Chapter 2
Locality and time scales of the underlying non-degenerate stochastic system: Freidlin-Wentzell theory

This chapter is devoted to substantiate the concept of *locality* (*metastability*, *quasi-deterministic approximation*) to be used. The framework here is set up by the SDE

$$\begin{aligned} dX_t^{\varepsilon,x} &= b\,(X_t^{\varepsilon,x})\;dt \,+\, \sqrt{\varepsilon}\;\sigma\,(X_t^{\varepsilon,x})\;\,dW_t \qquad\qquad (t \geq 0)\,, \\ X_0^{\varepsilon,x} &= x\;\in \mathbb{R}^d \end{aligned} \tag{2.1}$$

in $\mathbb{R}^d$. Here, the coefficients are functions $b \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d})$ and $\varepsilon \geq 0$ parametrizes the intensity of $(W_t)_{t\geq 0}$ which denotes a Brownian motion in $\mathbb{R}^d$ on[1] a standard filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. Hence, for any $\varepsilon > 0$ and $x \in \mathbb{R}^d$, the stochastic process $X^{\varepsilon,x}$ solving (2.1) is a diffusion with drift b and covariance $\varepsilon\, a$, i.e. the generator of X^ε is given by the differential operator

$$\mathcal{G}^\varepsilon f \;:=\; \sum_{i=1}^d b_i \,\frac{\partial f}{\partial x_i} \;+\; \frac{\varepsilon}{2} \sum_{i,j=1}^d a_{ij}\,\frac{\partial^2 f}{\partial x_i\,\partial x_j} \qquad\qquad (f \in C^2(\mathbb{R}^d, \mathbb{R}))\,,$$

where the positive semi-definite matrix a is defined by $a := \sigma\,\sigma^*$.

An important special case of equation (2.1) is given by the gradient SDE

$$dX_t^{\varepsilon,x} \;=\; -\,\nabla U\,(X_t^{\varepsilon,x})\;dt \,+\, \sqrt{\varepsilon}\;\,dW_t\;, \qquad\qquad X_0^{\varepsilon,x} \;=\; x\;\in \mathbb{R}^d\;, \tag{2.2}$$

where the drift $b := -\nabla U$ is given by a potential $U \in C^\infty(\mathbb{R}^d, \mathbb{R})$, as for example in figure 1, and where for simplicity $\sigma \equiv \mathrm{id}_{\mathbb{R}^d}$; also cf. example 1.5.3.

The above equations are stochastic differential equations in the Itô sense. If σ is constant, as for example in (2.2), then (2.1) coincides with its Stratonovich version. However, this is not true in general; more precisely,

[1] More precisely, W is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ which is supposed to be standard, i.e. to satisfy "the usual conditions", see e.g. Hackenbroch and Thalmaier [Hb-Th 94, 3.3.].

W. Siegert, *Local Lyapunov Exponents*.
Lecture Notes in Mathematics 1963.

replacing the Itô differential dW_t in (2.1) by the Stratonovich integral $\circ dW_t$, correction terms would need to be taken into account; this issue will be further commented on in remark 2.4.13.

The behavior of the above SDEs (2.2) and (2.1) has been the subject of a tremendous amount of ongoing research. Its foundations have been laid out by Smoluchowski, Einstein, Langevin, Andronov et al. [And-Pon-Vi 33] and Kramers [Kr 40]; see e.g. also Chandrasekhar et al. [Cha-Kac-Smo 86]. The corresponding mathematical theory of large deviations is due to Freidlin and Wentzell; see e.g. [We-Fr 70] and [Fr-We 98]. In their work (2.1) describes the deterministic differential system

$$\frac{dX_t^0}{dt} = b\left(X_t^0\right) \qquad (t \geq 0) ,$$

subject to the small (as $\varepsilon \to 0$) random perturbations $\sqrt{\varepsilon}\,\sigma\, dW_t$; furthermore, the exponential rate for large deviations of X^ε from the deterministic trajectory X^0 is calculated. Freidlin and Wentzell also deduce that the long-time behavior of the stochastic system $X^{\varepsilon,x}$ has a deterministic component which can be described by a hierarchy of cycles in case that the drift b has finitely many stable attractors; for example, one can think of a potential function U for SDE (2.2) with finitely many wells as in figure 1. By virtue of this hierarchy of cycles one can assign to (Lebesgue almost-) every initial value x and each typical time scale $\mathrm{e}^{\zeta/\varepsilon}$ a certain subset $K_{\mu(x,\zeta)}$ of the state space $\mathbb{R}^d$ such that $X^{\varepsilon,x}$ spends most of the time until $\mathrm{e}^{\zeta/\varepsilon}$ in a neighborhood of $K_{\mu(x,\zeta)}$. This set $K_{\mu(x,\zeta)}$ will be called *metastable* with respect to the point x and the time scale $\mathrm{e}^{\zeta/\varepsilon}$. Another feature of this phenomenon is that the transition probability

$$\mathbb{P}_x \left\{ X^\varepsilon_{\mathrm{e}^{\zeta/\varepsilon}} \in \ . \right\}$$

has different limits for different choices of the time scale and these limits do depend on the initial value x. This fact is expressed by saying that $X^{\varepsilon,x}$ has *sublimiting distributions* supported by the sets $K_{\mu(x,\zeta)}$. As already mentioned, these results are due to Freidlin and Wentzell (see [Fr 77], [Fr 00] and [Fr-We 98, § 6.6]).

The existence of "metastable states" corresponding to initial conditions and time scales as described above is abbreviated by speaking of *metastability* or *locality*. Furthermore, the asymptotic behavior of X^ε on the times scales will be described by basins of attraction, a hierarchy of cycles, metastable states, exit rates, rotation rates and so on. Although X^ε is a stochastic process all the previous objects are non random. This technique of reducing the stochastic dynamics of the SDE (2.1) to a deterministic description is called *quasi-deterministic approximation* for the long-time behavior of the dynamical system perturbed by small noise (Freidlin [Fr 00]).

Note that the terminology of "sublimiting distributions" emphasizes the asymptotic behavior on time scales $\mathrm{e}^{\zeta/\varepsilon}$. In contrast the *limiting distribution*, also called the *invariant* or *stationary measure*, is the limit of the transition probabilities and hence depicts the asymptotics of large times for a fixed noise intensity ε. In order to highlight this difference between the sublimiting and the limiting distribution also the latter object will be discussed in detail; see section 2.2.

In this chapter an outline of the theory of large deviations, exit probabilities for non-degenerate systems and metastability shall be given. However, in order not to overburden the scope of this book, we allow ourselves only to sketch or even omit some of the proofs; when doing so, references to the underlying work by Freidlin and Wentzell as well as to Dembo and Zeitouni [De-Zt 98] are given.

The terminus "metastable state" has been coined in statistical physics: The empirical description of "metastable thermodynamic phases" is characterized by the following properties, listed by Penrose and Lebowitz [Per-Leb 71]:

1) "Only one thermodynamic phase is present",
2) "a system that starts in this state is likely to take a long time to get out" and
3) "once the system has gotten out, it is unlikely to return".

Physically speaking, it is the goal of the current chapter to detect points K_μ in $\mathbb{R}^d$ featuring such heuristic properties in the framework set up by the SDE (2.1).

The importance of such SDEs (2.1) and (2.2) for applications shall then be illustrated in the concluding section of this chapter by discussing some sample systems.

2.1 Preliminaries and assumptions

Let the matrix $a \equiv \sigma\,\sigma^*$ be invertible. The *action functional* for (2.1) is defined by $\frac{1}{\varepsilon}\, I_{0T}^x$, where the functional I_{0T}^x is given by

$$I_{0T}^x \;:\; C([0,T],\mathbb{R}^d) \longrightarrow \bar{\mathbb{R}}_+\,,$$

$$I_{0T}^x(f) := \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \Big|\, a(f_s)^{-1/2}\,\Big[\dot f_s - b(f_s)\Big]\Big|^2\, ds & ,\ f \text{ absolutely continuous} \\ & \qquad\text{and } f_0 = x\,, \\ \infty & ,\ \text{otherwise}\ . \end{cases}$$

Here, $a(x)^{-1/2}$ denotes the (unique) $d \times d$-matrix whose square is the positive definite matrix $a(x)^{-1} \equiv [\sigma(x)\,\sigma(x)^*]^{-1}$. Also $a(x)^{-1/2}$ is a symmetric matrix; hence, one can rewrite the integrand as

$$\Big\langle\ \dot f_s - b(f_s)\,,\ a(f_s)^{-1}\ \Big[\dot f_s - b(f_s)\Big]\ \Big\rangle\quad ;$$

see e.g. Freidlin [Fr 68] on factorization theorems and how differentiability properties are preserved.

The above functional I^x_{0T} depends on b, σ, x and a fixed time horizon $T > 0$, but not on ε; it is the rate function for the large deviation principle for $X^{\varepsilon,x}$ as we will later see. We also define I^{xy}_{0T} as the restriction of I^x_{0T} to all paths f such that $f(T) = y$. The *quasipotential* for the SDE (2.1) is then defined as

$$\begin{aligned} V \,:\, \mathbb{R}^d \times \mathbb{R}^d &\longrightarrow \mathbb{R}_+\,, \\ V(x,y) \;&:=\; \inf_{T>0}\, I^{xy}_{0T}(\,.\,) \\ &\equiv\; \inf \left\{ I^x_{0T}(f) \,:\, f \in C([0,T],\mathbb{R}^d)\,,\, f_0 = x\,,\, f_T = y\,,\, T>0 \right\}; \end{aligned}$$

it describes the "minimal cost" of forcing X^ε to connect x and y eventually.

Now the assumptions are listed under which (2.1) is investigated:

Assumption 2.1.1 (on the coefficients of SDE (2.1)). The drift $b \in C^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and the diffusion term $\sigma \in C^\infty(\mathbb{R}^d,\mathbb{R}^{d\times d})$ are supposed to further satisfy:

(S) For all $\varepsilon > 0$ and $x \in \mathbb{R}^d$, there exists a unique strong, non-exploding solution $X^{\varepsilon,x}$ to the SDE (2.1).

(E) σ takes its values in the invertible matrices, $\sigma^{-1} \in C^\infty(\mathbb{R}^d,\mathbb{R}^{d\times d})$, and $a \equiv \sigma\sigma^*$ is strictly positive definite, i.e. there is a constant $c > 0$ such that

$$c\,|x_2|^2 \;\leq\; \langle\, a(x_1)x_2, x_2\,\rangle \;\leq\; c^{-1}\,|x_2|^2 \qquad (x_1,x_2 \in \mathbb{R}^d)\,.$$

(K) There exist finitely many points $K_1,\dots,K_l \in \mathbb{R}^d$ such that

(K1) any trajectory $X^{0,x}_t$ (except those in a finite union of lower dimensional submanifolds of $\mathbb{R}^d$) of the deterministic system

$$dX^0_t \;=\; b\left(X^0_t\right)\,dt$$

is attracted to one of the points K_i as $t \to \infty$;

(K2) each K_i is stable, i.e. there exist open sets $U_i \supset K_i$ such that U_i is attracted to K_i under the deterministic motion X^0.

(V) $\lim\limits_{|y|\to\infty} V(0,y) \;=\; \infty\,.$

(G) The system is *generic* in the sense that there are no symmetries in the function V. More precisely, it is assumed that the minima and maxima in equations (2.11)–(2.17) below are attained at only one point; this means (in the terminology coined below) that the main state, the rotation rate and the exit rate of cycles of any order shall be well defined.

Remark 2.1.2 (on the set of assumptions 2.1.1).

on (S): If $\sigma(x) = \mathrm{id}_{\mathbb{R}^d}$ for all $x \in \mathbb{R}^d$, a sufficient condition for **(S)** is e.g. that for some constant $c > 0$ and all $x \in \mathbb{R}^d$,

$$\langle x, b(x) \rangle \;\le\; c\left(1 + |x|^2\right) \;;$$

see Stroock and Varadhan [Str-Vdh 97, Th. 10.2.2]. Since **(K)** forces all trajectories to converge to one of the finitely many attractors K_i, one might think of a drift b for which $\langle x, b(x) \rangle < 0$ outside some sufficiently large ball.

on (E): Condition **(E)** ("ellipticity" of (2.1)) makes sure that the generator $\mathcal{G}^\varepsilon$ is an elliptic differential operator and also bounds the covariance a from above. **(E)** makes sure that the Markov transition probabilities $P_t^\varepsilon(x, .)$ have Lebesgue densities; together with assumptions **(B)** defined below, **(E)** guarantees the existence of an invariant probability measure ρ^ε with Lebesgue density $p^\varepsilon(x) > 0$ to which the Markov transition probabilities converge; see the next section. The crucial point here is that we will use time scales $T(\varepsilon)$ "below which" this limiting distribution ρ^ε is observed, so this invariant measure does not play a decisive role in the subsequent analysis; however, we will use it to emphasize the different behaviors of the transition probabilities in the limiting and the sublimiting cases, respectively; see sections 2.2 and 2.5. From an applications' point of view the existence of an invariant probability is a reasonable property; see section 2.6.

on (E): Due to assumption **(E)**, V is indeed a well defined function in $\mathbb{R}_+$.

on (K): This assumption, taken from Freidlin [Fr 77], allows to approximate the behavior of X^ε by the one of a certain Markov chain whose finite state space is given by the union of small spheres around the points K_i . It is this construction which backs the exit time law 2.5.3 for cycles.

For those initial conditions x for which $X^{0,x}$ does not converge to one of the points $K_1, \ldots, K_l$, there exist unstable equilibria $K_{l+1}, \ldots, K_{l'}$ to which the respective trajectories converge; these trajectories separate the domains of attraction $D_1, \ldots, D_l$ of the respective points $K_1, \ldots, K_l$ and therefore form separatrices of the deterministic dynamical system X^0 ; hence, these trajectories (and their initial values) constitute a finite union of lower dimensional submanifolds of $\mathbb{R}^d$ separating the domains $D_1, \ldots, D_l$.

Conversely, one could start with all equilibria $K_1, \ldots, K_{l'}$ of the deterministic dynamical system X^0 and then single out the stable ones $K_1, \ldots, K_l$, where $l \le l'$; see Freidlin [Fr 00, p.337].

on (K): The sets K_i need not be one point sets: Instead one can also allow

$$K_1, \ldots, K_l \subset \mathbb{R}^d$$

to be just compact sets. However, since the concepts of the current chapter will be applied to the case where the K_i's are one point sets, we formulate this assumption as it will be used.

In this more general situation, where the K_i's are compact sets, the statements of this chapter would hold true, if in **(K)** we add two more subassumptions:

(K3) if $x, y \in K_i$, then $x \sim y \ \ (1 \le i \le l)$;
(K4) if $x \in K_i$ and $y \in K_j$, then $x \nsim y \ \ (i \neq j)$;

thereby we made use of the following equivalence relation which is induced by the quasipotential V:

$$x \sim y \ \ :\Longleftrightarrow \ \ V(x,y) = V(y,x) = 0 \, .$$

In this general situation these sets K_i are for example the limit cycles of the ODE $\dot{x} = b(x)$; consider e.g. the drift

$$b(x) \ := \ \begin{pmatrix} -x_2 - x_1 \left[(x_1^2 + x_2^2)^2 - 3(x_1^2 + x_2^2) + 2 \right] \\ x_1 - x_2 \left[(x_1^2 + x_2^2)^2 - 3(x_1^2 + x_2^2) + 2 \right] \end{pmatrix} \ ,$$

which is taken from Jetschke [Je 89, p.55]; note that this drift decomposes orthogonally as

$$b \ = \ -\nabla U + \mathrm{L}$$

(see 2.4.5 on the implications for the quasipotential), where

$$U(x) \ := \ \frac{1}{6}\,|x|^6 \ - \ \frac{3}{4}\,|x|^4 \ + \ |x|^2 \qquad \text{and} \qquad \mathrm{L}(x) \ := \ \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad ;$$

in polar coordinates, $\varrho := |x|$ and $\alpha := \arctan \frac{x_2}{x_1}$, this ODE $\dot{x} = b(x)$ can be rewritten as

$$\begin{aligned} &\dot{\varrho} = -\,\varrho^5 \ + \ 3\,\varrho^3 \ - \ 2\,\varrho \,\Big(\equiv -\,U'(\varrho) \Big) \ \ = \ -\,\varrho\,(\varrho^2 - 1)\,(\varrho^2 - 2) \\ &\dot{\alpha} = 1 \quad ; \end{aligned}$$

since $U(x) = U(\varrho)$ only depends on the absolute value of the position, this implies that b satisfies **(K1)**–**(K4)** with the stable sets $K_1 := \{0\}$ and $K_2 := \{\, x \in \mathbb{R}^2 : |x| = 2 \,\}$. The invariant set $\{\, x \in \mathbb{R}^2 : |x| = 1 \,\}$ violates the stability criterion **(K2)** and hence constitutes the saddle for the radial motion.

An even simpler example for this phenomenon of a limit cycle is the two-dimensional family of drifts

$$b_\eta(x) \;:=\; \begin{pmatrix} -x_2 + x_1 \left[\eta - (x_1^2 + x_2^2)\right] \\ x_1 + x_2 \left[\eta - (x_1^2 + x_2^2)\right] \end{pmatrix}$$

which represents the normal form of the Hopf-bifurcation, where $\eta \in \mathbb{R}$ denotes the bifurcation parameter; cf. Guckenheimer and Holmes [Gu-Hl 83, p.146f.], Andronov et al. [And-Pon-Vi 33, §5] and Jetschke [Je 89, p.158;52,226]; a slight generalization of this system is considered by Leng et al. [Lg-SN-Tw 92, Ex.3]. Also in this case the drift decomposes orthogonally as

$$b_\eta \;=\; -\nabla U_\eta + \mathrm{L}\;,$$

where

$$U_\eta(x) \;:=\; \frac{1}{4}\,|x|^4 \;-\; \frac{\eta}{2}\,|x|^2 \qquad \text{and} \qquad \mathrm{L}(x) \;:=\; \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}\;;$$

in polar coordinates this ODE can be rewritten as

$$\begin{aligned} \dot\varrho &= -\,\varrho^3 \;+\; \eta\,\varrho\,\Big(\equiv -\,U_\eta'(\varrho)\Big) \;\;=\;\; \varrho\,(\eta - \varrho^2) \\ \dot\alpha &= 1 \quad ; \end{aligned}$$

again $U_\eta(x) = U_\eta(\varrho)$ only depends on the absolute value of the position, and b_η satisfies **(K)** with the only stable set

$$K(\eta) \;:=\; \begin{cases} \{0\}\;, & \eta \le 0 \\ \{\, x \in \mathbb{R}^2 : |x| = \sqrt{\eta}\,\}\;, & \eta > 0\;. \end{cases}$$

If $\eta \le 0$, the qualitative behavior of x_t resembles the motion in example 2.6.1; if $\eta > 0$, then the origin is a repelling fixed point.

on (K): A sample drift which violates the stability assumption **(K)** is the following “mock” Van der Pol-ODE given in polar coordinates by

$$\begin{aligned} \dot\varrho &= \varrho\,(1 - \varrho) \\ \dot\alpha &= 2 \;-\; \varrho\,\cos\alpha \quad ; \end{aligned}$$

cf. Zeeman [Ze 88, p.126].

on (V): This assumption assures that the sets $\{y \,:\, V(x,y) \,\le\, c\}$ which exhaust the state space $\mathbb{R}^d$ are all compact for any x and $c \ge 0$.

on (E) and (V): Instead of considering a diffusion on $\mathbb{R}^d$, we can also take a diffusion X^ε on a d-dimensional compact manifold M. In this case **(E)** and **(V)** are redundant except from assuming a to be positive definite on M. The (unique) stationary measure is already finite in this case; see example 2.6.11.

The compact setting is for example assumed by Wentzell and Freidlin [We-Fr 69] and Freidlin and Wentzell [Fr-We 98, §§6.4, 6.6].

One could equally well consider the "mixed" case with state space $M \times \mathbb{R}^d$ for a compact manifold M under suitably adopted assumptions.

on (G): As already mentioned, this assumption will be made precise below. In order to illustrate what might go wrong here, note that **(G)** is violated for example, if the potential function U in SDE (2.2) has wells of equal depth.

2.2 The limiting distribution (stationary measure)

In this section we collect some facts on the connection between SDEs and partial differential equations (PDEs) to illustrate the behavior of the transition probabilities and of invariant measures. Since the contents of this section is well documented in textbooks, we refrain from giving proofs here and refer instead to standard monographs as e.g. Friedman [Fri 75, Ch. 6], Friedman [Fri 76, Sec. 14.4], Khasminskii [Kh 80, Ch. III & IV], Stroock and Varadhan [Str-Vdh 97], Wentzell [We 79, §11.2] and Fleming and Rishel [Fl-Ris 75, Ch.V]. Also see the expositions by Freidlin and Wentzell [Fr-We 98, Ch.1 & 4], Khasminskii [Kh 60], Ichihara and Kunita [Ic-Ku 74], Andronov et al. [And-Pon-Vi 33], Bernstein [Bs 33], Horsthemke and Lefever [Hh-Lf 85, Sec. 4.4] and Hackenbroch and Thalmaier [Hb-Th 94, Sec. 6.6]; recent results are due to Pardoux and Veretennikov [Pd-Ver 01] for example.

In order to emphasize the difference between the "forward variables" and the "backward variables" in the "forward equation" and the "backward equation" to come, we allow for time-dependent coefficients in this section, i.e. we consider the following SDE in $\mathbb{R}^d$,

$$\begin{aligned} dX_t^{\varepsilon,x} &= b\left(t, X_t^{\varepsilon,x}\right)\, dt \;+\; \sqrt{\varepsilon}\; \sigma\left(t, X_t^{\varepsilon,x}\right)\; dW_t \qquad\qquad (t \geq s \geq 0)\,, \\ X_s^{\varepsilon,x} &= x \;\in \mathbb{R}^d \;\;, \end{aligned} \tag{2.3}$$

where the coefficients are functions $b \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^{d\times d})$ and where $s \geq 0$ denotes some initial time; assume that any $\sigma(t,\,.\,)$ satisfies the ellipticity condition **(E)** for some universal constant $c > 0$. For simplicity it is furthermore assumed that all coefficient functions b and $a \equiv (\sigma\sigma^*)$ as well as their first derivatives are bounded (with respect to the respective norms). The latter assumption can be relaxed (see Friedman [Fri 75, p.147]); however, since the main focus of this paper lies on the behavior on bounded domains and during time scales, this assumption does not restrict the scope of this section for illustrative purposes.

The partial differential operator associated with (2.3) is

$$\mathcal{G}^\varepsilon(s,x) \;:=\; \sum_{i=1}^d b_i(s,x)\,\frac{\partial}{\partial x_i} \;+\; \frac{\varepsilon}{2}\sum_{i,j=1}^d a_{ij}(s,x)\,\frac{\partial^2}{\partial x_i\,\partial x_j}\;.$$

In the sequel let $\mathbb{P}_{s,x}$ denote the probability measure $\mathbb{P}$ conditioned on events concerning the process X^ε starting in x at time $s \geq 0$; $\mathbb{E}_{s,x}$ then denotes the corresponding expected value operator.

The transition probabilities $P^\varepsilon_{s,x}\{X^\varepsilon_t \in .\}$ then possess densities with respect to the Lebesgue measure,

$$P^\varepsilon_{s,x}\{X^\varepsilon_t \in B\} \;=\; \int_B p^\varepsilon(s,x;t,y)\,dy \qquad (s<t\,,\; B\in\mathcal{B}(\mathbb{R}^d))\,.$$

These densities solve the *backward (parabolic) equation*

$$\begin{aligned}-\frac{\partial\, p^\varepsilon(s,x;t,y)}{\partial s} &= \sum_{i=1}^d b_i(s,x)\,\frac{\partial\, p^\varepsilon(s,x;t,y)}{\partial x_i} \;+\; \frac{\varepsilon}{2}\sum_{i,j=1}^d a_{ij}(s,x)\,\frac{\partial^2\, p^\varepsilon(s,x;t,y)}{\partial x_i\,\partial x_j}\\ &\equiv \mathcal{G}^\varepsilon(s,x)\; p^\varepsilon(s,x;t,y)\quad,\end{aligned}$$

a PDE with respect to the *backward* (i.e. past) *variables* (s,x). Equivalently, the densities solve the *forward (parabolic) equation*

$$\begin{aligned}\frac{\partial\, p^\varepsilon(s,x;t,y)}{\partial t} \;&=\; -\sum_{i=1}^d \frac{\partial}{\partial y_i}\,\bigl[b_i(t,y)\,p^\varepsilon(s,x;t,y)\bigr]\\ &\quad+\;\frac{\varepsilon}{2}\sum_{i,j=1}^d \frac{\partial^2}{\partial y_i\,\partial y_j}\,\bigl[a_{ij}(t,y)\,p^\varepsilon(s,x;t,y)\bigr]\\ &=:\;\bigl(\mathcal{G}^\varepsilon(t,y)\bigr)^*\;p^\varepsilon(s,x;t,y)\quad,\end{aligned}$$

a PDE with respect to the *forward* (i.e. future) *variables* (t,y); $(\mathcal{G}^\varepsilon)^*$ is the *formal adjoint operator* of $\mathcal{G}^\varepsilon$.

Moreover, the densities are the fundamental solutions of the backward and the forward equation.

In the time-homogeneous case, i.e. if b and σ are independent of t as in (2.1), one then gets for the transition densities

$$p^\varepsilon(s,x;t,y) \;=\; p^\varepsilon(0,x;t-s,y) \;=:\; p^\varepsilon_{t-s}(x,y) \qquad (s<t)\;,$$

from the above PDEs the *Kolmogorov backward equation*

$$\begin{aligned}\frac{\partial\, p^\varepsilon_t(x,y)}{\partial t} &= \sum_{i=1}^d b_i(x)\,\frac{\partial\, p^\varepsilon_t(x,y)}{\partial x_i} \;+\; \frac{\varepsilon}{2}\sum_{i,j=1}^d a_{ij}(x)\,\frac{\partial^2\, p^\varepsilon_t(x,y)}{\partial x_i\,\partial x_j}\\ &\equiv \mathcal{G}^\varepsilon(x)\; p^\varepsilon_t(x,y)\end{aligned}$$

and the equivalent *Kolmogorov forward equation (Fokker-Planck equation)*

$$\begin{aligned}\frac{\partial\, p_t^\varepsilon(x,y)}{\partial t} &= -\sum_{i=1}^{d} \frac{\partial}{\partial y_i}\big[b_i(y)\, p_t^\varepsilon(x,y)\big] + \frac{\varepsilon}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial y_i\, \partial y_j}\big[a_{ij}(y)\, p_t^\varepsilon(x,y)\big] \\ &\equiv \big(\mathcal{G}^\varepsilon(y)\big)^*\, p_t^\varepsilon(x,y)\,.\end{aligned}$$

Further on, consider the time-homogeneous case (2.1). A *stationary probability distribution for* X^ε is a probability measure ρ^ε on $\mathbb{R}^d$ which is *invariant* with respect to the Markov transition probabilities $(P_t^\varepsilon)_t$,

$$P_t^\varepsilon(x,\,.\,) := \mathbb{P}_x\{\, X_t^\varepsilon \in\ .\ \} \equiv \mathbb{P}\{\, X_t^{\varepsilon,x} \in\ .\ \} \qquad (t \ge 0, x \in \mathbb{R}^d)\ ,$$

in the sense that

$$\rho^\varepsilon(\,.\,) = \int_{\mathbb{R}^d} P_t^\varepsilon(x,\,.\,)\, \rho^\varepsilon(dx) \qquad (t \ge 0)\ ;$$

or equivalently

$$\int_{\mathbb{R}^d} f(x)\, \rho^\varepsilon(dx) = \int_{\mathbb{R}^d} T_t^\varepsilon f(x)\, \rho^\varepsilon(dx) \qquad (f \in C^c(\mathbb{R}^d, \mathbb{R}))\ ,$$

where

$$T_t^\varepsilon f(x) := \int_{\mathbb{R}^d} f(y)\, P_t^\varepsilon(x, dy)\ .$$

The stationary measure ρ^ε is the "equilibrium (limiting) distribution as $t \to \infty$", in the sense that for any $x \in \mathbb{R}^d$

$$T_t^\varepsilon f(x) \xrightarrow{t\to\infty} \int_{\mathbb{R}^d} f(y)\, \rho^\varepsilon(dy) \qquad (f \in C^b(\mathbb{R}^d, \mathbb{R}))$$

and

$$P_t^\varepsilon(x, B) \xrightarrow{t\to\infty} \rho^\varepsilon(B) \tag{2.4}$$

for all $B \in \mathcal{B}(\mathbb{R}^d)$ such that $\rho^\varepsilon(\partial B) = 0$.

