambda} = 0 \,, \qquad \lambda = \overline{1, l} \,, \tag{2.10}$$

where

$$MW_\sigma - Q_\sigma = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\sigma} - \frac{\partial T}{\partial q^\sigma} - Q_\sigma \,, \qquad \sigma = \overline{1, s} \,.$$

For linear nonholonomic constraints these equations have been obtained by Maggi in 1896 [355]. Later for nonlinear nonholonomic constraints, by means of the generalized principle of D'Alembert–Lagrange they have been generated by A. Przeborski [375]. Integrating equations (2.1), (2.10) with the given initial data, we can find the law of motion of the system

$$q^\sigma = q^\sigma(t) \,, \qquad \sigma = \overline{1, s} \,. \tag{2.11}$$

Multiplying equation (2.9) by the vectors $\boldsymbol{\varepsilon}_{l+\varkappa}$, $\varkappa = \overline{1, k}$, we obtain the second group of Maggi's equations

$$\big(MW_\sigma - Q_\sigma\big) \frac{\partial \dot{q}^\sigma}{\partial v_*^{l+\varkappa}} = \Lambda_\varkappa \,, \qquad \varkappa = \overline{1, k} \,. \tag{2.12}$$

For the given law of motion of system (2.11), the generalized reactions of nonholonomic constraints (2.1) can be determined as the time functions from the above equations. Formulas (2.12) do not give directly the quantities Λ_κ as the functions of $t, q, \dot{q}$. They can be found from the following equations

$$\boldsymbol{\varepsilon}^{l+\varkappa} \cdot \mathbf{W} = \chi_1^\varkappa(t, q, \dot{q}), \quad \mathbf{W}^K = \frac{1}{M} (\mathbf{Y}^K + \Lambda_\varkappa \boldsymbol{\nabla}' \varphi^\varkappa) \,.$$

Thus, for nonholonomic constraints the introduction of nonholonomic bases (2.4) permits us to obtain the two subspaces K and L. These subspaces turn out orthogonal to each other and in studying the problems in these subspaces it is convenient to make use of Maggi's equations (2.10) and (2.12).

Maggi's equations are highly convenient to consider the motion of nonholonomic systems. It is to be noted that they are valid for any nonholonomic constraints, including the nonlinear ones. Most of the well-known forms of equations, describing the motion of nonholonomic systems, can be obtained from these equations (for detail, see the next section), for example, *Chaplygin's equations*

$$\begin{gathered} \frac{d}{dt} \frac{\partial (T)}{\partial \dot{q}^\lambda} - \frac{\partial (T)}{\partial q^\lambda} + \frac{\partial T}{\partial \dot{q}^{l+\varkappa}} \left(\frac{\partial b_{\lambda^*}^{l+\varkappa}}{\partial q^\lambda} - \frac{\partial b_\lambda^{l+\varkappa}}{\partial q^{\lambda^*}} \right) \dot{q}^{\lambda^*} = Q_\lambda \,, \\ \lambda, \lambda^* = \overline{1, l} \,, \qquad \varkappa = \overline{1, k} \,, \end{gathered} \tag{2.13}$$

if the equations of constraints (2.1) are represented as

$$\dot q^{l+\varkappa} = b_\lambda^{l+\varkappa}(q^1, \dots, q^l)\dot q^\lambda\,, \qquad \lambda = \overline{1,l}\,, \quad \varkappa = \overline{1,k}\,, \tag{2.14}$$

or *the Hamel–Boltzmann equations*

$$\begin{gathered} \frac{d}{dt}\frac{\partial T^*}{\partial \dot\pi^\lambda} - \frac{\partial T^*}{\partial \pi^\lambda} + \frac{\partial T^*}{\partial \dot\pi^{l+\varkappa}}\gamma_{\lambda(l+\varkappa)\lambda^*}\dot\pi^{\lambda^*} = \widetilde Q_\lambda\,, \\ \lambda, \lambda^* = \overline{1,l},, \qquad \varkappa = \overline{1,k}\,, \qquad l = s-k\,, \end{gathered} \tag{2.15}$$

for the nonholonomic constraints of the form

$$\varphi^\varkappa(t,q,\dot q) \equiv a_\sigma^{l+\varkappa}(q)\dot q^\sigma = 0, \qquad \varkappa = \overline{1,k}\,, \quad \sigma = \overline{1,s}\,, \tag{2.16}$$

if in place of formulas (2.2), (2.3) there are introduced the following relations between the time derivatives of the generalized coordinates $q^1, \dots, q^s$ and of the quasicoordinates $\pi^1, \dots, \pi^s$:

$$\dot\pi^\rho = a_\sigma^\rho(q)\dot q^\sigma\,, \quad \dot q^\sigma = b_\rho^\sigma(q)\dot\pi^\rho\,, \qquad \rho, \sigma = \overline{1,s}\,. \tag{2.17}$$

In Chaplygin's equations the symbol (T) denotes, as usual [59], the kinetic energy in which the generalized velocities $\dot q^{l+\varkappa}$, $\varkappa = \overline{1,k}$, are replaced by relations (2.14). Similarly, in the Hamel–Boltzmann equations T^* denotes the kinetic energy if in it the quantities $\dot q^\sigma$, $\sigma = \overline{1,s}$, are replaced by their relations in unknowns $\dot\pi^\rho$, $\rho = \overline{1,s}$. Recall that the analytic representations of nonholonomic constraints (2.16) give k last quasivelocities $\dot\pi^{l+1}, \dots, \dot\pi^s$ in formulas (2.17). Besides, equations (2.15) involves the generalized forces $\widetilde Q_\lambda$, which correspond to the quasivelocities $\dot\pi^\lambda(\lambda = \overline{1,l})$:

$$\widetilde Q_\lambda = Q_\sigma \frac{\partial \dot q^\sigma}{\partial \dot\pi^\lambda}\,, \qquad \lambda = \overline{1,l}\,, \quad \sigma = \overline{1,s}\,, \tag{2.18}$$

and the objects of nonholonomicity $\gamma_{\lambda(l+\varkappa)\lambda^*}$

$$\begin{gathered} \gamma_{\lambda(l+\varkappa)\lambda^*} = b_\lambda^\sigma b_{\lambda^*}^\tau \left(\frac{\partial a_\sigma^{l+\varkappa}}{\partial q^\tau} - \frac{\partial a_\tau^{l+\varkappa}}{\partial q^\sigma} \right), \\ \lambda, \lambda^* = \overline{1,l}\,, \qquad \varkappa = \overline{1,k}\,, \qquad \sigma, \tau = \overline{1,s}\,. \end{gathered} \tag{2.19}$$

The derivatives $\partial T^*/\partial \pi^\lambda$ are computed by formulas

$$\frac{\partial T^*}{\partial \pi^\lambda} = \frac{\partial T^*}{\partial q^\sigma}\frac{\partial \dot q^\sigma}{\partial \dot\pi^\lambda}\,, \qquad \lambda = \overline{1,l}\,, \quad \sigma = \overline{1,s}\,. \tag{2.20}$$

The following equations [169, 314]

$$\frac{d}{dt}\frac{\partial T^*}{\partial v_*^\lambda} - \frac{\partial T^*}{\partial \pi^\lambda} - \frac{\partial T}{\partial \dot q^\sigma}\left(\frac{d}{dt}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} - \frac{\partial \dot q^\sigma}{\partial \pi^\lambda} \right) = \widetilde Q_\lambda\,, \qquad \lambda = \overline{1,l}$$

are more general than Chaplygin's equations. Whence it follows that in the case of linear stationary transformations (2.17) with respect to quasivelocities and generalized velocities we can obtain Chaplygin's equations. Therefore V. S. Novoselov calls the above equations *the equations of Chaplygin's type.*

Similarly, the equations more general than the Hamel–Boltzmann ones are *the equations of Hamel–Novoselov* [169, 314]

$$\frac{d}{dt}\frac{\partial T^*}{\partial v_*^\lambda} - \frac{\partial T^*}{\partial \pi^\lambda} + \frac{\partial T^*}{\partial v_*^\rho}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\left(\frac{d}{dt}\frac{\partial v_*^\rho}{\partial \dot q^\sigma} - \frac{\partial v_*^\rho}{\partial q^\sigma}\right) = \widetilde{Q}_\lambda\,, \qquad \lambda = \overline{1, l}\,,$$

which are obtained also for nonlinear constraints (2.1). V. S. Novoselov calls these equations the equations of the Voronets–Hamel type (for detail, see § 1 Chapter VII).

In the equations of Chaplygin, Hamel–Boltzmann, and those similar to them, the authors have made an attempt to discriminate the Lagrange operator. Then the rest of addends in the left-hand sides of equations characterize the nonholonomicity of system. Therefore in the case of the integrability of constraints the differential equations become usual Lagrange's equations of the second kind of holonomic mechanics. Equations (2.13), (2.15), and the similar ones are generated for the concrete forms of usually linear nonholonomic constraints of the type (2.14), (2.16) and therefore they are useful for solving the corresponding problems. As a rule, such equations make it possible to obtain the minimal number of equations of motion. For example, the left-hand sides of Chaplygin's equations (2.13) involve only the unknown $q^1, \dots, q^l$ and after integration of these equations the rest of the coordinates $q^{l+1}, \dots, q^s$ can be found from equations of constraints (2.14).

As distinct from this, Maggi's equations are valid, as is mentioned above, for any of nonholonomic constraints, including the constraints nonlinear in generalized velocities. An important point is that for generating differential equations of motion (2.10) we need to apply a single equitype technique to all problems: after the choice of the generalized coordinates $q^1, \dots, q^s$ the expressions for the left-hand sides of usual Lagrange's equations of the second kind are generated; the transition formulas (2.2) to nonholonomic variables are introduced, in which case the last of them take account of the relations of nonholonomic constraints by means of formulas (2.5); the inverse transformation (2.3) is found and after differentiating it with respect to new nonholonomic variables, the equations of motion (2.10) are generated. Here two remarks can be given which are useful from the computational point of view.

Firstly, for numerical integrating system (2.10) together with constraints (2.1) it is necessary previously to differentiate the latter with respect to time and to obtain the equations linear in generalized accelerations. These equations and Maggi's ones are the system of linear nonuniform algebraic equations in unknown $\ddot q^1, \dots, \ddot q^s$. Solving them, we obtain the system of differential equations convenient for numerical integration.

Secondly, in the case of nonlinear nonholonomic constraints (2.1) the obtaining of inverse transformations (2.3) from formulas (2.2) may turn out difficult. To avoid this we need to to compile the matrix of derivatives $(\partial v_*^\rho / \partial \dot q^\sigma)$,

$\rho, \sigma = \overline{1, s}$, using formulas (2.2), and to find then the inverse matrix $(\partial \dot{q}^\sigma / \partial v_*^\rho)$, $\rho, \sigma = \overline{1, s}$, the elements of which are used for generating Maggi's equations.

Consider one more type of equations of nonholonomic mechanics. In the case of ideal constraints (2.1) equation (2.9) can be represented as

$$M\mathbf{W} = \mathbf{Y} + \Lambda_\varkappa \frac{\partial \varphi^\varkappa}{\partial \dot{q}^\tau} \mathbf{e}^\tau \,. \tag{2.21}$$

Multiplying scalarly equation (2.21) by the vectors of fundamental basis $\mathbf{e}_\sigma$, $\sigma = \overline{1, s}$, of the original system of curvilinear coordinates, we obtain the following relation

$$MW_\sigma = Q_\sigma + \Lambda_\varkappa \frac{\partial \varphi^\varkappa}{\partial \dot{q}^\sigma} \,, \qquad \sigma = \overline{1, s} \,,$$

which can be rearranged in the form

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\sigma} - \frac{\partial T}{\partial q^\sigma} = Q_\sigma + \Lambda_\varkappa \frac{\partial \varphi^\varkappa}{\partial \dot{q}^\sigma} \,, \qquad \sigma = \overline{1, s} \,. \tag{2.22}$$

These equations are usually called *Lagrange's equations of the second kind with undetermined multipliers* [59]. Together with the equations of nonholonomic constraints (2.1) they give a system of $s + k$ differential equations in $s + k$ unknowns q^σ, $\sigma = \overline{1, s}$, $\Lambda_\varkappa$, $\varkappa = \overline{1, k}$. This is the reason why, like equations (2.22) of Chapter I, they can be called *Lagrange's equations of the first kind in generalized coordinates for nonholonomic systems* [28].

If the original system of coordinates are Cartesian, then we have

$$\begin{aligned} q^\sigma = y_\sigma \,, \qquad \mathbf{e}_\sigma = \mathbf{e}^\sigma = \mathbf{j}_\sigma \,, \qquad \sigma = \overline{1, s} \,, \\ \varphi^\varkappa(t, y, \dot{y}) = 0 \,, \qquad\qquad \varkappa = \overline{1, k} \,, \end{aligned}$$

and equations (2.22) take the form

$$M\ddot{y}_\sigma = Y_\sigma + \Lambda_\varkappa \frac{\partial \varphi^\varkappa}{\partial \dot{y}_\sigma} \,, \qquad \sigma = \overline{1, s} \,. \tag{2.23}$$

Equations (2.23) are usual Lagrange's equations of the first kind for nonholonomic constraints rearranged for representation point.

F. Udwadia and R. Kalaba [394. 1992] derived the equations of dynamics in the matrix form taking into consideration the presence of nonholonomic constraints up to the second order with the help of the Moore and Penrose generalized inverse. In their opinion "the equations of motion obtained in this paper appear to be the simplest and most comprehensive so far discovered".

Note that the partition of the whole s-dimensional space into the direct sum of the K-space and L-space by means of constraint equations (2.1) actually corresponds to the application of Moore and Penrose generalized inverse. (A more general case for the constraints that are linear in generalized accelerations is considered in Chapter IV). This partition led to expressions

(2.12) for generalized reactions. Substituting them into equations (2.22), we obtain

$$A_{\sigma\tau}(t,q,\dot q)\,\ddot q^\tau = B_\sigma(t,q,\dot q)\,,$$
$$A_{\sigma\tau} = M\left(g_{\sigma\tau} - g_{\sigma^*\tau}\,\frac{\partial \dot q^{\sigma^*}}{\partial v_*^{l+\varkappa}}\,\frac{\partial\varphi^\varkappa}{\partial \dot q^\sigma}\right),$$
$$B_\sigma = Q_\sigma - Q_{\sigma^*}\,\frac{\partial \dot q^{\sigma^*}}{\partial v_*^{l+\varkappa}}\,\frac{\partial\varphi^\varkappa}{\partial \dot q^\sigma} + M\,\Gamma_{\sigma^*,\alpha\beta}\dot q^\alpha\dot q^\beta\,\frac{\partial \dot q^{\sigma^*}}{\partial v_*^{l+\varkappa}}\,\frac{\partial\varphi^\varkappa}{\partial \dot q^\sigma} -$$
$$-M\,\Gamma_{\sigma,\alpha\beta}\dot q^\alpha\dot q^\beta\,,\qquad \sigma,\sigma^*,\tau=\overline{1,s}\,,\quad \alpha,\beta=\overline{0,s}\,,\quad \varkappa=\overline{1,k}\,.$$

These formulae do imply *the Udwadia–Kalaba equations*

$$\ddot q^\tau = A^{\tau\sigma}(t,q,\dot q)\,B_\sigma(t,q,\dot q)\,,\qquad \sigma,\tau=\overline{1,s}\,,$$

where $A^{\tau\sigma}$ are elements of the matrix inverse to the matrix $(A_{\sigma\tau})$. We note that these equations can be also derived with the help of the linear force transformation, which will be introduced in the next chapter, and elimination of generalized reaction forces $\Lambda_\varkappa$, $\varkappa=\overline{1,k}$, from equations (2.22) in a simillar way as it was described for holonomic systems in § 2 of Chapter I.

V. V. Rumyantsev [203] thinks that the most general equations of nonholonomic mechanics are *the Poincaré–Chetaev equations*. They were introduced by H. Poincaré [373] and N. G. Chetaev [247, 248, 292] for holonomic systems. The mathematical problems, concerning their structure, and their place in the new theory of Hamiltonian systems were considered by V. I. Arnol'd, V. V. Kozlov, A. I. Neishtadt [7], L. M. Markhashov [149], and others. In the sequel they were generalized to nonholonomic systems due to the works of L. M. Markhashov [149], V. V. Rumyantsev [203], and Fam Guen [229]. As is shown by V. V. Rumyantsev [203], all the rest of types of the equations of motion for nonholonomic mechanics with the linear nonholonomic constraints of the first kind can be obtained from the Poincaré–Chetaev equations. We assume that these constraints have the form

$$v_*^{l+\varkappa} = a_\sigma^{l+\varkappa}(t,q)\dot q^\sigma + a_0^{l+\varkappa}(t,q) = 0\,,\qquad \varkappa=\overline{1,k}\,,\quad \sigma=\overline{1,s}\,. \tag{2.24}$$

Arguing as in the work [203], we supplement equations (2.24) with the following relations

$$v_*^\lambda = a_\sigma^\lambda(t,q)\dot q^\sigma + a_0^\lambda(t,q)\,,\qquad \lambda=\overline{1,l}\,,\quad l=s-k\,,\quad \sigma=\overline{1,s}\,,$$

which implies that the generalized velocities can uniquely be represented as

$$\dot q^\sigma = b_\tau^\sigma(t,q)v_*^\tau + b_0^\sigma(t,q)\,,\qquad \sigma,\tau=\overline{1,s}\,.$$

Introducing, for short, the notions [203]

$$q^0=t\,,\quad \dot q^0=1\,,\quad v_*^0=1\,,\quad a_\alpha^0=b_\alpha^0=\delta_\alpha^0\,,\quad \alpha=\overline{0,s}\,,$$

we have

$$v_*^\alpha = a_\beta^\alpha \dot q^\beta\,, \quad \dot q^\beta = b_\alpha^\beta v_*^\alpha\,, \qquad \alpha,\beta=\overline{0,s}\,.$$

Denote the Lagrange function $L=T-\Pi$, which was found as the function of variables $t, q^\sigma, v_*^\sigma,\ \sigma=\overline{1,s}$, by $L^*(t,q,v_*)$. In these notions the Poincaré–Chetaev equations for nonholonomic systems with constraints (2.24) are the following [203]:

$$\frac{d}{dt}\frac{\partial L^*}{\partial v_*^\lambda} = c_{\mu\lambda}^\rho v_*^\mu \frac{\partial L^*}{\partial v_*^\rho} + c_{0\lambda}^\rho \frac{\partial L^*}{\partial v_*^\rho} + b_\lambda^\sigma \frac{\partial L^*}{\partial q^\sigma} + \widetilde{Q}_\lambda\,, \qquad \lambda,\mu=\overline{1,l},, \qquad \rho,\sigma=\overline{1,s}\,. \tag{2.25}$$

Here $\widetilde{Q}_\lambda = b_\lambda^\sigma Q_\sigma$ are generalized nonpotentional forces, corresponding to the quasivelocities $v_*^\lambda, \lambda=\overline{1,l}$, and $c_{\mu\alpha}^\rho$ and $c_{0\lambda}^\rho$ are the coefficients, given in the form

$$c_{\alpha\beta}^\rho = a_\gamma^\rho\left(\frac{\partial b_\beta^\gamma}{\partial q^\delta}b_\alpha^\delta - \frac{\partial b_\alpha^\gamma}{\partial q^\delta}b_\beta^\delta\right) = \left(\frac{\partial a_\gamma^\rho}{\partial q^\delta} - \frac{\partial a_\delta^\rho}{\partial q^\gamma}\right)b_\alpha^\gamma b_\beta^\delta\,, \qquad \alpha,\beta,\gamma,\delta=\overline{0,s}\,, \qquad \rho=\overline{1,s}\,. \tag{2.26}$$

As V. V. Rumyantsev emphasizes [203], the function L^*, entering into equations (2.25), depends, generally speaking, on all quasivelocities $v_*^\rho,\ \rho=\overline{1,s}$, and therefore the equations of constraints (2.24) $v_*^{l+\varkappa}=0$, $\varkappa=\overline{1,k}$, should be used only after the generation of equations (2.25). By (2.26) for holonomic constraints we have $c_{\alpha\beta}^{l+\varkappa}=0$, $\varkappa=\overline{1,k}$, $\alpha,\beta=\overline{0,s}$, and in this case the above remark does not refer to holonomic systems.