If a stationary probability distribution exists, assumption **(E)** implies its uniqueness and the existence of a Lebesgue density $p^\varepsilon(x) > 0$ of ρ^ε, the reason being that **(E)** entails *strong ellipticity* in the Lie-algebraic sense of Ichihara and Kunita [Ic-Ku 74, p.250]. The densities p_t^ε then converge to the "equilibrium distribution" in the sense that for all $x, y \in \mathbb{R}^d$

$$p_t^\varepsilon(x,y) \xrightarrow{t\to\infty} p^\varepsilon(y)\ ;$$

p^ε, being the Lebesgue density of ρ^ε, uniquely solves the respective autonomous differential equations

$$0 \;=\; \mathcal{G}^\varepsilon(x)\, p^\varepsilon(x)$$

(Kolmogorov backward equation) and

$$0 \;=\; (\mathcal{G}^\varepsilon(x))^*\, p^\varepsilon(x)$$

(Kolmogorov forward equation, Fokker-Planck equation) such that $\int p^\varepsilon(x)\, dx = 1$.

For any $f \in C^b(\mathbb{R}^d, \mathbb{R})$ and $\varepsilon \in (0, \varepsilon_0)$, the function

$$u_f^\varepsilon(t,x) \;:=\; \mathbb{E}_x\, f(X_t^\varepsilon) \equiv \mathbb{E}\,[\, f(X_t^{\varepsilon,x})\,]$$

is the unique solution of the Cauchy problem for (2.1):

$$\begin{aligned} &\frac{\partial u^\varepsilon}{\partial t} = \mathcal{G}^\varepsilon u^\varepsilon \;\equiv\; \sum_{i=1}^d b_i\, \frac{\partial u^\varepsilon}{\partial x_i} \;+\; \frac{\varepsilon}{2} \sum_{i,j=1}^d a_{ij}\, \frac{\partial^2 u^\varepsilon}{\partial x_i\, \partial x_j} \qquad (t>0), \\ &u^\varepsilon(0,\,.\,) = f. \end{aligned} \tag{2.5}$$

This function u_f^ε converges to the "equilibrium distribution as $t \to \infty$" as well, in the sense that for any $x \in \mathbb{R}^d$

$$u_f^\varepsilon(t,x) \;\xrightarrow{t\to\infty}\; \int_{\mathbb{R}^d} f(y)\, \rho^\varepsilon(dy)\;. \tag{2.6}$$

Concluding these remarks, we note that a general, sufficient condition for the existence of a stationary probability distribution ρ^ε is **(E)** together with

(B) There exists a bounded domain $S \subset \mathbb{R}^d$ with smooth boundary such that for all compact sets $K \subset \mathbb{R}^d$,

$$\sup_{x\in K} \mathbb{E}_x \tau_S^{\varepsilon,x} \;<\; \infty\;,$$

where $\tau_S^{\varepsilon,x}$ denotes the hitting time of S for the process $X^{\varepsilon,x}$ given by (2.1).

(for a proof see Khasminskii [Kh 80, Sec.IV.4] and [Kh 60]).

Remark 2.2.1 (Explicit formula for the stationary density in the gradient case). Suppose that the SDE under consideration is (2.2), i.e.: $\sigma = \mathrm{id}_{\mathbb{R}^d}$ and

$$b \;=\; -\nabla U$$

for some suitable potential function $U \in C^\infty(\mathbb{R}^d, \mathbb{R})$ (U being in $C^2(\mathbb{R}^d, \mathbb{R})$ would suffice here). Note that in dimension $d = 1$ any drift is the gradient of a potential function, $U(x) := -\int_0^x b(s)ds$. Then the above Fokker-Planck equation takes the form

$$(\mathcal{G}^\varepsilon)^* \, p^\varepsilon \;\equiv\; \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\frac{\partial U}{\partial x_i} \, p^\varepsilon \right) \;+\; \frac{\varepsilon}{2} \triangle p^\varepsilon \;=\; 0 \, ,$$

where $\triangle$ denotes the Laplace operator. A direct calculation of the respective derivatives shows that $x \mapsto \mathrm{e}^{-2\,U(x)/\varepsilon}$ fulfills the above PDE; hence, a stationary probability measure for (2.2) exists if and only if

$$N_\varepsilon \;:=\; \int_{\mathbb{R}^d} \mathrm{e}^{-\,2U(x)/\varepsilon} \, dx \;<\; \infty \, ,$$

in which case

$$p^\varepsilon(x) \;=\; N_\varepsilon^{-1} \, \mathrm{e}^{-\,2U(x)/\varepsilon} \qquad\qquad (x \in \mathbb{R}^d) \, .$$

Jacquot [Ja 92, p.347] shows that a sufficient condition for the finiteness of the normalization constant N_ε is the following:

(U) $\quad U \to \infty$ and $|\nabla U| \to \infty$ as $|x| \to \infty$.

Note, however, that nevertheless (**U**) does not imply that U has positive curvature for all sufficiently large x, as can be seen for example from $U(x) := \frac{x^2}{2} - 2\cos x$, where $x \in \mathbb{R}$.

Example 2.2.2 (Density p^ε for the potential function U_1). Consider for example (2.2) with the potential function U_1,

$$U_1(x) \;:=\; \frac{3}{2}\, x_1^4 \;-\; \frac{2}{3}\, x_1^3 \;-\; 3\, x_1^2 \;+\; x_1\, x_2 \;+\; \frac{3}{2}\, x_2^4 \;\; ,$$

as in (1.34) of example 1.5.3, where $c := 1$. The potential is sketched in the figure 2.1, where the different depths of the wells are emphasized by drawing level sets, too.

The density p^ε of the stationary measure has the following qualitative shape as depicted in the contour plot of figure 2.2: The peaks are located at

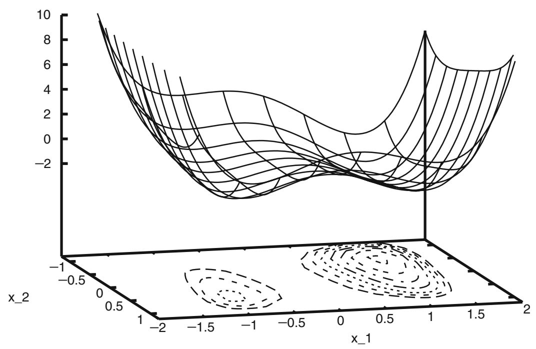

Fig. 2.1 The potential function $U_1(x_1, x_2) \;=\; \frac{3}{2}\, x_1^4 \;-\; \frac{2}{3}\, x_1^3 \;-\; 3\, x_1^2 \;+\; x_1\, x_2 \;+\; \frac{3}{2}\, x_2^4$

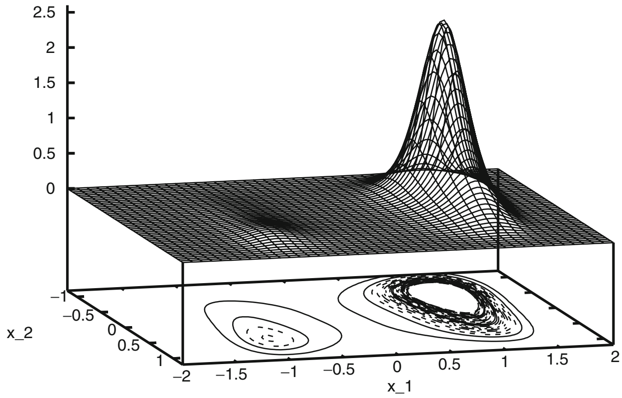

Fig. 2.2 Sketch of the density $p^\varepsilon \sim \mathrm{e}^{-2U_1/\varepsilon}$

the (local) minima of the potential and the mass concentrates at the global minimum in the small noise limit $\varepsilon \to 0$. This feature is characteristic for all potential SDEs (2.2): The local minima of U correspond to the local maxima of the density of the invariant distribution; these are the preferential states of the process X^ε as $t \to \infty$. Similarly, in the non-gradient case the system X^ε from (2.1) accumulates near the points $K_1, \ldots, K_l$; a corresponding statement for X^ε on time scales $T(\varepsilon)$ will appear later (see theorems 2.5.5 and 2.5.6).

For the Ornstein-Uhlenbeck process even the time-dependent transition densities can be obtained explicitly:

Example 2.2.3 (Ornstein-Uhlenbeck process, one-well potential function). Consider the special case of a linear drift $b(x) = -\beta x$ for a constant $\beta > 0$ in dimension $d = 1$ and choose $\sigma := 1$. Then the SDE (2.2) (or (2.1) respectively) becomes

$$dX_t^\varepsilon \;=\; -U'(X_t^\varepsilon)\, dt + \sqrt{\varepsilon}\, dW_t \;, \quad X_0^\varepsilon = x_0 \;,$$

where $U(x) := -\int_0^x b(y)dy = \frac{\beta}{2} x^2$ denotes a quadratic one-well potential. Then the transition probability densities $p_t^\varepsilon(x, \,.\,)$ are given by

$$p_t^\varepsilon(x, x_0) \;=\; \frac{1}{\sqrt{2\pi\, \sigma_t^2}}\; \exp\left(-\,\frac{(x - x_0\, \alpha_t)^2}{2\, \sigma_t^2}\right) \;,$$

where

$$\alpha_t \;:=\; \mathrm{e}^{-\beta\, t} \quad \text{and} \quad \sigma_t^2 \;:=\; \frac{\varepsilon}{2\beta}\, (1 - \mathrm{e}^{-2\beta t}) \;.$$

This follows from a direct verification of the Kolmogorov forward equation (Fokker-Planck equation). In other words, X_t^ε is normally distributed where the mean is given by the trajectory of the deterministic system, $X_t^0 = x_0 \mathrm{e}^{-\beta t}$, and the variance is σ_t^2.

Remark 2.2.4 (SDE of gradient type whose potential exhibits singularities). Consider the above SDE of gradient type (2.2)

$$dX_t^{\varepsilon,x} \ = \ -\nabla U\,(X_t^{\varepsilon,x})\ dt \ + \ \sqrt{\varepsilon}\ \ dW_t \ \ ;$$

in the previous situation the drift $b \equiv -\nabla U$ was supposed to be an element of $C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ (or at least $C^1(\mathbb{R}^d, \mathbb{R}^d)$); in particular, U is defined on the whole of $\mathbb{R}^d$ and the stationary density is calculated as in 2.2.1.

For some applications it is however necessary to diminish the state space $\mathbb{R}^d$ to an open set D. In the upcoming course of this exposition we will be concerned with the exit times τ_D^ε of X^ε from open sets $D \subset \mathbb{R}^d$. However, one could also force the system X^ε not to leave D at all. For this purpose fix a potential function

$$U \in C(\mathbb{R}^d, \mathbb{R} \cup \{+\infty\})$$

such that for

$$D \ := \ \left\{ \, x \in \mathbb{R}^d \ : \ U(x) < \infty \, \right\}$$

the restriction of U to D is C^1 ,

$$U\big|_D \in C^1(D, \mathbb{R}) \ .$$

Further assume that

$$\int_D |\nabla U(x)|^2 \, \mathrm{e}^{-4\,U(x)/\varepsilon} \, dx \ < \ \infty$$

and

$$N_\varepsilon \ := \ \int_{\mathbb{R}^d} \mathrm{e}^{-\,2U(x)/\varepsilon} \, dx \ < \ \infty \ ,$$

where the convention

$$\mathrm{e}^{-\,2U(\,.\,)/\varepsilon} \ := \ 0 \quad \text{on} \ \ \complement D$$

is used. Then

$$p^\varepsilon(x) \ = \ N_\varepsilon^{-1} \ \mathrm{e}^{-\,2U(x)/\varepsilon}$$

is the Lebesgue density of a probability measure ρ^ε and there exists a process $(X_t^\varepsilon)_{t\geq 0}$ with initial distribution $\mathbb{P} \circ (X_0^\varepsilon)^{-1} = \rho^\varepsilon$ which is a weak[2] solution of the gradient SDE (2.2),

$$dX_t^\varepsilon \ = \ -\nabla U\,(X_t^\varepsilon)\ dt \ + \ \sqrt{\varepsilon}\ dW_t \ \ ,$$

up to a terminal time T_∞ for which $\mathbb{P}_{\rho^\varepsilon}\{T_\infty < \infty\} = 0$; here $\mathbb{P}_{\rho^\varepsilon}$ denotes the law of $(X_t^\varepsilon)_{t\geq 0}$ starting with the stationary distribution ρ^ε and T_∞ being terminal means that it is the limit of an increasing sequence of stopping times.[3]

[2] See e.g. Hackenbroch and Thalmaier [Hb-Th 94, 6.42].

[3] See e.g. Hackenbroch and Thalmaier [Hb-Th 94, 6.16].

This result is due to Meyer and Zheng [My-Zh 85] and has been cited in its above form by Kunz [Kz 02, p.16f.,27].

We finish this section by quoting a general result which does not contain an assertion on the stationary measure. However, it suitably concludes the above considerations on PDEs corresponding to uniformly non-degenerate SDEs (2.3) and will be used later in sketching the proof of theorem 3.1.6. A reference for this theorem is Friedman [Fri 75, p.146f.] among others.

Theorem 2.2.5. *Consider the SDE*

$$\begin{aligned} dX_t^x &= b\left(t, X_t^x\right)\, dt \, + \, \sigma\left(t, X_t^x\right)\, dW_t \qquad\qquad (t \geq s \geq 0)\,, \\ X_s^x &= x \, \in \mathbb{R}^d \ , \end{aligned}$$

where the coefficients are functions $b \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^{d\times d})$; assume that any $\sigma(t,\,.\,)$ satisfies the ellipticity condition **(E)** *for some universal constant $c > 0$. The partial differential operator corresponding to this SDE is*

$$\mathcal{G}(s,x) \ := \ \sum_{i=1}^{d} b_i(s,x)\,\frac{\partial}{\partial x_i} \, + \, \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(s,x)\,\frac{\partial^2}{\partial x_i\,\partial x_j} \ .$$

Fix some bounded, open domain $D \subset \mathbb{R}^d$ with smooth boundary ∂D, some time horizon $T > 0$ and let

$$\theta_D \ := \ \min\Big(\tau_D(s,x)\,,\, T \Big)\ ,$$

where

$$\tau_D(s,x) \ := \ \inf\{t \geq 0 \, : \, X_t^x \notin D\}$$

denotes the first exit time[4] *of $X_\bullet^x$ from D. Furthermore, let $\psi \in C(\partial Q, \mathbb{R})$ be some continuous function*[5] *defined on the boundary of $Q := (0,T) \times D$.*

Then the boundary value problem

$$\begin{aligned} \left(\frac{\partial}{\partial s} + \mathcal{G}(s,x)\right)\, v &= 0 \quad \text{on } Q \\ v &= \psi \quad \text{on } (\{T\} \times D) \cup ([0,T] \times \partial D) \end{aligned}$$

is uniquely solved by

$$v(s,x) \ := \ \mathbb{E}_{s,x}\Big(\psi\left(\theta_D, X_{\theta_D}\right) \Big) \ .$$

[4] Since D is open, $\tau^{\varepsilon,x}$ is a stopping time with respect to the underlying standard filtration $(\mathcal{F}_t)_{t\geq 0}$; see e.g. Hackenbroch and Thalmaier [Hb-Th 94, 3.12.(ii)].

[5] This function $\psi(s,x)$ is not to be confused with the stochastic process $\psi_t^\varepsilon(\omega)$ as defined in equation (1.4). Since these two objects will not be considered simultaneously, there is no ambiguity.

2.3 The large deviations principle

This section is intended to sketch the fundamental principles for the SDE (2.1) from which the exit time law shall be deduced in the next section. Standard references underlying the following exposition are the monographs by Freidlin and Wentzell [Fr-We 98] and Dembo and Zeitouni [De-Zt 98].

At the beginning of this chapter the action functional and the quasipotential have already been mentioned (see p.55f.) in order to set up the assumptions 2.1.1. This section clarifies that the action functional indeed provides the rate function of a large deviation principle; in the next section the importance of the quasipotential in exit time considerations will be accounted for, thus illustrating its interpretation as cost function.

The general setup for a large deviation principle is that of a family $\{\mu_\varepsilon\}_{\varepsilon>0}$ of probability measures on a space E which we assume for simplicity to be a separable metric space equipped with its completed Borel-σ-Algebra $\mathcal{B}(E)$; the goal is to characterize the behavior of $\{\mu_\varepsilon\}_{\varepsilon>0}$ on $\mathcal{B}(E)$ as $\varepsilon \to 0$ in terms of a rate function I on an exponential scale.

Definition 2.3.1 (Large Deviation Principle (LDP)). Let E be a separable metric space with its (completed) Borel-σ-algebra $\mathcal{B}(E)$.

1) A function $I : E \to [0, \infty]$ is a *rate function*, if it is lower semicontinuous, i.e. if the level sets
$$\{x \in E : I(x) \le \alpha\}$$
are closed; since E is metric, an equivalent condition is that for all $x \in E$,
$$\liminf_{x_n \to x} I(x_n) \;\ge\; I(x)\,.$$
A rate function I is *good*, if the level sets $\{x \in E : I(x) \le \alpha\}$ are compact.

2) A family $\{\mu_\varepsilon\}_{\varepsilon>0}$ of probability measures on $\mathcal{B}(E)$ satisfies the *large deviation principle (LDP)* with rate function I, if
$$-\inf_{\Psi^\circ} I \;\le\; \liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(\Psi) \;\le\; \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(\Psi) \;\le\; -\inf_{\overline{\Psi}} I \tag{2.7}$$
for all $\Psi \in \mathcal{B}(E)$, where Ψ° and $\overline{\Psi}$ denote the topological interior and closure of the set Ψ, respectively.

A large deviation principle is “pushed forward” by continuous mappings; this is the content of the following proposition. For the straightforward verification we refer to Freidlin and Wentzell [Fr-We 98, p.81] and Dembo and Zeitouni [De-Zt 98, p.127]. The latter reference also contains more general versions of the contraction principle such as the “almost continuous case”, where the function F is measurable and a suitable limit of continuous functions; see [De-Zt 98, p.133].

Proposition 2.3.2 (Contraction principle - continuous case). *Let* $\mathrm{F} : E^1 \to E^2$ *be a continuous function between separable metric spaces* E^1, E^2 *and let* $I^{(1)} : E^1 \to [0, \infty]$ *be a good rate function. Then*

$$I^{(2)} : E^2 \to [0, \infty]\,, \qquad I^{(2)}(x) := \inf_{\mathrm{F}^{-1}(\{x\})} I^{(1)}(\,.\,)$$

(where $\inf \varnothing \equiv \infty$*) is a good rate function on* E^2 *.*

If, in addition, $I^{(1)}$ *governs a LDP for a family of probability measures* $\{\nu_\varepsilon\}_{\varepsilon>0}$ *on* E^1*, then* $I^{(2)}$ *governs a LDP for* $\{\mu_\varepsilon\}_{\varepsilon>0} := \{\, \nu_\varepsilon \circ \mathrm{F}^{-1}\,\}_{\varepsilon>0}$ *on* E^2*.*

Proposition 2.3.2 is used in proving the following theorem 2.3.4. Beforehand, the corresponding notation is fixed:

Notation 2.3.3. Let $T > 0$ be an arbitrary time horizon. Then the following function spaces over the time interval $[0, T]$ are defined:

$$C_x := C_x([0,T], \mathbb{R}^d) := \{f : [0,T] \to \mathbb{R}^d \text{ continuous, } f_0 = x\} \qquad (x \in \mathbb{R}^d)$$

and

$$H_1 := H_1([0,T], \mathbb{R}^d) := \left\{ \int_0^{\cdot} g_s\, ds \;:\; g \in L^2([0,T], \mathbb{R}^d) \right\},$$

the absolutely continuous functions starting in 0 with square integrable derivative. Furthermore, let $\mathrm{pr}_{[0,T]}$ denote the restriction map

$$\mathrm{pr}_{[0,T]} : (\mathbb{R}^d)^{[0,\infty)} \to (\mathbb{R}^d)^{[0,T]}\,, \qquad \mathrm{pr}_{[0,T]}(f) := f\big|_{[0,T]} \;.$$

Theorem 2.3.4 (LDP for strong solutions of SDE). *Let* $X^{\varepsilon,x}$ *be the solution of (2.1),*

$$dX_t^{\varepsilon,x} = b(X_t^{\varepsilon,x})\, dt + \sqrt{\varepsilon}\, \sigma(X_t^{\varepsilon,x})\, dW_t \;, \qquad X_0^{\varepsilon,x} = x \;,$$

where $b \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ *and* $\sigma \in C^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d})$ *are assumed to be bounded together with their first derivatives. Let* μ_ε *denote the law of* $X^{\varepsilon,x}$ *on* $C_x([0,T], \mathbb{R}^d)$ *for a fixed time* $T > 0$ *,*

$$\mu_\varepsilon := \mathbb{P} \circ \left(\mathrm{pr}_{[0,T]} \circ X_\bullet^{\varepsilon,x} \right)^{-1} \;.$$

Then $\{\mu_\varepsilon\}_{\varepsilon>0}$ *satisfies the LDP with good rate function*

$$I(f) \equiv I_{[0,T],x}(f) := \inf_{\{\, g \in H_1 \,:\, f_t = x + \int_0^t b(f_s)\, ds + \int_0^t \sigma(f_s)\, \dot g_s\, ds \,\}} \frac{1}{2} \int_0^T |\, \dot g_t \,|^2\, dt \;.$$

If $a \equiv \sigma\sigma^*$ is strictly positive definite, as in assumption **(E)**, *then this rate function is the action functional I^x_{0T} as defined on p.55, i.e.*

$$I(f) \;=\; \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \;\Big|\, a(f_s)^{-1/2}\,[\dot{f}_s - b(f_s)]\,\Big|^2 \;ds & ,\ f - x \in H_1 \\ \infty & ,\ f - x \notin H_1 \ . \end{cases}$$

Sketch of Proof. 1) Consider the case that $b(\,.\,) = 0$, $\sigma(\,.\,) = \mathrm{id}_{\mathbb{R}^d}$ and $x = 0$ first, i.e. $X^{\varepsilon,x}_t = \sqrt{\varepsilon} W_t$. In this case the assertion is due to Schilder's theorem (Dembo and Zeitouni [De-Zt 98, p.185f.]): The law of $\sqrt{\varepsilon} W_t$ on $C_0([0,T], \mathbb{R}^d)$,

$$\nu_\varepsilon := \mathbb{P} \circ (\mathrm{pr}_{[0,T]} \circ (\sqrt{\varepsilon}\, W_\bullet))^{-1} \ ,$$

satisfies the LDP with good rate function

$$I^{BM}_{[0,T],0}(g) \;:=\; \begin{cases} \frac{1}{2}\int_0^T |\dot{g}_s|^2 \,ds & ,g \in H_1 \\ \infty & ,g \notin H_1 \ . \end{cases}$$

2) Now consider a general drift b and initial condition x, but again let $\sigma(\,.\,) = \mathrm{id}_{\mathbb{R}^d}$,

$$dX^{\varepsilon,x}_t \;=\; b(X^{\varepsilon,x}_t)\,dt \;+\; \sqrt{\varepsilon}\,dW_t \ , \qquad\qquad X^{\varepsilon,x}_0 = x \ .$$

For any $g \in C_0$, the integral equation

$$f_t = x + \int_0^t b(f_s)ds + g_t \qquad (t \in [0,T])$$

admits a unique continuous solution f ; this gives rise to a well-defined mapping

$$\mathrm{F} : C_0 \to C_x \ , \quad \mathrm{F}(g) := f \ .$$

The map F is itself continuous by the Lipschitz continuity of b and Gronwall's Lemma. Thus the contraction principle 2.3.2 applies and yields an LDP, as

$$\mathrm{pr}_{[0,T]} \circ X^{\varepsilon,x}_\bullet \;=\; \mathrm{F}\left(\mathrm{pr}_{[0,T]} \circ \sqrt{\varepsilon}\, W_\bullet\right)$$

by the SDE for $X^{\varepsilon,x}$ which implies that $\mu_\varepsilon := \nu_\varepsilon \circ \mathrm{F}^{-1}$ is the law of $X^{\varepsilon,x}$ on C_x . Due to the contraction principle 2.3.2 and Schilder's theorem the corresponding rate function is

$$\begin{aligned} I(f) &= \inf_{\{g \in C_0\,:\, f = \mathrm{F}(g)\}} I^{BM}_{[0,T],0}(g) \\ &= \inf_{\{\, g \in H_1 \,:\, f(t) = x + \int_0^t b(f_s)\,ds + g_t \,\}} \frac{1}{2}\int_0^T \,|\,\dot{g}_t\,|^2 \;dt \ ; \end{aligned}$$

in the last equation it has been used that $g \in H_1$ if and only if $f = \mathrm{F}(g) \in H_1 + x$; in this case one obtains that $\dot{g}_t = \dot{f}_t - b(f_t)$ which yields together with the last equation that

$$I(f) \;=\; \frac{1}{2}\int_0^T \left|\, \dot{f}_t - b(f_t) \right|^2 \, dt \;\; ;$$

otherwise, if $f = \mathrm{F}(g) \notin H_1 + x$, then $g \notin H_1$ and $I(f) = \infty$ due to Schilder's theorem.

3) Finally consider a general diffusion coefficient σ; in this case the map F defined analogously as above as $\mathrm{F}(g) := f$ via

$$f_t = x + \int_0^t b(f_s)\, ds + \int_0^t \sigma(f_s)\dot{g}_s \, ds \qquad (t \in [0,T])$$

is not necessarily continuous. Therefore, one approximates $X^{\varepsilon,x}$ by $X^{\varepsilon,m,x}$,

$$dX_t^{\varepsilon,m,x} \;=\; b\left(X^{\varepsilon,m,x}_{\frac{\lfloor mt \rfloor}{m}}\right) dt \;+\; \sqrt{\varepsilon}\,\sigma\left(X^{\varepsilon,m,x}_{\frac{\lfloor mt \rfloor}{m}}\right) dW_t \;\; , \qquad X_0^{\varepsilon,m,x} = x$$

and uses the "almost continuous version" of the contraction principle which had already mentioned before; see Dembo and Zeitouni [De-Zt 98, p.214f., 133]. □

Further details concerning the above results are contained in the theorems 4.1.1 and 5.3.2. by Freidlin and Wentzell [Fr-We 98], in Wentzell and Freidlin [We-Fr 70, §1] as well as in Dembo and Zeitouni [De-Zt 98, Sec.5.6].

Remark 2.3.5. Consider the situation of the above theorem and assume that a is strictly positive definite. Then the rate function (action functional)

$$I(f) = I_{0T}^x(f) \equiv \begin{cases} \frac{1}{2}\int_0^T \left| a(f_s)^{-1/2}\left[\dot{f}_s - b(f_s)\right]\right|^2 ds & , f \text{ absolutely continuous and } f_0 = x, \\ \infty & , \text{otherwise} \end{cases}$$

vanishes if and only if f is a solution path of the ODE $\dot{x} = b(x)$ on the time interval $[0,T]$; the action functional hence weighs the deviation of a path f from being a deterministic solution in the L^2-norm.

The following continuity consequence of the LDP 2.3.4 will be used in the proof of the exit time law 2.4.6 (more precisely, in the lemmas 2.4.8, 2.4.9 and 2.4.10); it states that the above LDP also holds true uniformly with respect to the initial condition. For a proof see Dembo and Zeitouni [De-Zt 98, Cor.5.6.15].

A similar assertion will appear later in the case of degenerate SDEs; see corollary 3.2.1.

Theorem 2.3.6 (Uniform asymptotics). *Consider the situation of the above theorem 2.3.4 and fix some compact set $K \subset \mathbb{R}^d$. Then it follows for all open sets $G \subset C([0,T], \mathbb{R}^d)$,*

$$\liminf_{\varepsilon \to 0} \varepsilon \log \inf_{y \in K} \mathbb{P}\{X_\bullet^{\varepsilon,y} \in G\} \;\geq\; -\sup_{y \in K} \inf_{f \in G} I_{[0,T],y}(f)\,,$$

and for all closed sets $F \subset C([0,T], \mathbb{R}^d)$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log \sup_{y \in K} \mathbb{P}\{X_\bullet^{\varepsilon,y} \in F\} \;\leq\; -\inf_{y \in K,\, f \in F} I_{[0,T],y}(f)\,.$$

2.4 Exit probabilities for non-degenerate systems

In this section we are interested in the noise-induced exit (exit time, exit location) from a neighborhood of an equilibrium point of the corresponding deterministic system. Throughout this section we consider the SDE (2.1)

$$dX_t^{\varepsilon,x} \;=\; b(X_t^{\varepsilon,x})\,dt \;+\; \sqrt{\varepsilon}\,\sigma(X_t^{\varepsilon,x})\,dW_t\;, \qquad X_0^{\varepsilon,x} = x$$

under the assumptions 2.1.1. Again, the law of $(X_t^{\varepsilon,x})_{t \geq 0}$ is denoted by $\mathbb{P}_x$ and $\mathbb{E}_x$ is the corresponding expected value.

Notation 2.4.1 (First exit time). Let D be a bounded, open domain in $\mathbb{R}^d$ with smooth boundary ∂D. Then the random variable

$$\tau^\varepsilon \;\equiv\; \tau^{\varepsilon,x} \;\equiv\; \tau_D^{\varepsilon,x} \;:=\; \inf\{t \geq 0 \;:\; X_t^{\varepsilon,x} \notin D\}$$

denotes the first exit time of $X_\bullet^{\varepsilon,x}$ from D. Since D is open, $\tau^{\varepsilon,x}$ is a stopping time[6] with respect to the underlying standard filtration $(\mathcal{F}_t)_{t \geq 0}$.

Remark 2.4.2. The first exit time τ^ε and the first exit location can be characterized for any ε in terms of solutions of PDEs involving the generator $\mathcal{G}^\varepsilon$ of X^ε :

1) $f_1(t,x) \;:=\; \mathbb{P}_x\{\tau^\varepsilon \leq t\}$ is the unique solution of

$$\begin{aligned} \big(\mathcal{G}^\varepsilon f\big)(t,x) &= \frac{\partial f}{\partial t}(t,x)\;,\; t > 0\,,\; x \in D \\ f(t,x) &= \begin{cases} 0 & ,\; t = 0\,,\; x \in D \\ 1 & ,\; t > 0\,,\; x \in \partial D\;; \end{cases} \end{aligned}$$

let $Q := (0,\infty) \times D$; then f_1 is continuous at all points of $\overline{Q} \setminus \{(0,x) : x \in \partial D\}$.

[6] See e.g.: Hackenbroch and Thalmaier [Hb-Th 94, 3.12.(ii)].

2) $f_2(x) := \mathbb{E}_x \tau^\varepsilon$ is the unique solution of

$$\begin{aligned} \mathcal{G}^\varepsilon f &= -1 \text{ , on } D \\ f &= 0 \quad \text{, on } \partial D \text{ .} \end{aligned}$$

3) For any $g \in C(\partial D, \mathbb{R})$, $f_3(x) := \mathbb{E}_x(g(X^\varepsilon_{\tau^\varepsilon}))$ is the unique solution of

$$\begin{aligned} \mathcal{G}^\varepsilon f &= 0 \text{ , on } D \\ f &= g \text{ , on } \partial D \text{ ;} \end{aligned}$$

f_2 and f_3 are continuous on $\overline{D}$.

These differential equations are similar in spirit to the ones investigated in section 2.2; they are well known: 1) can be found e.g. in Friedman [Fri 76, p.347] and Freidlin and Wentzell [Fr-We 98, p.107], 2) and 3) are cited from Dembo and Zeitouni [De-Zt 98, p.222]; also see Hackenbroch and Thalmaier [Hb-Th 94, Sec.6.6]. The above PDEs are difficult to solve, especially in higher dimensions, and will not be used in the sequel. Instead, the asymptotic behavior of τ^ε as $\varepsilon \to 0$ is investigated by means of the LDP for X^ε.

Theorem 2.4.3 (Consequence of the LDP 2.3.4 for the first exit time). *Let D be a bounded, open domain in $\mathbb{R}^d$ with smooth boundary ∂D and first-exit time*

$$\tau^{\varepsilon,x} \equiv \inf\{t \geq 0 \,:\, X^{\varepsilon,x}_t \notin D\} \text{ ,}$$

where $X^{\varepsilon,x}$ is the solution of the SDE (2.1) under assumption **(E)**, *starting in $x \in D$. Furthermore, let I be the rate function (action functional) for $X^{\varepsilon,x}$ on $C_x([0,T],\mathbb{R}^d)$; see 2.3.4. Then it follows for $t \in [0,T]$ that*

$$\begin{aligned} \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_x\{\tau^\varepsilon \leq t\} &= -\inf \left\{ I_{[0,T],x}(f) : f \in C_x([0,T],\mathbb{R}^d),\ \underset{s\in[0,t]}{\exists} \quad f(s) \notin D \right\} \\ &\equiv -\inf\left\{V(x,y;s) : s \in [0,t], y \notin D\right\}, \end{aligned}$$

where

$$V(x,y;s) := \inf\{\, I_{[0,s],x}(f) \,:\, f \in C_x([0,s],\mathbb{R}^d)\,,\ f(s) = y\} \text{ .}$$

Proof. For simplicity, the proof shall be given only for the case that σ is a constant (invertible) matrix; in doing so we follow Freidlin and Wentzell (theorem 4.1.2 and example 3.3.5 in [Fr-We 98]) who consider $\sigma = \mathrm{id}_{\mathbb{R}^d}$. For the general case the reader is referred to Wentzell and Freidlin [We-Fr 70, Th.2.1] and Friedman [Fri 76, Th.14.4.1].

By theorem 2.3.4 the LDP holds for the law μ_ε of $X^{\varepsilon,x}$ on C_x with rate function $I \equiv I_{[0,T],x}$, so (2.7) applies and it is thus left to verify:

$$\inf_{\Psi^\circ} I_{[0,T],x} \le \inf_{\overline{\Psi}} I_{[0,T],x} \quad ,$$

where

$$\Psi := \left\{ f \in C_x([0,T], \mathbb{R}^d) : \underset{s\in[0,t]}{\exists} \quad f(s) \notin D \right\} .$$

As $\complement\Psi$ is open in C_x , Ψ is closed, $\Psi = \overline{\Psi}$. Furthermore,

$$\inf_{\Psi} I_{[0,T],x} < \infty ,$$

because for some (even for any) fixed $y \in \partial D$, $f \in \Psi$, where $f(u) := x + \frac{u}{t}(y - x)$, and $I_{[0,T],x}(f) < \infty$ due to the assumption **(E)**.

Suppose that the above claim were false, i.e. that

$$\inf_{\Psi^\circ} I_{[0,T],x} > \inf_{\Psi} I_{[0,T],x} \quad .$$

Hence, there exists $\varphi \in \Psi \setminus \Psi^\circ$ such that $\inf_{\Psi^\circ} I > I(\varphi)$. In particular, φ is absolutely continuous, since $I(\varphi) < \infty$. As $\varphi \in \Psi$, there exists $s \le t$ such that $\varphi(s) \notin D$.

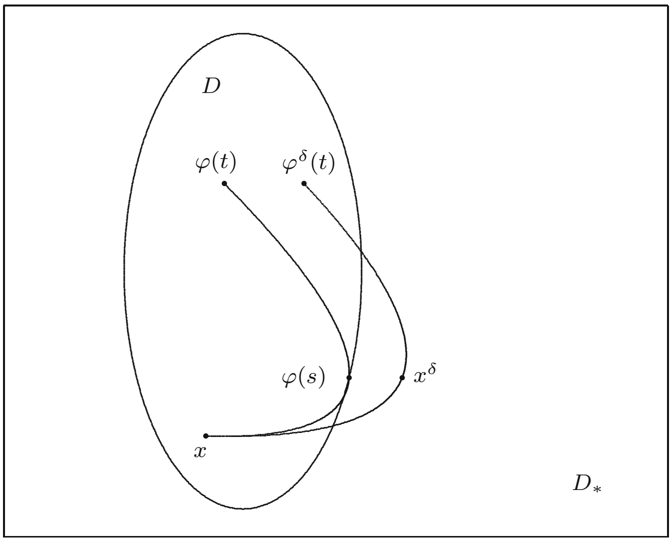

Fig. 2.3 Sketch of φ and φ^δ within the domains D and D_*

Now for any $\delta > 0$ fix some $x^\delta \in \complement\overline{D} \cap B(\varphi(s), \delta)$, where $B(\varphi(s), \delta)$ denotes the open ball with center $\varphi(s)$ and radius δ . The function

$$\varphi^\delta(r) := \varphi(r) + \frac{r}{s}\left(x^\delta - \varphi(s)\right) \qquad (r \in [0,T])$$

is also absolutely continuous and belongs to Ψ°. Hence, the proof is completed, if it is shown that $I(\varphi^\delta) \xrightarrow{\delta\to 0} I(\varphi)$, in contradiction to the choice of φ as $\inf_{\Psi^\circ} I > I(\varphi)$. For this purpose define the auxiliary functions

$$A_r^\delta := a^{-\frac{1}{2}}\left[\dot{\varphi}^\delta(r) - b(\varphi^\delta(r))\right] \qquad (r \in [0,T])$$

and

$$B_r := a^{-\frac{1}{2}}\left[\dot{\varphi}(r) - b(\varphi(r))\right] \qquad (r \in [0,T]) \ ,$$

where $a \equiv \sigma\sigma^*$, in order to get

$$\begin{aligned}
&I(\varphi^\delta) - I(\varphi) \\
&= \frac{1}{2}\int_0^T \left| a^{-\frac{1}{2}}\left[\dot{\varphi}^\delta(r) - b(\varphi^\delta(r))\right]\right|^2 - \left| a^{-\frac{1}{2}}\left[\dot{\varphi}(r) - b(\varphi(r))\right]\right|^2 dr \\
&\equiv \int_0^T \frac{1}{2}\left\{\left| A_r^\delta \right|^2 - \left| B_r \right|^2\right\} dr \\
&= \int_0^T \left\{\langle A_r^\delta, B_r\rangle - |B_r|^2 + \frac{1}{2}\left(|A_r^\delta|^2 + |B_r|^2 - \langle A_r^\delta, B_r\rangle - \langle B_r, A_r^\delta\rangle\right)\right\} dr \\
&= \int_0^T \left\{\langle (A_r^\delta - B_r), B_r\rangle + \frac{1}{2}\left\langle (A_r^\delta - B_r), (A_r^\delta - B_r)\right\rangle\right\} dr \\
&= \int_0^T \left\langle \left(A_r^\delta - B_r\right), B_r\right\rangle dr + \frac{1}{2}\int_0^T \left|A_r^\delta - B_r\right|^2 dr \\
&\equiv \left\langle (A^\delta - B), B\right\rangle_{L^2([0,T],\mathbb{R}^d)} + \frac{1}{2}\left|A^\delta - B\right|^2_{L^2([0,T],\mathbb{R}^d)} \ .
\end{aligned}$$

Therefore the claim has been reduced to showing that

$$A^\delta - B \xrightarrow{\delta\to 0} 0 \ \text{ in } \ L^2([0,T],\mathbb{R}^d) \ ;$$

a sufficient condition for this assertion is that

$$A_r^\delta - B_r \xrightarrow{\delta\to 0} 0 \ \text{ uniformly with respect to } r \in [0,T] \, .$$

For this purpose fix an open domain $D_* \subset \mathbb{R}^d$ such that

$$D_* \supset D \cup \{\varphi(r) : r \in [0,T]\} \cup \{\varphi^\delta(r) : r \in [0,T], \delta \in (0,1]\} \ ;$$

since $\varphi^\delta \xrightarrow{\delta\to 0} \varphi$ uniformly on $[0,T]$, this domain D_* can be chosen to be bounded. Since b has been assumed to be C^∞ in (2.1), there is a (local)

Lipschitz constant $C < \infty$ such that

$$|\, b(w_1) - b(w_2)\,| \;\le\; C\,|w_1 - w_2| \qquad\qquad (w_1, w_2 \in \overline{D_*}) \;\;.$$

Therefore, using that $\dot\varphi^\delta(r) = \dot\varphi(r) + \frac{1}{s}(x^\delta - \varphi(s))$, it follows altogether that

$$\begin{aligned}
\left|A_r^\delta - B_r\right| \;\;&\equiv\;\; \left|\, a^{-\frac{1}{2}}\,\left[\dot\varphi^\delta(r) - b(\varphi^\delta(r))\right] \;-\; a^{-\frac{1}{2}}\left[\dot\varphi\,(r) \;-\; b(\,\varphi(r))\right] \,\right| \\
&=\;\; \Big|\; \frac{1}{s}\, a^{-\frac{1}{2}}\,(x^\delta - \varphi(s)) \;+\; a^{-\frac{1}{2}}\,\dot\varphi(r) \;-\; a^{-\frac{1}{2}}\,\dot\varphi(r) \\
&\qquad\qquad -\; a^{-\frac{1}{2}}\, b(\varphi^\delta(r)) \;+\; a^{-\frac{1}{2}}\, b(\,\varphi(r)) \;\;\Big| \\
&\le\;\; \frac{1}{s}\,\left\| a^{-\frac{1}{2}} \right\| \;\left|\, x^\delta - \varphi(s)\,\right| \;+\; \left\| a^{-\frac{1}{2}} \right\| \;\left|\, b(\varphi^\delta(r)) \;-\; b(\varphi(r))\,\right| \\
&\le\;\; \left\| a^{-\frac{1}{2}} \right\| \left(\frac{1}{s}\,\left|\, x^\delta - \varphi(s)\,\right| \;+\; C\,\left|\, \varphi^\delta(r) \;-\; \varphi(r)\,\right| \right) \\
&\xrightarrow{\delta\to 0}\; 0 \;\;,
\end{aligned}$$

uniformly with respect to $r \in [0, T]$. □

The above theorem yields information about the distribution of exit times of non-degenerate stochastic systems. The corresponding assertion concerning degenerate systems by Hernández-Lerma will appear later in 3.1.7. Theorem 2.4.3 furthermore motivates the following definition which had been anticipated at the beginning of this chapter (see p.56) for formulating assumption **(V)** in 2.1.1; the corresponding cost function in the context of degenerate systems will then appear in 3.1.2.

Definition 2.4.4 (Quasipotential). Let I be the good rate function for the solution X^ε of (2.1) as provided by theorem 2.3.4 . Then

$$\begin{aligned}
&V : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_{>0} \longrightarrow \mathbb{R}_+ \;, \\
&\qquad\qquad V(x, y; s) \;:=\; \inf\left\{\, I_{[0,s],x}(f) \;:\; f \in C_x([0,s], \mathbb{R}^d)\,,\, f(s) = y \right\}
\end{aligned}$$

denotes the cost of forcing the system X^ε to connect x and y in time s (in the sense of theorem 2.4.3) . The function

$$\begin{aligned}
&V : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}_+ \;, \\
&\qquad V(x, y) \;:=\; \inf_{s>0}\, V(x, y; s)
\end{aligned}$$

is the *quasipotential of y with respect to x* ; it is considered as the cost of forcing the system X^ε to connect x and y eventually (see theorem 2.4.6). For a point $\mathcal{O} \in \mathbb{R}^d$,

$$V : \mathbb{R}^d \longrightarrow \mathbb{R}_+ \,,$$
$$V(y) \;:=\; V(\mathcal{O}, y)$$

denotes the *quasipotential of the system* X^ε *(with respect to* $\mathcal{O}$*)*.

The meaning of "quasipotential" is clarified in the following proposition which is cited from Freidlin and Wentzell [Fr-We 98, p.118f.]:

Proposition 2.4.5. *Let* D *be a bounded open domain in* $\mathbb{R}^d$*, suppose that* $\sigma = \mathrm{id}_{\mathbb{R}^d}$ *and let the drift* $b \in C(\mathbb{R}^d, \mathbb{R}^d)$ *derive from a potential* U *with an orthogonal component* L *on* $\overline{D}$ *and let* b *have a unique equilibrium in* D*, i.e. suppose that*

$$b(x) = -(\nabla U)(x) + \mathrm{L}(x) \qquad\qquad (x \in \overline{D}) \,,$$

where $U \in C^1(\overline{D}, \mathbb{R})$ *and* $\mathrm{L} \in C(\mathbb{R}^d, \mathbb{R}^d)$ *are functions such that*

$$\big\langle (\nabla U)(x), \mathrm{L}(x) \big\rangle \;=\; 0 \qquad\qquad (x \in \overline{D})$$

and for some $\mathcal{O} \in D$*, one can state that* $U(\mathcal{O}) = 0$ *as well as*

$$U(x) > 0 \quad \text{ and } \quad (\nabla U)(x) \neq 0 \qquad\qquad (x \in \overline{D} \setminus \{\mathcal{O}\}) \,.$$

Then for all $x \in \overline{D}$ *for which* $U(x) \le \min_{\partial D} U$*, the quasipotential* V *with respect to* $\mathcal{O}$ *is given by:*

$$V(x) \;\equiv\; V(\mathcal{O}, x) \;=\; 2\,U(x) \,.$$

If in addition $U \in C^2$*,* $\mathrm{L} \in C^1$ *and* $x \in \overline{D}$ *is some point, then the rate function* $I_{(-\infty,T],\mathcal{O}}$ *, defined analogously to* $I_{[0,T],\mathcal{O}}$*, has a unique extremal* φ *on the set*

$$\left\{ f \in C\big((-\infty, T], \mathbb{R}^d\big) \;:\; \lim_{s \to -\infty} f(s) = \mathcal{O} \,,\; f(T) = x \right\} \,;$$

furthermore, this extremal φ *is the solution of the ODE*

$$\dot\varphi(s) = +\big(\nabla U\big)(\varphi(s)) + \mathrm{L}(\varphi(s)) \qquad (s \in (-\infty, T]) \,,$$
$$\varphi(T) = x \,.$$

In the general case, $\sigma(\,.\,) \not\equiv \mathrm{id}_{\mathbb{R}^d}$, the above proposition remains true, if D is endowed with the Riemannian metric

$$ds^2 \;:=\; \sum_{i,j=1}^{d} \big(\, a(x)^{-1} \,\big)_{ij} \; dx^i \, dx^j$$

and $\langle \;,\; \rangle$ as well as ∇ now denote the scalar product and the Riemannian gradient with respect to this metric, respectively; see Freidlin and Wentzell [We-Fr 72, Th.1].

Further properties of the quasipotential V are the following (see Freidlin and Wentzell [Fr-We 98, Ch.4]):

$V(\mathcal{O}, .)$ is Lipschitz continuous, but not necessarily differentiable ([Fr-We 98, p.108,119]). The function $V(x, y, s)$ satisfies a Hamilton-Jacobi equation ([Fr-We 98, p.107]). In case that $V(\mathcal{O}, .)$ is continuously differentiable, a corresponding Jacobi equation follows; from this Jacobi variational equation a decomposition for b as in proposition 2.4.5 can be deduced ([Fr-We 98, p.119]). Furthermore, in the situation of proposition 2.4.5 one obtains that $V(\mathcal{O}, x) > 0$ if and only if $x \neq \mathcal{O}$ (see Day and Darden [Day-Dar 85, Cor.2]).

Next the fundamental exit law for non-degenerate systems will be discussed. It concerns the noise induced first exit of $X^{\varepsilon,x}$, given by (2.1),

$$dX_t^{\varepsilon,x} = b(X_t^{\varepsilon,x})\, dt + \sqrt{\varepsilon}\, \sigma(X_t^{\varepsilon,x})\, dW_t, \qquad X_0^{\varepsilon,x} = x \in D$$

from a bounded, open domain $D \subset \mathbb{R}^d$ with smooth boundary ∂D. For this purpose, it is not necessary to impose the set of requirements 2.1.1 to the full extent. Instead, the following assumptions are underlying:

(A1) There exists a unique stable equilibrium[7] point $\mathcal{O} \in D$ of the deterministic system

$$dX_t^{0,x} = b(X_t^{0,x})\, dt\,, \qquad X_0^{0,x} = x\,, \tag{2.8}$$

to which D is attracted.[8]

(A2) All trajectories of the deterministic system starting in ∂D converge to $\mathcal{O}$ (as $t \to \infty$).

(A3) $\overline{V} := \inf_{\partial D} V(\mathcal{O}, .) < \infty\,.$

(A4) There exist $K, \rho_0 > 0$ such that for all $\rho \leq \rho_0$ and all $x, y \in \mathbb{R}^d$ for which

$$|x - z| + |y - z| \leq \rho \quad \text{for some } z \in \partial D \cup \{\mathcal{O}\}\,,$$

there is a function $\mathrm{u} \equiv \mathrm{u}_{\rho;x,y} \in L^2([0, T_\rho], \mathbb{R}^d)$ such that $\|\mathrm{u}\|_\infty < K$ and $\mathrm{k}(T_\rho) = y$, where

$$\mathrm{k}(t) := x + \int_0^t b(\mathrm{k}(s))\, ds + \int_0^t \sigma(\mathrm{k}(s))\, \mathrm{u}(s)\, ds\ ,$$

and where $T_\rho \geq 0$ is a time such that $T_\rho \xrightarrow{\rho \to 0} 0$.

---

[7] $\mathcal{O}$ is an *(asymptotically) stable equilibrium point* of the deterministic dynamical system, if for any neighborhood B_1 of $\mathcal{O}$ there exists another neighborhood $B_2 \subset B_1$ such that all trajectories of the deterministic system starting in B_2 converge to $\mathcal{O}$ (as $t \to \infty$) without leaving B_1; of course, $b(\mathcal{O}) = 0$.

[8] D is *attracted to* $\mathcal{O}$, if all trajectories of the deterministic system $X^{0,x}$ starting in D converge to (the equilibrium position) $\mathcal{O}$ (as $t \to \infty$) without leaving D.

In the situation described by the assumptions 2.1.1 the above requirements **(A1)**-**(A4)** are fulfilled, if for some $i \in \{1, \ldots, l\}$, the domain D satisfies that $\mathcal{O} := K_i \in D$ and $\overline{D} \subset D_i$, where

$$D_i \ := \ \{x \in \mathbb{R}^d \ : \ X_t^{0,x} \xrightarrow{t \to \infty} K_i\}$$

denotes the domain of attraction of K_i under the deterministic motion X^0. The fact that necessarily $\overline{D} \subset D_i$, is due to **(A1)** and **(A2)** which exclude any other equilibrium (i.e. any other element in $\{K_1, \ldots, K_{l'}\} \setminus \{K_i\}$) from being in $\overline{D}$. **(A4)** is implied by **(E)**, see Dembo and Zeitouni [De-Zt 98, p.224]. In general, **(A2)** prevents that $\langle b(x), N(x) \rangle \ = \ 0$, $\forall x \in \partial D$, where $N(x)$ is the outer normal to ∂D at x; in this situation ∂D is a *characteristic boundary*; for studies on this case see Day [Day 90] and the references therein. Dembo and Zeitouni [De-Zt 98, Cor.5.7.16] investigate the situation when **(A2)** is skipped, but when the boundary is not necessarily characteristic.

The above assumptions **(A1)**–**(A4)** are taken from Dembo and Zeitouni [De-Zt 98, p.221ff.]; this reference is also underlying to the subsequent discussion. Since this section is intended to provide the argumentation in outlines, some of the proofs will only be sketched and the reader is referred to Dembo and Zeitouni [De-Zt 98, Sec.5.7] for details. These results are due to Freidlin and Wentzell; see [We-Fr 70, §3] and [Fr-We 98, §§4.2,4.4].

Theorem 2.4.6. *Let the assumptions* **(A1)**-**(A4)** *be satisfied and let*

$$\tau^{\varepsilon,x} \ \equiv \ \inf\{t \geq 0 \ : \ X_t^{\varepsilon,x} \notin D\} \ ,$$

denote the first exit time of $X^{\varepsilon,x}$, given by (2.1), from a bounded, open domain $D \subset \mathbb{R}^d$ with smooth boundary ∂D. Then it follows

1) for the first exit time: For all $x \in D$ and $\delta > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{P}_x \left\{ \mathrm{e}^{(\overline{V} - \delta)/\varepsilon} < \tau^\varepsilon < \mathrm{e}^{(\overline{V} + \delta)/\varepsilon} \right\} \ = \ 1 \ ,$$

and for all $x \in D$,

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_x \tau^\varepsilon \ = \ \overline{V} \ ;$$

2) for the first exit position: If $N \subset \partial D$ is a closed set for which $\inf_N V(\mathcal{O}, \,.\,) > \overline{V}$, then for any $x \in D$

$$\lim_{\varepsilon \to 0} \mathbb{P}_x \left\{ X_{\tau^\varepsilon}^\varepsilon \in N \right\} \ = \ 0 \ ;$$

thus, if $V(\mathcal{O}, \,.\,)$ has a unique minimum z^* on ∂D, then for any $x \in D$ and $\delta > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{P}_x \left\{ \left| X_{\tau^\varepsilon}^\varepsilon \ - \ z^* \right| \ < \ \delta \right\} \ = \ 1 \, .$$

The proof of Theorem 2.4.6 relies on the following lemmas. Here, $B_\rho(\mathcal{O})$ and $S_\rho(\mathcal{O})$ will denote the closed ball and the sphere around $\mathcal{O}$ with radius ρ, respectively; furthermore, the radii of all balls and spheres appearing are chosen so small such that the balls and spheres are contained in D.

Lemma 2.4.7 (Continuity of V given (A4)). *Assume condition* **(A4)**. *Then there exits for any $\delta > 0$ a sufficiently small $\rho > 0$ such that*

$$\sup_{x,y \in B_\rho(\mathcal{O})} \inf_{t\in[0,1]} V(x,y,t) < \delta \tag{2.9}$$

as well as

$$\sup_{\left\{ x,y\in\mathbb{R}^d :\ \inf\limits_{z\in\partial D}(|x-z|+|y-z|)\le\rho \right\}} \inf_{t\in[0,1]} V(x,y,t) < \delta\,. \tag{2.10}$$

Proof. Given $x, y \in \mathbb{R}^d$ for which $|x-z|+|y-z| \le \rho$ for some $z \in \partial D \cup \{\mathcal{O}\}$, let k, u and K, T_ρ be the functions and constants made available by **(A4)**. Due to theorem 2.3.4,

$$I_{[0,t],x}(\mathrm{k}) = \inf_{\{\, g\in H_1 \,:\, \mathrm{k}(s) = x + \int_0^s b(\mathrm{k}(r))\,dr + \int_0^s \sigma(\mathrm{k}(r))\,\dot g(r)\,dr \,\}} \frac{1}{2}\int_0^t |\,\dot g(s)\,|^2\ ds\,.$$

Hence, **(A4)** implies that

$$\begin{aligned} V(x,y;T_\rho) &\equiv \inf\left\{ I_{[0,T_\rho],x}(f)\ :\ f\in C_x([0,T_\rho],\mathbb{R}^d)\,,\ f(T_\rho)=y \right\} \\ &\le I_{[0,T_\rho],x}(\mathrm{k}) \\ &\le \frac{1}{2}\int_0^{T_\rho} |\,\mathrm{u}(s)\,|^2\ ds \\ &\le K^2\,T_\rho/2\ , \end{aligned}$$

which can become arbitrarily small for an appropriate choice of ρ, again due to **(A4)**. □

Next, five lemmas are formulated from which 2.4.6 then can be proved. Here, lemma 2.4.7 is needed for 2.4.8 and 2.4.10. The LDP for X^ε will be used in terms of theorem 2.3.6 in the proofs of the first three of these lemmas. In doing so, the boundedness conditions on b und σ in theorem 2.3.6 (see 2.3.4) are tacitly assumed to be satisfied. This is no restriction, since the system is only examined until its first exit time τ^ε from D which only[9] depends on the values of b and σ on D.