Equations (2.25), supplemented with the equations

$$\dot q^\sigma = b_\lambda^\sigma(t,q)v_*^\lambda + b_0^\sigma(t,q)\,, \qquad \lambda=\overline{1,l}\,, \quad \sigma=\overline{1,s}\,,$$

are the closed system of equations in unknowns q^σ, $\sigma=\overline{1,s}$, and v_*^λ, $\lambda=\overline{1,l}$. The number of sought independent variables is minimal and the differential equations in unknowns as q^σ as v_*^λ are of the first kind. This is the advantage of equations (2.25) in contrast with Maggi's equations.

The Hamel–Novoselov and Poincaré–Chetaev equations are also considered in Chapter VII, where they are obtained by three different approaches.

Finally, we give some important remarks.

In studying the motion of rigid body linear nonholonomic constraint (2.24) occurs, in particular, in the case when the projection of the velocity $\mathbf{v}$ of the point M of rigid body on the direction of the unit vector $\mathbf{j}$ of body is equal to zero by virtue of its interaction with another body. This example of nonholonomic constraint is the most routine one. Therefore we consider it in more detail. We shall show that the assumption that the considered constraint is ideal means that the force, applied to the point M of body in the result of its interaction with another body, is equal to $\Lambda\mathbf{j}$ if the equation of constraint is as follows

$$\varphi = \mathbf{v}\cdot\mathbf{j} = a_\sigma(t,q)\dot q^\sigma + a_0(t,q) = 0\,, \qquad \sigma=\overline{1,s}\,, \quad s\leqslant 6\,.$$

Here $\mathbf{v}$ is the velocity of the point M and q^σ,$\sigma = \overline{1,s}$, are generalized coordinates of rigid body. By assumption, the constraint is linear and therefore the unit vector $\mathbf{j}$ can only depend on the time t and on the generalized coordinates q^σ, $\sigma = \overline{1,s}$, but not on the generalized velocities $\dot{q}^\sigma$, $\sigma = \overline{1,s}$.

Equations (2.22) implies that for the proof of such assertion it is sufficient to show that the generalized forces $\mathcal{R}_\sigma$, corresponding to the force $\Lambda\mathbf{j}$, can be represented in the form

$$\mathcal{R}_\sigma = \Lambda \frac{\partial \varphi}{\partial \dot{q}^\sigma}\,.$$

Really, by definition, we have

$$\mathcal{R}_\sigma = \Lambda \mathbf{j} \cdot \frac{\partial \mathbf{r}}{\partial q^\sigma}\,,$$

where $\mathbf{r} = \mathbf{r}(t, q)$ is the radius vector of the point M. The velocity $\mathbf{v}$ of the point M is as follows

$$\mathbf{v} = \frac{\partial \mathbf{r}}{\partial t} + \frac{\partial \mathbf{r}}{\partial q^\sigma} \dot{q}^\sigma\,.$$

Hence

$$\frac{\partial \mathbf{r}}{\partial q^\sigma} = \frac{\partial \mathbf{v}}{\partial \dot{q}^\sigma}\,, \qquad \sigma = \overline{1,s}\,,$$

and we have

$$\mathcal{R}_\sigma = \Lambda \mathbf{j} \cdot \frac{\partial \mathbf{v}}{\partial \dot{q}^\sigma}\,, \qquad \sigma = \overline{1,s}\,.$$

The vector $\mathbf{j}$ is independent of the variables $\dot{q}^\sigma$, $\sigma = \overline{1,s}$. In this case the quantities $\mathcal{R}_\sigma$, $\sigma = \overline{1,s}$, can be represented as

$$\mathcal{R}_\sigma = \Lambda \frac{\partial (\mathbf{v} \cdot \mathbf{j})}{\partial \dot{q}^\sigma} = \Lambda \frac{\partial \varphi}{\partial \dot{q}^\sigma}\,, \quad \sigma = \overline{1,s}\,,$$

which was to be proved.

Consider another example from the dynamics of rigid body, related to the problem on the controllable motion of rigid body but not the problem on its rolling or sliding motion. We assume that it is necessary to realize the free motion, of rigid body, such that the projection of the vector of instantaneous angular velocity $\boldsymbol{\omega}$ on the fixed axis $\mathbf{j}$ is a given function of time t and the generalized coordinates q^σ, $\sigma = \overline{1,6}$. Thus equation of linear nonholonomic constraint (2.24) is assumed to be given in the form

$$\varphi = \boldsymbol{\omega} \cdot \mathbf{j} + a_0(t, q) = 0\,. \tag{2.27}$$

We shall show that from equations (2.22) it follows that the "ideal"realization of such program of motion is possible by means of an additional system of forces such that its force resultant is equal to zero and the resultant moment about the center of mass is equal to $\Lambda\mathbf{j}$.

Suppose, $\boldsymbol{\rho}_\nu$ are the radius-vectors of the points of application of the additional forces $\mathbf{F}_\nu$, the number of which is equal to n. By definition, we have

$$\mathcal{R}_\sigma = \sum_{\nu=1}^{n} \mathbf{F}_\nu \cdot \frac{\partial \boldsymbol{\rho}_\nu}{\partial q^\sigma} = \sum_{\nu=1}^{n} \mathbf{F}_\nu \cdot \frac{\partial \mathbf{v}_\nu}{\partial \dot{q}^\sigma} = \sum_{\nu=1}^{n} \mathbf{F}_\nu \cdot \frac{\partial}{\partial \dot{q}^\sigma}(\mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_\nu)\,. \qquad (2.28)$$

Here $\mathbf{v}_C$ is a velocity of the center of mass and $\mathbf{r}_\nu$ is a radius vector of the point of application of the additional force $\mathbf{F}_\nu$ relative to the center of mass.

By virtue of the problem setting we have

$$\sum_{\nu=1}^{n} \mathbf{F}_\nu = 0\,, \quad \sum_{\nu=1}^{n} \mathbf{r}_\nu \times \mathbf{F}_\nu = \Lambda\mathbf{j}\,. \qquad (2.29)$$

Therefore relations (2.27) and (2.28) yield

$$\mathcal{R}_\sigma = \sum_{\nu=1}^{n} \mathbf{F}_\nu \cdot \frac{\partial(\boldsymbol{\omega} \times \mathbf{r}_\nu)}{\partial \dot{q}^\sigma} = \sum_{\nu=1}^{n} \mathbf{F}_\nu \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^\sigma} \times \mathbf{r}_\nu =$$
$$= \sum_{\nu=1}^{n} (\mathbf{r}_\nu \times \mathbf{F}_\nu) \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^\sigma} = \Lambda\mathbf{j} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}^\sigma} = \Lambda \frac{\partial \varphi}{\partial \dot{q}^\sigma}\,.$$

In the above proof the fact that the unit vector $\mathbf{j}$ is that of fixed frame is not used. We need only in the fact that this vector is independent of the generalized velocities $\dot{q}^\sigma$, $\sigma = \overline{1,6}$. It can be a vector, which is of a given dependence of time and generalized coordinates, and therefore it can, in particular, be the unit vector, rigidly connected with body.

The essential distinction between the considered problem and the problem on the rolling or sliding motion of rigid body is that the validity of constraint (2.27) can be provided by different families of the additional forces $\mathbf{F}_\nu, \nu = \overline{1,n}$, satisfying condition (2.29) while in the problem on the rolling or sliding motion the validity of constraint is provided by the one additional force $\Lambda\mathbf{j}$, applied to the contact point M. It is also important that the generation of this force as the function of the variables $t, q, \dot{q}_\sigma$, $\sigma = \overline{1,s}$, is given by the contact interaction of two bodies. This force can be eliminated and the motion can be found from the equations of constraints and, for example, from Maggi's equations, which do not involve the Lagrange multipliers. For controllable motion, the generation of the moment $\Lambda\mathbf{j}$ is realized by the control system and only after applying the found control moment $\Lambda\mathbf{j}$ the motion can satisfy equation (2.27).

§ 3. The generation of the most usual forms of equations of motion of nonholonomic systems from Maggi's equations

We obtain now the mentioned above forms of equations of motion of nonholonomic systems from Maggi's equations.

Chaplygin's and Voronets' equations. Suppose that on the system are imposed the stationary linear nonholonomic constraints, the equations of which take the form

$$\dot q^{l+\varkappa} = \beta^{l+\varkappa}_\lambda(q)\dot q^\lambda, \qquad \lambda=\overline{1,l}, \quad \varkappa=\overline{1,k}\,. \tag{3.1}$$

Then, assuming

$$v^\lambda_* = \dot q^\lambda, \quad \lambda=\overline{1,l},$$
$$v^{l+\varkappa}_* = \dot q^{l+\varkappa} - \beta^{l+\varkappa}_\lambda(q)\dot q^\lambda, \quad \varkappa=\overline{1,k}\,,$$

we have

$$\frac{\partial \dot q^\mu}{\partial v^\lambda_*} = \delta^\mu_\lambda = \begin{cases} 1, & \mu=\lambda\,, \\ 0, & \mu\neq\lambda\,, \end{cases} \qquad \lambda,\mu=\overline{1,l}\,,$$
$$\frac{\partial \dot q^{l+\varkappa}}{\partial v^\lambda_*} = \beta^{l+\varkappa}_\lambda, \qquad \lambda=\overline{1,l}, \quad \varkappa=\overline{1,k}\,.$$

From these relations it follows that for nonholonomic constraints, given by (3.1), Maggi's equations (2.10) can be represented as

$$\begin{gathered} Mw_\lambda + Mw_{l+\varkappa}\beta^{l+\varkappa}_\lambda = Q_\lambda + Q_{l+\varkappa}\beta^{l+\varkappa}_\lambda\,, \\ \lambda=\overline{1,l}, \quad \varkappa=\overline{1,k}\,. \end{gathered} \tag{3.2}$$

Suppose that the kinetic energy T is independent of the generalized coordinates $q^{l+\varkappa}$ and $Q_{l+\varkappa}=0$ ($\varkappa=\overline{1,k}$). Then equations (3.2) have the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q^\lambda} - \frac{\partial T}{\partial q^\lambda} + \beta^{l+\varkappa}_\lambda \frac{d}{dt}\frac{\partial T}{\partial \dot q^{l+\varkappa}} = Q_\lambda, \qquad \lambda=\overline{1,l}\,. \tag{3.3}$$

Transform equation (3.3). By means of equation of constraints (3.1), we eliminate all velocities $\dot q^{l+\varkappa}$ from the relation for the kinetic energy T, and denote by T_* the obtained expression for kinetic energy. In this case the relations hold

$$\frac{\partial T_*}{\partial \dot q^\lambda} = \frac{\partial T}{\partial \dot q^\lambda} + \frac{\partial T}{\partial \dot q^{l+\varkappa}}\frac{\partial \dot q^{l+\varkappa}}{\partial \dot q^\lambda} = \frac{\partial T}{\partial \dot q^\lambda} + \frac{\partial T}{\partial \dot q^{l+\varkappa}}\beta^{l+\varkappa}_\lambda\,, \tag{3.4}$$

$$\begin{gathered} \frac{\partial T_*}{\partial q^\lambda} = \frac{\partial T}{\partial q^\lambda} + \frac{\partial T}{\partial \dot q^{l+\varkappa}}\frac{\partial \dot q^{l+\varkappa}}{\partial q^\lambda} = \frac{\partial T}{\partial q^\lambda} + \frac{\partial T}{\partial \dot q^{l+\varkappa}}\frac{\partial \beta^{l+\varkappa}_\mu}{\partial q^\lambda}\dot q^\mu\,, \\ \lambda,\mu=\overline{1,l}\,. \end{gathered} \tag{3.5}$$

We assume that the coefficients $\beta_\lambda^{l+\varkappa}$ are independent of $q^{l+\varkappa}$, $\varkappa = \overline{1,k}$. Then, differentiating in time relation (3.4), we obtain

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot q^\lambda} = \frac{d}{dt}\frac{\partial T}{\partial \dot q^\lambda} + \beta_\lambda^{l+\varkappa}\frac{d}{dt}\frac{\partial T}{\partial \dot q^{l+\varkappa}} + \frac{\partial T}{\partial \dot q^{l+\varkappa}}\frac{d\beta_\lambda^{l+\varkappa}}{dt} =$$
$$= \frac{d}{dt}\frac{\partial T}{\partial \dot q^\lambda} + \beta_\lambda^{l+\varkappa}\frac{d}{dt}\frac{\partial T}{\partial \dot q^{l+\varkappa}} + \frac{\partial T}{\partial \dot q^{l+\varkappa}}\frac{\partial \beta_\lambda^{l+\varkappa}}{\partial q^\mu}\dot q^\mu\,, \tag{3.6}$$
$$\lambda, \mu = \overline{1,l}\,.$$

Computing the quantities $d(\partial T/\partial \dot q^\lambda)/dt$ and $\partial T/\partial q^\lambda$ by formulas (3.6) and (3.5) and substituting them into equations (3.3), we get

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot q^\lambda} - \frac{\partial T_*}{\partial q^\lambda} - \frac{\partial T}{\partial \dot q^{l+\varkappa}}\left(\frac{\partial \beta_\lambda^{l+\varkappa}}{\partial q^\mu} - \frac{\partial \beta_\mu^{l+\varkappa}}{\partial q^\lambda}\right)\dot q^\mu = Q_\lambda\,, \tag{3.7}$$
$$\varkappa = \overline{1,k}\,, \qquad \lambda, \mu = \overline{1,l}\,.$$

These equations were obtained by *S. A. Chaplygin* [239].

Now we eliminate the dependent velocities $\dot q^{l+1}, \dot q^{l+2}, ..., \dot q^{l+k}$ from the expressions $\partial T/\partial \dot q^{l+\varkappa}$ in equations (3.7), using equations of constraints (3.1). Then we get the system of l equations in unknown functions $q^1, q^2, ..., q^l$. Thus, Chaplygin's equations permit us to determine $q^1(t), q^2(t), ..., q^l(t)$ independently of constraints (3.1) and to find then the rest of $q^{l+1}(t), q^{l+2}(t), ...,$ $q^{l+k}(t)$ from equations (3.1).

Suppose, the coefficients $\beta_\lambda^{l+\varkappa}$ satisfy the following conditions

$$\frac{\partial \beta_\mu^{l+\varkappa}}{\partial q^\lambda} - \frac{\partial \beta_\lambda^{l+\varkappa}}{\partial q^\mu} = 0\,, \qquad \varkappa = \overline{1,k}\,, \qquad \lambda, \mu = \overline{1,l}\,. \tag{3.8}$$

According to the assumption that the coefficients $\beta_\lambda^{l+\varkappa}$ are independent of $q^{l+\varkappa}$ ($\varkappa = \overline{1,k}$) this implies that they take the form

$$\beta_\lambda^{l+\varkappa} = \frac{\partial U^{l+\varkappa}}{\partial q^\lambda}\,, \qquad \lambda = \overline{1,l}\,, \qquad \varkappa = \overline{1,k}\,. \tag{3.9}$$

Here $U^{l+\varkappa}$ are the functions of coordinates $q^1, q^2, ..., q^l$. Substituting relations (3.9) into equations (3.1), we obtain

$$q^{l+\varkappa} = U^{l+\varkappa}(q^1, q^2, ..., q^l)\,, \qquad \varkappa = \overline{1,k}\,.$$

Thus, the coordinates $q^{l+\varkappa}$ result from the rest. Therefore if conditions (3.8) are satisfied the motion is described by usual Lagrange's equations.

Now we generate the equations of motion, obtained by P. V. Voronets [41. 1901]. Consider a mechanical system with constraints given in the form (3.1) without the additional assumptions, which arrive to Chaplygin's equations.

In the case when the kinetic energy T depends on all coordinates Maggi's equations (3.2) are the following

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q^\lambda} - \frac{\partial T}{\partial q^\lambda} + \left(\frac{d}{dt}\frac{\partial T}{\partial \dot q^{l+\varkappa}} - \frac{\partial T}{\partial q^{l+\varkappa}}\right)\beta_\lambda^{l+\varkappa} = Q_\lambda + Q_{l+\varkappa}\beta_\lambda^{l+\varkappa}\,, \qquad \varkappa = \overline{1,k}\,, \qquad \lambda = \overline{1,l}\,. \tag{3.10}$$

Arguing as above, we reduce these equations to the Voronets ones. Relations (3.5) preserve their form. In accordance with that the coefficients $\beta_\lambda^{l+\varkappa}$ depend now on all q^σ, relations (3.6) take the form

$$\begin{gathered}\frac{d}{dt}\frac{\partial T_*}{\partial \dot q^\lambda} = \frac{d}{dt}\frac{\partial T}{\partial \dot q^\lambda} + \beta_\lambda^{l+\varkappa}\frac{d}{dt}\frac{\partial T}{\partial \dot q^{l+\varkappa}} + \frac{\partial T}{\partial \dot q^{l+\varkappa}}\frac{\partial \beta_\lambda^{l+\varkappa}}{\partial q^\mu}\dot q^\mu + \\ + \frac{\partial T}{\partial \dot q^{l+\varkappa}}\frac{\partial \beta_\lambda^{l+\varkappa}}{\partial q^{l+\nu}}\beta_\mu^{l+\nu}\dot q^\mu\,, \qquad \varkappa,\nu = \overline{1,k}\,, \quad \lambda,\mu = \overline{1,l}\,.\end{gathered} \tag{3.11}$$

In the considered case, together with (3.5) and (3.11) we need to account for the following relations

$$\beta_\lambda^{l+\varkappa}\frac{\partial T_*}{\partial q^{l+\varkappa}} = \beta_\lambda^{l+\varkappa}\left(\frac{\partial T}{\partial q^{l+\varkappa}} + \frac{\partial T}{\partial \dot q^{l+\nu}}\frac{\partial \beta_\mu^{l+\nu}}{\partial q^{l+\varkappa}}\dot q^\mu\right).$$

This relation, together with relations (3.5) and (3.11), permits us to represent equations (3.10) as

$$\begin{gathered}\frac{d}{dt}\frac{\partial T_*}{\partial \dot q^\lambda} - \frac{\partial T_*}{\partial q^\lambda} - \beta_\lambda^{l+\varkappa}\frac{\partial T_*}{\partial q^{l+\varkappa}} - \frac{\partial T}{\partial \dot q^{l+\varkappa}}\beta_{\lambda\mu}^{l+\varkappa}\dot q^\mu = \\ = Q_\lambda + Q_{l+\varkappa}\beta_\lambda^{l+\varkappa}\,, \quad \lambda,\mu = \overline{1,l}\,, \quad \varkappa = \overline{1,k}\,,\end{gathered} \tag{3.12}$$

where

$$\beta_{\lambda\mu}^{l+\varkappa} = \frac{\partial \beta_\lambda^{l+\varkappa}}{\partial q^\mu} - \frac{\partial \beta_\mu^{l+\varkappa}}{\partial q^\lambda} + \frac{\partial \beta_\lambda^{l+\varkappa}}{\partial q^{l+\nu}}\beta_\mu^{l+\nu} - \frac{\partial \beta_\mu^{l+\varkappa}}{\partial q^{l+\nu}}\beta_\lambda^{l+\nu}\,.$$

Equations (3.12) are called *Voronets' equations*. The equations of motion (3.12) together with equations of constraints (3.1) are the system of differential equations for obtaining the functions $q^\sigma(t), \sigma = \overline{1,s}$.