[9] See e.g. Hackenbroch and Thalmaier [Hb-Th 94, 6.22].

Lemma 2.4.8 (Uniform lower bound on the exit probability for starts near $\mathcal{O}$). *Assume the set of conditions* **(A)**. *For any $\eta > 0$ there is then a $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0]$, there exists $T_0 < \infty$ for which*

$$\liminf_{\varepsilon \to 0} \varepsilon \log \inf_{x \in B_\rho(\mathcal{O})} \mathbb{P}_x \{\tau^\varepsilon \le T_0\} \;>\; -\big(\overline{V} + \eta\big) \;.$$

Proof. Given $\eta > 0$, apply lemma 2.4.7 for $\delta := \frac{\eta}{6}$, to get $\rho_0 > 0$ such that (2.9) and (2.10) hold for ρ_0 — and hence also for all $\rho \in (0, \rho_0]$. Fix such a ρ and an arbitrary $x \in B_\rho(\mathcal{O})$: (2.9) provides a path ψ^x and $t_x \in [0,1]$ such that

$$\psi^x(0) \;=\; x\,, \qquad \psi^x(t_x) \;=\; \mathcal{O}\,, \qquad I_{[0,t_x],x}(\psi^x) \;\le\; \delta < \frac{\eta}{3} \;\;;$$

due to **(A3)** there are a path ψ^0, $t_0 > 0$ and $z \in \partial D$ such that

$$\psi^0(0) \;=\; \mathcal{O}\,, \qquad \psi^0(t_0) \;=\; z\,, \qquad I_{[0,t_0],\mathcal{O}}(\psi^0) \;\le\; \overline{V} \;+\; \frac{\eta}{6} \;\;;$$

for this choice of z, (2.10) yields a ψ^z, $t_z \in [0,1]$ and $y \notin \overline{D}$ for which $\operatorname{dist}(y, \partial D) = \rho$ and

$$\psi^z(0) \;=\; z\,, \qquad \psi^z(t_z) \;=\; y\,, \qquad I_{[0,t_z],z}(\psi^z) \;\le\; \delta \equiv \frac{\eta}{6} \;\;;$$

finally, let $\psi^y := X^{0,y}$ denote the solution curve of the deterministic system (2.8) started in y and considered until time $t_y := 2 - (t_x + t_z)$,

$$\psi^y(0) \;=\; y\,, \qquad I_{[0,2-(t_x+t_z)],y}(\psi^y) \;=\; 0 \;\;.$$

Juxtaposing ψ^x, ψ^0, ψ^z and ψ^y results in a path ϕ^x which is defined on $[0, T_0]$, where $T_0 \;:=\; t_x + t_0 + t_z + t_y \;\equiv\; t_0 + 2$ and for which

$$I_{[0,T_0],x}(\phi^x) \;\le\; I(\psi^x) \;+\; I(\psi^0) \;+\; I(\psi^z) \;+\; I(\psi^y) \;<\; \overline{V} \;+\; \eta\,.$$

Using these functions ϕ^x, $x \in B_\rho(\mathcal{O})$, the set

$$\Psi \;:=\; \bigcup_{x \in B_\rho(\mathcal{O})} \Big\{\psi \in C([0,T_0], \mathbb{R}^d) \;:\; \|\,\psi - \phi^x\|_\infty \;<\; \frac{\rho}{2}\Big\}$$

is open and $\{\, X^{\varepsilon,x} \in \Psi \,\} \;\subset\; \{\, \tau^{\varepsilon,x} \le T_0 \,\}$. Therefore it follows from theorem 2.3.6 that

$$\begin{aligned}\liminf_{\varepsilon \to 0} \varepsilon \log \inf_{x \in B_\rho(\mathcal{O})} \mathbb{P}_x \{\tau^\varepsilon \le T_0\} &\ge \liminf_{\varepsilon \to 0} \varepsilon \log \inf_{x \in B_\rho(\mathcal{O})} \mathbb{P}\{X^{\varepsilon,x}_\bullet \in \Psi\} \\ &\ge - \sup_{x \in B_\rho(\mathcal{O})} \inf_{\phi \in \Psi} I_{[0,T_0],x}(\phi)\end{aligned}$$

$$\geq - \sup_{x \in B_\rho(\mathcal{O})} I_{[0,T_0],x}(\phi^x)$$
$$> -(\overline{V} + \eta) \, .$$

$\square$

Lemma 2.4.9 (X^ε cannot stay in D arbitrarily long without approaching $\mathcal{O}$). *Assume the set of conditions* **(A)**. *Then we have for any* $\rho > 0$,

$$\lim_{t\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in D} \mathbb{P}_x \left\{ \sigma_\rho^\varepsilon > t \right\} \; = \; -\infty \, ,$$

where σ_ρ^ε *denotes the* first hitting time of either ∂D or a small neighborhood of $\mathcal{O}$,

$$\sigma_\rho^{\varepsilon,x} \; := \; \inf\{t \geq 0 \, : \, X_t^{\varepsilon,x} \in B_\rho(\mathcal{O}) \cup \partial D\} \; .$$

Sketch of Proof. For all $x \in B_\rho(\mathcal{O})$, $\sigma_\rho^{\varepsilon,x} = 0$; thus, only initial values $x \in D \setminus B_\rho(\mathcal{O})$ are of relevance. The set of functions which do not leave the closure of the latter set,

$$\Psi_t \; := \; \left\{ \psi \in C([0,t], \mathbb{R}^d) \, : \, \psi(s) \in \overline{D \setminus B_\rho(\mathcal{O})} \text{ for all } s \in [0,t] \right\} \qquad (t > 0)$$

is closed and $\{ \, \sigma^{\varepsilon,x} > t \, \} \subset \{ \, X^{\varepsilon,x} \in \Psi_t \, \}$ for $x \in D \setminus B_\rho(\mathcal{O})$. Hence, theorem 2.3.6 yields

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in\overline{D\setminus B_\rho(\mathcal{O})}} \mathbb{P}_x \left\{ \sigma^\varepsilon > t \right\} \leq \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in\overline{D\setminus B_\rho(\mathcal{O})}} \mathbb{P}_x \left\{ X_\bullet^\varepsilon \in \Psi_t \right\}$$
$$\leq - \inf_{\phi\in\Psi_t} I_{[0,t],\phi(0)}(\phi) \, ,$$

which reduces the claim of the lemma to proving that the right hand side diverges ($t \to \infty$).

Via **(A2)**, a Gronwall argument (b is C^∞, hence Lipschitz on $\overline{D}$) and the compactness of $\overline{D \setminus B_\rho(\mathcal{O})}$ one can get $T < \infty$ such that for all $x \in \overline{D \setminus B_\rho(\mathcal{O})}$, the solution $\phi^x(t) := X_t^{0,x}$ of the deterministic system (2.8) is contained in the ball $B_{2\rho/3}(\mathcal{O})$ for all $t \geq T$.

In order to obtain a contradiction suppose the divergence $\inf_{\phi\in\Psi_t} I_{[0,t],\phi(0)}(\phi) \xrightarrow{t\to\infty} \infty$ were wrong; so imagine that there exists an $M < \infty$ such that for all $n \in \mathbb{N}$, there is some $\psi^n \in \Psi_{nT}$ for which $I_{[0,nT],\psi^n(0)}(\psi^n) \leq M$; merely considering times nT is no restriction, since I is additive. Dissecting ψ^n into n pieces and using the additivity of I again, one obtains $\phi^n \in \Psi_T$ such that $I_{[0,T],\phi^n(0)}(\phi^n) \leq M/n \xrightarrow{n\to\infty} 0$. As the rate function I is good, $\Psi_T \cap \{I \leq 1\}$ is compact, providing a limit point $\phi^* \in \Psi_T$ of $(\phi^n)_n$. I being lower semicontinuous, $I_{[0,T],\phi^*(0)}(\phi^*) = 0$ follows and ϕ^* is necessarily a solution curve of the deterministic system (2.8). Due to the definition of T this implies that $\phi^*(T) \in B_{2\rho/3}(\mathcal{O})$, contradicting the fact that $\phi^*(T) \notin B_\rho(\mathcal{O})^\circ$, as an element of Ψ_T. $\square$

Lemma 2.4.10 (Bound on the probability of leaving D before further approaching $\mathcal{O}$). *Assume the set of conditions* **(A)**. *Then for all closed sets $N \subset \partial D$,*

$$\limsup_{\rho\to 0} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in S_{2\rho}(\mathcal{O})} \mathbb{P}_y \left\{ X^{\varepsilon}_{\sigma^{\varepsilon}_{\rho}} \in N \right\} \le -\inf_N V(\mathcal{O}, \,.\,) .$$

Sketch of Proof. Define $V_{N,\delta} := \min[\inf_N V(\mathcal{O}, \,.\,) - \delta \,, \frac{1}{\delta}]$ for $\delta > 0$. Due to the previous lemma 2.4.9 there exists $T < \infty$ such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in S_{2\rho}(\mathcal{O})} \mathbb{P}_y \left\{ \sigma^{\varepsilon}_{\rho} > T \right\} < -V_{N,\delta} .$$

Now, one applies theorem 2.3.6 to the closed set

$$\Phi := \left\{ \phi \in C([0,T], \mathbb{R}^d) : \phi(t) \in N \text{ for some } t \in [0,T] \right\}$$

to see that $-V_{N,\delta}$ also bounds the exponential growth rate of $\sup_{y\in S_{2\rho}(\mathcal{O})} \mathbb{P}_y\{X^{\varepsilon}_{\bullet} \in \Phi\}$ from above (as $\varepsilon \to 0$), where $\rho \equiv \rho(\delta)$ derives from (2.9). The same bound on the exponential rate holds true for $\mathbb{P}_y\big\{X^{\varepsilon}_{\sigma^{\varepsilon}_{\rho}} \in N\big\} \le \mathbb{P}_y\{X^{\varepsilon}_{\bullet} \in \Phi\} + \mathbb{P}_y \left\{\sigma^{\varepsilon}_{\rho} > T\right\}$. Finally, take $\delta \to 0$. □

The final two lemmas are not based on the large deviations principle.

Remarkably, the assertion of the next lemma is not uniform with respect to the initial point; in contrast, the other lemmas contain uniformity information. This is why theorem 2.4.6 does not hold uniformly on D (but only uniformly on compact subsets of D).

Lemma 2.4.11 (The probability of approaching $\mathcal{O}$ without leaving D is large). *Assume the set of conditions* **(A)**. *Then it follows for any $\rho > 0$ and $x \in D$ that*

$$\lim_{\varepsilon\to 0} \mathbb{P}_x \left\{ X^{\varepsilon}_{\sigma^{\varepsilon}_{\rho}} \in B_{\rho}(\mathcal{O}) \right\} = 1 .$$

Sketch of Proof. Since $\sigma^{\varepsilon,x}_{\rho} = 0$ for $x \in B_{\rho}(\mathcal{O})$, fix $x \in D \setminus B_{\rho}(\mathcal{O})$. Again, there is $T < \infty$ such that $X^{0,x}_t \in B_{\rho/2}(\mathcal{O})$ for $t \ge T$. Now it holds for

$$\Delta := \min \left[\operatorname{dist}\big(\{\phi^x(t) : t \in [0,T]\}, \partial D\big) \,,\, \rho \right] > 0$$

that $\left\{X^{\varepsilon}_{\sigma^{\varepsilon}_{\rho}} \in \partial D\right\} \subset \left\{\|X^{\varepsilon}_{\bullet} - X^{0}_{\bullet}\|_{[0,T]} > \Delta/2\right\}$ and the probability of the latter event can be estimated from above by means of a Gronwall argument and the Burkholder-Davis-Gundy maximal inequality.[10] This upper bound converges to 0 as $\varepsilon \to 0$. By the definition of $\sigma^{\varepsilon}_{\rho}$ this is the converse of the claim. See Dembo and Zeitouni [De-Zt 98, p.234f.] for details. □

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[10] See e.g. Dembo and Zeitouni [De-Zt 98, E.3] or Hackenbroch and Thalmaier [Hb-Th 94, 4.63].

Lemma 2.4.12 (Upper bound on the distance of X^ε from its starting point). *Assume the set of conditions* **(A)**. *For all $\rho, c > 0$ there exists $T \equiv T(\rho, c) < \infty$ such that*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in D} \mathbb{P}\left\{ \sup_{t\in[0,T]} |X_t^{\varepsilon,x} - x| \geq \rho \right\} < -c\,.$$

Sketch of Proof. Also in this case a Gronwall argument is applied to $|X_t^{\varepsilon,x} - x|$. By the time change theorem[11] of martingale theory the upper bound, hence obtained, can be further modified such that the statement is seen to hold true. See Dembo and Zeitouni [De-Zt 98, p.235f.] for details. □

Proof of Theorem 2.4.6:

1) Let $x \in D$ and $\delta > 0$ be fixed. First, the bounds in probability for $\tau^{\varepsilon,x}$,

$$\mathbb{P}_x\left\{ \tau^\varepsilon \geq \mathrm{e}^{\left(\overline{V}+\delta\right)/\varepsilon} \right\} \xrightarrow{\varepsilon\to 0} 0$$

and

$$\mathbb{P}_x\left\{ \tau^\varepsilon \leq \mathrm{e}^{\left(\overline{V}-\delta\right)/\varepsilon} \right\} \xrightarrow{\varepsilon\to 0} 0\ ,$$

will be proved; afterwards, the assertion on $\mathbb{E}_x\tau^\varepsilon$ will then follow from these arguments.

(a) $\boxed{\tau^{\varepsilon,x} < \mathrm{e}^{(\overline{V}+\delta)/\varepsilon}}$ Fix $\eta > 0$. Lemma 2.4.8 yields $\rho, \varepsilon_0 > 0$ and $T_0 < \infty$ such that

$$\inf_{x\in B_\rho(\mathcal{O})} \mathbb{P}_x\left\{\tau^\varepsilon \leq T_0\right\} > \mathrm{e}^{-\left(\overline{V}+\frac{\eta}{2}\right)/\varepsilon} \qquad (\varepsilon < \varepsilon_0)\,.$$

With this choice of ρ lemma 2.4.9 provides some $T_1 < \infty$ such that

$$\sup_{x\in D} \mathbb{P}_x\left\{\sigma_\rho^\varepsilon > T_1\right\} < \mathrm{e}^{-\frac{\eta}{4}\frac{1}{\varepsilon}} \qquad (\varepsilon < \varepsilon_0)\,.$$

Furthermore, choose ε_0 sufficiently small such that

$$\mathrm{e}^{\eta/(2\,\varepsilon)} - \mathrm{e}^{\eta/(4\,\varepsilon)} \geq 1 \qquad (\varepsilon < \varepsilon_0)\,.$$

Setting $T := T_0 + T_1$, the definitions of σ_ρ^ε and τ^ε, the strong Markov property and the previous string of estimates imply that for all $\varepsilon < \varepsilon_0$:

[11] See e.g. Dembo and Zeitouni [De-Zt 98, E.2] or Hackenbroch and Thalmaier [Hb-Th 94, 5.24].

$$\begin{aligned}\underline{q}^\varepsilon \ := \ \inf_{x\in D} \mathbb{P}_x\left\{\tau^\varepsilon \le T\right\} &\ge \inf_{x\in D} \mathbb{P}_x\left\{\sigma_\rho^\varepsilon \le T_1\,,\ \tau^{\varepsilon, X^\varepsilon_{\sigma^\varepsilon_\rho}} \le T_0\right\}\\ &\ge \inf_{x\in D} \mathbb{P}_x\left\{\sigma_\rho^\varepsilon \le T_1\right\}\ \cdot \inf_{x\in B_\rho(\mathcal{O})} \mathbb{P}_x\left\{\tau^\varepsilon \le T_0\right\}\\ &> \left(1-\mathrm{e}^{-\eta/(4\,\varepsilon)}\right)\ \mathrm{e}^{-\left(\overline{V}+\frac{\eta}{2}\right)/\varepsilon}\\ &\ge \mathrm{e}^{-\eta/(2\,\varepsilon)}\ \mathrm{e}^{-\left(\overline{V}+\frac{\eta}{2}\right)/\varepsilon} \quad = \quad \mathrm{e}^{-\left(\overline{V}+\eta\right)/\varepsilon}\ .\end{aligned}$$

An iteration of the strong Markov property[12] implies that

$$\sup_{x\in D} \mathbb{P}_x\{\tau^\varepsilon > kT\} \ \le\ (1-\underline{q}^\varepsilon)^k \qquad (k\in\mathbb{N})\ ;$$

more precisely, for $k=1$ this is just the definition of $\underline{q}^\varepsilon$; for $k>1$,

$$\begin{aligned}\mathbb{P}_x\{\tau^\varepsilon > kT\} &= \mathbb{P}_x\{\tau^\varepsilon > (k-1)T\,,\ \tau^{\varepsilon, X^\varepsilon_{(k-1)T}} > T\}\\ &= \mathbb{E}\left[1_{\{\tau^{\varepsilon,x}>(k-1)T\}}\cdot\left(1-\mathbb{E}^{\mathcal{F}_{(k-1)T}}\left(H\circ X^{\varepsilon,x}_{(k-1)T+\bullet}\right)\right)\right],\end{aligned}$$

where H is defined on the path space by $H := 1_{\{\tau\le T\}}$ for $\tau(f) := \inf\{t\ge 0 : f_t\notin D\}$, i.e. $\tau^{\varepsilon,x}\equiv\tau\circ X^{\varepsilon,x}_\bullet$. With $(TH)(z) := \mathbb{E}(H\circ X^{\varepsilon,z}_\bullet)$ the strong Markov property[13] implies:

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_{(k-1)T}}\left(H\circ X^{\varepsilon,x}_{(k-1)T+\bullet}\right)\\ &\quad = \ (TH)\circ X^{\varepsilon,x}_{(k-1)T}\ \equiv\ \mathbb{E}\left(1_{\{\tau\le T\}}\circ X^{\varepsilon,z}_\bullet\right)\Big|_{z=X^{\varepsilon,x}_{(k-1)T}}\ ;\end{aligned}$$

plugging this into the previous equation one gets :

$$\begin{aligned}\mathbb{P}_x\{\tau^\varepsilon > kT\} &\le \mathbb{E}\left[1_{\{\tau^{\varepsilon,x}>(k-1)T\}}\cdot\left(1\ -\ \inf_{z\in D}\ \mathbb{E}\left(1_{\{\tau\le T\}}\circ X^{\varepsilon,z}_\bullet\right)\right)\right]\\ &\equiv \left(1-\underline{q}^\varepsilon\right)\ \mathbb{P}_x\{\tau^\varepsilon > (k-1)T\}\end{aligned}$$

and thus by the induction assumption (IA):

$$\sup_{x\in D}\mathbb{P}_x\{\tau^\varepsilon > kT\}\ \le\ \left(1-\underline{q}^\varepsilon\right)\ \sup_{x\in D}\mathbb{P}_x\{\tau^\varepsilon > (k-1)T\}\ \overset{\text{IA}}{\le}\ \left(1-\underline{q}^\varepsilon\right)^k\ .$$

This induction result yields together with the previously obtained bound on $\underline{q}^\varepsilon$:

[12] Such an iterative application of the strong Markov property will also appear in the last chapter which is the reason for us to calculate details explicitly here.

[13] See e.g. Hackenbroch and Thalmaier [Hb-Th 94, 6.32 & 6.41].

$$\sup_{x\in D} \mathbb{E}_x \tau^\varepsilon \le \sup_{x\in D} T \sum_{k\in\mathbb{N}_0} \mathbb{P}_x\{\tau^\varepsilon > kT\} \le T \sum_{k\in\mathbb{N}_0} (1-\underline{q}^\varepsilon)^k$$
$$= \frac{T}{\underline{q}^\varepsilon} \le T\, \mathrm{e}^{(\overline{V}+\eta)/\varepsilon}, \tag{†}$$

the upper bound on the mean exit time. For $\eta := \frac{\delta}{2}$, Chebyshev's inequality implies:

$$\sup_{x\in D} \mathbb{P}_x\{\tau^\varepsilon \ge \mathrm{e}^{(\overline{V}+\delta)/\varepsilon}\} \le \mathrm{e}^{-(\overline{V}+\delta)/\varepsilon} \sup_{x\in D} \mathbb{E}_x \tau^\varepsilon \le T\, \mathrm{e}^{-\delta/(2\varepsilon)} \xrightarrow{\varepsilon\to 0} 0\,.$$

(b) $\boxed{\tau^{\varepsilon,x} > \mathrm{e}^{(\overline{V}-\delta)/\varepsilon}}$ Fix $\rho > 0$ (not necessarily as above) such that $S_{2\rho}(\mathcal{O}) \subset D$ and define

$$\begin{aligned}
\theta_0 &:= 0\,, \\
\tau_m^{\varepsilon,x} &:= \inf\{\, t \ge \theta_m^{\varepsilon,x} : X_t^{\varepsilon,x} \in B_\rho(\mathcal{O}) \cup \partial D\,\} && (m\in\mathbb{N}_0)\,, \\
\theta_{m+1}^{\varepsilon,x} &:= \begin{cases} \inf\{\, t \ge \tau_m^{\varepsilon,x} : X_t^{\varepsilon,x} \in S_{2\rho}(\mathcal{O})\,\} & , X_{\tau_m}^{\varepsilon,x} \in B_\rho(\mathcal{O}) \\ \infty & , X_{\tau_m}^{\varepsilon,x} \in \partial D \end{cases} && (m\in\mathbb{N}_0)\,.
\end{aligned}$$

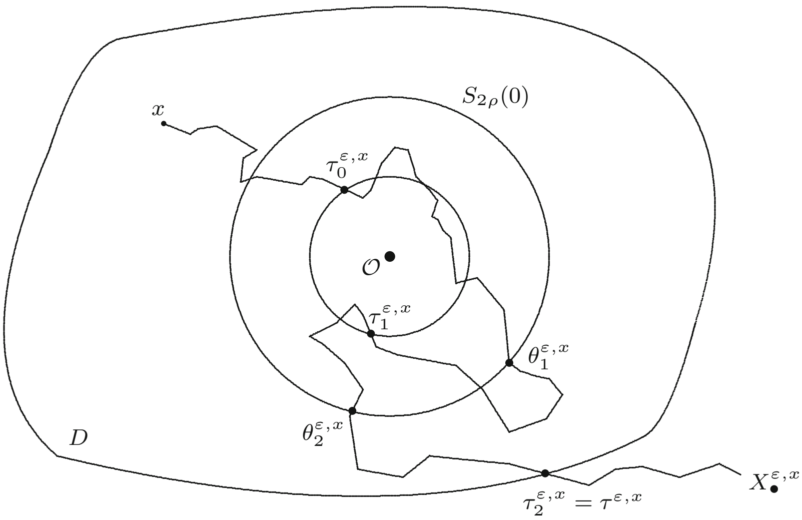

Fig. 2.4 Sketch of the stopping times $\tau_m^{\varepsilon,x}$ and $\theta_{m+1}^{\varepsilon,x}$.

From these definitions, it follows that on $\{\tau^{\varepsilon,x} = \tau_m^{\varepsilon,x}\}$ for $m \ge 1$, $X_\bullet^{\varepsilon,x}$ hits $B_\rho(\mathcal{O})$ before it hits ∂D. Thus for all $m \ge 1$,

$$\mathbb{P}_x\{\tau^\varepsilon = \tau_m^\varepsilon\} \le \sup_{y\in S_{2\rho}(\mathcal{O})} \mathbb{P}_y\left\{X^\varepsilon_{\sigma_\rho^\varepsilon} \in \partial D\right\}.$$

By applying lemma 2.4.10 with $N := \partial D$, ρ and $\varepsilon_0 > 0$ can be chosen such that

$$\sup_{y \in S_{2\rho}(\mathcal{O})} \mathbb{P}_y \left\{ X^\varepsilon_{\sigma^\varepsilon_\rho} \in \partial D \right\} \ < \ \mathrm{e}^{\left(-\overline{V}+\frac{\delta}{2}\right)/\varepsilon}$$

for all $\varepsilon < \varepsilon_0$. Consequently, $\rho > 0$ has been fixed such that for all $m \in \mathbb{N}$ and $\varepsilon < \varepsilon_0$,

$$\sup_{x \in D} \mathbb{P}_x\{\tau^\varepsilon = \tau^\varepsilon_m\} \ < \ \mathrm{e}^{-\left(\overline{V}-\frac{\delta}{2}\right)/\varepsilon} . \tag{††}$$

From the above definitions it also follows that if $\theta^{\varepsilon,x}_m - \tau^{\varepsilon,x}_{m-1} \le T_2$ (where $T_2 < \infty$ is some parameter to be specified in a moment), then $X^{\varepsilon,x}_\bullet$ must cover the distance ρ between $B_\rho(\mathcal{O})$ and $S_{2\rho}(\mathcal{O})$ in time T_2, i.e.

$$\mathbb{P}_x \left\{ \theta^\varepsilon_m - \tau^\varepsilon_{m-1} \le T_2 \right\} \le \sup_{y \in D} \mathbb{P}\left\{ \sup_{t \in [0,T_2]} |X^{\varepsilon,y}_t - y| \ \ge \ \rho \right\} \qquad (m \in \mathbb{N}).$$

Furthermore, using lemma 2.4.12 with the above choice of ρ and $c := \overline{V} + \delta/2$, one can choose $T_2 := T(\rho, c) < \infty$ and modify $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,

$$\sup_{y \in D} \mathbb{P}\left\{ \sup_{t \in [0,T_2]} |X^{\varepsilon,y}_t - y| \ \ge \ \rho \right\} \ < \ \mathrm{e}^{-\left(\overline{V}-\frac{\delta}{2}\right)/\varepsilon} .$$

Consequently, $T_2 < \infty$ has been fixed such that for all $m \in \mathbb{N}$ and $\varepsilon < \varepsilon_0$:

$$\sup_{x \in D} \mathbb{P}_x\left\{ \theta^\varepsilon_m - \tau^\varepsilon_{m-1} \le T_2 \right\} \ < \ \mathrm{e}^{-\left(\overline{V}-\frac{\delta}{2}\right)/\varepsilon} . \tag{†††}$$

Assume for the moment that $\tau^{\varepsilon,x} \le k\,T_2$ for some $k \in \mathbb{N}$: By definition there is a random variable $m \in \mathbb{N}_0$ such that $\tau^{\varepsilon,x} = \tau^{\varepsilon,x}_m$; here, either $m \le k$ or $m > k$, but in the latter case $X^{\varepsilon,x}_\bullet$ has already performed $m > k$ hits of $B_\rho(\mathcal{O})$, (see figure 2.4); if the lengths of these excursion time intervals $[\tau^{\varepsilon,x}_n, \tau^{\varepsilon,x}_{n+1}]$ were $\ge T_2$ for all $n = 0, \ldots, m-1\,(\ge k)$, then

$$k\,T_2 \ < \ m\,T_2 \ \le \ \tau^{\varepsilon,x}_0 + \sum_{n=0}^{m-1} \left(\tau^{\varepsilon,x}_{n+1} - \tau^{\varepsilon,x}_n\right) \ = \ \tau^{\varepsilon,x} \ \le \ k\,T_2 ,$$

a contradiction. Therefore it follows for all $k \in \mathbb{N}_0$ that

$$\{\, \tau^{\varepsilon,x} \le k\,T_2 \,\} \ \subset \ \bigcup_{m=0}^{k} \{\, \tau^{\varepsilon,x} = \tau^{\varepsilon,x}_m \,\} \ \cup \ \bigcup_{m=1}^{k} \{\, \theta^{\varepsilon,x}_m - \tau^{\varepsilon,x}_{m-1} \le T_2 \,\}$$

and thus

$$\begin{aligned}\mathbb{P}_x\{\,\tau^\varepsilon \le k\,T_2\,\} &\le \sum_{m=0}^{k} \mathbb{P}_x\{\,\tau^\varepsilon = \tau^\varepsilon_m\,\} \;+\; \sum_{m=1}^{k} \mathbb{P}_x\left\{\,\theta^\varepsilon_m - \tau^\varepsilon_{m-1} \le T_2\,\right\} \\ &\le \mathbb{P}_x\{\,\tau^\varepsilon = \tau^\varepsilon_0\,\} \;+\; 2\,k\,\mathrm{e}^{-\left(\overline{V}-\frac{\delta}{2}\right)/\varepsilon}\end{aligned}$$

for all $\varepsilon < \varepsilon_0$, where in the last estimate, (††) and (†††) have been used. Now plug in

$$k \;:=\; \left\lfloor T_2^{-1}\mathrm{e}^{\left(\overline{V}-\delta\right)/\varepsilon} \right\rfloor + 1$$

($\lfloor\,.\,\rfloor$ denoting the integer part) to get:

$$\begin{aligned}\mathbb{P}_x\left\{\tau^\varepsilon \le \mathrm{e}^{\left(\overline{V}-\delta\right)/\varepsilon}\right\} &\le \mathbb{P}_x\{\,\tau^\varepsilon \le k\,T_2\,\} \\ &\le \mathbb{P}_x\{\,\tau^\varepsilon = \tau^\varepsilon_0\,\} \;+\; 2\,k\,\mathrm{e}^{-\left(\overline{V}-\frac{\delta}{2}\right)/\varepsilon} \\ &\le \mathbb{P}_x\{\,X^\varepsilon_{\sigma^\varepsilon_\rho} \notin B_\rho(\mathcal{O})\,\} \;+\; 4\,T_2^{-1}\,\mathrm{e}^{-\delta/(2\varepsilon)} \quad \xrightarrow{\varepsilon\to 0} \; 0\end{aligned}$$

by lemma 2.4.11. Altogether, the first claim concerning the first exit time is proven.

(c) $\boxed{\lim\limits_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_x\tau^\varepsilon \;=\; \overline{V}}$ The upper bound, $\limsup\limits_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_x\tau^\varepsilon \;\le\; \overline{V}$, has already been verified in (†). For the lower bound note that Chebyshev's inequality states that

$$\mathbb{P}_x\{\tau^\varepsilon > \mathrm{e}^{(\overline{V}-\delta)/\varepsilon}\} \le \mathrm{e}^{-(\overline{V}-\delta)/\varepsilon}\,\mathbb{E}_x\tau^\varepsilon$$

and thus, as $\mathbb{P}_x\{\tau^\varepsilon > \mathrm{e}^{(\overline{V}-\delta)/\varepsilon}\} \to 1$, it follows that for any $\delta > 0$,

$$\varepsilon \log \mathbb{E}_x\tau^\varepsilon \;\ge\; \left(\overline{V}-\delta\right) \;+\; \varepsilon \log \mathbb{P}_x\left\{\tau^\varepsilon > \mathrm{e}^{(\overline{V}-\delta)/\varepsilon}\right\} \;\xrightarrow{\varepsilon\to 0}\; \left(\overline{V}-\delta\right) \;+\; 0\,.$$

2) $\boxed{\lim\limits_{\varepsilon\to 0} \mathbb{P}_x\left\{X^\varepsilon_{\tau^\varepsilon} \in N\right\} \;=\; 0}$ Here, $N \subset \partial D$ denotes a fixed closed set such that

$$V_N \;:=\; \inf\nolimits_N V(\mathcal{O},\,.\,) \;>\; \overline{V}\;\;;$$

if $\inf_N V(\mathcal{O},\,.\,) = \infty$, then some $V_N \in (\overline{V},\infty)$ is fixed instead.

Again, for any $\rho > 0$ (to be specified later),

$$\mathbb{P}_x\left\{X^\varepsilon_{\tau^\varepsilon} \in N\right\} \;\le\; \mathbb{P}_x\left\{X^\varepsilon_{\sigma^\varepsilon_\rho} \notin B_\rho(\mathcal{O})\right\} \;+\; \sup_{y\in B_\rho(\mathcal{O})} \mathbb{P}_y\left\{X^\varepsilon_{\tau^\varepsilon} \in N\right\}\;.$$

By lemma 2.4.11 the first summand on the right tends to 0 as $\varepsilon \to 0$ for any choice of ρ. Thus it remains to be verified that also

$$\sup_{y\in B_\rho(\mathcal{O})} \mathbb{P}_y\left\{X^\varepsilon_{\tau^\varepsilon}\in N\right\} \xrightarrow{\varepsilon\to 0} 0\ .$$

Fix $\eta\in\left(0,\frac{V_N-\overline{V}}{3}\right)$ and choose $\rho,\varepsilon_0>0$ according to lemma 2.4.10 such that

$$\sup_{z\in S_{2\rho}(\mathcal{O})} \mathbb{P}_z\{X^\varepsilon_{\sigma^\varepsilon_\rho}\in N\} \;\le\; \mathrm{e}^{-(V_N-\eta)/\varepsilon} \tag{‡}$$

for all $\varepsilon<\varepsilon_0$. From the definition of $\tau^{\varepsilon,x}_m$ and the strong Markov property it follows that for all $T_3\in\mathbb{R}_+$ and $\kappa\in\mathbb{N}_0$,

$$\sup_{z\in D}\mathbb{P}_z\left\{\tau^\varepsilon_\kappa\;\le\;\kappa\, T_3\right\}\le\kappa\,\sup_{z\in D}\mathbb{P}\left\{\sup_{t\in[0,T_3]}|X^{\varepsilon,z}_t-z|\;\ge\;\rho\right\}\ ;$$

on the other hand, by applying lemma 2.4.12 with the above choice of ρ and $c:=V_N-\eta$, one can choose $T_3\equiv T_3(\rho,V_N,\eta)<\infty$ such that the latter term is further estimated as

$$\sup_{z\in D}\mathbb{P}\left\{\sup_{t\in[0,T_3]}|X^{\varepsilon,z}_t-z|\;\ge\;\rho\right\}\;\le\;\mathrm{e}^{-(V_N-\eta)/\varepsilon}$$

for all $\varepsilon<\varepsilon_0$. Consequently, there is $T_3<\infty$ such that for all $\varepsilon<\varepsilon_0$ and $\kappa\in\mathbb{N}_0$:

$$\sup_{z\in D}\mathbb{P}_z\left\{\tau^\varepsilon_\kappa\;\le\;\kappa\, T_3\right\}\;\le\;\kappa\,\mathrm{e}^{-(V_N-\eta)/\varepsilon}\ . \tag{‡‡}$$

Since[14] $\left\{\tau^{\varepsilon,y}>\tau^{\varepsilon,y}_{m-1}\right\}\in\mathcal{F}_{\tau^{\varepsilon,y}_{m-1}}\subset\mathcal{F}_{\theta^{\varepsilon,y}_m}$ and $\tau^{\varepsilon,y}_m\equiv\theta^{\varepsilon,y}_m+\sigma^{\varepsilon,X^{\varepsilon,y}_{\theta^\varepsilon_m}}_\rho$, the strong Markov property implies here that for any $y\in D$:

$$\begin{aligned}
\mathbb{P}_y\left(\left\{X^\varepsilon_{\tau^\varepsilon_m}\in N\right\}\cap\left\{\tau^\varepsilon>\tau^\varepsilon_{m-1}\right\}\right) &= \int_{\left\{\tau^{\varepsilon,y}>\tau^{\varepsilon,y}_{m-1}\right\}} d\mathbb{P}\;\mathbb{E}^{\mathcal{F}_{\theta^{\varepsilon,y}_m}}1_{\left\{X^{\varepsilon,y}_{\tau^\varepsilon_m}\in N\right\}}\\
&= \int_{\left\{\tau^{\varepsilon,y}>\tau^{\varepsilon,y}_{m-1}\right\}} d\mathbb{P}\;\mathbb{P}\left\{X^{\varepsilon,z}_{\sigma^\varepsilon_\rho}\in N\right\}\Big|_{z=X^{\varepsilon,y}_{\theta^\varepsilon_m}(\,.\,)}\\
&\le \mathbb{P}_y\left\{\tau^\varepsilon>\tau^\varepsilon_{m-1}\right\}\cdot\sup_{z\in S_{2\rho}(\mathcal{O})}\mathbb{P}_z\left\{X^\varepsilon_{\sigma^\varepsilon_\rho}\in N\right\}.
\end{aligned}$$

Now it follows for all $y\in B_\rho(\mathcal{O})$, $\kappa\in\mathbb{N}$ and $\varepsilon<\varepsilon_0$:

$$\begin{aligned}
\mathbb{P}_y\left\{X^\varepsilon_{\tau^\varepsilon}\in N\right\} &\le \sum_{m=1}^{\kappa}\mathbb{P}_y\left\{X^\varepsilon_{\tau^\varepsilon_m}\in N\,,\,\tau^\varepsilon>\tau^\varepsilon_{m-1}\right\}\;+\;\mathbb{P}_y\left\{\tau^\varepsilon>\tau^\varepsilon_\kappa\right\}\\
&\le \kappa\sup_{z\in S_{2\rho}(\mathcal{O})}\mathbb{P}_z\left\{X^\varepsilon_{\sigma^\varepsilon_\rho}\in N\right\}+\mathbb{P}_y\left\{\tau^\varepsilon>\kappa\, T_3\right\}+\mathbb{P}_y\left\{\tau^\varepsilon_\kappa\le\kappa\, T_3\right\}\\
&\le 2\kappa\,\mathrm{e}^{-(V_N-\eta)/\varepsilon}\;+\;\mathbb{P}_y\left\{\tau^\varepsilon>\kappa\, T_3\right\}\ ;
\end{aligned}$$

[14] See e.g. Hackenbroch and Thalmaier [Hb-Th 94, p.93f.].

the first inequality is a general decomposition of the set $\{X^\varepsilon_{\tau^\varepsilon} \in N\}$ in which it has been used that $\tau^{\varepsilon,y} > \tau_0^{\varepsilon,y} = 0$ for all $y \in B_\rho(\mathcal{O})$ by definition; the second step follows from the previous application of the strong Markov property and the last inequality is due to (‡) and (‡‡). Further reducing ε_0 (if needed) such that (†) holds true for all $\varepsilon < \varepsilon_0$, using Chebyshev's inequality and plugging in the integer part of $\mathrm{e}^{\left(\overline{V}+2\eta\right)/\varepsilon}$,

$$\kappa := \left\lfloor \mathrm{e}^{\left(\overline{V}+2\eta\right)/\varepsilon} \right\rfloor ,$$

and using that $\overline{V} - V_N + 3\eta < 0$ due to the choice of η, the previous estimate implies:

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \sup_{y\in B_\rho(\mathcal{O})} \mathbb{P}_y \left\{X^\varepsilon_{\tau^\varepsilon} \in N\right\} \\
&\qquad\le \limsup_{\varepsilon\to 0} \left(2\kappa\, \mathrm{e}^{-(V_N-\eta)/\varepsilon} + \frac{T}{\kappa\, T_3}\, \mathrm{e}^{\left(\overline{V}+\eta\right)/\varepsilon} \right) \\
&\qquad\le \limsup_{\varepsilon\to 0} \left(2\mathrm{e}^{\left(\overline{V}-V_N+3\eta\right)/\varepsilon} + \frac{T}{T_3\left(\mathrm{e}^{\eta/\varepsilon} - \mathrm{e}^{-\left(\overline{V}+\eta\right)/\varepsilon}\right)} \right) \quad = \quad 0.
\end{aligned}$$

As has been observed above, this concludes the proof of the claim, $\lim\limits_{\varepsilon\to 0} \mathbb{P}_x \left\{X^\varepsilon_{\tau^\varepsilon} \in N\right\} = 0$.

$\boxed{\lim\limits_{\varepsilon\to 0} \mathbb{P}_x\left\{\left|X^\varepsilon_{\tau^\varepsilon} - z^*\right| < \delta\right\} = 1}$ follows hereby with $N := \{z \in \partial D : |z - z^*| \ge \delta\}$. □

Remark 2.4.13 (the corresponding Stratonovich SDE). Instead of proposing the Itô-SDE (2.1) as above, it is also intriguing to consider the corresponding Stratonovich-SDE,

$$dX_t^{\varepsilon,x} = b\left(X_t^{\varepsilon,x}\right) dt + \sqrt{\varepsilon}\,\sigma\left(X_t^{\varepsilon,x}\right) \circ dW_t\,, \qquad X_0^{\varepsilon,x} = x\,,$$

where "$\circ$" denotes the stochastic integral in the sense of Stratonovich.[15] This equation is equivalent to the Itô-SDE

$$dX_t^{\varepsilon,x} = \left[b\left(X_t^{\varepsilon,x}\right) + \frac{\varepsilon}{2}\,\sigma\left(X_t^{\varepsilon,x}\right)\, D\sigma\left(X_t^{\varepsilon,x}\right)\right] dt + \sqrt{\varepsilon}\,\sigma\left(X_t^{\varepsilon,x}\right) dW_t\,,$$

where the coordinates of $\sigma(x)\, D\sigma(x)$ are given by

$$\big(\sigma(x)\, D\sigma(x)\big)_k := \sum_{i,j=1}^{d} \sigma_{ij}(x)\, \frac{\partial \sigma_{kj}}{\partial x_i}(x)$$

for $k = 1, \ldots, d$. Hence, one also needs to cope with an ε-dependent drift

[15] See e.g. Arnold [Ar 98, Ch.2] or Hackenbroch and Thalmaier [Hb-Th 94, Ch.5].

$$b^{\varepsilon}(x) := b(x) + \frac{\varepsilon}{2}\,\sigma(x)\,D\sigma(x)$$

in the Itô equation. Large deviation results on such equations can also be obtained: Freidlin and Wentzell (see [Fr-We 98, p.154f.] and [We-Fr 70, p.7ff.]) prove that the action functional is the same as for (2.1), if the coordinates of b^{ε} converge to those of b uniformly in x as $\varepsilon \to 0$.

Therefore it would be conceptually equivalent to investigate the above Stratonovich-SDE instead of (2.1), as Freidlin [Fr 00] does indeed; also see Carmona and Freidlin [Cm-Fr 03] who assume the coefficients to be (globally) bounded. In general, one then needs to assure that

$$\frac{\varepsilon}{2}\,\|\,D\sigma\left(X_t^{\varepsilon,x}\right)\,\sigma\left(X_t^{\varepsilon,x}\right)\,\| \xrightarrow{\varepsilon\to 0} 0$$

uniformly with respect to x. It seems as if this assumption should be added to Freidlin's [Fr 00] premises.

However, the rationale of our work is to restrict the assumptions on b and σ to a minimum in order not to overburden the technical apparatus for installing locality into the treatment of linear differential systems: Since in most applications (see section 2.6), where $\sigma = \mathrm{id}_{\mathbb{R}^d}$, the Itô-SDE and the Stratonovich-SDE coincide, it seems reasonable to stick to the Itô-SDE.

2.5 Sublimiting distributions: Metastability and quasi-deterministic behavior

This subsection describes the concept of metastability, made precise by the convergence of the system to the sublimiting distributions on the respective time scales; the exposition here follows Freidlin [Fr 77], [Fr 00]; also see Freidlin and Wentzell [Fr-We 98, Ch. 6] as well as Carmona and Freidlin [Cm-Fr 03].

It is required throughout the section that the standing assumptions 2.1.1 are satisfied.

For describing the behavior of X^{ε} a hierarchy of "cycles" in the state space, together with their rotation and exit rate as well as their main state and the entrance point to the next cycle is needed. More precisely, the cycles will be defined as certain subsets of

$$\mathfrak{L} := \{1, \ldots, l\},$$

equipped with a cyclic order which expresses the order of transition between the domains of $(K_i)_{i\in\mathfrak{L}}$. In this construction the constants

$$V_{ij} := V(K_i, K_j) \qquad (i, j \in \mathfrak{L})$$

will play a central role. Due to **(K)** the entries of this matrix $(V_{ij})_{i,j\in\mathfrak{L}}$ are zero on the diagonal and strictly positive otherwise. In case that the sets K_i

do not consist of one point only, but are compact subsets of $\mathbb{R}^d$, this definition of V_{ij} would read

$$V_{ij} := V(x,y) \text{ for some } x \in K_i \text{ and } y \in K_j \qquad (i,j \in \mathfrak{L}) ;$$

this definition then does not depend on the choices of x and y due to assumption **(K3)** in remark 2.1.2. Hence, the matrix $(V_{ij})_{i,j\in\mathfrak{L}}$ is well defined also in this case.

One more definition is needed (Freidlin and Wentzell [Fr-We 98, p.177]) now.

Definition 2.5.1 (arrows, W-graphs on $\mathfrak{L}$). Let $\mathfrak{L} := \{1,\dots,l\}$ be as above (or an arbitrary finite set whose elements are labeled as $1,\dots,l$).

(a) An *arrow* $(\alpha \to \beta)$ is an ordered tuple $(\alpha,\beta) \in \mathfrak{L}^2$ where α and β are called the *initial point* and the *endpoint* of the arrow, respectively.

(b) A set $\mathfrak{g}$ of arrows is called a *graph on* $\mathfrak{L}$.

(c) Let a subset $W \subset \mathfrak{L}$ be fixed; then a set $\mathfrak{g}$ of arrows is called a W*-graph on* $\mathfrak{L}$, if

1) $(\alpha \to \beta) \in \mathfrak{g}$ implies that $\alpha \in \mathfrak{L} \setminus W$ and $\alpha \neq \beta$;
2) for any $\alpha \in \mathfrak{L}\setminus W$ there exists exactly one element of $\mathfrak{g}$ whose initial point is α;
3) for any $\alpha \in \mathfrak{L}\setminus W$ there exists a sequence in $\mathfrak{g}$ leading from α into W (i.e. a sequence $\alpha \to \alpha_1 \to \alpha_2 \to \cdots \to \alpha_{m-1} \to \alpha_m$ of arrows contained in $\mathfrak{g}$ such that $\alpha_m \in W$) .

Given the initial two assumptions 1) and 2), 3) is equivalent to the following one:

4) There are no closed loops in $\mathfrak{g}$
(i.e. for any sequence $\alpha_0 \to \alpha_1 \to \alpha_2 \to \cdots \to \alpha_{m-1} \to \alpha_m$ of arrows contained in $\mathfrak{g}$, it follows that $\alpha_0 \neq \alpha_m$) .

Let

$$G_W(\mathfrak{L}) := \{\, \mathfrak{g} : \mathfrak{g}\ W\text{-graph on } \mathfrak{L}\}$$

denote the set of all W-graphs on $\mathfrak{L}$; if W consists of one element i only, one defines

$$G_i(\mathfrak{L}) := G_{\{i\}}(\mathfrak{L}) ,$$

the set of i*-graphs on* $\mathfrak{L}$. In other words, $G_i(\mathfrak{L})$ is the set of all sets $\mathfrak{g}$ of arrows $(\alpha \to \beta)$ such that $\alpha \in \mathfrak{L} \setminus \{i\}$, to each initial point α exactly one element of $\mathfrak{g}$ is attached and by forming sequences $\alpha_0 \to \alpha_1 \to \alpha_2 \to \cdots \to \alpha_{m-1} \to \alpha_m$ of arrows in $\mathfrak{g}$ no closed paths are possible: $\alpha_0 \neq \alpha_m$.

Example 2.5.2 (Simple cases of i-graphs). Let $\mathfrak{L} \equiv \{1, \ldots, l\}$. Then one obtains for
$\underline{l=1:}$ $G_1(\mathfrak{L}) = \emptyset$,
$\underline{l=2:}$ $G_1(\mathfrak{L}) = \big\{ \{(2 \to 1)\} \big\}$ and $G_2(\mathfrak{L}) = \big\{ \{(1 \to 2)\} \big\}$, and
$\underline{l=3:}$ $G_1(\mathfrak{L}) = \big\{ \{(2 \to 1), (3 \to 1)\}, \{(2 \to 1), (3 \to 2)\}, \{(3 \to 1), (2 \to 3)\} \big\}$.

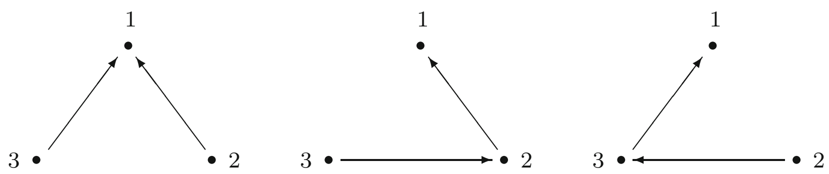

Fig. 2.5 Visualization of the 1-graphs on $\{1, 2, 3\}$

The Hierarchy of cycles

The hierarchy shall describe the transitions of $X^{\varepsilon,x}$ between the different domains of attraction which belong to $K_1, \ldots, K_l$. Successively the k-cycles will be defined. The cycles of any rank k are subsets of $\mathfrak{L}$ equipped with a cyclic order. Furthermore, to any cycle there will be assigned the main state, the rotation rate and the exit rate. The notation and terminology coined here will be illustrated in example 2.6.2 for the instructive situation of a two-well potential function.

$\underline{k=0}$: A 0-*cycle* is an element of $\mathfrak{L}$, i.e. the set of all 0-cycles is defined as $\mathcal{C}^{(0)} := \mathfrak{L}$.

$$M : \mathcal{C}^{(0)} \to \mathfrak{L}\,, \qquad M(i) := i$$

defines the *main state* of a 0-cycle; the *rotation rate* R and the *stationary distribution rate* m_i vanish on 0-cycles by definition,

$$R(i) := 0 \qquad \text{and} \qquad m_i(i) := 0 \qquad (i \in \mathcal{C}^{(0)} \equiv \mathfrak{L})\ ,$$

respectively. The *exit rate* $\mathcal{E}(i)$ of a 0-cycle i is given by the mapping

$$\mathcal{E} : \mathcal{C}^{(0)} \to (0, \infty]\,, \qquad \mathcal{E}(i) := \begin{cases} \min\limits_{J \in \mathfrak{L}\setminus\{i\}} V_{iJ} & , \; l \equiv |\mathfrak{L}| \geq 2\,, \\ \infty & , \; l = 1\,. \end{cases} \tag{2.11}$$

The meaning of the terminus "exit rate" (which will be also defined for cycles of higher order) will be explained in theorem 2.5.3; it states that $\mathcal{E}$ yields the precise exponential rates for the exit time of the respective domains, analogously to the exit time law 2.4.6.

Now if $l \geq 2$ (i.e. if the situation is non-trivial for what follows) and if $i \in \mathfrak{L}$ is fixed, one can choose $J \equiv J(i) \in \mathfrak{L} \setminus \{i\}$ such that this minimum (2.11) is attained; by **(G)** it is actually postulated that this J is unique; thus there is a well-defined mapping

$$J : \mathcal{C}^{(0)} \to \mathcal{C}^{(0)} \ ,$$

implicitly given by

$$V_{i,J(i)} = \mathcal{E}(i) \equiv \min_{J \in \mathfrak{L} \setminus \{i\}} V_{iJ} \ .$$

This definition of J is also expressed by saying that $J(i)$ *follows after* i.

$\underline{k=1}$: Applying the mapping J successively yields a sequence $\{J^r(i)\}_{r \in \mathbb{N}_0}$ in $\mathcal{C}^{(0)}$ for each $i \in \mathcal{C}^{(0)}$. As $\mathcal{C}^{(0)} \equiv \mathfrak{L}$ is a finite set, this sequence returns to itself at step

$$r(i) \ := \ \min\{\, r : J^n(i) = J^r(i) \text{ for some } n < r \,\} \qquad\quad (i \in \mathcal{C}^{(0)}) \,.$$

At the step $r(i)$ the sequence becomes periodic. Therefore it decomposes into 0-cycles $i, J(i), \ldots, J^{n(i)-1}(i)$ and the closed loop $\{J^{n(i)}(i), J^{n(i)+1}(i), \ldots, J^{r(i)}(i) = J^{n(i)}(i)\}$. If the cyclic order of the loop (which is simply a subset of $\mathfrak{L}$) shall be emphasized, it is written as succession of arrows,

$$J^{n(i)}(i) \to J^{n(i)+1}(i) \to \cdots \to J^{r(i)}(i) = J^{n(i)}(i) \ ;$$

this loop does not contain $i, J(i), \ldots, J^{n(i)-1}(i)$ by definition. We conclude that each sequence $\{J^r(i)\}_{r \in \mathbb{N}_0}$ is characterized by the objects

$$i, J(i), \ldots, J^{n(i)-1}(i) \,, \ \Big(J^{n(i)}(i) \to J^{n(i)+1}(i) \to \cdots \to J^{r(i)}(i) = J^{n(i)}(i) \Big)$$

which are called the 1-*cycles generated by* i. Let $\mathcal{C}^{(1)}$ denote the set of all 1-cycles generated by some $i \in \mathcal{C}^{(0)}$; its elements are called 1-*cycles*. Now

$$M : \mathcal{C}^{(1)} \to \mathfrak{L} \ ,$$

implicitly given by the requirement

$$V_{M(C),J(M(C))} = \max_{i \in C} V_{i,J(i)} \tag{2.12}$$

defines the *main state* of a 1-cycle C; again, the uniqueness of such an element $M(C)$ and hence the well-definedness of the main state mapping M is postulated in assumption 2.1.1**(G)**. The *rotation rate* R of 1-cycles is given by

$$R : \mathcal{C}^{(1)} \to \mathbb{R}_+ \,, \qquad R(C) := \max_{i \in C} V_{i,J(i)} \ , \tag{2.13}$$

the *stationary distribution rate* m_C for a 1-cycle C is the map[16]

$$m_C : C \to \mathbb{R}_+ \,, \qquad m_C(i) := R(C) - V_{i,J(i)}$$

and the *exit rate* $\mathcal{E}(C)$ of a 1-cycle C is defined by

$$\mathcal{E} : \mathcal{C}^{(1)} \to (0,\infty], \quad \mathcal{E}(C) := \begin{cases} \min\limits_{i\in C, j\notin C} (m_C(i) + V_{ij}) & , \left|\mathcal{C}^{(1)}\right| \geq 2, \\ \infty & , \left|\mathcal{C}^{(1)}\right| = 1. \end{cases} \tag{2.14}$$

If $\mathcal{C}^{(1)}$ consists of one element only, this is equal to $\mathfrak{L}$, considered as a point set; in this case the recursive definition of cycles stops. Otherwise, if $\left|\mathcal{C}^{(1)}\right| \geq 2$, there is a cyclic order on these 1-cycles: Fix an arbitrary $C \in \mathcal{C}^{(1)}$ and define $i^*, j^* \in \mathfrak{L}$ by the requirement

$$m_C(i^*) + V_{i^*j^*} \;=\; \min_{i\in C, j\notin C} (m_C(i) + V_{ij}) \;\equiv\; \mathcal{E}(C) \,.$$

By **(G)** we assume that i^* and j^* are unique, since they are uniquely determined by (2.14). Now let $J(C) \in \mathcal{C}^{(1)}$ be the (unique) 1-cycle which contains j^*. This procedure provides a well-defined mapping

$$J : \mathcal{C}^{(1)} \to \mathcal{C}^{(1)}$$

and one says that *the 1-cycle $J(C)$ follows after the 1-cycle C.*

$\underline{k > 1}$: Assume that the set $\mathcal{C}^{(k)}$ of k-cycles has already been defined and that the main state map M, the rotation rate R, the stationary distribution rate m, the exit rate $\mathcal{E}$ and the "follow"-map J have been extended to $\mathcal{C}^{(k)}$. Iterating J yields a sequence $\{J^r(C)\}_{r\in\mathbb{N}_0}$ in $\mathcal{C}^{(k)}$ for each $C \in \mathcal{C}^{(k)}$. As $\mathcal{C}^{(k)}$ is finite, this sequence returns to itself at step

$$r(C) \;:=\; \min\{\, r : J^n(C) = J^r(C) \text{ for some } n < r \,\} \qquad\qquad (C \in \mathcal{C}^{(k)}) \,.$$