In the case of constrained motion of system acted by forces, which have a potential, equations (3.12) take the form

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot q^\lambda} - \frac{\partial (T_* + U)}{\partial q^\lambda} - \beta_\lambda^{l+\varkappa}\frac{\partial (T_* + U)}{\partial q^{l+\varkappa}} - \frac{\partial T}{\partial \dot q^{l+\varkappa}}\beta_{\lambda\mu}^{l+\varkappa}\dot q^\mu = 0\,, \qquad \lambda,\mu = \overline{1,l}\,, \quad \varkappa = \overline{1,k}\,.$$

In the partial case when the coordinates $q^{l+1}, q^{l+2}, ..., q^{l+k}$, corresponding to the eliminated velocities, do not enter into the relations for kinetic and potential energies in explicit form and also into the equations of constraints, Voronets' equations (3.12) coincide with Chaplygin's equations (3.7).

The equations in quasicoordinates (the Hamel–Novoselov, Voronets–Hamel, and Poincaré–Chetaev equations). As is known, the projections of the vector of instantaneous angular velocity $\boldsymbol{\omega}$ on the fixed axes cannot be regarded as the derivatives with respect to certain new angles, which uniquely determine the position of rigid body. Similarly, it may turn out that the quantities v_*^ρ, which are a one-to-one function of the generalized velocities $\dot q^\sigma$, cannot be regarded as derivatives with respect to the certain new coordinates q_*^ρ. Therefore the quantities v_*^ρ are called *quasivelocities* and the variables π^ρ, given by formulas

$$\pi^\rho = \int_t^{t_0} v_*^\rho dt\,,$$

are called *quasicoordinates.*

In the relation for the kinetic energy T the generalized velocities $\dot q^\sigma$ are changed by the quasivelocities v_*^ρ. The function thus obtained is denoted by T^*. Consider, which form can have Maggi's equations, represented as

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q^\sigma} - \frac{\partial T}{\partial q^\sigma} - Q_\sigma\right)\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} = 0\,, \qquad \sigma = \overline{1,s}\,, \quad \lambda = \overline{1,l}\,, \tag{3.13}$$

when used the function T^*.

Taking into account the relations

$$\frac{\partial T^*}{\partial v_*^\lambda} = \frac{\partial T}{\partial \dot q^\sigma}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\,, \qquad \frac{\partial T^*}{\partial q^\sigma} = \frac{\partial T}{\partial q^\sigma} + \frac{\partial T}{\partial \dot q^\rho}\frac{\partial \dot q^\rho}{\partial q^\sigma}\,,$$
$$\rho,\sigma = \overline{1,s}\,, \quad \lambda = \overline{1,l}\,,$$

we have

$$\begin{aligned}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q^\sigma}\right)\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q^\sigma}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\right) - \\ -\frac{\partial T}{\partial \dot q^\sigma}\frac{d}{dt}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} &= \frac{d}{dt}\frac{\partial T^*}{\partial v_*^\lambda} - \frac{\partial T}{\partial \dot q^\sigma}\frac{d}{dt}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\,,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\frac{\partial T}{\partial q^\sigma}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} &= \frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\left(\frac{\partial T^*}{\partial q^\sigma} - \frac{\partial T}{\partial \dot q^\rho}\frac{\partial \dot q^\rho}{\partial q^\sigma}\right) = \\ &= \frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\frac{\partial T^*}{\partial q^\sigma} - \frac{\partial T}{\partial \dot q^\rho}\frac{\partial \dot q^\rho}{\partial q^\sigma}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\,.\end{aligned} \tag{3.15}$$

In the double sum in the right-hand side of relation (3.15) we exchange the indices of summing ρ and σ. As a result we have

$$\frac{\partial T}{\partial q^\sigma}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} = \frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\frac{\partial T^*}{\partial q^\sigma} - \frac{\partial T}{\partial \dot q^\sigma}\frac{\partial \dot q^\sigma}{\partial q^\rho}\frac{\partial \dot q^\rho}{\partial v_*^\lambda}\,. \tag{3.16}$$

Consider the operator

$$\frac{\partial}{\partial \pi^\rho} = \frac{\partial \dot q^\sigma}{\partial v_*^\rho}\frac{\partial}{\partial q^\sigma}\,, \qquad \rho,\sigma = \overline{1,s}\,. \tag{3.17}$$

Under the assumption $v_*^\rho = \dot{\pi}^\rho = \dot{q}_*^\rho$ it passes into the operator of partial derivative with respect to the new coordinate q_*^ρ since we have

$$\frac{\partial \dot{q}^\sigma}{\partial v_*^\rho}\frac{\partial}{\partial q^\sigma} = \frac{\partial \dot{q}^\sigma}{\partial \dot{q}_*^\rho}\frac{\partial}{\partial q^\sigma} = \frac{\partial q^\sigma}{\partial q_*^\rho}\frac{\partial}{\partial q^\sigma} = \frac{\partial}{\partial q_*^\rho}\,.$$

By (3.17) relation (3.16) takes the form

$$\frac{\partial T}{\partial q^\sigma}\frac{\partial \dot{q}^\sigma}{\partial v_*^\lambda} = \frac{\partial T^*}{\partial \pi^\lambda} - \frac{\partial T}{\partial \dot{q}^\sigma}\frac{\partial \dot{q}^\sigma}{\partial \pi^\lambda}\,.$$

According to relation (3.14) this implies that Maggi's equations (3.13) take the form

$$\begin{gathered}\frac{d}{dt}\frac{\partial T^*}{\partial v_*^\lambda} - \frac{\partial T^*}{\partial \pi^\lambda} - \frac{\partial T}{\partial \dot{q}^\sigma}\left(\frac{d}{dt}\frac{\partial \dot{q}^\sigma}{\partial v_*^\lambda} - \frac{\partial \dot{q}^\sigma}{\partial \pi^\lambda}\right) = Q_\lambda^*\,,\\ \sigma = \overline{1,s}\,, \qquad \lambda = \overline{1,l}\,.\end{gathered} \tag{3.18}$$

Here

$$Q_\lambda^* = Q_\sigma \frac{\partial \dot{q}^\sigma}{\partial v_*^\lambda}\,. \tag{3.19}$$

Equations (3.18) are called, sometimes, *Chaplygin's type equations* [169].

Consider the partial case when the generalized velocities $\dot{q}^\sigma$ and the quasivelocities v_*^ρ are related by the following linear uniform stationary relations

$$\begin{gathered}v_*^\rho = \alpha_\sigma^\rho(q)\dot{q}^\sigma\,, \qquad \dot{q}^\sigma = \beta_\rho^\sigma(q)v_*^\rho\,,\\ \rho,\sigma = \overline{1,s}\,,\end{gathered} \tag{3.20}$$

and the equations of constraints are the following

$$v_*^{l+\varkappa} \equiv \alpha_\sigma^{l+\varkappa}(q)\dot{q}^\sigma = 0\,, \qquad \varkappa = \overline{1,k}\,. \tag{3.21}$$

In this case, using relations (3.20) and operator (3.17) and taking into account that after performing the operations of differentiation it can be assumed that $v_*^{l+\varkappa} = 0$ ($\varkappa = \overline{1,k}$), we have

$$\begin{gathered}\frac{d}{dt}\frac{\partial \dot{q}^\sigma}{\partial v_*^\lambda} = \frac{d}{dt}\beta_\lambda^\sigma(q) = \frac{\partial \beta_\lambda^\sigma}{\partial q^\rho}\dot{q}^\rho = \frac{\partial \beta_\lambda^\sigma}{\partial q^\rho}\beta_\mu^\rho v_*^\mu =\\ = v_*^\mu \frac{\partial \dot{q}^\rho}{\partial v_*^\mu}\frac{\partial \beta_\lambda^\sigma}{\partial q^\rho} = v_*^\mu \frac{\partial \beta_\lambda^\sigma}{\partial \pi^\mu}\,, \qquad \rho,\sigma = \overline{1,s}\,, \quad \lambda,\mu = \overline{1,l}\,;\\ \frac{\partial \dot{q}^\sigma}{\partial \pi^\lambda} = \frac{\partial \dot{q}^\rho}{\partial v_*^\lambda}\frac{\partial \dot{q}^\sigma}{\partial q^\rho} = \frac{\partial \dot{q}^\rho}{\partial v_*^\lambda}\frac{\partial \beta_\mu^\sigma}{\partial q^\rho}v_*^\mu =\\ = v_*^\mu \frac{\partial \beta_\mu^\sigma}{\partial \pi^\lambda}\,, \qquad \rho,\sigma = \overline{1,s}\,, \qquad \lambda,\mu = \overline{1,l}\,.\end{gathered}$$

Then equations (3.18) take the form

$$\begin{gathered}\frac{d}{dt}\frac{\partial T^*}{\partial v_*^\lambda} - \frac{\partial T^*}{\partial \pi^\lambda} - \frac{\partial T}{\partial \dot{q}^\sigma}\left(\frac{\partial \beta_\lambda^\sigma}{\partial \pi^\mu} - \frac{\partial \beta_\mu^\sigma}{\partial \pi^\lambda}\right)v_*^\mu = Q_\lambda^*\,,\\ \sigma = \overline{1,s}\,, \qquad \lambda,\mu = \overline{1,l}\,.\end{gathered} \tag{3.22}$$

These equations are usually called *Chaplygin's equations in quasicoordinates* [166, 169]. Note that equations (3.18) and (3.22) should be considered together with the equations of nonholonomic constraints.

Equations (3.18) and (3.22) involve as the function T^* as the function T. We reduce now Maggi's equations (3.13) to the form that involves the function T^* only. The following relations

$$\frac{\partial T}{\partial \dot q^\sigma} = \frac{\partial T^*}{\partial v_*^\rho}\frac{\partial v_*^\rho}{\partial \dot q^\sigma}\,, \qquad \rho,\sigma = \overline{1,s}\,,$$

yield the relation

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q^\sigma}\right)\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} = \frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\frac{d}{dt}\left(\frac{\partial T^*}{\partial v_*^\rho}\frac{\partial v_*^\rho}{\partial \dot q^\sigma}\right) =$$
$$= \left(\frac{d}{dt}\frac{\partial T^*}{\partial v_*^\rho}\right)\frac{\partial v_*^\rho}{\partial \dot q^\sigma}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} + \frac{\partial T^*}{\partial v_*^\rho}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\frac{d}{dt}\frac{\partial v_*^\rho}{\partial \dot q^\sigma}\,.$$

Since

$$\frac{\partial v_*^\rho}{\partial \dot q^\sigma}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} = \delta_\lambda^\rho = \begin{cases} 1\,, & \rho = \lambda\,,\\ 0\,, & \rho \neq \lambda\,,\end{cases}$$

we have

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q^\sigma}\right)\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} = \frac{d}{dt}\frac{\partial T^*}{\partial v_*^\lambda} + \frac{\partial T^*}{\partial v_*^\rho}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\frac{d}{dt}\frac{\partial v_*^\rho}{\partial \dot q^\sigma}\,. \tag{3.23}$$

Taking into account the relations

$$\frac{\partial T}{\partial q^\sigma} = \frac{\partial T^*}{\partial q^\sigma} + \frac{\partial T^*}{\partial v_*^\rho}\frac{\partial v_*^\rho}{\partial q^\sigma}$$

and operator (3.17), we obtain

$$\frac{\partial T}{\partial q^\sigma}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} = \frac{\partial T^*}{\partial \pi^\lambda} + \frac{\partial T^*}{\partial v_*^\rho}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\frac{\partial v_*^\rho}{\partial q^\sigma}\,.$$

Then from the above and formulas (3.19) and (3.23) it follows that Maggi's equations (3.13) can be represented in the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial v_*^\lambda} - \frac{\partial T^*}{\partial \pi^\lambda} + \frac{\partial T^*}{\partial v_*^\rho}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\left(\frac{d}{dt}\frac{\partial v_*^\rho}{\partial \dot q^\sigma} - \frac{\partial v_*^\rho}{\partial q^\sigma}\right) = Q_\lambda^*\,, \tag{3.24}$$
$$\rho,\sigma = \overline{1,s}\,, \qquad \lambda = \overline{1,l}\,.$$

Equations (3.18) and (3.24) can be applied to both holonomic and nonholonomic systems with as the linear with respect to velocities ideal constraints as the nonlinear ones. In the case when the time does not enter into the kinetic energy and the equations of constraints in explicit form, equations (3.18) and (3.24) were obtained by G. Hamel [314] and in the general case by V. S. Novoselov [169]. Therefore we shall call these equations *the Hamel–Novoselov ones.*

In the case when the quasivelocities are defined by formulas (3.20) and the constraints are given by equations (3.21) we have

$$\begin{gathered}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\frac{d}{dt}\frac{\partial v_*^\rho}{\partial \dot q^\sigma} = \beta_\lambda^\sigma \frac{d\alpha_\sigma^\rho}{dt} = \beta_\lambda^\sigma \frac{\partial \alpha_\sigma^\rho}{\partial q^\tau}\dot q^\tau = \beta_\lambda^\sigma \beta_\mu^\tau \frac{\partial \alpha_\sigma^\rho}{\partial q^\tau} v_*^\mu\,,\\ \frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\frac{\partial v_*^\rho}{\partial q^\sigma} = \beta_\lambda^\sigma \frac{\partial \alpha_\tau^\rho}{\partial q^\sigma}\dot q^\tau = \beta_\lambda^\sigma \beta_\mu^\tau \frac{\partial \alpha_\tau^\rho}{\partial q^\sigma} v_*^\mu\,,\\ \rho,\sigma,\tau = \overline{1,s}\,,\qquad \lambda,\mu = \overline{1,l}\,.\end{gathered}$$

Then (3.24) takes the form

$$\begin{gathered}\frac{d}{dt}\frac{\partial T^*}{\partial v_*^\lambda} - \frac{\partial T^*}{\partial \pi^\lambda} + c_{\lambda\mu}^\rho v_*^\mu \frac{\partial T^*}{\partial v_*^\rho} = Q_\lambda^*\,,\\ c_{\lambda\mu}^\rho = \left(\frac{\partial \alpha_\sigma^\rho}{\partial q^\tau} - \frac{\partial \alpha_\tau^\rho}{\partial q^\sigma}\right)\beta_\lambda^\sigma \beta_\mu^\tau\,,\\ \rho,\sigma,\tau = \overline{1,s}\,,\qquad \lambda,\mu = \overline{1,l}\,.\end{gathered} \tag{3.25}$$

In the case $l = s$ these equations and the relations for the coefficients $c_{\sigma\tau}^\rho$ were obtained first by P. V. Voronets in 1901 [41]. In 1904 for $l < s$ these results were obtained once more by G. Hamel [313]. Therefore these equations are usually called *the Voronets–Hamel equations* but G. Hamel itself called them the Euler–Lagrange equations. We remark that in the literature they are also called *the Hamel–Boltzmann equations.*

Together with the works of P. V. Voronets, H. Poincaré obtains [373] the equations, which are highly close to equations (3.25). *Poincaré's equations* correspond to the case when in equations (3.25) for $l = s$ the coefficients $c_{\sigma\tau}^\rho$ are constant and the forces are expressed via the forcing function U:

$$Q_\tau^* = \beta_\tau^\sigma \frac{\partial U}{\partial q^\sigma}\,,\qquad \sigma,\tau = \overline{1,s}\,.$$

In this case equations (3.25) can be represented in the form, suggested by H. Poincaré:

$$\begin{gathered}\frac{d}{dt}\frac{\partial L^*}{\partial v_*^\tau} = c_{\sigma\tau}^\rho v_*^\sigma \frac{\partial L^*}{\partial v_*^\rho} + \beta_\tau^\sigma \frac{\partial L^*}{\partial q^\sigma}\,,\qquad L^*(q, v_*) = T^* + U\,,\\ \rho,\sigma,\tau = \overline{1,s}\,.\end{gathered} \tag{3.26}$$

When generated equations of motion (3.26) H. Poincaré made use of the group theory. The approach of Poincaré was developed then in the works of N. G. Chetaev, L. M. Markhashov, V. V. Rumyantsev, and Fam Guen. They generalized Poincaré's equations to the case when the coefficients $c_{\sigma\tau}^\rho$ are not constant and the motion is acted by as potential as nonpotential forces. Besides, V. V. Rumyantsev considered the case of nonlinear first-order nonholonomic constraints. These equations, describing the motion of nonholonomic systems, are called *the Poincaré–Chetaev–Rumyantsev equations.* For detail, see Chapter VII.

§ 4. The examples of applications of different kinds equations of nonholonomic mechanics

E x a m p l e II.1. *The motion of a double-mass system with holonomic and nonholonomic constraints* (the application of Maggi's equations). Consider in the horizontal plane Oxy the motion of two points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ with masses m. They are connected by a rigid rod whose mass is ignored and length is $2l$ (Fig. II. 1,a). The other similar examples of the systems of finite numbers of material particles with nonholonomic constraints are considered in the work [84]. The short runner with clinches (a skate) is horizontally fixed at the middle point C of the rod at right angles. The runner has a knife-edge and therefore it allows the displacement without friction along the knife-edge but it does not allow the motion in perpendicular direction. We assume that since the runner is of sufficiently small length and has clinches, the system can freely rotate about its center.

The following holonomic constraint

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (2l)^2$$

is imposed on the motion of points. Then the position of system is uniquely defined by three parameters. As the generalized coordinates we shall regard the Cartesian coordinates x, y of the middle point of rod and the angle θ between the direction of the rod and the axis Oz:

$$q^1 = x\,, \qquad q^2 = \theta\,, \qquad q^3 = y\,. \tag{4.1}$$

Then we have

$$\begin{aligned} \dot x_1 &= \dot x + \dot\theta l \sin\theta\,, \qquad & \dot y_1 &= \dot y - \dot\theta l \cos\theta\,, \\ \dot x_2 &= \dot x - \dot\theta l \sin\theta\,, \qquad & \dot y_2 &= \dot y + \dot\theta l \cos\theta\,. \end{aligned} \tag{4.2}$$

Now we obtain the equation of nonholonomic constraint. Since the runner is at the point C of the middle of rod, this point may have only the velocity perpendicular to the axis of rod. The projections of velocity of any of two points of rigid body on the straight line, passing through these points, are equal. Since there exists a skate, the velocity of middle point of rod $\mathbf{v}$ has no a projection on the axis of rod and therefore the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the points M_1 and M_2 also have no this projection. This condition can be represented as

$$\frac{\dot x_1}{\dot x_2} = \frac{\dot y_1}{\dot y_2}\,.$$

With (4.2) this implies that

$$\dot\theta(\dot x \cos\theta + \dot y \sin\theta) = 0\,.$$

The above equation is satisfied for $\dot\theta = 0$ or under the condition

$$\dot x \cos\theta + \dot y \sin\theta = 0\,. \tag{4.3}$$

In the case $\dot\theta = 0$ the angle θ is constant and therefore we have a translational motion for the linear displacement of the point C. Such a motion is realized for a long runner, which opposes the rotation of system round the point C. Since we consider the case of short runner, the nonholonomic constraint is given by (4.3). The represented in Fig. II. 1,a system with nonholonomic constraint (4.3) may explain, in particular, the motion, on one skate, of the vertically standing figure-skater and in the case $\dot\theta = 0$ the motion of the skater on racing skates.

Note that constraint (4.3) is satisfied for $\dot\theta = 0$ and for $\dot\theta \neq 0$. In this case we cannot assume that the equation $\dot\theta(\dot x\cos\theta + \dot y\sin\theta) = 0$ is more general than equation (4.3) and therefore we cannot treat its as the example of non-linear nonholonomic constraint. Similarly, the functional relations, obtained in the other examples of the work [84], cannot also be regarded as nonlinear nonholonomic constraints.

For generation of equations of motion we obtain first the relation for the kinetic energy T. By (4.2) we have

$$T = \frac{m}{2}\left(\dot x_1^2 + \dot y_1^2 + \dot x_2^2 + \dot y_2^2\right) = m\left(\dot x^2 + \dot y^2 + \dot\theta^2 l^2\right).$$

This implies the relation

$$MW_1 = 2m\ddot x\,, \qquad MW_2 = 2ml^2\ddot\theta\,, \qquad MW_3 = 2m\ddot y\,, \tag{4.4}$$

where $M = 2m$ is a mass of representation point.

In accordance with the general theory the new velocities v_*^1, v_*^2, v_*^3 are introduced by formulas $v_*^1 = \dot q^1 \equiv \dot x$, $v_*^2 = \dot q^2 \equiv \dot\theta$, $v_*^3 = \dot x\cos\theta + \dot y\sin\theta$. Then we obtain

$$\dot x \equiv \dot q^1 = v_*^1\,, \qquad \dot\theta \equiv \dot q^2 = v_*^2\,, \qquad \dot y \equiv \dot q^3 = \frac{v_*^3 - v_*^1\cos\theta}{\sin\theta}\,. \tag{4.5}$$

By (4.4), (4.5) Maggi's equations (2.10) take the form

$$\begin{gathered} 2m\ddot x - Q_1 + (2m\ddot y - Q_3)(-\operatorname{ctg}\theta) = 0\,, \\ 2ml^2\ddot\theta - Q_2 = 0\,. \end{gathered} \tag{4.6}$$

We remark that the second equation coincides with usual Lagrange's equation of the second kind, which corresponds to the generalized coordinate θ since in the equation of nonholonomic constraint (4.3) is lacking the velocity $\dot\theta$.

System of equations (4.6) must be supplemented by the equation of constraint (4.3). Differentiating it in time, we obtain

$$\ddot x\cos\theta - \dot x\dot\theta\sin\theta + \ddot y\sin\theta + \dot y\dot\theta\cos\theta = 0\,. \tag{4.7}$$

Solving the system of equations (4.6) and (4.7) as the system of algebraic linear nonhomogeneous equations in unknowns $\ddot x, \ddot y, \ddot\theta$ and representing the obtained results as the system of six first-order differential equations, we have

$$\dot{x} = v_x\,, \quad \dot{y} = v_y\,, \quad \dot{\theta} = \omega_z\,,$$
$$\dot{v}_x = \omega_z(v_x \sin\theta - v_y \cos\theta)\cos\theta + (Q_1 \sin\theta - Q_3 \cos\theta)\sin\theta/(2m)\,,$$
$$\dot{v}_y = \omega_z(v_x \sin\theta - v_y \cos\theta)\sin\theta - (Q_1 \sin\theta - Q_3 \cos\theta)\cos\theta/(2m)\,,$$
$$\dot{\omega}_z = Q_2/(2ml^2)\,.$$

This normal form of the system of differential equations is convenient to use the numerical integration methods.

For the computation of generalized reaction of nonholonomic constraint by formula (2.12) we have now the following relation

$$\Lambda = (2m\ddot{y} - Q_3)/\sin\theta\,.$$

Consider the motion of system acted by the force $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$, imposed at the point C, and when there exists the moment $\mathbf{N} = N_z\mathbf{k}$. Besides, we also take into account the resistance forces $\mathbf{F}_1^{\mathrm{resist}} = -\mu\mathbf{v}_1$, $\mathbf{F}_2^{\mathrm{resist}} = -\mu\mathbf{v}_2$ ($\mu = \mathrm{const}$), applied at the points M_1, M_2 (Fig. II. 1,*a*), in which case to generalized coordinates (4.1) correspond the following generalized forces:

$$Q_1 \equiv Q_x = F_x - 2\mu\dot{x}\,, \quad Q_2 \equiv Q_\theta = N_z - 2\mu l^2\dot{\theta}\,, \quad Q_3 \equiv Q_y = F_y - 2\mu\dot{y}\,.$$

For concrete computation we assumed that $m = 7$ *kg*, $l = 1$ *m*, $\mu = 0.6$ *N·s/m*, $F_x = F_y = 2$ *N*. In Fig. II. 1,*b* are given three trajectories, which the point C traces in the time 15 *s* for $N_{z1} = 1$ *N·m*, $N_{z2} = 0.65$ *N·m*, $N_{z3} = 0.3$ *N·m*. The initial data are zero.

E x a m p l e II.2. *The motion of figure-skater* (the application of Chaplygin's equations). Now we apply Chaplygin's equations to the solution of the

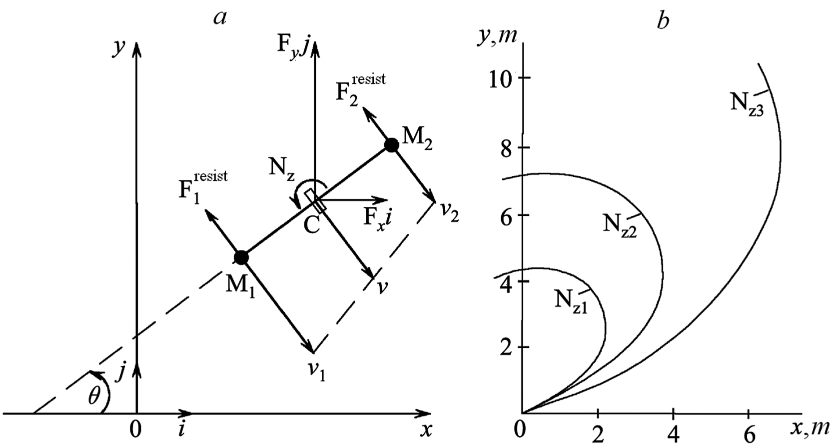

Fig. II. 1

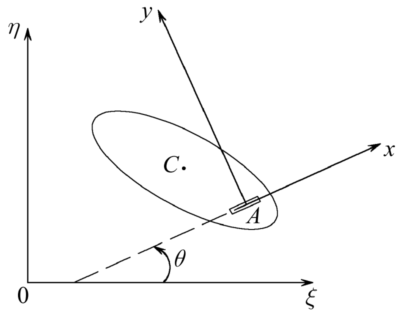

Fig. II. 2

following problem: find the motion of lop-sided figure-skater on the short skate A (Fig. II. 2).

We introduce the moving and stationary coordinates; $Axyz$ and $O\xi\eta\zeta$, respectively,. The motion is acted by the resistance force $\mathbf{F}_{\text{resist}} = -\kappa_1 \mathbf{v}_C$ and the drag torque $\mathbf{N}_{\text{resist}} = -\kappa_2 \boldsymbol{\omega}$, C is a center of mass of figure-skater.

Since the figure-skater can move only along a skate going round at a time, the constraint, imposed on the system considered, consists in that the velocity of the point A is always directed along the moving axis Ax, i. e. its projection v_{Ay} on the axis Ay is equal to zero at each moment of time. Denote the unit vectors of stationary coordinates $O\xi\eta\zeta$ by $\mathbf{i}_1$, $\mathbf{j}_1$, $\mathbf{k}_1$ and the coordinates of a center of gravity in the stationary coordinates by ξ_C, η_C. The coordinates of a center of gravity in the moving coordinates $Axyz$ are assumed to be the following: $x_C = \alpha$, $y_C = \beta$.

As the generalized coordinates of system we regard the coordinates of the point A and the angle between the axes Ax and $O\xi$, namely

$$q^1 = \xi \,, \qquad q^3 = \eta \,, \qquad q^2 = \theta \,.$$

We obtain now the equation of constraint. Represent the constraint in terms of projections of the vector $\mathbf{v}_A$ on the fixed axis $O\xi\eta$, taking into account that

$$\mathbf{v}_A = v_{A\xi}\mathbf{i}_1 + v_{A\eta}\mathbf{j}_1 = \dot{\xi}\mathbf{i}_1 + \dot{\eta}\mathbf{j}_1 \,.$$

The projection of the vector $\mathbf{v}_A$ on the axis Ay has the form

$$v_{Ay} = -\dot{\xi}\sin\theta + \dot{\eta}\cos\theta \,.$$

Then the equation of constraint $v_{Ay} = 0$ can be written as

$$\varphi(t, q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \dot{q}^3) \equiv -\dot{\xi}\sin\theta + \dot{\eta}\cos\theta = 0 \,. \tag{4.8}$$

The kinetic energy is determined by the König theorem:

$$T = \frac{1}{2}M\left[(\dot{\xi} - \dot{\theta}(\alpha\sin\theta + \beta\cos\theta))^2 + (\dot{\eta} + \dot{\theta}(\alpha\cos\theta - \beta\sin\theta))^2 + k_C^2\dot{\theta}^2\right], \tag{4.9}$$

where k_C is a radius of inertia of body relative to the axis, passing through the center of gravity and perpendicularly to the plane of motion, M is a mass of system.

After the transformation in accordance with the equation of constraint, the relation for kinetic energy takes the form

$$(T)=\frac{1}{2}M\left[(\dot\xi-\dot\theta(\alpha\sin\theta+\beta\cos\theta))^2+(\dot\xi\,\mathrm{tg}\,\theta+\dot\theta(\alpha\cos\theta-\beta\sin\theta))^2+k_C^2\dot\theta^2\right].$$

Now we write Chaplygin's equation in unknown coordinate ξ:

$$\frac{d}{dt}\frac{\partial(T)}{\partial\dot\xi}-\frac{\partial(T)}{\partial\xi}+\frac{\partial T}{\partial\dot\eta}\left[\left(\frac{\partial b_1^3}{\partial\xi}-\frac{\partial b_1^3}{\partial\xi}\right)\dot\xi+\left(\frac{\partial b_2^3}{\partial\xi}-\frac{\partial b_1^3}{\partial\theta}\right)\dot\theta\right]=Q_\xi\,. \tag{4.10}$$

By the above notions Chaplygin's equation for constraint takes the form

$$\dot q^3=b_1^3\dot q^1+b_2^3\dot q^2\,,\qquad b_1^3=\mathrm{tg}\,\theta\,,\quad b_2^3=0\,.$$

In this case equation (4.10) can be represented as

$$\frac{d}{dt}\frac{\partial(T)}{\partial\dot\xi}+\frac{\partial T}{\partial\dot\eta}\left(-\frac{\partial b_1^3}{\partial\theta}\right)\dot\theta=Q_\xi\,.$$

Using the relation for kinetic energy, we obtain

$$\ddot\xi+\dot\xi\dot\theta\,\mathrm{tg}\,\theta-\ddot\theta\beta\cos\theta-\dot\theta^2\alpha\cos\theta=\frac{Q_\xi\cos^2\theta}{M}\,.$$

We generate now the equation of motion over the coordinate θ. Performing similar numerical computation, we obtain

$$\gamma^2\cos^2\theta\ddot\theta-\beta\cos\theta\ddot\xi+(\alpha\cos\theta-\beta\sin\theta)\dot\xi\dot\theta=\frac{Q_\theta\cos^2\theta}{M}\,,$$

where $\gamma^2=\alpha^2+\beta^2+k_C^2$.

The generalized forces, acting on the system, are the following

$$Q_\xi=-\kappa_1\dot\xi\,,\qquad Q_\eta=-\kappa_1\dot\eta\,,\qquad Q_\theta=-\kappa_2\dot\theta\,. \tag{4.11}$$

Finally, we obtain a system of differential equations in Chaplygin's form, describing the motion of a figure-skater in the case when its center of mass lies not above the skate:

$$\begin{gathered}\ddot\xi+\dot\xi\dot\theta\,\mathrm{tg}\,\theta-\ddot\theta\beta\cos\theta-\dot\theta^2\alpha\cos\theta=-\kappa_1\dot\xi\cos^2\theta/M\,,\\ \gamma^2\cos^2\theta\ddot\theta-\beta\cos\theta\ddot\xi+(\alpha\cos\theta-\beta\sin\theta)\dot\xi\dot\theta=-\kappa_2\dot\theta\cos^2\theta/M\,,\\ \dot\eta=\dot\xi\,\mathrm{tg}\,\theta\,.\end{gathered} \tag{4.12}$$

Note that the considered motion of a figure-skater is one of possible interpretations of the motion of Chaplygin's sledge. One more problem, related to Chaplygin's sledge, is considered in Appendix D.

E x a m p l e II.3. *The motion of figure-skater* (application of Maggi's equations). We generate Maggi's equations for the problem considered in Example II. 2. The frames of reference and the generalized coordinates are introduced as above. Then the relations for the kinetic energy T and the covariant components of generalized forces Q_ξ, Q_θ, Q_η are given by formulas (4.9) and (4.11). The equation of constraint (4.8) has the form

$$\dot\xi \operatorname{tg}\theta - \dot\eta = 0\,. \tag{4.13}$$

We introduce the new nonholonomic variables in the following way:

$$v_*^1 = \dot\xi\,, \qquad v_*^2 = \dot\theta\,, \qquad v_*^3 = \dot\xi \operatorname{tg}\theta - \dot\eta\,.$$

Having performed the change of the old variables to the new ones, we obtain the following inverse transformation

$$\dot\xi = v_*^1\,, \qquad \dot\theta = v_*^2\,, \qquad \dot\eta = v_*^1 \operatorname{tg}\theta - v_*^3\,.$$

Using these formulas, we can compute the derivatives:

$$\begin{aligned}
&\frac{\partial \dot q^1}{\partial v_*^1} = 1\,, \qquad && \frac{\partial \dot q^2}{\partial v_*^1} = 0\,, \qquad && \frac{\partial \dot q^3}{\partial v_*^1} = \operatorname{tg}\theta\,,\\
&\frac{\partial \dot q^1}{\partial v_*^2} = 0\,, && \frac{\partial \dot q^2}{\partial v_*^2} = 1\,, && \frac{\partial \dot q^3}{\partial v_*^2} = 0\,,\\
&\frac{\partial \dot q^1}{\partial v_*^3} = 0\,, && \frac{\partial \dot q^2}{\partial v_*^3} = 0\,, && \frac{\partial \dot q^3}{\partial v_*^3} = 1\,.
\end{aligned}$$

Using the computed coefficients in Maggi's equations (2.10) and performing some simplifications, we obtain the differential equations of motion for the system

$$\begin{gathered}
\ddot\xi + \ddot\eta \operatorname{tg}\theta - \ddot\theta \frac{\beta}{\cos\theta} - \dot\theta^2 \frac{\alpha}{\cos\theta} = -\frac{\kappa_1}{M}(\dot\xi + \dot\eta \operatorname{tg}\theta)\,,\\
\gamma^2\ddot\theta + \ddot\eta(\alpha \operatorname{tg}\theta - \beta\sin\theta) - \ddot\xi(\alpha\sin\theta + \beta\cos\theta) = -\frac{\kappa_2}{M}\dot\theta\,.
\end{gathered} \tag{4.14}$$

These equations should be integrated together with equation of constraint (4.13).

Compare the obtained results with those, get in Example II. 2. Using the method of Chaplygin, we change in system (4.14) the quantities $\dot\eta$ and $\ddot\eta$ to their expressions from the equation of nonholonomic constraint (4.13). Then we have

$$\begin{gathered}
\ddot\xi + \operatorname{tg}\theta\left(\ddot\xi \operatorname{tg}\theta + \dot\xi\dot\theta\frac{1}{\cos^2\theta}\right) - \ddot\theta\frac{\beta}{\cos\theta} - \dot\theta^2\frac{\alpha}{\cos\theta} = -\frac{\kappa_1}{M}(\dot\xi + \dot\xi \operatorname{tg}^2\theta)\,,\\
\gamma^2\ddot\theta + \left(\ddot\xi \operatorname{tg}\theta + \dot\xi\dot\theta\frac{1}{\cos^2\theta}\right)(\alpha\cos\theta - \beta\sin\theta) -\\
-\ddot\xi(\alpha\sin\theta + \beta\cos\theta) = -\frac{\kappa_2}{M}\dot\theta\,.
\end{gathered}$$

After the transformations we arrive at the system

$$\ddot{\xi}\frac{1}{\cos^2\theta} + \dot{\xi}\dot{\theta}\frac{\operatorname{tg}\theta}{\cos^2\theta} - \ddot{\theta}\frac{\beta}{\cos\theta} - \dot{\theta}^2\frac{\alpha}{\cos\theta} = -\frac{\kappa_1}{M}\dot{\xi}\,,$$
$$\gamma^2\ddot{\theta} - \ddot{\xi}\frac{\beta}{\cos\theta} + \dot{\xi}\dot{\theta}\frac{(\alpha\cos\theta - \beta\sin\theta)}{\cos^2\theta} = -\frac{\kappa_2}{M}\dot{\theta}\,.$$

It is easily remarked that multiplying these equations by $\cos^2\theta$, we obtain Chaplygin's equations (4.12), generated in Example II. 2.

Thus, Maggi's equations give a more simple method to find the equations of motion than Chaplygin's equations, in which case it is not required that the mechanical system satisfies additional conditions. It is sufficient only to generate the relations for the kinetic energy and generalized forces, to choice rationally new nonholonomic variables, to find the derivatives of inverse transformation, and to construct the linear combinations of the Lagrange operators. Besides, by equation (2.12) we can easily write the relations for generalized reactions of nonholonomic constraints. For the considered problem we obtain

$$\frac{\Lambda}{M} = \ddot{\xi}\operatorname{tg}\theta + \dot{\xi}\dot{\theta}\frac{1}{\cos^2\theta} + \ddot{\theta}(\alpha\cos\theta - \beta\sin\theta) - \dot{\theta}^2(\alpha\sin\theta + \beta\cos\theta) + \frac{\kappa_1}{M}\dot{\xi}\operatorname{tg}\theta\,.$$

In Fig. II. 3 are given the results of numerical integrating the system of differential equations for 10 s. Here we assumed that

$$\gamma^2 = 0.07\ m^2\,,\quad \kappa_1/M = 1\ s^{-1}\,,\quad \kappa_2/M = 0.02\ m^2\cdot s^{-1}\,,$$
$$\xi(0) = 0\,,\quad \dot{\xi}(0) = 5\ m\cdot s^{-1}\,,\quad \eta(0) = 0\,,\quad \dot{\eta}(0) = 0\,,$$
$$\theta(0) = 0\,,\quad \dot{\theta}(0) = 12.5\ s^{-1}\,,\quad \alpha = 0\,,\quad \beta = 0\,.$$

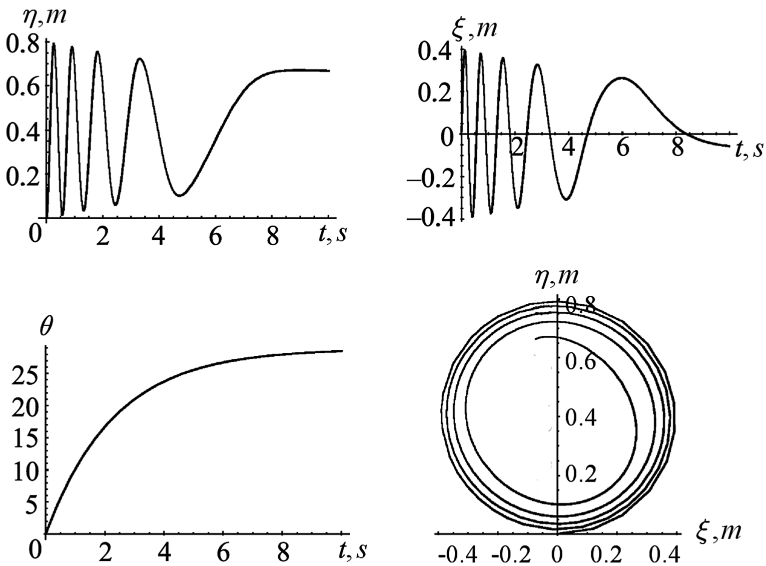

Fig. II. 3

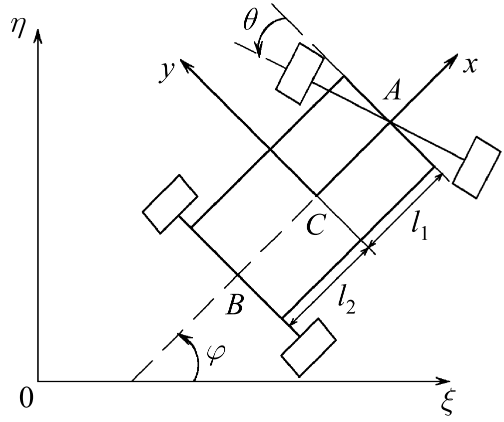

Fig. II. 4

E x a m p l e II.4. *The motion of car in a sweep* (the application of the Hamel–Boltzmann equations). Consider a motion of car (Fig. II. 4), consisting of a body of the mass M_1 and a front axis with the mass M_2. Suppose, they have the moments of inertia J_1 and J_2 about the vertical axes through their centers of mass, respectively. The front axis can rotate about their vertical axis through its center of mass. The masses of wheels and backward axis, regarded as separate parts, are assumed to be negligible. The motion of car is subject to the force $F_1(t)$, acting along its longitudinal axis Cx, and to the moment $L_1(t)$, rotating the front axis. In this case $F_1(t)$, $L_1(t)$ are the given functions of time. In addition, we take into account the resistance force $F_2(v_C)$, acting in the direction opposite to the direction of the velocity $\mathbf{v}_C$ of the center of mass C of body, the drag torque $L_2(\dot{\theta})$, which is applied to the front axis and is opposite to the angular velocity of its rotating, and the righting moment $L_3(\theta)$. A similar scheme was considered in the work [132] as the simplified mathematical model for the car motion on a sweep. At present it can be of interest when studying wheeled robot vehicles [146-148, 423].