Therefore the sequence decomposes into

$$\begin{aligned} &C, J(C), \ldots, J^{n(C)-1}(C) \,, \\ &\quad \left(J^{n(C)}(C) \to J^{n(C)+1}(C) \to \cdots \to J^{r(C)}(C) = J^{n(C)}(C)\right) \end{aligned}$$

which are called the $(k+1)$*-cycles generated by* C. $\mathcal{C}^{(k+1)}$ denotes the set of $(k+1)$-cycles, i.e. of all $(k+1)$-cycles generated by some $C \in \mathcal{C}^{(k)}$. For a

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[16] Freidlin [Fr 00, p.338] defines the stationary distribution rate m_C for a 1-cycle C with the opposite sign, i.e. $m_C(i) := V_{i,J(i)} - R(C)$; however, this would result in a negative rate; here, we use the corrected definition of m_C (equivalently of $\mathcal{E}(C)$) for 1-cycles from Carmona and Freidlin [Cm-Fr 03, (2.2)].

given $C \in \mathcal{C}^{(k+1)}$ the requirement

$$\min_{\mathfrak{g} \in G_M(C)} \sum_{(\alpha\to\beta)\in\mathfrak{g}} V_{\alpha\beta} \;=\; \min_{i\in C} \min_{\mathfrak{g} \in G_i(C)} \sum_{(\alpha\to\beta)\in\mathfrak{g}} V_{\alpha\beta} \tag{2.15}$$

uniquely determines $M \equiv M(C) \in C(\subset \mathfrak{L})$ thus extending the *main state* map

$$M : \mathcal{C}^{(k+1)} \to \mathfrak{L}$$

on $(k+1)$-cycles; recall that $G_i(C)$ denotes the set of i-Graphs on C as in definition 2.5.1. The *rotation rate* R of $(k+1)$-cycles is given by

$$R : \mathcal{C}^{(k+1)} \to \mathbb{R}_+\,, \qquad R(C) := \max\{\, \mathcal{E}(C') : C' \in \mathcal{C}^{(k)}, C' \subset C\,\}\,; \tag{2.16}$$

the *stationary distribution rate* m_C for a $(k+1)$-cycle C is the map

$$m_C : C \to \mathbb{R}_+\,, \qquad m_C(i) := \min_{\mathfrak{g}\in G_i(C)} \sum_{(\alpha\to\beta)\in\mathfrak{g}} V_{\alpha\beta} \;-\; \min_{\mathfrak{g}\in G_{M(C)}(C)} \sum_{(\alpha\to\beta)\in\mathfrak{g}} V_{\alpha\beta}$$

and the *exit rate* $\mathcal{E}(C)$ of a $(k+1)$-cycle C is defined by

$$\mathcal{E} : \mathcal{C}^{(k+1)} \to (0,\infty]\,, \qquad \mathcal{E}(C) \;:=\; \begin{cases} \min\limits_{i\in C, j\notin C} (m_C(i) + V_{ij}) & ,\ \left|\mathcal{C}^{(k+1)}\right| \geq 2\,, \\ \infty & ,\ \left|\mathcal{C}^{(k+1)}\right| = 1\,. \end{cases} \tag{2.17}$$

If $\mathcal{C}^{(k+1)}$ consists of one element only, this is equal to $\mathfrak{L}$ as a point set and the recursive definition of cycles stops. Otherwise, if $\left|\mathcal{C}^{(k+1)}\right| \geq 2$, there is a cyclic order on the $(k+1)$-cycles again: Fix an arbitrary $C \in \mathcal{C}^{(k+1)}$ and define $i^*, j^* \in \mathfrak{L}$ by the requirement

$$m_C(i^*) + V_{i^* j^*} \;=\; \min_{i\in C, j\notin C} (m_C(i) + V_{ij}) \;\equiv\; \mathcal{E}(C)\,,$$

where $m_C(i)$ denotes the newly defined stationary distribution rate for $(k+1)$-cycles. By **(G)** we assume that i^* and j^* are unique, since they are uniquely determined by (2.17). Now let $J(C) \in \mathcal{C}^{(k+1)}$ be the unique $(k+1)$-cycle which contains j^*; i^* is called the *exit point* of C and j^* is the *entrance point* of $J(C)$. This procedure now provides a well-defined map

$$J : \mathcal{C}^{(k+1)} \to \mathcal{C}^{(k+1)}$$

and we say that *the* $(k+1)$*-cycle* $J(C)$ *follows after the* $(k+1)$*-cycle* C. □

Example 2.6.2 shall give an illustration for the above definitions. There, a two well potential function is considered and the previous construction of the hierarchy of cycles is seen to terminate at $k = 1$. For a discussion of the three well situation which corresponds to a maximal degree $k = 2$ see Freidlin [Fr 00, p.344].

Next, an explanation of the above terminology "exit rates" shall be cited. For this purpose let

$$D_i \;:=\; \{x \in \mathbb{R}^d \,:\, X_t^{0,x} \xrightarrow{t\to\infty} K_i\} \qquad\qquad (i \in \mathfrak{L})$$

denote the domain of attraction of K_i under the deterministic motion X^0 as before which is an open subset of $\mathbb{R}^d$. Furthermore, define the open set

$$D(C) \;:=\; \left(\bigcup_{i\in C} \overline{D}_i\right)^{\circ}$$

for cycles C. The set $D(C)$ is connected, since by taking the closures $\overline{D}_i$ also the lower dimensional submanifolds are taken into account which are attracted to one of the points $K_{l+1}\,,\,\dots\,,\,K_{l'}$ and which thus separate the domains $D_1,\dots,D_l$; see remark 2.1.2 and assumption 2.1.1 **(K1)**. Hence, investigating the open set $\widetilde{D}(C) := \bigcup_{i\in C} D_i$ instead[17] would result in considering a disconnected set. Its topological components are the sets D_i, separated by those lower dimensional submanifolds which are attracted to one of the points $K_{l+1}\,,\,\dots\,,\,K_{l'}$. Therefore, the corresponding exit time from $\widetilde{D}(C)$ coincides with the exit time from the domain D_i containing the initial condition x. However, this is not the time of interest in the following discussion; rather one is concerned with the exit time from the whole cycle C in the sense that $X^{\varepsilon,x}$ exits the domain "spanned" by the domains D_i for which $i \in C$ and which also includes the separatrices between them.

Theorem 2.5.3 (The exit rates). *Let*

$$\tau_C^{\varepsilon,x} \;:=\; \inf\{\, t>0 : X_t^{\varepsilon,x} \notin D(C)\,\}$$

denote the first exit time of $X^{\varepsilon,x}$ from $D(C)$. Then for any $x \in D(C)$,

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_x \tau_C^{\varepsilon} \;=\; \mathcal{E}(C)$$

and for all $\delta > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}_x \left\{ \mathrm{e}^{(\mathcal{E}(C)-\delta)/\varepsilon} < \tau_C^{\varepsilon} < \mathrm{e}^{(\mathcal{E}(C)+\delta)/\varepsilon} \right\} \;=\; 1\,.$$

This generalization of theorem 2.4.6 is taken from Freidlin [Fr 00, p.339], Carmona and Freidlin [Cm-Fr 03, p.61f.] and Freidlin and Wentzell [Fr-We 98, Th.6.6.2], respectively. Being technically very involved the proof of this theorem is beyond the scope of this book: It is necessary to consider balls around the points K_i for which stopping times are then defined analogously as in the proof of theorem 2.4.6 above; figure 2.4 then depicts the case for $l = 1$. For

[17] As Freidlin [Fr 00] and Carmona and Freidlin [Cm-Fr 03] do

the arguments of the proof see Freidlin and Wentzell [Fr-We 98, Ch.6]; also see Freidlin [Fr 77].

Note that the exit rates $\mathcal{E}(\,.\,)$ considered here coincide with the rates $C(\,.\,)$ as used in the formulation of theorem 6.6.2 by Freidlin and Wentzell [Fr-We 98]: More precisely, let π be some cycle and define

$$C(\pi) \;:=\; A(\pi) \;-\; \min_{i\in\pi}\, \min_{\mathfrak{g}\in G_i(\pi)} \sum_{(\alpha\to\beta)\in\mathfrak{g}} V_{\alpha\beta} \;, \tag{2.18}$$

where

$$A(\pi) \;:=\; \min_{\mathfrak{g}\in G_{\mathfrak{L}\setminus\pi}(\mathfrak{L})} \sum_{(\alpha\to\beta)\in\mathfrak{g}} V_{\alpha\beta} \tag{2.19}$$

(see p.199 and p.201 in Freidlin and Wentzell [Fr-We 98]). Now $G_{\mathfrak{L}\setminus\pi}(\mathfrak{L})$ consists of those graphs whose arrows take their courses in π and whose last endpoint is in $\mathfrak{L}\setminus\pi$; this is the same as considering graphs in π terminating in $i\in\pi$ (elements of $G_i(\pi)$ in other words) and proceeding to some $j\notin\pi$ subsequently. Hence, unwinding the definitions of graphs and main states yields that for any cycle π of order $k>1$,

$$\begin{aligned}
C(\pi) &\equiv \min_{\mathfrak{g}\in G_{\mathfrak{L}\setminus\pi}(\mathfrak{L})} \sum_{(\alpha\to\beta)\in\mathfrak{g}} V_{\alpha\beta} - \min_{i\in\pi}\, \min_{\mathfrak{g}\in G_i(\pi)} \sum_{(\alpha\to\beta)\in\mathfrak{g}} V_{\alpha\beta} \\
&= \min_{i\in\pi, j\notin\pi} \left(\min_{\mathfrak{g}\in G_i(\pi)} \sum_{(\alpha\to\beta)\in\mathfrak{g}} V_{\alpha\beta} + V_{ij} \right) - \min_{\mathfrak{g}\in G_{M(\pi)}(\pi)} \sum_{(\alpha\to\beta)\in\mathfrak{g}} V_{\alpha\beta} \\
&= \min_{i\in\pi, j\notin\pi} \left(\min_{\mathfrak{g}\in G_i(\pi)} \sum_{(\alpha\to\beta)\in\mathfrak{g}} V_{\alpha\beta} - \min_{\mathfrak{g}\in G_{M(\pi)}(\pi)} \sum_{(\alpha\to\beta)\in\mathfrak{g}} V_{\alpha\beta} + V_{ij} \right) \\
&\equiv \min_{i\in\pi, j\notin\pi} \left(m_\pi(i) + V_{ij}\right) \\
&\equiv \mathcal{E}(\pi) \;.
\end{aligned}$$

Therefore the above theorem coincides with Theorem 6.6.2 by Freidlin and Wentzell [Fr-We 98]. Furthermore, note that the compactness assumption on the state space which is underlying § 6.6 by Freidlin and Wentzell [Fr-We 98] is replaced here by assumption **(V)**, i.e. by considering the compact set $\{y : V(0,y) \le \tilde{V}\}$ for some large $\tilde{V}$ instead which X^ε leaves only with an exponentially small probability; see Freidlin [Fr 77].

The support of the sublimiting distribution

Using the hierarchy of cycles we shall now define the *metastable state* $K_{\mu(x,\zeta)}$ for an initial value x and a time scale $\mathrm{e}^{\zeta/\varepsilon}$. Again the presentation follows

Freidlin and Wentzell ([Fr 77], [Fr 00] and [Fr-We 98, § 6.6]). An illustration will be provided in example 2.6.2 .

Fix an initial value

$$x \in \bigcup_{i=1}^{l} D_i \quad .$$

This set has full Lebesgue measure by assumption 2.1.1 **(K)**. Choose $i(x) \in \mathfrak{L}$ such that $x \in D_{i(x)}$ and let $C^{(k)}(x)$ denote the (unique) element of $\mathcal{C}^{(k)}$ such that $i(x) \in C^{(k)}(x)$,

$$i(x) \equiv C^{(0)}(x) \subset C^{(1)}(x) \subset \cdots \subset C^{(\kappa-1)}(x) \subset C^{(\kappa)} = \mathfrak{L}\,,$$

where $\kappa < \infty$ denotes the maximal rank in this sequence. This yields the finite sequences

$$E_k(x) := \mathcal{E}\left(C^{(k)}(x)\right) \quad \text{and} \quad R_k(x) := R\left(C^{(k)}(x)\right) \qquad (k = 0, 1, \ldots, \kappa)$$

of the corresponding exit and rotation rates, respectively. For these rates it follows that

$$\begin{aligned} 0 \equiv R_0(x) < E_0(x) \equiv V_{i(x),J(i(x))} &\leq R_1(x) \leq E_1(x) \leq \cdots \\ &\leq R_k(x) \leq E_k(x) \leq \cdots \\ &\leq R_\kappa(x) \leq E_\kappa(x) \equiv \infty \end{aligned}$$

Now fix

$$\zeta \in \mathbb{R}_{>0} \setminus \bigcup_{k=0}^{\kappa} \{R_k(x), E_k(x)\} \tag{2.20}$$

and define

$$m^* \equiv m^*(x) \in \{-1, 0, 1, \ldots, \kappa - 1\}$$

by

$$E_{m^*}(x) < \zeta < E_{m^*+1}(x)\,,$$

where we put $E_{-1}(x) := 0$ for definiteness. There are two cases to distinguish:

$\underline{\zeta > R_{m^*+1}(x)}$: In this case define $\mu(x,\zeta) := M\left(C^{(m^*+1)}(x)\right)$.
$\underline{\zeta < R_{m^*+1}(x)}$: Since $R_{m^*+1}(x) \equiv \max\{\mathcal{E}(C') : C' \in \mathcal{C}^{(m^*)}, C' \subset C^{(m^*+1)}(x)\}$, there is a

$$\hat{C}^{(m^*)} \in \mathcal{C}^{(m^*)} \quad \text{such that} \quad \hat{C}^{(m^*)} \subset C^{(m^*+1)}(x) \quad \text{and } \mathcal{E}(\hat{C}^{(m^*)}) > \zeta\,;$$

we also assume that $\hat{C}^{(m^*)}$ is the first m^*-cycle satisfying this statement which follows after $C^{(m^*)}(x)$ in $C^{(m^*+1)}(x)$. Again there are two cases to distinguish:

$\underline{\zeta > R(\hat{C}^{(m^*)})}$: In this case define $\mu(x,\zeta) := M(\hat{C}^{(m^*)})$.

$\underline{\zeta < R(\hat{C}^{(m^*)})}$: By the same argument as above there is a

$$\hat{C}^{(m^*-1)} \in \mathcal{C}^{(m^*-1)} \qquad \text{such that} \qquad \hat{C}^{(m^*-1)} \subset \hat{C}^{(m^*)}, \quad \mathcal{E}(\hat{C}^{(m^*-1)}) > \zeta$$

and $\hat{C}^{(m^*-1)}$ is the first (m^*-1)-cycle which follows after the (m^*-1)-cycle containing the entrance point of $\hat{C}^{(m^*)}$. Again there are two cases to distinguish:

$\underline{\zeta > R(\hat{C}^{(m^*-1)})}$: In this case define $\mu(x,\zeta) := M(\hat{C}^{(m^*-1)})$.
$\underline{\zeta < R(\hat{C}^{(m^*-1)})}$: We proceed successively as in the previous step till we find

$$\hat{C}^{(m^*-n)} \in \mathcal{C}^{(m^*-n)} \qquad \text{such that} \qquad \mathcal{E}(\hat{C}^{(m^*-n)}) \; > \; \zeta \; > \; R(\hat{C}^{(m^*-n)})\,.$$

Since $R(\,.\,) = 0$ on 0-cycles, this condition is fulfilled after a finite number $n \leq m^*$ of further steps; then define $\mu(x,\zeta) := M(\hat{C}^{(m^*-n)})$.

By the previous distinction of the possible cases one gets a well defined index function

$$\mu \; : \; \left(\bigcup_{i=1}^{l} D_i \right) \times \left(\mathbb{R}_{>0} \setminus \bigcup_{k=0}^{\kappa} \{R_k(x), E_k(x)\} \right) \longrightarrow \mathfrak{L}\,. \tag{2.21}$$

Definition 2.5.4 (Metastable states). Fix $x \in \bigcup\limits_{i=1}^{l} D_i$ and $\zeta > 0$, where ζ is not contained in a finite exceptional set depending on x (see (2.20)). Let $T(\varepsilon)$ be a time scale which is *logarithmically equivalent to* $\mathrm{e}^{\zeta/\varepsilon}$, abbreviated as

$$T(\varepsilon) \asymp \mathrm{e}^{\zeta/\varepsilon}$$

in the sequel, i.e. let T be a function

$$T : (0, \varepsilon_0) \to \mathbb{R}_{>0} \;\; ,$$

where $\varepsilon_0 > 0$, such that

$$0 \; < \; \lim_{\varepsilon \to 0} \varepsilon \, \log T(\varepsilon) \; = \; \zeta.$$

Then the *metastable state for the initial value x and the time scale $T(\varepsilon)$* is

$$K_{\mu(x,\zeta)} \;,$$

where $\mu(x,\zeta) \in \mathfrak{L}$ denotes the index function defined in (2.21).

The following theorem clarifies the behavior of X^ε on the time scales $T(\varepsilon)$.

Theorem 2.5.5 (Sublimiting distribution). *Let the assumptions 2.1.1 be satisfied and consider the solution X^ε of SDE (2.1). Fix an initial value $x \in \bigcup_{i=1}^{l} D_i$ and a time scale $T(\varepsilon) \asymp \mathrm{e}^{\zeta/\varepsilon}$, where $\zeta > 0$ is not contained in a finite exceptional set depending on x (see (2.20)). Then for all open sets $B \subset \mathbb{R}^d$,*

$$P^\varepsilon_{T(\varepsilon)}(x, B) \;\equiv\; \mathbb{P}_x \left\{ X^\varepsilon_{T(\varepsilon)} \in B \right\} \xrightarrow{\varepsilon\to 0} \begin{cases} 0 & , \; K_{\mu(x,\zeta)} \notin B\,, \\ 1 & , \; K_{\mu(x,\zeta)} \in B\,. \end{cases}$$

In particular, $X^{\varepsilon,x}_{T(\varepsilon)}$ converges in probability to the respective metastable state,

$$X^{\varepsilon,x}_{T(\varepsilon)} \xrightarrow[\varepsilon\to 0]{\mathbb{P}} K_{\mu(x,\zeta)}\;.$$

Furthermore, for all functions $f \in C^b(\mathbb{R}^d, \mathbb{R})$,

$$u^\varepsilon_f\big(T(\varepsilon), x\big) \xrightarrow{\varepsilon\to 0} f\left(K_{\mu(x,\zeta)} \right)\;,$$

where $u^\varepsilon_f(t,x) \equiv \mathbb{E}_x\, f(X^\varepsilon_t)$ denotes the solution of the Cauchy problem (2.5) for (2.1).

The first and the third claim are a reformulation of Freidlin's [Fr 77] theorem. The statement on the convergence in probability follows from the first part simply by taking $B_\eta := \complement\, \overline{B\left(K_{\mu(x,\zeta)}, \eta\right)}$ for arbitrary $\eta > 0$ and hence

$$\mathbb{P}\left\{ \,|\, X^{\varepsilon,x}_{T(\varepsilon)} \;-\; K_{\mu(x,\zeta)} \,|\; > \; \eta \right\} \;\equiv\; P^\varepsilon_{T(\varepsilon)}(x, B_\eta) \xrightarrow{\varepsilon\to 0} 0\,.$$

Note that in citing Freidlin's [Fr 77] theorem the notation from (2.18) and (2.19) is used again: $C(\pi)$ as defined in (2.18) equals Freidlin's [Fr 77] "$c(\pi)$" and $A(\pi)$ as given by (2.19) coincides with Freidlin's [Fr 77] "$\mathfrak{A}(\pi)$" and "$\mathcal{A}(\pi)$". Furthermore, note that Freidlin [Fr 77] formally assumes the coefficient functions b_i and a_{ij} of the SDE (of the generator $\mathcal{G}^\varepsilon$) to be globally bounded. This assumption, however, is not used in the course of the paper and can be done away; see Freidlin [Fr 00]. For additional discussions see Freidlin and Wentzell [Fr-We 98, p.202f.]; further arguments and proofs can also be found in the work by Li and Qian [Li-Qi 99] and [Li-Qi 98].

This theorem in particular shows that the transition probabilities have different limits for different choices of the time scale and these limits do depend on the initial value x. Since the resultant convergence in probability implies weak convergence, one gets that the distribution of $X^{\varepsilon,x}_{T(\varepsilon)}$ converges weakly to the Dirac measure at the respective metastable state,

$$\mathbb{P} \circ \left(X^{\varepsilon,x}_{T(\varepsilon)} \right)^{-1} \xrightarrow[\varepsilon\to 0]{w} \delta_{K_{\mu(x,\zeta)}}.$$

This fact is expressed by calling $K_{\mu(x,\zeta)}$ the *support of the sublimiting distributions* for x and the time scale $T(\varepsilon) \asymp e^{\zeta/\varepsilon}$, since in this sense $\delta_{K_{\mu(x,\zeta)}}$ is the *sublimiting distribution*, in contrast to the limiting distribution ρ^ε (section 2.2) for which it was obtained in equations (2.4) and (2.6) that

$$P_t^\varepsilon(x, B) \xrightarrow{t\to\infty} \rho^\varepsilon(B)$$

and

$$u_f^\varepsilon(t, x) \xrightarrow{t\to\infty} \int_{\mathbb{R}^d} f(y)\, \rho^\varepsilon(dy)$$

independently of x.

Since the above hierarchy of cycles and the corresponding main states, rotation rates and exit rates — which all describe the behavior of X^ε — do not depend on chance, the long-time evolution of this system has an intrinsic deterministic component; this is called the *quasi-deterministic approximation of* X^ε (Freidlin [Fr 00]). More precisely, Freidlin's [Fr 00] theorem 1 on the quasi-deterministic behavior of the solution X^ε of the SDE (2.2) states the following, where $\mathbb{L}$ denotes the Lebesgue measure on $\mathbb{R}$:

Theorem 2.5.6 (Quasi-deterministic behavior). *Let the assumptions 2.1.1 be satisfied. Fix* $x \in \bigcup_{i=1}^{l} D_i$ *and a time scale* $T(\varepsilon) \asymp e^{\zeta/\varepsilon}$*, where* $\zeta > 0$ *is not contained in a finite exceptional set depending on* x *(see (2.20)). Then for any* $c, \Gamma > 0$

$$\mathbb{L}\Big\{\, t \in [0, \Gamma] \;:\; \Big|\, X^{\varepsilon,x}_{t\,T(\varepsilon)} \;-\; K_{\mu(x,\zeta)} \Big| > c \,\Big\} \;\xrightarrow[\varepsilon\to 0]{\mathbb{P}}\; 0 \;\; .$$

Sketch of Proof. According to the definition of $\mu(x,\zeta)$ in the distinction of cases preceding (2.21), there exist $m^* \equiv m^*(x) \in \mathbb{N}_0 \cup \{-1\}$, a number $s \in \{0, \ldots, m^*+1\}$ of steps and a cycle

$$\hat{C}^{(m^*+1-s)} \in \mathcal{C}^{(m^*+1-s)} \quad \text{such that} \quad \mathcal{E}(\hat{C}^{(m^*+1-s)}) \;>\; \zeta \;>\; R(\hat{C}^{(m^*+1-s)})$$

and

$$\mu(x,\zeta) \;=\; M(\hat{C}^{(m^*+1-s)}) \;\; .$$

Applying theorem 2.5.5 to the time scale $\widetilde{T}(\varepsilon) := t_0\, T(\varepsilon) \asymp e^{\zeta/\varepsilon}$, for some $t_0 > 0$, yields that for $\eta > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}_x \Big\{\, X^\varepsilon_{t\,T(\varepsilon)} \notin B(K_{\mu(x,\zeta)}, \eta) \;\text{ for all } t \in [0, t_0] \Big\} \;=\; 0\,.$$

The exit rate theorem 2.5.3 directly implies that for all $\delta \in (0, \mathcal{E}(\hat{C}^{(m^*+1-s)}) - \zeta)$ and all $y \in B(K_{\mu(x,\zeta)}, \eta)$,

$$\lim_{\varepsilon\to 0} \mathbb{P}_y \Big\{\, \Gamma\, T(\varepsilon) < e^{(\mathcal{E}(\hat{C}^{(m^*+1-s)}) - \delta)/\varepsilon} < \tau^\varepsilon_{\hat{C}^{(m^*+1-s)}} \Big\} \;=\; 1 \;\; ,$$

since $\mathcal{E}(\hat{C}^{(m^*+1-s)}) > \zeta$; in other words,

$$\lim_{\varepsilon\to 0} \mathbb{P}_x \left\{ X^{\varepsilon}_{t\,T(\varepsilon)} \in D(\hat{C}^{(m^*+1-s)}) \text{ for all } t \in [t_0, \Gamma] \right\} = 1 \, .$$

By the definition of the rotation rate, it follows that $\mathcal{E}(C') \leq R(\hat{C}^{(m^*+1-s)}) < \zeta$ for all cycles $C' \subsetneq \hat{C}^{(m^*+1-s)}$. The exit rate theorem 2.5.3 therefore implies that for all $\delta \in (0, \zeta - R(\hat{C}^{(m^*+1-s)}))$, $z \in D(C')$ and $t_1 > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}_z \left\{ \tau^{\varepsilon}_{C'} < \mathrm{e}^{(\mathcal{E}(C')+\delta)/\varepsilon} < t_1\, T(\varepsilon) \right\} = 1$$

and hence,

$$\lim_{\varepsilon\to 0} \mathbb{P}_z \left\{ X^{\varepsilon}_{t\,T(\varepsilon)} \in D(J(C')) \text{ for some } t \in (0, t_1) \right\} = 1 \, .$$

Now fix $\eta > c$; then the closed ball $B_c(K_{\mu(x,\zeta)})$ is contained in the open ball $B(K_{\mu(x,\zeta)}, \eta)$. Piecing together the previous time scales $t_0 T(\varepsilon) > \exp(R(\hat{C}^{(m^*+1-s)})/\varepsilon)$ by using the Markov property, it can be seen that the system $X^{\varepsilon,x}$ mostly stays near $K_{\mu(x,\zeta)}$ during the interval $[0, \Gamma T(\varepsilon)]$. The system also visits cycles $C' \subsetneq \hat{C}^{(m^*+1-s)}$; however, the lengths of these excursions is of exponential order smaller than ζ. Following this sketch of proof the claim can be deduced; see Freidlin [Fr 00, p.340]. □

Freidlin [Fr 00, Cor.1] also notes the following corollary. It provides the decisive connection between the LDP and the quasi-deterministic approximation defined above.

Corollary 2.5.7 (Boundedness in probability I). *Let the assumptions 2.1.1 be satisfied. Fix* $x \in \bigcup_{i=1}^{l} D_i$ *and a time scale* $T(\varepsilon) \asymp \mathrm{e}^{\zeta/\varepsilon}$*, where* $\zeta > 0$ *is not contained in a finite exceptional set depending on* x *(see (2.20)). Then for the set*

$$\mathbb{F}_{x,\zeta} := \{ \, y \in \mathbb{R}^d \, : \, V(x,y) \leq \zeta \, \} \, ,$$

it follows that for any $\Gamma > 0$

$$\lim_{\varepsilon\to 0} \mathbb{P}_x \left\{ X^{\varepsilon}_{tT(\varepsilon)} \notin \mathbb{F}_{x,\zeta} \text{ for some } t \in [0, \Gamma] \right\} = 0 \, .$$

Furthermore, one obtains that for any $p > 0$

$$\left\| X^{\varepsilon}_{\cdot\, T(\varepsilon)} - K_{\mu(x,\zeta)} \right\|_{L^p([0,\Gamma],\mathbb{L})} \xrightarrow[\varepsilon\to 0]{\mathbb{P}} 0 \, .$$

Sketch of Proof. The set $\mathbb{F}_{x,\zeta}$ is compact (hence bounded) due to assumption **(V)**. By fixing a continuation of b and σ outside the open set $\mathbb{F}^{\delta}_{x,\zeta} := \{y \in \mathbb{R}^d : \mathrm{dist}(y, \mathbb{F}_{x,\zeta}) < \delta\}$ for some δ, the large deviation principle

2.3.4 can be applied. This approach provides the asymptotic behavior for the exit time from $\mathbb{F}^\delta_{x,\zeta}$ in terms of the rate function and, hence also, in terms of the quasipotential as in theorem 2.4.3, since the first exit time from $\mathbb{F}^\delta_{x,\zeta}$ does not depend on the specific continuation of b and σ on $\complement\mathbb{F}^\delta_{x,\zeta}$. This proves the first statement.