We generate the Hamel–Boltzmann equations for the study of the motion of this system.

The motion of car in the horizontal plane is considered in the fixed coordinates $O\xi\eta\zeta$. The car position is given by the following generalized coordinates: $q^1 = \varphi$ is an angle between the longitudinal axis of car Cx and the axis $O\xi$, $q^2 = \theta$ is an angle between the front axis and the perpendicular to the axis Cx, and $q^3 = \xi_C$, $q^4 = \eta_C$ are coordinates of the point C.

On the motion of car the two nonholonomic constraints are imposed which express that the sideways sliding motion of the backward and front axes of car is missed. Their equations can be written similarly to formula (4.8) from Example II. 2:

$$\begin{gathered} -\dot{\xi}_B \sin\varphi + \dot{\eta}_B \cos\varphi = 0\,, \\ -\dot{\xi}_A \sin(\varphi+\theta) + \dot{\eta}_A \cos(\varphi+\theta) = 0\,. \end{gathered} \tag{4.15}$$

Here ξ_A,η_A,ξ_B,η_B are the coordinates of the centers of mass for the front and backward axes of car. We assume that the distances between the centers of mass of these axes and the center of gravity of the body of car are equal to l_1 and l_2, respectively. Then the equations of nonholonomic constraints (4.15) take the form

$$\begin{aligned} \varphi^1 &\equiv -\dot{\xi}_C \sin\varphi + \dot{\eta}_C \cos\varphi - l_2\dot{\varphi} = 0\,, \\ \varphi^2 &\equiv -\dot{\xi}_C \sin(\varphi+\theta) + \dot{\eta}_C \cos(\varphi+\theta) + l_1\dot{\varphi}\cos\theta = 0\,. \end{aligned} \tag{4.16}$$

We introduce the quasivelocities by formulas

$$\begin{aligned} \dot{\pi}^1 = \dot{\varphi}\,, \qquad \dot{\pi}^2 &= \dot{\theta}\,, \\ \dot{\pi}^3 = -\dot{\xi}_C \sin\varphi + \dot{\eta}_C &\cos\varphi - l_2\dot{\varphi}\,, \\ \dot{\pi}^4 = -\dot{\xi}_C \sin(\varphi+\theta) + \dot{\eta}_C &\cos(\varphi+\theta) + l_1\dot{\varphi}\cos\theta\,, \end{aligned} \tag{4.17}$$

i. e. in formulas (2.17) the coefficients $a^\rho_\sigma(q)$, $\rho,\sigma = \overline{1,4}$, have the form

$$\begin{gathered} a^1_1 = 1\,, \quad a^2_2 = 1\,, \quad a^3_1 = -l_2\,, \quad a^3_3 = -\sin\varphi\,, \quad a^3_4 = \cos\varphi\,, \\ a^4_2 = l_1\cos\theta\,, \quad a^4_3 = -\sin(\varphi+\theta)\,, \quad a^4_4 = \cos(\varphi+\theta)\,. \end{gathered}$$

To formulas (4.17) corresponds the inverse transformation

$$\begin{aligned} \dot{q}^1 &\equiv \dot{\varphi} = \dot{\pi}^1\,, \qquad \dot{q}^2 \equiv \dot{\theta} = \dot{\pi}^2\,, \\ \dot{q}^3 &\equiv \dot{\xi}_C = b^3_1\dot{\pi}^1 + b^3_3\dot{\pi}^3 + b^3_4\dot{\pi}^4\,, \\ \dot{q}^4 &\equiv \dot{\eta}_C = b^4_1\dot{\pi}^1 + b^4_3\dot{\pi}^3 + b^4_4\dot{\pi}^4\,, \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} b^3_1 &= (l_1\cos\varphi\cos\theta + l_2\cos(\varphi+\theta)/\sin\theta\,, \\ b^3_3 &= \cos(\varphi+\theta)/\sin\theta\,, \qquad b^3_4 = -\cos\varphi/\sin\theta\,, \\ b^4_1 &= (l_1\sin\varphi\cos\theta + l_2\sin(\varphi+\theta))/\sin\theta\,, \\ b^4_3 &= \sin(\varphi+\theta)/\sin\theta\,, \qquad b^4_4 = -\sin\varphi/\sin\theta\,. \end{aligned}$$

The rest of coefficients a^ρ_σ and b^σ_ρ, are equal to zero.

Thus, in transformation (2.17) the matrices (a^ρ_σ) and (b^σ_ρ) are obtained. Now we can compute the coefficients of nonholonomicity by formulas (2.19). We have

$$\begin{aligned} &\gamma_{133} = -\gamma_{331} = b^3_3\cos\varphi + b^4_3\sin\varphi\,, \\ &\gamma_{134} = -\gamma_{431} = b^3_4\cos\varphi + b^4_4\sin\varphi\,, \\ &\gamma_{241} = -\gamma_{142} = l_1\sin\theta + b^3_1\cos(\varphi+\theta) + b^4_1\sin(\varphi+\theta)\,, \\ &\gamma_{143} = -\gamma_{341} = \gamma_{243} = -\gamma_{342} = b^3_3\cos(\varphi+\theta) + b^4_3\sin(\varphi+\theta)\,, \\ &\gamma_{144} = -\gamma_{441} = \gamma_{244} = -\gamma_{442} = b^3_4\cos(\varphi+\theta) + b^4_4\sin(\varphi+\theta)\,. \end{aligned} \tag{4.19}$$

The rest of quantities $\gamma_{\lambda(l+\varkappa)\lambda^*}$, are equal to zero.

The kinetic energy of system consists of the kinetic energies of the body and the front axis and is computed by formula

$$2T = M^*(\dot{\xi}_C^2 + \dot{\eta}_C^2) + J^*\dot{\varphi}^2 + J_2\dot{\theta}^2 + 2J_2\dot{\varphi}\dot{\theta} + 2M_2 l_1\dot{\varphi}(-\dot{\xi}_C\sin\varphi + \dot{\eta}_C\cos\varphi)\,,$$
$$M^* = M_1 + M_2\,, \qquad J^* = J_1 + J_2 + M_2 l_1^2\,. \tag{4.20}$$

The generalized forces, acting on the car, can be represented as

$$\begin{aligned}
&Q_1 \equiv Q_\varphi = 0\,,\\
&Q_2 \equiv Q_\theta = L_1(t) - L_2(\dot{\theta}) - L_3(\theta)\,,\\
&Q_3 \equiv Q_{\xi_C} = F_1(t)\cos\varphi - F_2(v_C)\dot{\xi}_C/v_C\,,\\
&Q_4 \equiv Q_{\eta_C} = F_1(t)\sin\varphi - F_2(v_C)\dot{\eta}_C/v_C\,, \quad v_C = \sqrt{\dot{\xi}_C^2 + \dot{\eta}_C^2}\,.
\end{aligned} \tag{4.21}$$

Then by (2.18) we have

$$\widetilde{Q}_1 = (F_1(t)\cos\varphi - F_2(v_C)\dot{\xi}_C/v_C)b_1^3 + (F_1(t)\sin\varphi - F_2(v_C)\dot{\eta}_C/v_C)b_1^4\,,$$
$$\begin{aligned}
&\widetilde{Q}_2 = L_1 - L_2 - L_3\,,\\
&\widetilde{Q}_3 = (F_1(t)\cos\varphi - F_2(v_C)\dot{\xi}_C/v_C)b_3^3 + (F_1(t)\sin\varphi - F_2(v_C)\dot{\eta}_C/v_C)b_3^4\,,\\
&\widetilde{Q}_4 = (F_1(t)\cos\varphi - F_2(v_C)\dot{\xi}_C/v_C)b_4^3 + (F_1(t)\sin\varphi - F_2(v_C)\dot{\eta}_C/v_C)b_4^4\,.
\end{aligned} \tag{4.22}$$

Using formulas (4.18) and (4.20) we can construct the relation for T^*:

$$\begin{aligned}
2T^* = \dot{\pi}_1^2\Big(&M^*\big((\beta_1^3)^2 + (\beta_1^4)^2\big) +\\
&+ J^* + M_2 l_1^2 + 2M_2 l_1\big(\beta_1^4\cos\varphi - \beta_1^3\sin\varphi\big)\Big) +\\
&+ \dot{\pi}_2^2 J_2 + \dot{\pi}_3^2\Big(M^*\big((\beta_3^3)^2 + (\beta_3^4)^2\big)\Big) + \dot{\pi}_4^2\Big(M^*\big((\beta_4^3)^2 + (\beta_4^4)^2\big)\Big) +\\
&+ \dot{\pi}_1\dot{\pi}_2 2J_2 + \dot{\pi}_1\dot{\pi}_3\Big(2M^*\big(\beta_1^3\beta_3^3 + \beta_1^4\beta_3^4\big) + 2M_2 l_1\big(\beta_3^4\cos\varphi - \beta_3^3\sin\varphi\big)\Big) +\\
&+ \dot{\pi}_1\dot{\pi}_4\Big(2M^*\big(\beta_1^3\beta_4^3 + \beta_1^4\beta_4^4\big) + 2M_2 l_1\big(\beta_4^4\cos\varphi - \beta_4^3\sin\varphi\big)\Big) +\\
&+ \dot{\pi}_3\dot{\pi}_4\Big(2M^*\big(\beta_3^3\beta_4^3 - \beta_3^4\beta_4^4\big)\Big)\,.
\end{aligned}$$

Omitting some tedious calculations and using formulas (2.20), (4.19), and (4.22), we obtain the Hamel–Boltzmann equations (2.15) for the problem considered:

$$\left[J^* + M_2 l_1^2 + 2M_2 l_1 l_2 + M^* \left(\frac{l_2^2 + l_1^2 \cos^2\theta + 2l_1 l_2 \cos^2\theta}{\sin^2\theta} \right) \right] \ddot{\varphi} + J_2 \ddot{\theta} -$$
$$- \frac{(l_1 + l_2)^2 M^* \cos\theta}{\sin^3\theta} \dot{\varphi} \dot{\theta} =$$
$$= \left(F_1(t) \cos\varphi - \frac{F_2(v_C)\dot{\xi}_C}{v_C} \right) b_1^3 + \left(F_1(t) \sin\varphi - \frac{F_2(v_C)\dot{\eta}_C}{v_C} \right) b_1^4 ,$$
$$J_2(\ddot{\varphi} + \ddot{\theta}) = L_1(t) - L_2(\dot{\theta}) - L_3(\theta) . \tag{4.23}$$

Note that we need to solve this system together with equations of constraints (4.16).

As an example, we consider the motion of the hypothetical compact motor car with

$$M_1 = 1000\ kg\,, \quad M_2 = 110\ kg\,, \quad J_1 = 1500\ kg \cdot m^2\,,$$
$$J_2 = 30\ kg \cdot m^2\,, \qquad l_1 = 0.75\ m\,, \qquad l_2 = 1.65\ m$$

and with the following power characteristics:

$$F_1(t) = 2500\ N\,, \quad F_2(v_C) = \kappa_2 v_C\,, \quad \kappa_2 = 100\ N \cdot s \cdot m^{-1}\,,$$
$$L_1(t) = 15\ N \cdot m\,, \quad L_2(\dot{\theta}) = \kappa_1 \dot{\theta}\,, \quad \kappa_1 = 0.5\ N \cdot m \cdot s\,,$$
$$L_3(\theta) = \kappa_3 \theta\,, \quad \kappa_3 = 100\ N \cdot m\,.$$

The results of the numerical solution of nonlinear system of differential equations (4.16), (4.23) are given in Fig. II. 5. In computing we make use of the following initial data

$$\varphi(0) = 0\,, \quad \dot{\varphi}(0) = 0\,, \quad \theta(0) = \pi/180\ rad\,, \quad \dot{\theta}(0) = 0\,, \quad \xi_C(0) = 0\,,$$
$$\dot{\xi}_C(0) = 0.00176856\ m \cdot s^{-1}, \quad \eta_C(0) = 0\,, \quad \dot{\eta}_C(0) = 0.000018008\ m \cdot s^{-1}\,.$$

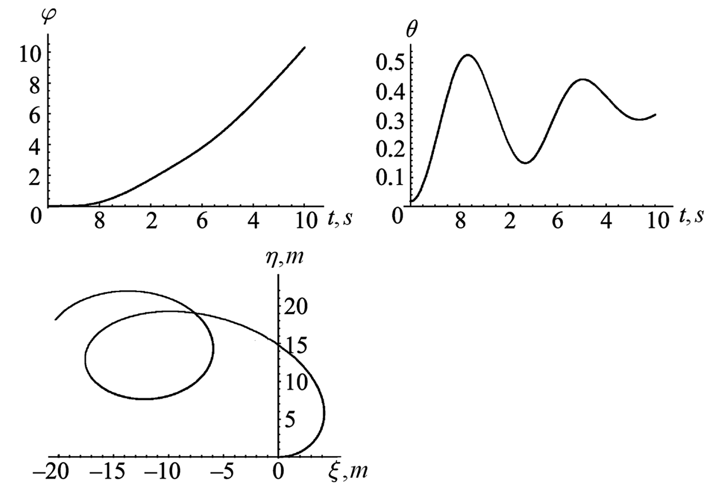

Fig. II. 5

E x a m p l e II.5. *The turning movement of a car* (the application of Maggi's equations and the Lagrange equations of the first kind). Consider now the motion of car from Example II. 4 by means of Maggi's equations. We make use of the same curvilinear coordinates. Then the equations of constraints take the form (4.16) and the kinetic energy and the generalized forces are given by formulas (4.20) and (4.21), respectively.

We introduce the following new nonholonomic variables

$$v_*^1 = \dot\varphi\,, \qquad v_*^2 = \dot\theta\,,$$
$$v_*^3 = -l_2\dot\varphi - \dot\xi_C \sin\varphi + \dot\eta_C \cos\varphi\,,$$
$$v_*^4 = l_1\dot\varphi\cos\theta - \dot\xi_C\sin(\varphi+\theta) + \dot\eta_C\cos(\varphi+\theta)$$

and write the inverse transformation

$$\begin{aligned} &\dot q^1 \equiv \dot\varphi = v_*^1\,, \qquad \dot q^2 \equiv \dot\theta = v_*^2\,,\\ &\dot q^3 \equiv \dot\xi_C = \beta_1^3 v_*^1 + \beta_3^3 v_*^3 + \beta_4^3 v_*^4\,,\\ &\dot q^4 \equiv \dot\eta_C = \beta_1^4 v_*^1 + \beta_3^4 v_*^3 + \beta_4^4 v_*^4\,, \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} &\beta_1^3 = (l_1\cos\varphi\cos\theta + l_2\cos(\varphi+\theta)/\sin\theta\,,\\ &\beta_3^3 = \cos(\varphi+\theta)/\sin\theta\,, \qquad \beta_4^3 = -\cos\varphi/\sin\theta\,,\\ &\beta_1^4 = (l_1\sin\varphi\cos\theta + l_2\sin(\varphi+\theta))/\sin\theta\,,\\ &\beta_3^4 = \sin(\varphi+\theta)/\sin\theta\,, \qquad \beta_4^4 = -\sin\varphi/\sin\theta\,. \end{aligned} \tag{4.25}$$

In this case the first Maggi's equation has the form

$$(MW_1 - Q_1)\frac{\partial\dot q^1}{\partial v_*^1} + (MW_3 - Q_3)\frac{\partial\dot q^3}{\partial v_*^1} + (MW_4 - Q_4)\frac{\partial\dot q^4}{\partial v_*^1} = 0\,. \tag{4.26}$$

Since the equations of constraints do not involve the velocity $\dot\theta$, the second Maggi's equation is Lagrange's equation of the second kind:

$$MW_2 - Q_2 = 0\,. \tag{4.27}$$

The relations MW_σ can be computed in terms of the kinetic energy by formulas

$$MW_\sigma = \frac{d}{dt}\frac{\partial T}{\partial\dot q^\sigma} - \frac{\partial T}{\partial q^\sigma}\,, \qquad \sigma = \overline{1,4}\,.$$

Then by (4.20), (4.21), (4.24), (4.25) equations of motion of car (4.26), (4.27) can be represented as

$$\begin{gathered} \big[J^* + M_2 l_1(l_1 - \beta_1^3\sin\varphi + \beta_1^4\cos\varphi)\big]\ddot\varphi + J_2\ddot\theta + (M^*\beta_1^3 - M_2 l_1\sin\varphi)\ddot\xi_C +\\ +(M^*\beta_1^4 + M_2 l_1\cos\varphi)\ddot\eta_C = M_2 l_1\dot\varphi^2(\beta_1^3\cos\varphi + \beta_1^4\sin\varphi) +\\ +\big[F_1(t)\cos\varphi - F_2(v_C)\dot\xi_C/v_C\big]\beta_1^3 + \big[F_1(t)\sin\varphi - F_2(v_C)\dot\eta_C/v_C\big]\beta_1^4\,,\\ J_2(\ddot\theta + \ddot\varphi) = L_1(t) - L_2(\dot\theta) - L_3(\theta)\,. \end{gathered} \tag{4.28}$$

If the initial data and the analytic representations of the functions $F_1(t)$, $F_2(v_C)$, $L_1(t)$, $L_2(\dot{\theta})$, $L_3(\theta)$ are given, then after numerical integrating nonlinear system of differential equations (4.16), (4.28), we find the law of motion of car

$$\varphi = \varphi(t)\,, \qquad \theta = \theta(t)\,, \qquad \xi_C = \xi_C(t), \qquad \eta_C = \eta_C(t)\,. \tag{4.29}$$

Now we can determine generalized reactions. For the second group of Maggi's equations we have

$$\begin{aligned}
\Lambda_1 &= (MW_3 - Q_3)\frac{\partial \dot{q}^3}{\partial v_*^3} + (MW_4 - Q_4)\frac{\partial \dot{q}^4}{\partial v_*^3}\,,\\
\Lambda_2 &= (MW_3 - Q_3)\frac{\partial \dot{q}^3}{\partial v_*^4} + (MW_4 - Q_4)\frac{\partial \dot{q}^4}{\partial v_*^4}\,,
\end{aligned}$$

or the same in the expanded form:

$$\begin{aligned}
\Lambda_1 =& [M^*\ddot{\xi}_C - M_2 l_1(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi) - F_1(t)\cos\varphi + F_2(v_C)\dot{\xi}_C/v_C]\beta_3^3 +\\
&+ [M^*\ddot{\eta}_C + M_2 l_1(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) - F_1(t)\sin\varphi + F_2(v_C)\dot{\eta}_C/v_C]\beta_3^4\,,\\
\Lambda_2 =& [M^*\ddot{\xi}_C - M_2 l_1(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi) - F_1(t)\cos\varphi + F_2(v_C)\dot{\xi}_C/v_C]\beta_4^3 +\\
&+ [M^*\ddot{\eta}_C + M_2 l_1(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) - F_1(t)\sin\varphi + F_2(v_C)\dot{\eta}_C/v_C]\beta_4^4\,.
\end{aligned}$$

Taking into account (4.29), we obtain the law of variation of the generalized reactions $\Lambda_i = \Lambda_i(t)$, $i = 1, 2$. Using these functions we can study the conditions, under which nonholonomic constraints (4.16) are satisfied. If the reaction forces are equal to the Coulomb friction forces, then these constraints need not be satisfied and the car can begin to slide along their axes. Note that in Appendix D the motion of nonholonomic systems in the absence of reaction forces of nonholonomic constraints is considered, and in Appendix E the turning movement of a car with possible side slipping of its wheels is studied.

Thus, Maggi's equations can be generated in the same easy manner as Lagrange's equations of the second kind. For ideal nonholonomic constraints, Maggi's equations decompose into two groups. The first group, together with the equations of constraints, permits one to find the law of motion of a nonholonomic system and then the generalized reactions can be determined from the second group. We notice that for generating the Hamel–Boltzmann equations it is required much greater calculations than for Maggi's equations.

It is of interest to compare the obtained Maggi's equations (4.28) with the Hamel–Boltzmann equations (4.23). The second equations of these systems are coincide. If we obtain the relations $\ddot{\xi}_C$ and $\ddot{\eta}_C$ from the equations of constraints and substitute them into the first equation of system (4.28), then we obtain the first equation of system (4.23).

We could also write the Lagrange equations of the first kind in curvilinear coordinates (see equations (2.22) of the present Chapter). In the problem considered they have a form:

$$J^*\ddot{\varphi} + J_2\ddot{\theta} - M_2 l_1 \ddot{\xi}_C \sin\varphi + M_2 l_1 \ddot{\eta}_C \cos\varphi = -\Lambda_1 l_2 + \Lambda_2 l_1 \cos\theta\,,$$
$$J_2(\ddot{\theta} + \ddot{\varphi}) = L_1(t) - L_2(\dot{\theta}) - L_3(\theta)\,,$$
$$M^*\ddot{\xi}_C - M_2 l_1 \ddot{\varphi} \sin\varphi - M_2 l_1 \dot{\varphi}^2 \cos\varphi =$$
$$= F_1 \cos\varphi - L_2(\dot{\theta}) - \Lambda_1 \sin\varphi - \Lambda_2 \sin(\varphi + \theta)\,,$$
$$M^*\ddot{\eta}_C + M_2 l_1 \ddot{\varphi} \cos\varphi - M_2 l_1 \dot{\varphi}^2 \sin\varphi =$$
$$= F_1 \sin\varphi - k_2 \dot{\eta}_C + \Lambda_1 \cos\varphi + \Lambda_2 \cos(\varphi + \theta)\,.$$

The four equations given include four unknown generalized coordinates and two unknown Lagrange multipliers, so they have to be solved together with the equations of constraints (4.16). This is characteristic (peculiar) for the Lagrange equations of the first kind. If we differentiate in time the equations of constraints and with the help of them eliminate generalized reaction forces, then we get Maggi's equations of motion (4.28), as well as formulas for defining Λ_1 и Λ_2.

E x a m p l e II.6. *The rolling of ellipsoid on a rough plane* (the generation of Maggi's equations). It is to be noted that the specific form of Maggi's equations depends essentially on the choice of the variables v_*^ρ. By successful choice we can considerably simplify calculations, concerning the reduction of the problem to the system of differential equations, represented in normal form.

As an example, consider the rolling of homogeneous rigid body of ellipsoidal form on the fixed plane. The center of ellipsoid coincident with centroid is taken as the origin of the moving coordinates $Cxyz$, the axes of which are rigidly fixed with the axes of body (Fig. II. 6). We assume that the plane π, on which the ellipsoid rolls, coincides with the plane $O\xi\eta$ of the fixed system of coordinates $O\xi\eta\zeta$. Denote by ξ, η, ζ the coordinates of the center of

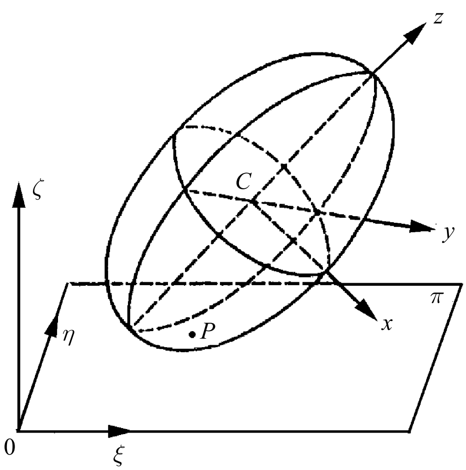

Fig. II. 6

ellipsoid relative in the fixed frame. The velocity of the contact point p can be computed by formula

$$\mathbf{v}_p = \mathbf{v}_C + \boldsymbol{\omega} \times \overrightarrow{CP}.$$

For the rolling without sliding the velocity of the point p is equal to zero and therefore the equation of constraint can be represented as

$$\mathbf{v}_C + \boldsymbol{\omega} \times \overrightarrow{CP} = \dot{\xi}\mathbf{i}_\xi + \dot{\eta}\mathbf{i}_\eta + \dot{\zeta}\mathbf{i}_\zeta + \begin{vmatrix} \mathbf{i}_\xi & \mathbf{i}_\eta & \mathbf{i}_\zeta \\ \omega_\xi & \omega_\eta & \omega_\zeta \\ \xi_0 & \eta_0 & \zeta_0 \end{vmatrix} = 0\,. \tag{4.30}$$

Here ξ_0, η_0, ζ_0 are the coordinates of the points P in the frame with $\xi_1\eta_1\zeta_1$, the axes of which ξ_1, η_1, ζ_1 are parallel to the axes ξ, η, ζ of the fixed coordinates, respectively. It can be shown that the quantities ξ_0, η_0, ζ_0 are computed by formulas

$$\begin{aligned} -\,\xi_0\zeta &= (a^2 - b^2)\sin\theta\cos\psi\sin\varphi\cos\varphi + \\ &\qquad + (c^2 - a^2\sin^2\varphi - b^2\cos^2\varphi)\sin\psi\cos\theta\sin\theta\,, \\ -\,\eta_0\zeta &= (a^2 - b^2)\sin\psi\sin\theta\sin\varphi\cos\varphi + \\ &\qquad + (a^2\sin^2\varphi + b^2\cos^2\varphi - c^2)\cos\psi\cos\theta\sin\theta\,, \end{aligned}$$

$$\zeta_0 = -\zeta = -\sqrt{a^2\sin^2\theta\sin^2\varphi + b^2\sin^2\theta\cos^2\varphi + c^2\cos^2\theta}\,.$$

Here a, b, c are the semiaxis of ellipsoid; ψ, θ, φ are the Euler angles, giving the orientation of the system of coordinates $Cxyz$ with respect to the frame with $\xi_1\eta_1\zeta_1$.

Vector equation (4.30) is equivalent to three scalar equations for nonholonomic constraints:

$$\begin{aligned} \varphi^1 &\equiv \dot{\xi} + \omega_\eta\zeta_0 - \omega_\zeta\eta_0 = 0\,, \\ \varphi^2 &\equiv \dot{\eta} + \omega_\zeta\xi_0 - \omega_\xi\zeta_0 = 0\,, \\ \varphi^3 &\equiv \dot{\zeta} + \omega_\xi\eta_0 - \omega_\eta\xi_0 = 0\,. \end{aligned} \tag{4.31}$$

As generalized Lagrangian coordinates we regard the coordinates ξ, η, ζ of the center of mass and the Euler angles ψ, θ, φ. The numerical computation of the kinetic energy of ellipsoid in these coordinates is based on applying the König theorem. Then we have

$$T = \frac{M}{2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) + \frac{J_\omega\omega^2}{2}\,.$$

The quantity $J_\omega\omega^2$ can be represented as

$$J_\omega\omega^2 = A\omega_x^2 + B\omega_y^2 + C\omega_z^2\,,$$

where A, B, C are the moments of inertia of the ellipsoid about the axes x, y, z, respectively. Since the ellipsoid, by assumption, is a homogeneous rigid body, we get

$$A = \frac{M(b^2 + c^2)}{5}\,, \qquad B = \frac{M(c^2 + a^2)}{5}\,, \qquad C = \frac{M(a^2 + b^2)}{5}\,.$$

The projections $\omega_x, \omega_y, \omega_z$ of the vector $\boldsymbol{\omega}$ on the axes of moving coordinates $Cxyz$ are the following:

$$\omega_x = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi\,,$$
$$\omega_y = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi\,,$$
$$\omega_z = \dot{\psi}\cos\theta + \dot{\varphi}\,.$$

These formulas allow one to compute the covariant components of the vector $M\mathbf{W}$

$$MW_\xi = M\ddot{\xi}\,, \qquad MW_\eta = M\ddot{\eta}\,, \qquad MW_\zeta = M\ddot{\zeta}\,,$$
$$MW_\varphi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\varphi}} - \frac{\partial T}{\partial\varphi}\,,\ MW_\psi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\psi}} - \frac{\partial T}{\partial\psi}\,,\ MW_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}\,.$$

Since the explicit relations for W_φ, W_ψ and W_θ are lengthy, they are omitted.

The quantities $\omega_\xi, \omega_\eta, \omega_\zeta$, entering into the equations of constraints (4.31), take the form

$$\omega_\xi = \dot{\varphi}\sin\psi\sin\theta + \dot{\theta}\cos\psi\,,$$
$$\omega_\eta = -\dot{\varphi}\cos\psi\sin\theta + \dot{\theta}\sin\psi\,,$$
$$\omega_\zeta = \dot{\varphi}\cos\theta + \dot{\psi}\,.$$

In this case if we assume that $v_*^1 = \dot{\xi}$, $v_*^2 = \dot{\eta}$, $v_*^3 = \dot{\zeta}$, $v_*^{3+\varkappa} = \varphi^\varkappa$, $\varkappa = \overline{1,3}$, then by reason of the complicated dependence of the functions $\varphi^\varkappa$ on the velocities $\dot{q}^\sigma$ the relations $\partial\dot{q}^\sigma/\partial v_*^\lambda$ turn out highly awkward and therefore final Maggi's equations are also complicated. The task turn out more simple if as the free variables v_*^λ we choose the angular velocities $\omega_\xi, \omega_\eta, \omega_\zeta$. It can be shown that if the quasivelocities v_*^ρ are given by formulas

$$v_*^1 = \omega_\xi\,, \qquad v_*^2 = \omega_\eta\,, \qquad v_*^3 = \omega_\zeta\,,$$
$$v_*^4 = \dot{\xi} + \omega_\eta\zeta_0 - \omega_\zeta\eta_0\,,\ v_*^5 = \dot{\eta} + \omega_\zeta\xi_0 - \omega_\xi\zeta_0\,,\ v_*^6 = \dot{\zeta} + \omega_\xi\eta_0 - \omega_\eta\xi_0\,,$$

then we get

$$\frac{\partial\dot{\xi}}{\partial\omega_\xi} = 0\,, \quad \frac{\partial\dot{\eta}}{\partial\omega_\xi} = \zeta_0\,, \quad \frac{\partial\dot{\zeta}}{\partial\omega_\xi} = -\eta_0\,,$$
$$\frac{\partial\dot{\varphi}}{\partial\omega_\xi} = \frac{\sin\psi}{\sin\theta}\,, \quad \frac{\partial\dot{\psi}}{\partial\omega_\xi} = -\frac{\sin\psi\cos\theta}{\sin\theta}\,, \quad \frac{\partial\dot{\theta}}{\partial\omega_\xi} = \cos\psi\,,$$
$$\frac{\partial\dot{\xi}}{\partial\omega_\eta} = -\zeta_0\,, \quad \frac{\partial\dot{\eta}}{\partial\omega_\eta} = 0\,, \quad \frac{\partial\dot{\zeta}}{\partial\omega_\eta} = \xi_0\,,$$
$$\frac{\partial\dot{\varphi}}{\partial\omega_\eta} = -\frac{\cos\psi}{\sin\theta}\,, \quad \frac{\partial\dot{\psi}}{\partial\omega_\eta} = \frac{\cos\psi\cos\theta}{\sin\theta}\,, \quad \frac{\partial\dot{\theta}}{\partial\omega_\eta} = \sin\psi\,,$$
$$\frac{\partial\dot{\xi}}{\partial\omega_\zeta} = \eta_0\,, \quad \frac{\partial\dot{\eta}}{\partial\omega_\zeta} = -\xi_0\,, \quad \frac{\partial\dot{\zeta}}{\partial\omega_\zeta} = 0\,,$$

$$\frac{\partial\dot\varphi}{\partial\omega_\zeta} = 0\,, \quad \frac{\partial\dot\psi}{\partial\omega_\zeta} = 1\,, \quad \frac{\partial\dot\theta}{\partial\omega_\zeta} = 0\,.$$

Substituting these relations into Maggi's equations, we obtain them in explicit form.

This example demonstrates that the problems on the rolling of one body on the surface of the other are complicated even under the assumption that the constraint, given by equation (4.30), is ideal. The dynamics of the rigid body, contacting with a rigid surface, is considered in the treatise of A. P. Markeev [143]. The new theory of the interaction of a rolling rigid body with a deformable surface is supposed by V. F. Zhuravlev [70].

E x a m p l e II.7. *Equations of motion and the technical realization of the Appell–Hamel problem* (the generation of Maggi's equations and Lagrange's equations of the first kind in the case of nonlinear nonholonomic constraints).

The example of P. Appell [270, 271] on the motion of a special nonholonomic system (Fig. II. 7, a) was of fundamental importance for the development of Analytic mechanics. This problem has intensively been considered, especially in journal "Rendiconti del circolo matematico di palermo"(1911–1912). Some works were due to E. Delassus. He considered the example of Appell in more detail in the work [298] and in his book [299]. This problem was also studied by G. Hamel [315, p. 502–505]. Until the present time the problem of Appell–Hamel is of interest for scientists (see, for example, the works [274, 376, 408]).

In the example of Appell–Hamel the motion of a disk with knife-edge on the horizontal plane $O\xi\eta$ is considered. The horizontal axis of disk through its center C is fixed in a weightless frame, the stubs of which can slide on the plane without friction (Fig. II. 7, a). The frame prevents the overturn of disk. The disk is rigidly fixed with a coaxial cylinder. The nonstretchable thread, which is turned over the two blocks fixed with the frame, is winded on the cylinder. To the end of thread the mass m is hung, the descent of which results in the rolling of disk. The axis of the mass descend is spaced at ρ from the contact point B of the disk with horizontal plane. We also assume that the frame is fixed with the parallel to BC smooth rail, preventing the swinging of mass. The disk and cylinder have the radii a and b, respectively.

Denote the angle between the plane of the rolling of disk and the axis $O\xi$ by θ, the angle of rotation of the wheel about its axis by φ, the coordinates of the mass m by x, y, z, the coordinates of the point B by ξ, η. Obviously, the coordinates are related as

$$x = \xi + \rho\cos\theta\,, \qquad y = \eta + \rho\sin\theta\,. \tag{4.32}$$

On the motion of system it is imposed the linear nonholonomic constraints

$$\dot\xi = a\dot\varphi\cos\theta\,, \qquad \dot\eta = a\dot\varphi\sin\theta\,, \tag{4.33}$$

$$\dot z = -b\dot\varphi\,. \tag{4.34}$$

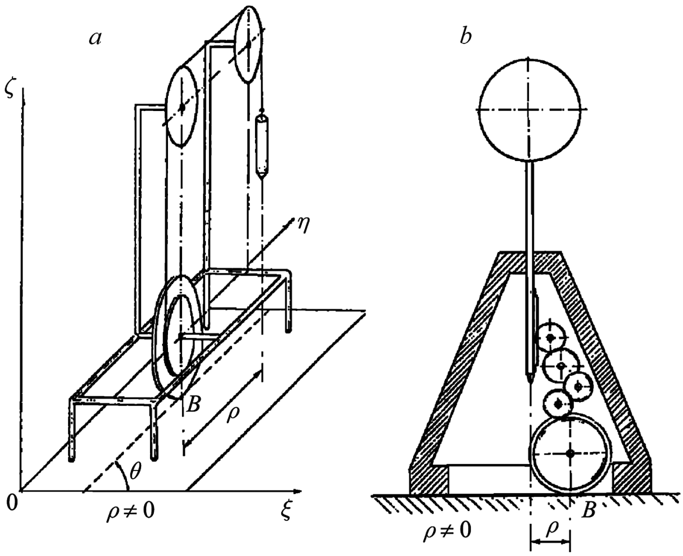

Fig. II. 7

Taking into account constraints (4.33) and (4.34), G. Hamel generates the equations of motion for the considered system [315]. Further, he analyzes the limiting case as $\rho \to 0$. In this case it is necessary to consider only the change of the coordinates x, y, z of the mass m, in which case the following nonlinear nonholonomic constraint occurs:

$$\varphi^1 \equiv \dot{x}^2 + \dot{y}^2 - \frac{a^2}{b^2}\dot{z}^2 = 0\,. \tag{4.35}$$

P. Appell also considered a similar passage to the limit, introducing the parameter α, which is the quotient of the moment of inertia of the disk about its diameter to the quantity ρ.

The problem of Appell–Hamel was considered most completely and in more detail by Yu. I. Neimark and N. A. Fufaev in the paper [164] entering into the book [166], which became the classical monograph on the nonholonomic mechanics. They notice [166, p. 227, 228] that "... the system, considered by P. Appell and G. Hamel, with nonlinear nonholonomic constraints can be obtained from the nonholonomic system with linear constraints by means of the passage to the limit as $\rho \to 0$. However in this case we have the depression of the system of differential equations, i. e. their degeneracy and therefore it is not known in advance wether the motions of the limit system ($\rho = 0$) coincide with the limit motions of nondegenerate system as $\rho \to 0$. Therefore the question remains open: to what extent the equations of motion for degenerated system describe properly the motion of the original system with the vanishingly small ρ?". The authors performed the "investigation, which was based on the study of the motions of nondegenerate system for

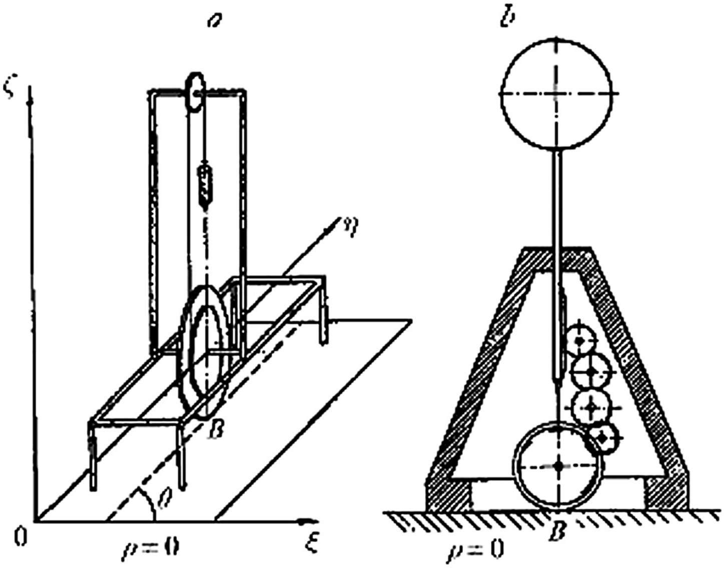

Fig. II. 8

$\rho > 0$ and $\rho < 0$, the limit motions of nondegenerate system as $|\rho| \to 0$, and the motions of degenerated system. This investigation implies that the motions of degenerated system differs essentially from the limit motions and therefore the example of nonholonomic system with nonlinear nonholonomic constraints is incorrect".