The second claim can be proven from the first statement and theorem 2.5.6 by a similar argument as will be performed in theorem 2.5.10 below. □

The subsequent conclusion has also been drawn by Freidlin [Fr 00, Cor.2]:

Corollary 2.5.8 (Boundedness in probability II). *Let the assumptions 2.1.1 be satisfied. Fix* $x \in \bigcup_{i=1}^{l} D_i$ *and a time scale* $T(\varepsilon) \asymp \mathrm{e}^{\zeta/\varepsilon}$, *where* $\zeta > 0$ *is not contained in a finite exceptional set depending on* x *(see (2.20)). Furthermore, suppose that*

$$\zeta \; < \; V_{\mu(x,\zeta),J(\mu(x,\zeta))} \;\; .$$

Then for the set

$$\mathbb{H}_{x,\zeta} \; := \; \{ \, y \in \mathbb{R}^d \, : \, V(K_{\mu(x,\zeta)}, y) \leq \zeta \, \} \; ,$$

it follows that for any $\Gamma > 0$ *and* $c \in (0, \Gamma]$

$$\lim_{\varepsilon \to 0} \; \mathbb{P}_x \left\{ \, X^\varepsilon_{tT(\varepsilon)} \in \mathbb{H}_{x,\zeta} \;\; \text{for all } t \in [c, \Gamma] \, \right\} \;\; = \;\; 1 \;\; .$$

Sketch of Proof. First note that also the set $\mathbb{H}_{x,\zeta}$ is bounded due to assumption **(V)**. As in the sketch of proof for theorem 2.5.6 $\mu(x,\zeta)$ is the main state of the cycle corresponding to ζ. Hence, any neighborhood of $K_{\mu(x,\zeta)}$ is hit before $c\,T(\varepsilon)$ with probability converging to 1. However, as in corollary 2.5.7 above, the bound on the quasipotential on the respective set $\mathbb{H}_{x,\zeta}$ (or $\mathbb{F}_{x,\zeta}$) provides the time scale on which this set cannot be left. More precisely, the exit time law 2.4.6 for $D := \mathbb{H}^\delta_{x,\zeta} := \{y \in \mathbb{R}^d : V(K_{\mu(x,\zeta)}, y) < \zeta + \delta\}$, for small $\delta > 0$, and $\mathcal{O} := K_{\mu(x,\zeta)}$ implies, since $\overline{V} := \inf_{\partial \mathbb{H}^\delta_{x,\zeta}} V(K_{\mu(x,\zeta)}, \,.\,) = \zeta + \delta$, that for all $y \in D$ and $\delta > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{P}_y \left\{ \mathrm{e}^{\zeta/\varepsilon} < \tau^\varepsilon_{\mathbb{H}^\delta_{x,\zeta}} \right\} \; = \; 1 \;\; ;$$

in other words,

$$\lim_{\varepsilon \to 0} \mathbb{P}_x \left\{ \, X^\varepsilon_{t\,T(\varepsilon)} \in \mathbb{H}^\delta_{x,\zeta} \;\; \text{for all } t \in [c, \Gamma] \, \right\} \; = \; 1 \; .$$

The assumption that the claim is false can be led to a contradiction this way. □

Remark 2.5.9. As has already been remarked before, the property of K_i consisting of one single point might be relaxed: If K_i were a compact set, such that also **(K3)** and **(K4)** are satisfied, then the assertions would hold true, if suitably modified.

For instance, theorem 2.5.6 would state the following: For any $c, \Gamma > 0$

$$\mathbb{L}\Big\{ t \in [0,\Gamma] \,:\, \mathrm{dist}\Big(X^{\varepsilon,x}_{tT(\varepsilon)} \,,\, K_{\mu(x,\zeta)} \Big) > c \Big\} \xrightarrow[\varepsilon\to 0]{\mathbb{P}} 0 \;,$$

where $\mathrm{dist}(x,B) \equiv \inf_{y\in B}|x-y|$. Also see Freidlin [Fr 77, Remark 1] and [Fr 00].

Now we prove a consequence of the metastability which is not contained in the workings by Freidlin and Wentzell. This theorem is of central importance for the investigations of local Lyapunov exponents to come. Its final assertion can be regarded as a "sublimiting ergodic theorem".

Theorem 2.5.10 (Consequences of the metastability of $X^{\varepsilon,x}$). *Let the assumptions 2.1.1 be satisfied. Fix* $x \in \bigcup_{i=1}^{l} D_i$ *and a time scale* $T(\varepsilon) \asymp \mathrm{e}^{\zeta/\varepsilon}$, *where* $\zeta > 0$ *is not contained in a finite exceptional set depending on* x *(see (2.20)). Moreover, consider some continuous function* $f \in C(\mathbb{R}^d,\mathbb{K})$. *Then for all open sets* $B \subset \mathbb{K}$,

$$\mathbb{P}_x\Big\{ f\big(X^{\varepsilon}_{T(\varepsilon)}\big) \in B \Big\} \xrightarrow{\varepsilon\to 0} \begin{cases} 0 & ,\ f(K_{\mu(x,\zeta)}) \notin B\,, \\ 1 & ,\ f(K_{\mu(x,\zeta)}) \in B\,. \end{cases}$$

In particular, $f(X^{\varepsilon,x}_{T(\varepsilon)})$ *converges in probability,*

$$f\big(X^{\varepsilon,x}_{T(\varepsilon)}\big) \xrightarrow[\varepsilon\to 0]{\mathbb{P}} f(K_{\mu(x,\zeta)})\,.$$

Since the resultant convergence in probability implies weak convergence, one gets that the distribution of $f(X^{\varepsilon,x}_{T(\varepsilon)})$ *converges weakly to the Dirac measure at the corresponding point,*

$$\mathbb{P}\circ\Big(f(X^{\varepsilon,x}_{T(\varepsilon)})\Big)^{-1} \xrightarrow[\varepsilon\to 0]{w} \delta_{f(K_{\mu(x,\zeta)})} \quad .$$

Furthermore,

$$\frac{1}{T(\varepsilon)}\int_0^{T(\varepsilon)} f\big(X^{\varepsilon,x}_u\big)\,du \;=\; \int_0^1 f\big(X^{\varepsilon,x}_{t\,T(\varepsilon)}\big)\,dt \xrightarrow[\varepsilon\to 0]{\mathbb{P}} f(K_{\mu(x,\zeta)})\,.$$

Proof. The first statement follows from theorem 2.5.5 together with the continuity of the function f. The second and the third statements follow immediately from the first one. It remains to verify the fourth assertion:

For this purpose we fix an arbitrary $\eta > 0$; then

$$
\begin{aligned}
&\mathbb{P}\left\{ \left| \frac{1}{T(\varepsilon)} \int_0^{T(\varepsilon)} f\big(X_u^{\varepsilon,x}(\,.\,)\big)\, du \,-\, f\left(K_{\mu(x,\zeta)}\right) \right| \,>\, \eta \right\} \\
&\qquad = \mathbb{P}\left\{ \left| \int_0^1 f\big(X_{t\cdot T(\varepsilon)}^{\varepsilon,x}(\,.\,)\big)\, dt \,-\, f\left(K_{\mu(x,\zeta)}\right) \right| \,>\, \eta \right\} \\
&\qquad \le \mathbb{P}\left\{ \int_0^1 \left| f\big(X_{t\cdot T(\varepsilon)}^{\varepsilon,x}(\,.\,)\big) \,-\, f\left(K_{\mu(x,\zeta)}\right) \right| dt \,>\, \eta \right\} \\
&\qquad = \mathbb{P}\left\{ \tau_R^\varepsilon(\,.\,) \,\le\, 1 \;,\; \int_0^1 \left| f\big(X_{t\cdot T(\varepsilon)}^{\varepsilon,x}(\,.\,)\big) \,-\, f\left(K_{\mu(x,\zeta)}\right) \right| dt \,>\, \eta \right\} \\
&\qquad\quad + \mathbb{P}\left\{ \tau_R^\varepsilon(\,.\,) \,>\, 1 \,,\; \int_0^1 \left| f\big(X_{t\cdot T(\varepsilon)}^{\varepsilon,x}(\,.\,)\big) \,-\, f\left(K_{\mu(x,\zeta)}\right) \right| dt \,>\, \eta \right\} \\
&\qquad \equiv A^\varepsilon \,+\, B^\varepsilon \;,
\end{aligned}
$$

where

$$
A^\varepsilon \;:=\; \mathbb{P}\left\{ \tau_R^\varepsilon(\,.\,) \,\le\, 1 \;,\; \int_0^1 \left| f\big(X_{t\cdot T(\varepsilon)}^{\varepsilon,x}(\,.\,)\big) \,-\, f\left(K_{\mu(x,\zeta)}\right) \right| dt \,>\, \eta \right\}
$$

and

$$
B^\varepsilon \;:=\; \mathbb{P}\left\{ \tau_R^\varepsilon(\,.\,) \,>\, 1 \,,\; \int_0^1 \left| f\big(X_{t\cdot T(\varepsilon)}^{\varepsilon,x}(\,.\,)\big) \,-\, f\left(K_{\mu(x,\zeta)}\right) \right| dt \,>\, \eta \right\} \;,
$$

and where

$$
\tau_R^\varepsilon \;:=\; \inf\left\{ t > 0 \;:\; \left| X_{t\cdot T(\varepsilon)}^{\varepsilon,x}(\,.\,) - K_{\mu(x,\zeta)} \right| \ge R \right\} \qquad\qquad (\varepsilon, R > 0)
$$

denotes the first exit time of $X_{t\cdot T(\varepsilon)}^{\varepsilon,x}(\,.\,)$ from the open ball $B\left(K_{\mu(x,\zeta)}, R\right)$ with center $K_{\mu(x,\zeta)}$ and radius R.

It needs to be shown that $A^\varepsilon \xrightarrow{\varepsilon\to 0} 0$ and $B^\varepsilon \xrightarrow{\varepsilon\to 0} 0$ which is done in the sequel: First, we have that

$$
A^\varepsilon \;\le\; \mathbb{P}\{\, \tau_R^\varepsilon \,\le\, 1 \,\} \xrightarrow{\varepsilon\to 0} 0\,,
$$

if R is sufficiently large; namely, choose R sufficiently large such that $\mathbb{F}_{x,\zeta} \subset B(K_{\mu(x,\zeta)}, R)$ and use that X^ε is bounded in probability (corollary 2.5.7 with $\Gamma = 1$).

For estimating B^ε fix $\omega \in \{\tau_R^\varepsilon \,>\, 1\}$ and $c > 0$ in order to get:

$$
\begin{aligned}
&\int_0^1 \left| f\big(X_{t\cdot T(\varepsilon)}^{\varepsilon,x}(\omega)\big) \,-\, f\left(K_{\mu(x,\zeta)}\right) \right| dt \\
&= \int\limits_{\left\{ t\in[0,1] \,:\, \left| X_{t\cdot T(\varepsilon)}^{\varepsilon,x}(\omega) - K_{\mu(x,\zeta)} \right| > c \right\}} \left| f\big(X_{t\cdot T(\varepsilon)}^{\varepsilon,x}(\omega)\big) \,-\, f\left(K_{\mu(x,\zeta)}\right) \right| dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\left\{ t\in[0,1] \,:\, \left| X^{\varepsilon,x}_{t\cdot T(\varepsilon)}(\omega) - K_{\mu(x,\zeta)} \right| \le c \right\}} \left| f\big(X^{\varepsilon,x}_{t\cdot T(\varepsilon)}(\omega)\big) - f\left(K_{\mu(x,\zeta)}\right) \right| dt \\
&\le \max\nolimits_{B(K_{\mu(x,\zeta)},R)} \left| f - f\left(K_{\mu(x,\zeta)}\right)\right| \cdot \mathbb{L}\Big\{ t\in[0,1] : |X^{\varepsilon,x}_{t\cdot T(\varepsilon)}(\omega) - K_{\mu(x,\zeta)}| > c \Big\} \\
&\quad + \max\nolimits_{\overline{B(K_{\mu(x,\zeta)},c)}} \left| f - f\left(K_{\mu(x,\zeta)}\right) \right| \quad ;
\end{aligned}
$$

now if c is chosen sufficiently small such that the latter summand is bounded by $\eta/2$,

$$
\max\nolimits_{\overline{B(K_{\mu(x,\zeta)},c)}} \left| f - f\left(K_{\mu(x,\zeta)}\right) \right| < \frac{\eta}{2}
$$

(f is continuous by assumption) and if one defines

$$
\mathfrak{m} := \max\nolimits_{B(K_{\mu(x,\zeta)},R)} \left| f - f\left(K_{\mu(x,\zeta)}\right) \right| ,
$$

then it follows from the previously obtained estimate that

$$
\begin{aligned}
B^{\varepsilon} \quad &\equiv \quad \mathbb{P}\left\{ \tau^{\varepsilon}_{R}(\,.\,) > 1 \,,\, \int_0^1 \left| f\big(X^{\varepsilon,x}_{t\cdot T(\varepsilon)}(\,.\,)\big) - f\left(K_{\mu(x,\zeta)}\right) \right| dt > \eta \right\} \\
&\le \quad \mathbb{P}\Bigg\{ \omega : \tau^{\varepsilon}_{R}(\omega) > 1 \,, \\
&\qquad\qquad \mathfrak{m} \cdot \mathbb{L}\Big\{ t\in[0,1] : |X^{\varepsilon,x}_{t\cdot T(\varepsilon)}(\omega) - K_{\mu(x,\zeta)}| > c \Big\} + \frac{\eta}{2} > \eta \Bigg\} \\
&\le \quad \mathbb{P}\left\{ \mathbb{L}\{ t\in[0,1] : |X^{\varepsilon,x}_{t\cdot T(\varepsilon)}(\,.\,) - K_{\mu(x,\zeta)}| > c \} > \frac{\eta}{2\,\mathfrak{m}} \right\} \\
&\xrightarrow{\varepsilon\to 0} 0
\end{aligned}
$$

by the stochastic convergence stated in theorem 2.5.6 with $\Gamma = 1$. □

Remark 2.5.11 (Further concepts of metastability). The concept of *metastability* used here is the one as described by Freidlin [Fr 00]. Other authors also address related phenomena and offer different routes to metastability. We only mention a few of them.

Cassandro et al. [Cs-Ga-Ol-Va 84] choose the "pathwise approach": Here, the Curie-Weiss model (an Ising spin system) and the contact process (taking its values in the power set of $\mathbb{Z}$), two Markov processes with discrete state spaces, are investigated; time averages are taken along each path and these averages are then shown to converge to measure valued jump processes. The same group, Galves et al. [Ga-Ol-Va 87], also applies this rationale to the gradient SDE (2.2) for a potential function with two wells. Surveys on this approach are given by Vares [Va 96] and Olivieri and Vares [Ol-Va 05].

Bovier et al. [Bv-Ec-Gd-Kn 04] and [Bv-Gd-Kn 05] develop a potential theoretic approach: By calculating capacities metastable exit times can be deduced. These authors are then able to obtain precise prefactors for the exit times, too.

Huisinga et al. [Hui-Mey-Sch 04] study metastability using the "V-uniform ergodicity" of the process; this is a spectral theoretic assumption from which these authors deduce that the generator admits a spectral gap and the eigenfunctions provide a decomposition of the state space into "almost-absorbing subsets".

Remark 2.5.12 (Simulated annealing). A stochastic process which is closely related to the gradient SDE (2.2) is the "simulated annealing process", i.e. the solution process X of the SDE

$$dX_t \;=\; -\,\nabla U\,(X_t)\;dt \;+\; \sqrt{\varepsilon(t)}\;dW_t \qquad\qquad (t \geq 0)\,,$$

where $\varepsilon(t) \to 0$ is a *time-varying* noise-intensity. For the time dependent "temperature" $\varepsilon(t)$ one can then consider cooling schedules

$$\varepsilon(t) \;:=\; \varepsilon_c(t) \;\equiv\; \frac{c}{\log(2+t)} \qquad\qquad (t \geq 0)$$

for different choices of $c > 0$. Hwang and Sheu [Hw-Sh 90, Th.3.3] prove that, under appropriate conditions, there is some cooling schedule such that X minimizes U; more precisely, there is a constant $d^* > 0$ such that for all $c > d^*$,

$$\mathbb{P}_{0,x}\{X_t^c \in \underline{S}\} \xrightarrow{t\to\infty} 1$$

uniformly with respect to initial conditions x in compact sets, where

$$\underline{S} \;:=\; \left\{\, z \in \mathbb{R}^d \;:\; U(z) = \min_{\mathbb{R}^d} U(\,.\,) \,\right\}$$

and where X^c denotes the solution of the above simulated annealing SDE for $\varepsilon(t) = \varepsilon_c(t)$. There are many more interesting features of the simulated annealing process and interesting connections to the spectral theory of the generator of (2.2) which are beyond the scope of this book. Instead, the reader is referred to Hwang and Sheu [Hw-Sh 90], Royer [Roy 89], Chiang et al. [Cg-Hw-Sh 87] and Geman and Hwang [Gem-Hw 86] among others.

2.6 Sample systems

The first example, taken from Freidlin and Wentzell [Fr-We 98, 4.3.2], lies at the core of our investigations. On the one hand it is an illustrative example for the Freidlin-Wentzell theory; on the other hand it alludes to the manner in which the local Lyapunov exponents shall generalize the Lyapunov exponents of linear ODEs with constant coefficients which are given by the real parts of the eigenvalues of the coefficient matrix; see 1.5.2.

Example 2.6.1 (Multi-dimensional Ornstein-Uhlenbeck process). Consider the SDE (2.1) with linear drift $b(x) := \mathbf{A}x$ and $\sigma = \mathrm{id}_{\mathbb{R}^d}$,

$$dX_t^{\varepsilon,x} \;=\; \mathbf{A}X_t^{\varepsilon,x}\,dt \,+\, \sqrt{\varepsilon}\;dW_t \;, \qquad\qquad X_0^{\varepsilon,x} \;=\; x \;\in \mathbb{R}^d \;,$$

where $\mathbf{A} \in \mathbb{R}^{d\times d}$ is a constant matrix. The origin $K_1 := 0$ is an equilibrium of b and assuming that $\mathbf{A}$ is normal and $\mathbf{A}+\mathbf{A}^*$ is negative definite (i.e. its eigenvalues, which are real by symmetry, are strictly negative), then one gets for $X^0 = \mathrm{e}^{\mathbf{A}t}$ (the propagator) that

$$\begin{aligned} |\,X_t^{0,x}\,|^2 \;=\; \langle\, \mathrm{e}^{\mathbf{A}t}\,x\,,\mathrm{e}^{\mathbf{A}t}\,x\,\rangle &= \langle\, \mathrm{e}^{\mathbf{A}^*t}\,\mathrm{e}^{\mathbf{A}t}\,x\,,x\,\rangle \\ &= \langle\, \mathrm{e}^{(\mathbf{A}+\mathbf{A}^*)t}\,x\,,x\,\rangle \\ &\le |\,x\,|^2\;\; \mathrm{e}^{\|\mathbf{A}+\mathbf{A}^*\|\,t} \;\;\xrightarrow{t\to\infty}\;\; 0\,, \end{aligned}$$

i.e. K_1 attracts $\mathbb{R}^d$, $D_1 = \mathbb{R}^d$. Now the vector field $b(x) \equiv \mathbf{A}x$ can be decomposed as

$$b \;=\; -\nabla U + \mathrm{L}\,,$$

where

$$U(x) \;:=\; -\,\frac{1}{4}\,\langle(\mathbf{A}+\mathbf{A}^*)x, x\rangle \qquad\text{and}\qquad \mathrm{L}(x) \;:=\; \frac{1}{2}\,(\mathbf{A}-\mathbf{A}^*)x\,.$$

More precisely, let $(e_k)_{k=1}^d$ denote the canonical basis of $\mathbb{R}^d$; then

$$\begin{aligned} -(\nabla U)(x) &\equiv +\,\frac{1}{4}\,\nabla_x\;\langle(\mathbf{A}+\mathbf{A}^*)x, x\rangle \\ &= \frac{1}{4}\sum_k e_k\,\frac{\partial}{\partial x_k}\,\sum_{i,j}(a_{ij}+a_{ji})\,x_j\,x_i \\ &= \frac{1}{4}\sum_{i,j,k} e_k\,(a_{ij}+a_{ji})\,(\delta_{kj}\,x_i+\delta_{ki}\,x_j) \\ &= \frac{1}{4}\sum_{i,j} e_j(a_{ij}+a_{ji})x_i + \frac{1}{4}\sum_{i,j} e_i(a_{ij}+a_{ji})\,x_j \;\;=\;\; \frac{1}{2}(\mathbf{A}+\mathbf{A}^*)x\,. \end{aligned}$$

The normality of $\mathbf{A}$ implies that this decomposition of b is orthogonal: For all $x \in \mathbb{R}^d$,

$$\begin{aligned} \big\langle -(\nabla U)(x)\,,\,\mathrm{L}(x)\big\rangle &= \left\langle\, \frac{1}{2}\,(\mathbf{A}+\mathbf{A}^*)x\,,\,\frac{1}{2}\,(\mathbf{A}-\mathbf{A}^*)x \right\rangle \\ &= \frac{1}{4}\,\big(\langle\mathbf{A}x,\mathbf{A}x\rangle \;+\; 0 \;-\; \langle\mathbf{A}^*x,\mathbf{A}^*x\rangle\big) \\ &= \frac{1}{4}\,\big(\langle\mathbf{A}^*\mathbf{A}x,x\rangle \;-\; \langle\mathbf{A}\mathbf{A}^*x,x\rangle\big) \;\;=\;\; 0\;\;. \end{aligned}$$

Proposition 2.4.5 thus implies for the quasipotential that $V(K_1, x) = 2\,U(x)$ for all $x \in \mathbb{R}^d$. Theorem 2.5.6 yields that for all initial values $x \in \mathbb{R}^d$, all time scales $T(\varepsilon) \asymp e^{\zeta/\varepsilon}$ for some parameter $\zeta > 0$ and all $c, \Gamma > 0$, the following holds true:

$$\mathbb{L}\left\{ t \in [0,\Gamma] \;:\; \left| X^{\varepsilon,x}_{t\,T(\varepsilon)} \right| > c \right\} \xrightarrow[\varepsilon\to 0]{\mathbb{P}} 0 \; .$$

Example 2.6.2 (Two well potential). Consider the gradient SDE (2.2) with a two well potential function U as sketched in the following figure 2.6, where $v < V$; for a numerical example see (1.34).

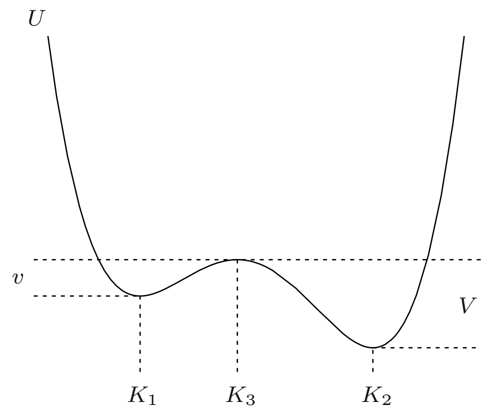

Fig. 2.6 A two well potential function $U : \mathbb{R}^d \to \mathbb{R}$ (cf. figure 1)

Here, the quasi-potential is calculated by means of proposition 2.4.5 as

$$\begin{aligned} V(K_1, K_3) &= 2\,(U(K_3) - U(K_1)) \;\equiv\; 2v \;, \\ V(K_2, K_3) &= 2\,(U(K_3) - U(K_2)) \;\equiv\; 2V \end{aligned}$$

and hence also $V(K_1, K_2) = 2v$ and $V(K_2, K_1) = 2V$, since following the deterministic path X^0 amounts to a vanishing rate function.

The assumption 2.1.1 **(K)** is satisfied with $l = 2$ and the stable points K_1, K_2; the saddle point K_3 ($l' = 3$) is sorted out, since it violates the stability requirement **(K2)**. In other words, we are concerned with

$$(V_{ij})_{i,j\in\mathfrak{L}} \;\equiv\; \big(V(K_i, K_j) \big)_{i,j\in\mathfrak{L}} \;=\; \begin{pmatrix} 0 & 2v \\ 2V & 0 \end{pmatrix} \quad ,$$

where $\mathfrak{L} := \{1, 2\}$. Furthermore, the respective domain of attraction of K_i is D_i, $i \in \mathfrak{L}$, where D_1 denotes the shallow well (around K_1) and where D_2 is the deep well (around K_2), respectively, both excluding K_3. The quasi-deterministic approximation is given by the following data here: The set of

0-cycles is

$$\mathcal{C}^{(0)} \;=\; \mathfrak{L} \;\equiv\; \{1,2\}\,.$$

On the 0-cycles the main state, the rotation rate and the stationary distribution rate are trivial, i.e. $M(i) = i$, $R(i) = 0$ and $m_i(i) = 0$, respectively, where $i \in \mathfrak{L}$. The exit rates are

$$\mathcal{E}(1) \;=\; V(K_1, K_2) \;=\; 2v \qquad \text{and} \qquad \mathcal{E}(2) \;=\; V(K_2, K_1) \;=\; 2V\,.$$

The "follow-mapping" J is given by

$$J(1) \;=\; 2 \qquad \text{and} \qquad J(2) \;=\; 1\,.$$

The possible 1-cycles are $(1 \to 2 \to 1)$ and $(2 \to 1 \to 2)$ which both describe the same cyclic order; hence,

$$\mathcal{C}^{(1)} \;=\; \big\{\,(1 \to 2 \to 1)\,\big\}$$

and the main state is

$$M\big(\,(1 \to 2 \to 1)\,\big) \;=\; 2$$

by (2.12). The rotation rate as defined in (2.13) is

$$R\big(\,(1 \to 2 \to 1)\,\big) \;\equiv\; \max_{i \in \{1,2\}} V_{i,J(i)} \;=\; V_{21} \;=\; 2V\,;$$

furthermore, the stationary distribution rate $m_{(1\to2\to1)}$ for the 1-cycle $(1 \to 2 \to 1)$ is given by

$$m_{(1\to2\to1)}(1) \;\equiv\; R\big((1 \to 2 \to 1)\big) \;-\; V_{12} \;=\; 2V \;-\; 2v \;=\; 2(V - v)$$

and

$$m_{(1\to2\to1)}(2) \;\equiv\; R\big((1 \to 2 \to 1)\big) \;-\; V_{21} \;=\; 2V \;-\; 2V \;=\; 0\,.$$

The exit rate of the 1-cycle $(1 \to 2 \to 1)$ is

$$\mathcal{E}\big((1 \to 2 \to 1)\big) \;=\; \infty$$

due to (2.14), since $\big|\mathcal{C}^{(1)}\big| = 1$. The latter fact also means that $\mathcal{C}^{(1)}$ is equal to $\mathfrak{L}$, considered as a point set. Hence, the recursive definition of cycles stops at $k = 1$ for the two well potential function.

Next, we use the above findings to determine the support of the sublimiting distribution, i.e. we calculate the metastable states corresponding to initial values and time scales as given by definition 2.5.4 and the distinction of cases preceding it.