Thus, when used the mentioned above passage to the limit, P. Appell and G. Hamel in place of the study of the original system investigated the degenerated system. We regard the motion of the obtained degenerated system as the separate problem of mechanics: there is the mass m with the coordinates x, y, z, on the motion of which is imposed nonlinear nonholonomic constraint (4.35). Note that in the model of P. Appell and in the corresponding model of V. S. Novoselov [171] when the mass is connected with the disk by the set of inertialess pinions (Fig. II. 7, *b*) the case $\rho = 0$ can easily be carried out technically (Fig. II. 8, *a, b*). However for $\rho = 0$ in the mentioned above models the satisfaction of constraints (4.33) remains essential. In this case the constraints yield the following relation, imposed on the projections of velocities of the mass m:

$$\dot{y} = \dot{x} \operatorname{tg} \theta \,. \tag{4.36}$$

Here we take into account that for $\rho = 0$ by virtue of formulas (4.32) we have $\dot{x} = \dot{\xi}$, $\dot{y} = \dot{\eta}$. In studying the degenerated system constraint (4.36) is not accounted and constraint (4.35) is only introduced such that, by assumption, the velocity of the center of disk can have any direction. This means that, by using constraints (4.35) only, we replace the motion of disk by the motion of ball.

Thus, in studying the degenerated system it should be required that constraint (4.36) is satisfied, i. e., together with the coordinates x, y, z, it is necessary to look to the change of the variable θ. The neglect of the masses of disk, frame, and blocks results in the degeneracy of system and therefore the variable θ turns out to be the "massless" coordinate. If this coordinate is ignored, then it is impossible to describe the motion of massless ball by means of the motion of massless disk.

The technical realization is obviously hard in the case when the connection between the velocity of descending the load and the velocity of the center of ball is provided by means of the nonstretchable thread or the system of pinions. However it is possible to study the motion of the mass m with the coordinates x, y, z under the condition that constraint (4.35) is satisfied only. For this purpose we consider the following control problem: the motion of material point with the mass m is realized in such a way that by virtue of (4.35) its vertical velocity is varied proportionally to the velocity of the motion of its trace in horizontal plane. The realization of such problem can be provided by means of the new technical means. We generate Maggi's equations and Lagrange's equations of the first kind, using exactly such problem setting.

So, we consider the problem on the space motion of material point with nonlinear nonholonomic constraint (4.35). In this case the generalized coordinates are the following

$$q^1 = x\,, \qquad q^2 = y\,, \qquad q^3 = z\,. \tag{4.37}$$

We introduce the following new nonholonomic variables:

$$v_*^1 = \dot{x}\,, \qquad v_*^2 = \dot{y}\,, \qquad v_*^3 = \dot{x}^2 + \dot{y}^2 - \frac{a^2}{b^2}\dot{z}^2\,. \tag{4.38}$$

In the Appell–Hamel problem Maggi's equations take the form

$$\begin{aligned}
(MW_1 - Q_1)\frac{\partial \dot{q}^1}{\partial v_*^1} + (MW_2 - Q_2)\frac{\partial \dot{q}^2}{\partial v_*^1} + (MW_3 - Q_3)\frac{\partial \dot{q}^3}{\partial v_*^1} &= 0\,,\\
(MW_1 - Q_1)\frac{\partial \dot{q}^1}{\partial v_*^2} + (MW_2 - Q_2)\frac{\partial \dot{q}^2}{\partial v_*^2} + (MW_3 - Q_3)\frac{\partial \dot{q}^3}{\partial v_*^2} &= 0\,,\\
(MW_1 - Q_1)\frac{\partial \dot{q}^1}{\partial v_*^3} + (MW_2 - Q_2)\frac{\partial \dot{q}^2}{\partial v_*^3} + (MW_3 - Q_3)\frac{\partial \dot{q}^3}{\partial v_*^3} &= \Lambda\,.
\end{aligned} \tag{4.39}$$

These relations involve the derivatives $\dfrac{\partial \dot{q}^\sigma}{\partial v_*^\rho}$, $\sigma, \rho = \overline{1,3}$, for numerical computation of which it is required the transformation inverse to transformation (4.38). However, the obtaining of such transformation is a difficult task since nonholonomic constraint (4.35) is nonlinear. Therefore we obtain these derivatives in the following way. Compute the matrix

$$(\alpha_\sigma^\rho) = \left(\frac{\partial v_*^\rho}{\partial \dot{q}^\sigma}\right), \qquad \sigma, \rho = \overline{1,3}\,.$$

By (4.38) we have

$$\begin{aligned}
&\alpha_1^1 = 1\,, \quad &&\alpha_2^1 = 0\,, \quad &&\alpha_3^1 = 0\,,\\
&\alpha_1^2 = 0\,, \quad &&\alpha_2^2 = 1\,, \quad &&\alpha_3^2 = 0\,,\\
&\alpha_1^3 = 2\dot{x}\,, \quad &&\alpha_2^3 = 2\dot{y}\,, \quad &&\alpha_3^3 = -2a^2 b^{-2}\dot{z}\,.
\end{aligned}$$

Find the matrix (β_ρ^σ) inverse to the matrix (α_σ^ρ). Then we obtain

$$\begin{aligned}
&\beta_1^1 = 1\,, \quad &&\beta_2^1 = 0\,, \quad &&\beta_3^1 = 0\,,\\
&\beta_1^2 = 0\,, \quad &&\beta_2^2 = 1\,, \quad &&\beta_3^2 = 0\,,\\
&\beta_1^3 = h^2\dot{x}/\dot{z}\,, \quad &&\beta_2^3 = h^2\dot{y}/\dot{z}\,, \quad &&\beta_3^3 = -h^2/(2\dot{z})\,, \quad h^2 = b^2/a^2\,.
\end{aligned} \tag{4.40}$$

It is important for us that $(\beta_\rho^\sigma) = \left(\dfrac{\partial \dot{q}^\sigma}{\partial v_*^\rho}\right)$.

For the considered problem we have

$$T = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/2\,, \qquad \Pi = mgz\,.$$

Then, using formulas (4.40), we can generate Maggi's equations (4.39):

$$\ddot{x} + (m\ddot{z} + mg)(h^2\dot{x}/\dot{z}) = 0\,, \tag{4.41}$$

$$m\ddot{y} + (m\ddot{z} + mg)(h^2\dot{y}/\dot{z}) = 0\,, \tag{4.42}$$

$$(m\ddot{z} + mg)(-h^2/(2\dot{z})) = \Lambda\,. \tag{4.43}$$

If we now consider Lagrange's equations of the first kind

$$m\ddot{x} = \Lambda 2\dot{x}\,, \qquad m\ddot{y} = \Lambda 2\dot{y}\,, \qquad m\ddot{z} = -mg + \Lambda(-2\dot{z}/h^2)\,, \tag{4.44}$$

then it is easily seen that they are the linear combination of equations (4.41)–(4.43).

We need to solve Lagrange's equations of the first kind (4.44) together with the equation of constraint (4.35), in which case the fact that the unknowns involve the reaction Λ makes to be rather sophisticate the solution. At the same time the determining of the motion itself from equations (4.35), (4.41), (4.42) turns out more simple since the reaction can be found from equation (4.43). In addition, using Maggi's equation, the form of reaction can be obtained also in original curvilinear coordinates (4.37).

In fact, in the case of ideal nonholonomic constraint (4.35) we have

$$\mathbf{R} = \mathbf{N} = \Lambda \boldsymbol{\nabla}' \varphi^1 = \Lambda \frac{\partial \varphi^1}{\partial \dot{q}^\tau}\mathbf{e}^\tau = (m\ddot{z} + mg)(-h^2/\dot{z})(\dot{x}\mathbf{i} + \dot{y}\mathbf{j} - h^{-2}\dot{z}\mathbf{k})\,. \tag{4.45}$$

By (4.45) the direction of horizontal constraint (4.35) is opposed to the horizontal component of velocity of the mass m, what is distinctive feature of the motion of ball. Consider also nonholonomic constraint (4.36), which is convenient to rewrite as

$$\varphi^2 \equiv \dot{y} - \dot{x}\,\mathrm{tg}\,\theta = 0\,. \tag{4.46}$$

Then, by reason of the existence of constraint (4.46), together with the reaction $\mathbf{R}$ given by (4.45), we also need to consider the reaction

$$\mathbf{R}^* = \Lambda^* \boldsymbol{\nabla}' \varphi^2 = \Lambda^*(-\,\mathrm{tg}\,\theta \mathbf{i} + \mathbf{j})\,.$$

The latter provides the satisfaction of constraint (4.46) and is distinctive feature of the motion of disk.

In the case of the massless coordinate θ the value of this angle for $\rho = 0$ does not enter into the system of equations of motion and therefore it is natural to consider the rolling of the ball in place of the disk. For $\rho = 0$ we might consider the rolling of the massless disk but in this case there must exists the mechanism for the disk to be oriented in the corresponding way. The indefiniteness, which occur in determining the angle θ, can be put out if either $\rho \neq 0$ is satisfied for the massless disk, frame, and blocks, either for $\rho = 0$ with due regard any of these masses. In place of the account of their masses we can evidently consider not the material point of the mass m but the body of the same mass, which, descending in the rail mentioned above, rotates together with the frame about the axis BC.

The examples of employing the Hamel–Novoselov equations are given, in particular, in the work [65] and the examples of the application of the Poincaré–Chetaev equations in the works [149, 203, 229].

§ 5. The Suslov–Jourdain principle

Consider the vector

$$\widehat{\mathbf{V}} = \widehat{v}^\sigma \mathbf{e}_\sigma\,, \qquad \widehat{v}^\sigma \equiv \dot{q}^\sigma\,, \qquad \sigma = \overline{1, s}\,. \tag{5.1}$$

In the general case this vector $\widehat{\mathbf{V}}$ differs from the velocity of representation point since

$$\mathbf{V} = \frac{\partial \mathbf{y}}{\partial t} + \frac{\partial \mathbf{y}}{\partial q^\sigma}\dot{q}^\sigma = \frac{\partial \mathbf{y}}{\partial t} + \widehat{v}^\sigma \mathbf{e}_\sigma\,.$$

Above the new variables v_*^ρ were introduced by formulas (2.2) in place of the variables $\widehat{v}^\sigma \equiv \dot{q}^\sigma$. By assumption, there is also inverse transformation (2.3). We emphasize that in the mentioned above transformations the time t and the coordinates q^σ are regarded as parameters. Introduce the variations $\delta'\widehat{v}^\sigma$ and $\delta' v_*^\rho$ of the variables $\widehat{v}^\sigma$ and v_*^ρ, defining them as the partial differentials of these functions, related as

$$\delta'\widehat{v}^\sigma = \frac{\partial \widehat{v}^\sigma}{\partial v_*^\rho}\delta' v_*^\rho\,, \quad \delta' v_*^\rho = \frac{\partial v_*^\rho}{\partial \widehat{v}^\sigma}\delta'\widehat{v}^\sigma\,, \qquad \rho, \sigma = \overline{1, s}\,. \tag{5.2}$$

Recall that in formulas (2.2) we make use of relations (2.5) and therefore

$$\delta' v_*^{l+\varkappa} = \frac{\partial \varphi^\varkappa}{\partial \widehat{v}^\sigma}\delta'\widehat{v}^\sigma = 0\,, \qquad \sigma = \overline{1,s}\,, \quad \varkappa = \overline{1,k}\,, \tag{5.3}$$

and formulas (5.2) take the form

$$\delta'\widehat{v}^\sigma = \frac{\partial \widehat{v}^\sigma}{\partial v_*^\lambda}\delta' v_*^\lambda\,, \quad \delta' v_*^\lambda = \frac{\partial v_*^\lambda}{\partial \widehat{v}^\sigma}\delta'\widehat{v}^\sigma\,, \qquad \sigma = \overline{1,s}\,, \quad \lambda = \overline{1,l}\,. \tag{5.4}$$

Consider the vector

$$\delta'\widehat{\mathbf{V}} = \delta'\widehat{v}^\sigma \mathbf{e}_\sigma = \frac{\partial \widehat{v}^\sigma}{\partial v_*^\lambda}\delta' v_*^\lambda \mathbf{e}_\sigma = \delta' v_*^\lambda \boldsymbol{\varepsilon}_\lambda\,, \tag{5.5}$$

and construct together with the vector $\widehat{\mathbf{V}}$, given by relation (5.1), the new vector

$$\widetilde{\mathbf{V}} = \widehat{\mathbf{V}} + \delta'\widehat{\mathbf{V}} = (\widehat{v}^\sigma + \delta'\widehat{v}^\sigma)\mathbf{e}_\sigma = (\dot{q}^\sigma + \delta'\widehat{v}^\sigma)\mathbf{e}_\sigma\,.$$

We substitute the coordinates $\dot{q}^\sigma + \delta'\widehat{v}^\sigma$ of the generalized velocity $\widetilde{\mathbf{V}}$ into the equations of constraints (2.1) and expand the functions $\varphi^\varkappa$ (as the function of the variables $\dot{q}^\sigma$ only) in the Taylor series in the neighborhood of the point with the coordinates $(q^1, \ldots, q^s)$, corresponding to time t:

$$\varphi^\varkappa(t, q, \dot{q} + \delta'\widehat{v}) = \varphi^\varkappa(t, q, \dot{q}) + \boldsymbol{\nabla}'\varphi^\varkappa \cdot \delta'\widehat{\mathbf{V}} + o(|\delta'\widehat{\mathbf{V}}|)\,, \qquad \varkappa = \overline{1,k}\,. \tag{5.6}$$

These relations imply that if for the point with the coordinates $(q^1, \ldots, q^s)$ at time t the generalized velocity $\widehat{\mathbf{V}}$ is kinematically possible, then with an accuracy to the first order the velocity $\widetilde{\mathbf{V}} = \widehat{\mathbf{V}} + \delta'\widehat{\mathbf{V}}$ is also kinematically possible under the condition

$$\boldsymbol{\nabla}'\varphi^\varkappa \cdot \delta'\widehat{\mathbf{V}} = 0\,, \qquad \varkappa = \overline{1,k}\,. \tag{5.7}$$

Thus, a set of the vectors $\delta'\widehat{\mathbf{V}}$, satisfying equation (5.7), describes the kinematically possible changes, of the generalized velocity $\widehat{\mathbf{V}}$, allowed by constraints at time t when the system is in the position $(q^1, \ldots, q^s)$. The arbitrary vector $\delta'\widehat{\mathbf{V}}$, satisfying relations (5.7), is called a variation of the generalized velocity $\widehat{\mathbf{V}}$.

Since the variations $\delta' v_*^\lambda$ are linear independent, the family of Maggi's equations (2.10) is equivalent to one equation

$$\big(MW_\sigma - Q_\sigma\big)\frac{\partial \dot{q}^\sigma}{\partial v_*^\lambda}\delta' v_*^\lambda = 0\,,$$

which by virtue of formulas (5.4) can be represented as

$$\big(MW_\sigma - Q_\sigma\big)\delta'\widehat{v}^\sigma = 0 \tag{5.8}$$

or by (5.5) in the vector form:

$$\big(M\mathbf{W} - \mathbf{Y}\big) \cdot \delta'\widehat{\mathbf{V}} = 0\,. \tag{5.9}$$

It is important for us that these equations are independent of the choice of free variables v_*^λ. They are obtained as a sequence of equations of motion (2.10) and therefore as a sequence of Newton's equation (2.9), written for ideal nonholonomic constraints with reaction (2.8). We remark that by formulas (5.9), (2.8), (2.9) we have

$$\mathbf{R} \cdot \delta'\widehat{\mathbf{V}} = 0\,, \tag{5.10}$$

i. e. the reaction of ideal nonholonomic constraints is orthogonal to the vector of variation of generalized velocity.

We obtain now Maggi's equations from relation (5.9) being regarded as the initial one. Since the vector $\delta'\widehat{\mathbf{V}}$ has the form (5.5), then scalar product (5.9) is as follows

$$\big(MW_\sigma - Q_\sigma\big)\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\delta' v_*^\lambda = 0\,,$$

which implies in accordance with linear independence of variations that $\delta' v_*^\lambda$, $\lambda = \overline{1,l}$. So, we arrive at Maggi's equations (2.10).

Thus, relation (5.9) can be regarded as a differential variational principle of mechanics, according to which for systems with ideal retaining nonholonomic constraints the scalar product of the vector of constraint reaction by the variation of generalized velocity is equal to zero. This principle has been formulated in 1908–1909 by P. Jourdain [326] and in 1900 by G. K. Suslov [218], who named it a universal equation of mechanics. That is why it is reasonable to refer this principle as *the Suslov–Jourdain principle.*

E x a m p l e II.8. *The equations of motion for Novoselov's reducer* (the generation of equations of motion by means of the Suslov–Jourdain principle). We generate the equations of motion for friction reducer, which was considered first by V. S. Novoselov [170]. The reducer (Fig. II. 9) transmits the rotation from shaft *1* to shaft *2* and consists of disk A, rigidly fixed with shaft *1*, wheel B, freely rotating on shaft *3*, shaft *2* with cylinder C, and a centrifugal regulator by the masses K and N and a spring with the deflection rate c_1. The motion of muff D of regulator with the help of a cable, turned over fixed blocks O_1 and O_2, and a spring with the deflection rate c_2 results in the displacement of shaft *3* with wheel B and changes the distance ρ between the average circle of wheel B and the axis of shaft *1*. Wheel B has the radius a. The sizes given are the following: $PN = NL = LK = KP = l$.

The position of the friction reducer is given by the following generalized coordinates: the rotation angles of shafts $q^1 = \varphi_1$ and $q^2 = \varphi_2$ and the distance $q^3 = x$ of muff D from joint L. As is shown in Fig. II. 9, the distance ρ is related to x as

$$x - \rho = c \equiv \text{const}\,.$$

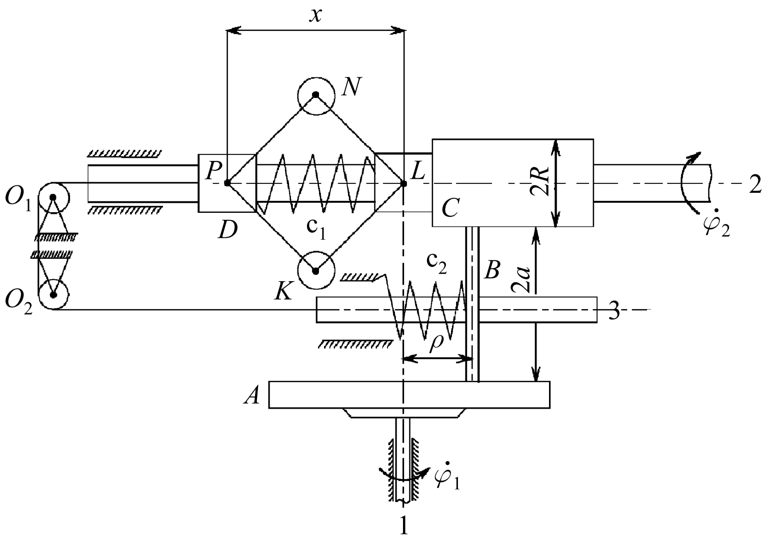

Fig. II. 9

We consider a system with the nonholonomic constraint

$$\varphi(t, q^1, q^2, q^3, \dot q^1, \dot q^2, \dot q^3) \equiv (x - c)\dot\varphi_1 - R\dot\varphi_2 = 0. \tag{5.11}$$

If the sliding is missed, then constraint (5.11) gives the condition that the rotational velocities of the points of contact of wheel B with disk A and cylinder C are equal.

The kinetic and potential energies are defined as

$$T = \frac{1}{2}\Big[J_A\dot\varphi_2^2 + J_C\dot\varphi_2^2 + m_D\dot x^2 + m_B\dot\rho^2 + J_B\frac{R^2}{a^2}\dot\varphi_2^2 +$$
$$+2m_N\left(\left(l^2 - \frac{x^2}{4}\right)\dot\varphi_2^2 + \frac{l^2\dot x^2}{4l^2 - x^2}\right)\Big],$$
$$\Pi = \frac{1}{2}c_1(\delta_1 + x - x_0)^2 + \frac{1}{2}c_2(\delta_2 + x_0 - x)^2 .$$

Here δ_1, δ_2 are the static deformations of springs with the deflection rates c_1 and c_2, respectively, x_0 is the static declination of muff D from joint L.

For this system the Suslov–Jourdain principle is as follows

$$(MW_1 - Q_1)\delta'\dot\varphi_1 + (MW_2 - Q_2)\delta'\dot\varphi_2 + (MW_3 - Q_3)\delta'\dot x = 0 . \tag{5.12}$$

The relation between the variations of velocities have the form

$$\frac{\partial\varphi}{\partial\dot\varphi_1}\delta'\dot\varphi_1 + \frac{\partial\varphi}{\partial\dot\varphi_2}\delta'\dot\varphi_2 = 0 . \tag{5.13}$$

Therefore in equation (5.12) the variations $\delta'\dot\varphi_2$ and $\delta'\dot x$ are independent. Express from relation (5.13) the variation $\delta'\dot\varphi_1$ in terms of $\delta'\dot\varphi_2$ and then from equation (5.12) we obtain

$$(MW_1 - Q_1)\frac{R}{x-c} + (MW_2 - Q_2) = 0\,, \tag{5.14}$$

$$MW_3 - Q_3 = 0\,. \tag{5.15}$$

Here $Q_1 = M_1,\ Q_2 = -M_2$ are the moments of forces, impressed upon shafts *1* and *2*, respectively, and $Q_3 = -\partial\Pi/\partial x$.

From the general theory it follows that the equations obtained coincide with Maggi's equations. We remark that the second of them is usual Lagrange's equation of the second kind since the coordinate x is holonomic.

Since

$$MW_\sigma = \frac{d}{dt}\frac{\partial T}{\partial \dot q^\sigma} - \frac{\partial T}{\partial q^\sigma}\,, \qquad \sigma = \overline{1,3}\,,$$

then equations (5.14) and (5.15) can be rewritten as

$$J_A\frac{R}{x-c}\ddot\varphi_1 + J(x)\ddot\varphi_2 - m_N x\dot x\dot\varphi_2 = M_1\frac{R}{x-c} - M_2\,,$$
$$m(x)\ddot x + \frac{1}{2}m_N x\dot\varphi_2^2 + + \frac{2l^2x}{(4l^2+x^2)^2}m_N\dot x^2 = c_1(-\delta_1 - x + x_0) + c_2(-x + x_0 + \delta_2)\,. \tag{5.16}$$

Here

$$J(x) = J_C + J_B\frac{R^2}{a^2} + \frac{1}{2}m_N(4l^2 - x^2)\,,$$
$$m(x) = m_B + m_D + \frac{2m_N l^2}{4l^2 - x^2}\,.$$

The equations of motion (5.16) together with equation of constraint (5.11) give a closed system for determining the functions $\varphi_1(t)$, $\varphi_2(t)$, $x(t)$.

It is to be noted that if we substitute equation of constraint (5.11), differentiated with respect to time, into the first equation of system (5.16), then the equations can be written in the Appell form. Such equations were also generated by A. I. Lur'e [135]. This example was also considered by Ya. L. Geronimus [47]. His results coincide with equations (5.16).

Example II.9. *The motion of mechanical system with a fluid flywheel* (the generation of the equations of motion for nonholonomic systems with the help of the Suslov–Jourdain principle and Lagrange's equation of the first kind in generalized coordinates). The fluid flywheel consists of two centrifugal wheels filled by oil: a driving torus and a turbine. The driving torus is fixed with the motor shaft and while rotating, by means of blades and centrifugal force, it speeds up the oil, which falls with a great velocity on the blades of turbine, setting the latter in motion. The turbine is placed on the shaft of consumer and therefore by means of the fluid flywheel the rotation is imparted from the leading shaft to the driven one, in which case the connection between them turns out nonrigid. At present the fluid flywheels gain a wide application in different powerful transmissions, in the starters of

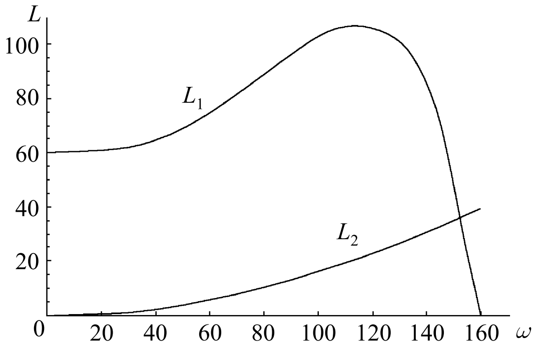

Fig. II. 10

gas turbines, the drives of pumps, the conveyors of hoisting machines, and so on. The study of transient processes in similar arrangements is of importance since the transient regime averages about sixty percents of their operation time.

Consider one of possible approaches to the study of transient processes in systems with fluid flywheel or with fluid converter, which differs from the fluid flywheel by an additional wheel (of reactor). We shall regard mechanical systems with hydrotransmissions as first-order nonholonomic systems. This permits us to eliminate the reaction and to obtain the equation of motion, which is to be integrated together with the equation of constraint.

Denote by ω_1 and L_1 an angle velocity and a moment generated by a motor, respectively, by J_1 a moment of inertia of impeller and driving parts of motor, by ω_2 and L_2 an angle velocity and a drag torque produced by customer, respectively, by J_2 a moment of inertia of a turbine and driven parts of device.

Suppose that for the racing of system the following characteristics of the motor $L_1 = L_1(\omega_1)$ and the customer $L_2 = L_2(\omega_2)$ hold which taken off for the regimes being steady-state (see. Fig. II. 10; these and all subsequent numerical data are taken from the work [106]). The quantities L_1, L_2 are given in Newton-meters, $(N{\cdot}m)$, ω_1, and ω_2 in the seconds in the minus first degree (s^{-1}), t in seconds (s).

The process of the racing of system can be partitioned into three stage. At the first stage after a starting of motor its moment L_1, applied to the driving parts of device, is used for their racing and for the racing of fluid in the fluid flywheel. When the flow occurs in the work chamber of fluid flywheel the moment L occurs on its fixed driving torus. At the end of the first stage the value of the moment L becomes sufficient for the initiation of driven parts (L_2 for $\omega_2 = 0$) and we have the second stage when the turbine begins to rotate with the ascending angle velocity ω_2.

The racing of system at the third stage is characterized by increasing the angle velocity ω_2 with deacceleration of flow, in which case the moment L, generated by the turbine, is greater than the moment, which the driving torus

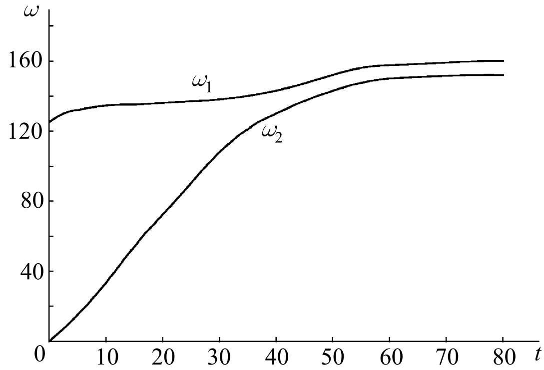

Fig. II. 11

imparts to a flow. At this stage the moment L generally decreases. When its value for the certain angle velocity ω_2 becomes equal to the moment of customer L_2, the racing is ended and the system operates in the steady-state regime for $L_1 = L = L_2$.

The analysis of experimental and computed investigations of racing processes for the systems with different relative moments of inertia $J = J_1/J_2$ and with different characteristics of a motor and a customer shows that in view of a great power of motor the angle velocity of driving shaft varies slightly and the angle velocity of driven shaft varies essentially at the initial period and tends asymptotically to a certain constant value as the mode of operation tends to the steady-state regime when we have $\omega_1 = \omega_2 + \text{const}$. For the racing of system with hydrotransmission the graphs of the functions $\omega_1 = \omega_1(t)$ and $\omega_2 = \omega_2(t)$ have a specific form shown in Fig. II. 11. These graphs demonstrate that for as nonstationary as stationary regimes we have $\omega_1 \neq \omega_2$. Therefore between the angle velocities of the driving and driven shafts there exists a certain functional relation, which can be regarded as a nonholonomic constraint. Since from the graphs of the functions $\omega_1 = \omega_1(t)$ and $\omega_2 = \omega_2(t)$ we can obtain the relation for the angle velocities as a function of time, then the equation of nonholonomic constraint can be represented as

$$\varphi(t, \omega_1, \omega_2) \equiv \omega_2 - i(t)\omega_1 = 0\,. \tag{5.17}$$

The kinetic energy of system is defined by the following relation

$$T = \frac{J_1 \omega_1^2}{2} + \frac{J_2 \omega_2^2}{2}\,.$$

Let us generate the equations of motion.

The Suslov–Jourdain principle for this system takes the form

$$(MW_1 - Q_1)\delta' \omega_1 + (MW_2 - Q_2)\delta' \omega_2 = 0\,. \tag{5.18}$$

The relation between the variations of angle velocities is given by the following formula

$$\frac{\partial \varphi}{\partial \omega_1}\delta' \omega_1 + \frac{\partial \varphi}{\partial \omega_2}\delta' \omega_2 = 0\,. \tag{5.19}$$

Hence from equation (5.18) with relations (5.17) and (5.19) we have

$$J_1 \frac{d\omega_1}{dt} + i(t) J_2 \frac{d\omega_2}{dt} = L_1 - i(t) L_2 \,. \tag{5.20}$$

Equation (5.20) together with equation of constraint (5.17) give the closed system for determining the functions $\omega_1(t)$ and $\omega_2(t)$.

Equation of motion (5.20) can also be obtained in a different way. We write Lagrange's equations of the first kind in generalized coordinates for nonholonomic system (2.22):

$$\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \omega_1} - \frac{\partial T}{\partial \varphi_1} &= L_1 + R_1 \,, \\ \frac{d}{dt}\frac{\partial T}{\partial \omega_2} - \frac{\partial T}{\partial \varphi_2} &= -L_2 + R_2 \,. \end{aligned} \tag{5.21}$$

Here the generalized reactions are defined as

$$R_1 = \Lambda \frac{\partial \varphi}{\partial \omega_1} = -i(t)\Lambda \,, \qquad R_2 = \Lambda \frac{\partial \varphi}{\partial \omega_2} = \Lambda \,.$$

Eliminating Λ from system (5.21), we obtain equation of motion (5.20).

When passing to the stationary regime, the relation $\omega_1 = \omega_2 + \mathrm{const}$ is satisfied, in which case we have

$$R_1|_{\omega_1 = \omega_2 + \mathrm{const}} = -\Lambda \frac{\partial \varphi}{\partial \omega_1}\bigg|_{\omega_1 = \omega_2 + \mathrm{const}} = -\Lambda \,,$$

i. e. $-R_1 = R_2$. The model suggested coincides with the model considered in the work [106] since in the equations of this work

$$L_1 = (J_1 + J^*) \frac{d\omega_1}{dt} + L \,, \qquad L = J_2 \frac{d\omega_2}{dt} + L_2 \,,$$

the correction $J^* \varepsilon_1$, which accounts for the moment, initiating the racing of flow in rotor blades, is equal to zero. In the above equations $R_1 = -R_2 = -L$. In the work [106] the quantity J^* and the moment L, transmitted by a fluid flywheel, are accounted empirically.

Having solved the system of equations (5.17), (5.20), we can find the reaction and, by that, the moment developed by the fluid flywheel.

The system of equations (5.17), (5.20) was numerically integrated by computer. The computing permits us to obtain the following relations: the behavior of the angle velocities of the driving and driven shafts in time, the change of the moments on the driving and driven shafts, the behavior of the moment, transmitted by fluid flywheel, in time. The certain results of computing are represented in Fig. II. 12.

Thus, the method suggested permits us, using the experimental data $\omega_1 = \omega_1(t)$, $\omega_2 = \omega_2(t)$, to describe nonstationary processes in systems with

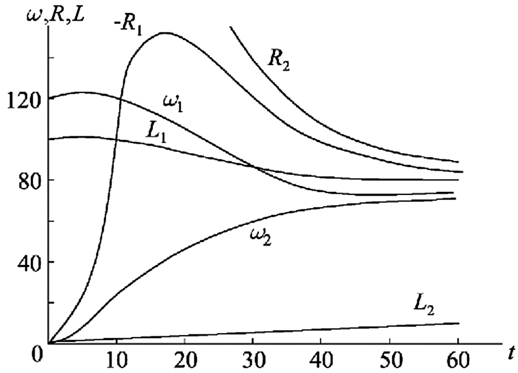

Fig. II. 12

hydrotransmission. In this case we take into account an error of experiment only while in the other methods, moreover, the additional inaccuracy occurs by reason of using the approximate theory for the account of hydrodynamic processes.

§ 6. The definitions of virtual displacements by Chetaev

As is noted in the review of the main stages of the development of nonholonomic mechanics, the definition of nonholonomic system was introduced first by Hertz [317] in 1894. He was the first to pay attention to the possible existence of such kinematical constraints, which do not impose some restrictions on the possibility of the passing of system from one position to another. The development of nonholonomic mechanics was essentially due to the work of E. Lindelöf [352], in which with the help of usual methods of holonomic mechanics there were obtained the incorrect equations of motion of nonholonomic system. In particular, S. A. Chaplygin points to this error and suggests his own method [239] to obtain the equations of motion. The error similar to Lindelöf's error was also made by C. Neumann [366], what was repeatedly remarked in the literature (see, for example, [41]). Further, in 1899, C. Neumann gives already the certainly valid equations of motion [366].

For the description of motions of nonholonomic systems, P. Appell, L. Boltzmann, V. Volterra, P. V. Voronets, G. Hamel, J. Gibbs, Bl. Dolaptschiew, G. A. Maggi, L. M. Markhashov, J. Nielsen, V. S. Novoselov, G. S. Pogosov, A. Przeborski, V. V. Rumyantsev, J. Schouten, Fam Guen, N. Ferrers, J. Tzénoff, S. A. Chaplygin, M. F. Shul'gin, and another authors suggest a number of different methods to generate the differential equations of motion. Some of them are considered, for example, in the treatises [59, 166]. At the time of intensive development of nonholonomic mechanics many scholars often obtained similar results and therefore the same forms of equations of motion have different names. The investigations, connected with the possible applications of these equations to more wide classes of nonholonomic constraints, are being continued to the present (see, for example, [370]).

In generating the equations of motion of nonholonomic systems, most authors made use of the D'Alembert–Lagrange principle that they extended to the case of the system under consideration. In this case they need to clarify what should be regarded as a virtual displacement for the given type of constraint. V. V. Kozlov [114, p. 60, 61] notes that "... for such method for constructing the dynamics, the hypotheses must involve the definition of virtual displacements"and "even in the simplest case of stationary integralable constraint the definition of virtual displacements is the independent hypothesis of dynamics". Exactly such hypothesis (see below (6.3)) was brightly formulated by N. G. Chetaev. He aims [245, p. 68] "... at the introduction of the notion of virtual displacement for nonlinear constraints in such a way that to save as D'Alembert's, as Gaussian principles ... ".

For the sake of generality, we consider an arbitrary mechanical system, the position of which is given by the generalized coordinates q^σ, $\sigma = \overline{1,s}$. Suppose that on this system is imposed the following nonlinear nonholonomic constraints

$$\varphi^{\varkappa}(t, q, \dot{q}) = 0\,, \qquad \varkappa = \overline{1,k}\,, \qquad k < s\,. \tag{6.1}$$

We remark that the nonholonomicity of these constraints makes itself evident in the fact that in spite of their existence the passage of system from any its position with the coordinates q_0^σ, $\sigma = \overline{1,s}$, to another with the coordinates q_1^σ, $\sigma = \overline{1,s}$, is kinematically possible.

According to N. G. Chetaev for the real motion of the system considered the D'Alembert–Lagrange principle

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\sigma} - \frac{\partial T}{\partial q^\sigma} - Q_\sigma\right)\delta q^\sigma = 0 \tag{6.2}$$

must be satisfied. We assume that the kinetic energy T has the form

$$T = \frac{M}{2} g_{\alpha\beta}(t, q)\dot{q}^\alpha \dot{q}^\beta\,, \qquad \alpha, \beta = \overline{0,s}\,, \quad q^0 = t\,, \quad \dot{q}^0 = 1$$

and the generalized forces Q_σ are given in terms of the functions of the time t, the coordinates q^σ, and the generalized velocities $\dot{q}^\sigma$ ($\sigma = \overline{1,s}$).

N. G. Chetaev also assume that the quantities δq^σ, entering into the D'Alembert–Lagrange principle (6.2), satisfy the following conditions

$$\frac{\partial \varphi^{\varkappa}}{\partial \dot{q}^\sigma}\delta q^\sigma = 0\,, \qquad \varkappa = \overline{1,k}\,. \tag{6.3}$$

Nonlinear nonholonomic constraints (6.1) such that conditions (6.3) are assumed to be valid, were named the constraints of the Chetaev type.

As is shown in the previous section, the general principle of nonholonomic mechanics is the Suslov–Jourdain principle (5.9) or (5.8) in which the variations of velocity must satisfy conditions (5.3). The virtual displacements, allowed by the constraints of the Chetaev type, are satisfied exactly such conditions (6.3). Therefore the generalized principle of D'Alembert–Lagrange, which in the case of Chetaev's postulate permits us to apply the usual D'Alembert–Lagrange principle (6.2) to the study of nonholonomic systems, coincides with the Suslov–Jourdain principle. The comparison of formulas (6.3) and (5.3) implies, in turn, that the virtual displacements $(\delta q^1, \dots, \delta q^s)$, introduced by Chetaev for nonlinear nonholonomic constraints, coincide with the variations of the generalized velocity $(\delta' \widehat{v}^1, \dots, \delta' \widehat{v}^s)$. Like the holonomic problems from formula (5.10) it follows that the reaction of constraints of the Chetaev type is orthogonal to the virtual displacements satisfying conditions (6.3).

Differential forms (6.3) show that the scalar products of the vectors $\delta \mathbf{y} = \delta q^\sigma \mathbf{e}_\sigma$ by the vectors $\boldsymbol{\nabla}' \varphi^\varkappa \equiv \boldsymbol{\varepsilon}^{l+\varkappa}$, $\varkappa = \overline{1, k}$, are equal to zero. Formulas (5.7) stress even this orthogonality.

Thus, conditions (6.3) suggested by N. G. Chetaev, which give the axiomatic definition of virtual displacements with nonlinear nonholonomic constraints, show the possibility of the transition, in the nonholonomic mechanics, from the vectors given on the manifold of possible positions of mechanical system to the vectors given on the manifold of possible velocities of system.

Relations (6.3) have played an important role in the development of nonholonomic mechanics. The different forms of conditions of type (6.3) were also introduced by the other famous scholars, for example, by J. W. Gibbs [309], P. Appell [265], and A. Przeborski [375]. J. Papastavridis [370. 1997, 2002] refers to conditions (6.3) as the Maurer–Appell–Chetaev–Hamel definition. Note, that much attention to this definition of virtual displacements with nonlinear nonholonomic constraints is paid in the works of Norwegian scientist L. Johnsen [324], which are not too famous.

We shall repeatedly remark further the role of Chetaev's type constraints for obtaining the cited results.

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