Firstly, fix $x \in D_1$. The sequence of cycles belonging to x is then

$$1 \;=\; i(x) \;\equiv\; C^{(0)}(x) \;\subset\; (1 \to 2 \to 1) \;\equiv\; C^{(1)}(x) \;=\; \mathfrak{L}$$

and the corresponding exit and rotation rates are

$$E_0(x) \;\equiv\; \mathcal{E}(1) \;=\; 2v \;\;, \qquad\qquad E_1(x) \;\equiv\; \mathcal{E}\big((1 \to 2 \to 1)\big) \;=\; \infty$$

and

$$R_0(x) \;\equiv\; 0 \;\;, \qquad\qquad R_1(x) \;\equiv\; R\big((1 \to 2 \to 1)\big) \;=\; 2V \;,$$

respectively, which we collect as

$$0 \;\equiv\; R_0(x) \;<\; 2v \;=\; E_0(x) \;\leq\; 2V \;=\; R_1(x) \;<\; \infty \;=\; E_1(x)\;.$$

Due to (2.20) we fix $\zeta \in \mathbb{R}_{>0} \setminus \{2v, 2V\}$ and then we have

$$m^* \;\equiv\; m^*(x) \;=\; \begin{cases} -1 & ,\text{if } \zeta < 2v\,, \\ 0 & ,\text{if } \zeta > 2v\,. \end{cases}$$

By virtue of the distinction of cases leading to (2.21) we immediately get that $\mu(x,\zeta) = 1$, if $m^* = -1$, i.e. if $\zeta < 2v$, since $\zeta > 0 = R_0(x)$. Otherwise, if $m^* = 0$, i.e. if $\zeta > 2v$, then $\mu(x,\zeta) = 2$, if also $\zeta > 2V = R_{m^*+1}(x)$; if $m^* = 0$ and if, moreover, $\zeta \in (2v, 2V)$, then we also get $\mu(x,\zeta) = 2$, since for $\hat{C}^{(m^*)} := 2$ we have $\mathcal{E}(\hat{C}^{(m^*)}) = 2V > \zeta$ and $\zeta > R(\hat{C}^{(m^*)}) \equiv 0$ then. All in all, we get for $x \in D_1$ and $\zeta \in \mathbb{R}_{>0} \setminus \{2v, 2V\}$ that $\mu(x,\zeta) = 1$, if $\zeta < 2v$ and $\mu(x,\zeta) = 2$ otherwise.

Secondly, fix $x \in D_2$. The sequence of cycles corresponding to x is then

$$2 \;=\; i(x) \;\equiv\; C^{(0)}(x) \;\subset\; (1 \to 2 \to 1) \;\equiv\; C^{(1)}(x) \;=\; \mathfrak{L}$$

and the respective exit and rotation rates are

$$E_0(x) \;\equiv\; \mathcal{E}(2) \;=\; 2V \;\;, \qquad\qquad E_1(x) \;\equiv\; \mathcal{E}\big((1 \to 2 \to 1)\big) \;=\; \infty$$

and

$$R_0(x) \;\equiv\; 0 \;\;, \qquad\qquad R_1(x) \;\equiv\; R\big((1 \to 2 \to 1)\big) \;=\; 2V \;,$$

which we again collect as

$$0 \;\equiv\; R_0(x) \;<\; 2V \;=\; E_0(x) \;\leq\; 2V \;=\; R_1(x) \;<\; \infty \;=\; E_1(x)\;.$$

On the basis (2.20) we fix $\zeta \in \mathbb{R}_{>0} \setminus \{2V\}$ and then we have

$$m^* \;\equiv\; m^*(x) \;=\; \begin{cases} -1 & ,\text{if } \zeta < 2V\,, \\ 0 & ,\text{if } \zeta > 2V\,; \end{cases}$$

both cases directly yield that $\mu(x,\zeta) = 2$.

Hence, we can summarize our above findings as follows: Fix some $\zeta \in \mathbb{R}_{>0} \setminus \{2V\}$, then

$$\mu(x,\zeta) = \begin{cases} 2 & , x \in D_2 \\ 2 & , x \in D_1 \text{ and } \zeta > 2v \\ 1 & , x \in D_1 \text{ and } \zeta < 2v \ . \end{cases}$$

Now fix an initial value $x \in \mathbb{R}^d \setminus \{K_3\}$ and a time scale $T(\varepsilon) \asymp \mathrm{e}^{\zeta/\varepsilon}$ for such a parameter $\zeta \in \mathbb{R}_{>0} \setminus \{2V\}$. Then it follows from theorem 2.5.6:

1) If $\underline{x \in D_1 \text{ and } \zeta < 2v}$, then for any $c, \Gamma > 0$,

$$\mathbb{L}\Big\{\, t \in [0,\Gamma] \ : \ \Big|\, X^{\varepsilon,x}_{t\,T(\varepsilon)} - K_1 \,\Big| \ > \ c \Big\} \ \xrightarrow[\varepsilon\to 0]{\mathbb{P}} \ 0 \ ;$$

2) in $\underline{\text{all other cases } (x \in D_1 \text{ and } \zeta > 2v\,;\ x \in D_2)}$ for any $c, \Gamma > 0$,

$$\mathbb{L}\Big\{\, t \in [0,\Gamma] \ : \ \Big|\, X^{\varepsilon,x}_{t\,T(\varepsilon)} - K_2 \,\Big| \ > \ c \Big\} \ \xrightarrow[\varepsilon\to 0]{\mathbb{P}} \ 0 \ .$$

□

In the remainder of this section several real-world situations will be discussed which can be modeled by the SDEs (2.2) and (2.1). In doing so, the conceptual scope of these equations from an applications' point of view shall be demonstrated.

As in equation (2.2) the diffusion coefficient of the SDE (2.1) will mostly be taken as $\sigma(\,.\,) = \mathrm{id}_{\mathbb{R}^d}$. From an applications' point of view this is not a severe restriction, but avoids technicalities; see e.g. 2.4.13.

Economic time series modeling

Example 2.6.3 (Vasicek model for the interest rates with several wells). The classical *Vasicek model* [Vc 77] describes the short term interest rate[18] as solution of the SDE

$$dX^{\varepsilon,x}_t \ = \ -\beta(X^\varepsilon_t - R)\,dt \ + \ \sqrt{\varepsilon}\,dW_t \ , \qquad X^{\varepsilon,x}_0 = x \ ,$$

where $\beta, R, x > 0$. This equation fits into the framework of the SDEs (2.1) and (2.2) via $b(x) := -\beta(x - R)$, $d = 1$ and $U(x) := -\int_0^x b(y)dy = \beta\,(\frac{x^2}{2} - Rx)$. U is a quadratic one-well potential and after a translation by R, X^ε coincides with the Ornstein-Uhlenbeck process as described in example 2.2.3. Therefore, the transition densities are known and

$$X^{\varepsilon,x}_t \ = \ R + (x - R)\mathrm{e}^{-\beta t} + \sqrt{\varepsilon}\int_0^t \mathrm{e}^{-\beta(t-u)}dW_u$$

[18] For an overview of interest rate models see e.g. Gibson et al. [Gs-Lh-Pr-Ta 99] and [Gs-Lh-Ta 01].

is normally distributed, where the mean is given by the trajectory of the deterministic system $X_t^{0,x}$. A characteristic feature of this process is the mean-reversion with respect to the one stable equilibrium point $K_1 := R$. Hence, the assumptions 2.1.1 hold with $l(= l') = 1$. Therefore, the previous large deviation considerations can be readily applied to the Vasicek interest rate model; see Callen et al. [Ca-Gv-Xu 00].

Using non-parametric tests, however, Aït-Sahalia [Aï 96] rejects such linear drift specifications for the short term interest rate. In order to better capture the underlying economic effects, regime switching models seem necessary (see Aït-Sahalia [Aï 96, p.397]). In the literature such spot rate shifts have mostly been modeled by replacing the above mean parameter R by a Markov chain $(R_t)_t$ with finite state space (see Landen [Lan 00]): E.g. if $(R_t)_t$ can attain two possible values $R^{(1)} < R^{(2)}$, this amounts to a low-mean regime and an (exceptional) high-mean regime. However, the resulting drift

$$b(t,x) := -\beta^{(1)} \left(x - R^{(1)}\right) 1_{\left\{R_t = R^{(1)}\right\}} - \beta^{(2)} \left(x - R^{(2)}\right) 1_{\left\{R_t = R^{(2)}\right\}},$$

$\beta^{(1)}, \beta^{(2)} > 0$, is discontinuous as time evolves. Aït-Sahalia, [Aï 96, p.397f.] and [Aï 99], suggests to take a "two-regime potential drift" instead, i.e. to consider the gradient SDE (2.2) (in dimension $d = 1$) with a potential function U as in figure 2.6 of example 2.6.2. Here, the well at $R^{(1)}$ is supposed to be deeper than the well of the exceptional level $R^{(2)}$. The Freidlin-Wentzell theory underlying the two-regime potential model (2.2) then allows to calculate the mean exit times from the regimes; see Aït-Sahalia [Aï 96]).

There is, however, one severe drawback of this model: As in the one-well case (the original Vasicek model) the interest rates may become negative. One possibility to overcome this disadvantage is to define the potential as ∞ on $\mathbb{R}_{\leq 0}$; see remark 2.2.4 above and the underlying references Meyer and Zheng [My-Zh 85] and Kunz [Kz 02].

Example 2.6.4 (Prices of energy commodities). Borovkova et al. [Br-Dh-Re-Tu 03] and Anderluh and Borovkova [Ad-Br 04] use the gradient SDE (2.2) to model prices of energy commodities (heating oil, gasoline) and agricultural commodities (coffee, cocoa, soybean).

Focusing on oil prices these authors estimate the potential function U and fit the model to the time series of daily closing prices of Brent North Sea oil from 1991 to 1998. As Borovkova et al. [Br-Dh-Re-Tu 03] note, the oil price was generally known to have several "preferred regions" at 14, 18 and 23 dollars per barrel during this period of time and most trading occurs there: The price "clusters" at these levels and deviates from them relatively briefly.

As in example 2.6.3 above this additive noise model can admit negative prices; again, see remark 2.2.4 above. □

More examples for the application of “metastability” to economic (or more generally: socio-political) systems can be found in the literature:

Weidlich [Wd 71] studies polarization phenomena in society (e.g. formation of opinion) akin to Ising spin models; in the continuous limit case this leads to Fokker-Planck equations on compact intervals.

Bouchaud and Cont [Bd-Ct 98] use the gradient SDE (2.2) with a cubic potential to model the motion of the instantaneous return u_t of a stock X_t. The exit of u_t from the one well is then interpreted as “crash” of the stock. However, Bouchaud and Cont [Bd-Ct 98] deduce that in this situation the stock price X_t diverges to $-\infty$. A more realistic model of crashes on financial markets therefore has to investigate, more precisely, to which regime X_t moves in case of a crash. It seems reasonable to use a gradient ansatz (2.2) in which the potential function is time-dependent and mostly attains the one-well shape (“regular regime”). In exceptional situations the potential might feature a second well; the regime of the latter well then represents the (lower) price level to which the stock might crash. The depth of the regular well then determines the crash probability and expected crash time by the exit time law. Such an approach to the investigation of crashes is supported by one-period trading models; see Gennotte and Leland [Ge-Ll 90]. This is work in progress.

Haag et al. [Hg-Wd-Mn 85] consider a macroeconomic potential function to study the structural change of an economy. This also leads to a time-dependent potential function. However, Haag et al. [Hg-Wd-Mn 85] do not add white noise, but a (deterministic) periodic forcing to model business cycle effects (e.g. Kondratieff cycles, ...). Here, one might alternatively investigate a time-periodic potential function upon which additive white noise is imposed. In this situation it would then be interesting to investigate the phenomenon of “stochastic resonance”; see Pavlyukevich [Pv 02] and Freidlin [Fr 00].

Models for climate systems

Example 2.6.5 (Thermohaline circulation). The Thermohaline circulation (THC) describes the density driven circulation of ocean water: The respective components are the temperature (“thermo”), as cold water is denser than warm water, and the salinity (“haline”), as saltwater is denser than freshwater. The present state of the THC in the Atlantic Ocean is that warm surface water from the equator area flows to the north, cools and sinks down near Greenland and Iceland; this cold deep water is then transported back to the south.

An approximation of this phenomenon can be described by the use of box models: The classical examples by Stommel [Sto 61] and Cessi [Ce 94] use two boxes for modeling the equatorial and polar basins of the North Atlantic Ocean, respectively[19]:

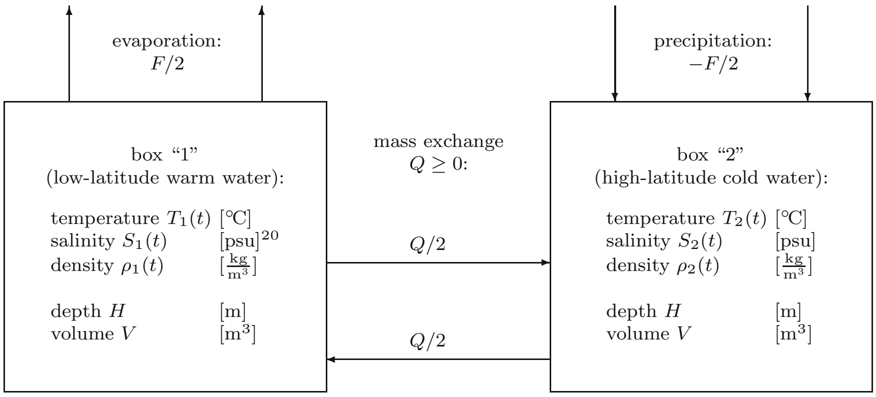

Fig. 2.7 Schematic depiction of the thermohaline circulation

The mass exchange function Q [s^{-1}] is supposed to depend only on the (non-dimensional) difference of the respective densities,

$$\rho \;:=\; \frac{\rho_1 \,-\, \rho_2}{\rho_0} \;:=\; \alpha_S\,(\,S_1 - S_2\,) \;-\; \alpha_T\,(\,T_1 - T_2\,)\;,$$

where ρ_0 denotes a reference density and α_S [psu^{-1}] and α_T [(°C)$^{-1}$] are constants. In the presence of a freshwater flux F [$m\,\mathrm{s}^{-1}$], which stems from evaporation and precipitation and influences the salinities, and assuming a relaxational forcing for the temperatures, the time-dependent quantities T_1, T_2, S_1 and S_2 are then governed by the differential equations

$$\begin{aligned}
\frac{d}{dt}\,T_1(t) &= -\,\frac{1}{t_r}\left(T_1(t) - \frac{\theta}{2}\right) \;-\; \frac{1}{2}\,Q\big(\rho(t)\big)\,\big(T_1(t) - T_2(t)\big)\\
\frac{d}{dt}\,T_2(t) &= -\,\frac{1}{t_r}\left(T_2(t) + \frac{\theta}{2}\right) \;-\; \frac{1}{2}\,Q\big(\rho(t)\big)\,\big(T_2(t) - T_1(t)\big)\\
\frac{d}{dt}\,S_1(t) &= \frac{F}{2H}\,S_0 \;-\; \frac{1}{2}\,Q\big(\rho(t)\big)\,\big(S_1(t) - S_2(t)\big)\\
\frac{d}{dt}\,S_2(t) &= -\,\frac{F}{2H}\,S_0 \;-\; \frac{1}{2}\,Q\big(\rho(t)\big)\,\big(S_2(t) - S_1(t)\big)\;;
\end{aligned}$$

here, the following constants are used: t_r [s] is the relaxation time for the temperatures towards $\pm\frac{\theta}{2}$ [°C] (if mass exchange were absent) and S_0 [psu] is a reference salinity.

[19] Square brackets are used here to indicate physical units (dimensions).

[20] "psu" denotes Practical Salinity Units, a scale for salinity.

From the above system of ODEs one gets that the salinity and temperature differences,

$$T \;:=\; T_1 \,-\, T_2 \qquad \text{and} \qquad S \;:=\; S_1 \,-\, S_2 \;,$$

are governed by the differential equations

$$\begin{aligned}
\frac{d}{dt}\,T(t) &= -\,\frac{1}{t_r}\,\big(T(t)-\theta\big) \;-\; T(t)\;Q\big(\alpha_S\,S(t) \;-\; \alpha_T\,T(t)\big)\\
\frac{d}{dt}\,S(t) &= \frac{F}{H}\,S_0 \;-\; S(t)\;Q\big(\alpha_S\,S(t) \;-\; \alpha_T\,T(t)\big)\;.
\end{aligned}$$

Now the exchange function is specified as

$$Q(\rho) \;:=\; \frac{1}{t_d} \,+\, \frac{q}{V}\,\rho^2 \;,$$

where t_d [s] is the “diffusive” time scale, q [$\mathrm{m}^3\,\mathrm{s}^{-1}$] is proportional to the pressure difference (Poiseuille’s law) and V [m^3] denotes the volume of one of the boxes. Note that Stommel [Sto 61], who pioneered the THC research by box models, used the exchange function

$$Q_{\mathrm{Sto}}\,(\rho) \;:=\; \frac{1}{t_d} \,+\, \frac{q}{V}\,|\,\rho\,| \;,$$

which we will not take into account for convenience, since it exhibits a plane at which it is not differentiable.

Reasonable choices for the constants in use are for example

$$t_r = 25 \text{ days},\;\; \theta = 20℃,\;\; F = 2.3 \;\text{ m year}^{-1},\;\; H = 4.5\cdot 10^3 \text{ m},\;\; S_0 = 35 \text{ psu},$$

$$\alpha_S \;=\; 0.75\cdot 10^{-3}\;\text{psu}^{-1}\,,\quad \alpha_T \;=\; 0.17\cdot 10^{-3}\,(℃)^{-1}\,,\quad \rho_0 \;=\; 1029\;\text{ kg m}^{-3}\,,$$

$$t_d \;=\; 219 \text{ years}\,,\quad q \;=\; 1.0\,\cdot\,10^{12}\;\;\mathrm{m}^3\;\mathrm{s}^{-1}\,,\quad V \;=\; 1.1\,\cdot\,10^{16}\;\;\mathrm{m}^3\,,$$

according to Cessi [Ce 94].

Introducing the dimensionless variables

$$x_1 \;:=\; \frac{T}{\theta} \;\equiv\; \frac{T_1 \,-\, T_2}{\theta} \qquad \text{and} \qquad x_2 \;:=\; \frac{\alpha_S\,S}{\alpha_T\,\theta} \;\equiv\; \frac{\alpha_S\,(S_1 \,-\, S_2)}{\alpha_T\,\theta}$$

and scaling time by t_d,

$$t_{\mathrm{new}} \;:=\; t_d\,\cdot\,t_{\mathrm{old}}\;,$$

the above ODEs are rewritten as

$$\begin{aligned}
\frac{d}{dt}\,x_1(t) &= -\,\alpha\,\big(x_1(t)-1\big) \;-\; x_1(t)\;\;\Big[1 \,+\, \nu^2\,\big(\,x_1(t) \;-\; x_2(t)\,\big)^2\,\Big] \qquad (2.22)\\
\frac{d}{dt}\,x_2(t) &= p \;-\; x_2(t)\;\;\Big[1 \,+\, \nu^2\,\big(\,x_1(t) \;-\; x_2(t)\,\big)^2\,\Big]\;,
\end{aligned}$$

where

$$\alpha := \frac{t_d}{t_r}\,, \quad \nu^2 := \frac{q\, t_d\, (\alpha_T \theta)^2}{V} \qquad \text{and} \qquad p := \frac{\alpha_S\, S_0\, t_d}{\alpha_T\, \theta\, H}\, F\,.$$

Using the above sample values for the parameters one would obtain the numerical approximations

$$\alpha = 3.20 \cdot 10^3\,, \quad \nu^2 = 7.3 \qquad \text{and} \qquad p = 0.9\,.$$

Note that the above ODE (2.22) can also be considered as deterministic slow-fast system, in which x_1 represents the fast variable and x_2 is the slow variable; see Berglund and Gentz [Bg-Gen 06].

Now we consider two ways of introducing stochastic dynamics in this system:

a) Let b denote the two-dimensional drift vector of this ODE system (2.22), i.e.

$$b(x_1, x_2) := \begin{pmatrix} -\,\alpha\,(x_1 - 1) - x_1 \left[1 + \nu^2\,(x_1 - x_2)^2\right] \\ p - x_2 \left[1 + \nu^2\,(x_1 - x_2)^2\right] \end{pmatrix}\,.$$

This drift fits into the setting imposed on (2.1); adding additive random perturbations of intensity $\sqrt{\varepsilon}$ one gets the following SDE for the temperature and salinity differences:

$$dX_t^\varepsilon = b\,(X_t^\varepsilon)\; dt + \sqrt{\varepsilon}\; dW_t\,.$$

For appropriate choices of the parameters this THC-drift b has three equilibria, two of which are stable and one is not; the former states are then the metastable states K_1 and K_2 of the THC. Note that the above design of the THC-drift is due to Cessi [Ce 94]. If one uses Stommel's choice [Sto 61]

$$Q_{\text{Sto}}\,(\rho) = \frac{1}{t_d} + \frac{q}{V}\,|\,\rho\,|$$

of the exchange function, the resulting drift after introducing dimension-less variables is

$$b_{\text{Sto}}\,(x_1, x_2) := \begin{pmatrix} -\,\alpha\,(x_1 - 1) - x_1\,[1 + \nu\,|\,x_1 - x_2\,|] \\ p - x_2\,[1 + \nu\,|\,x_1 - x_2\,|] \end{pmatrix}$$

which exhibits the same qualitative features, but also non-smoothness.

b) Another route to a stochastic equation of motion for the THC is to consider the freshwater flux as driving stochastic process, i.e. to add white noise to p, which will be described next: For this purpose we first replace (2.22) by a one-dimensional equation, which can be justified since for a large α the first summand in the x_1-equation forces the temperature difference x_1 to stay near 1. Hence, by substituting $x_1 = 1$ into (2.22) one gets the one-dimensional equation for the salinity difference x_2:

$$\begin{aligned}\frac{d}{dt}\, x_2(t) &= p \;-\; x_2(t)\;\left[1 \,+\, \nu^2\,\big(1 \,-\, x_2(t)\,\big)^2\right] \\ &= -\,U'\big(x_2(t)\big)\;,\end{aligned}$$

where integrating the one-dimensional drift leads to the potential function

$$U(x) \;:=\; \frac{\nu^2}{4}\,x^4 \;-\; \frac{2\,\nu^2}{3}\,x^3 \;+\; \frac{(1+\nu^2)}{2}\,x^2 \;-\; p\,x\,.$$

Adding small additive random perturbations of intensity $\sqrt{\varepsilon}$ of the freshwater flux, as announced, this leads to the SDE for the salinity difference,

$$dX_t^\varepsilon \;=\; -\,U'\left(X_t^\varepsilon\right)\,dt \;+\; \sqrt{\varepsilon}\;dW_t\;\;,$$

where W is a Brownian motion in $\mathbb{R}$. This SDE again fits into the setting of (2.1); depending on p and ν^2, the potential U has several local minima which are then the metastable states of the salinity difference of the THC. For further details we refer to Cessi [Ce 94].

More references on the THC are given by Imkeller and Monahan [Im-Mo 02]. □

Example 2.6.6 (El Niño-Southern Oscillation). El Niño is an anomaly of the sea surface temperature in the tropical Pacific Ocean. This short term phenomenon recurs on time scales of several (6–10) years and has important consequences such as reduced fishing in the eastern Pacific Ocean, low grain yields in south Asia and Australia and high crop yields in the North American prairies (see Hansen et al. [Hn-Hd-Jo 98] and the references therein). Furthermore, the hurricane activity in the western Atlantic Ocean is reduced during the season following the El Niño event and returns to normal only in the second summer following an El Niño event (see Gray [Gy 84]).

a) Wang et al. [Wg-Bc-Fg 99] describe the El Niño system by the SDE (2.1) in $\mathbb{R}^2$ with the drift function

$$b(x) \;:=\; \begin{pmatrix} a_1\,x_1 \;+\; a_2\,x_2 \;+\; a_3\,x_1\,(x_1 - c\,x_2) \;-\; 2\,x_1^3 \\ b\,(2x_2 - x_1) \;-\; 2\,x_2^3 \end{pmatrix}$$

and $\sigma(\,.\,) := \mathrm{id}_{\mathbb{R}^2}$. Here, the (nondimensional) variables x_1 and x_2 represent the anomalous sea surface temperature and the thermocline depth, respectively; $a_1, a_2, a_3, b, c \;\in\; \mathbb{R}$ are constants. Depending on the choice of the parameters this drift exhibits either a stable regime, a limit cycle regime or several equilibria. In the latter case the system can be adjusted such that there are three unstable fixed points and two stable equilibria of b; one of the latter points (the "warm" state) represents the El Niño situation.

For details such as typical values of the parameters see Wang et al. [Wg-Bc-Fg 99].

b) The multi-dimensional Ornstein-Uhlenbeck process of example 2.6.1 can also be used for describing the El Niño-Southern Oscillation, as has been carried out by Penland and Sardeshmukh [Pl-Sa 95b] and [Pl 96]. More precisely, consider the SDE (2.1) with linear drift $b(x) := \mathbf{A}x$,

$$dX_t^\varepsilon \;=\; \mathbf{A}X_t^{\varepsilon,x}\,dt \;+\; \sqrt{\varepsilon}\;\sigma\;dW_t\;, \qquad X_0^{\varepsilon,x} \;=\; x\;\in \mathbb{R}^d\;,$$

where $\mathbf{A}, \sigma \in \mathbb{R}^{d\times d}$ are constant matrices. Here, σ is additionally invertible, $\mathbf{A}$ again attracts $D_1 = \mathbb{R}^d$ towards the origin $K_1 := 0$ and the dimension is $d := 15$. If the eigenvalues (characteristic roots) are non-real, there are components of X^0 which show oscillatory behavior. In the randomly perturbed case $\varepsilon > 0$, X^ε exhibits random periodicity. One can then calculate the respective rotation numbers as in Arnold [Ar 98, Sec.6.5] for the stationary situation (i.e. for fixed $\varepsilon > 0$). This rotational behavior also implies random periodicity for the sea surface temperature which can be generated through X^ε.

For more references to the literature on stochastic (or deterministic) models of the El Niño phenomenon see Imkeller and Monahan [Im-Mo 02]. □

Example 2.6.7. A potential function with n wells is for example also given by

$$U(x) \;=\; \frac{1}{2}\,|x|^2 \;-\; \frac{1}{2}\sum_{i=1}^{n} k_i\;\exp\left(\,-\,\frac{|x-z_i|^2}{c_i}\,\right)\;,$$

where $x \in \mathbb{R}^d$; $k_i > 0$, $c_i > 0$ and $z_i \in \mathbb{R}^d$ are the parameters representing the depth, width and position of the i-th well. Teng et al. [Tg-Mo-Fy 04] use this potential as a model for the extratropical northern hemisphere atmosphere. □

The gradient SDE (2.2) is also used by Nicolis and Nicolis [Nc-Ni 81] and Sutera [Su 81] as a general model for climate transitions.

Further examples

Finally, we briefly list some more examples for the interested reader. Again, the goal is not to compile a complete list, but to hint at the various possibilities of application.

Example 2.6.8 (Saksaul tree population). Freidlin and Svetlosanov [Fr-Sv 76] consider (2.2) with the potential function

$$U(x) \;:=\; -\,\frac{a}{\kappa^2}\,x \;+\; \frac{a}{\kappa^3}\;\arctan(\,\kappa\cdot x\,) \;+\; \frac{l}{2}\,x^2\;,$$

where a, κ, l are real parameters such that $0 < \frac{2l\kappa}{a} < 1$. This potential function has a local minimum at $K_1 := 0$, a global minimum at

$$K_2 := \frac{a}{2l\kappa^2}\left(1 + \sqrt{1 - \left(\frac{2l\kappa}{a}\right)^2}\right)$$

and a saddle at

$$K_3 := \frac{a}{2l\kappa^2}\left(1 - \sqrt{1 - \left(\frac{2l\kappa}{a}\right)^2}\right) .$$

Choose an initial condition $x \in D_2$, the domain of attraction of K_2. Then the process $X^{\varepsilon,x}$ represents the population of the Saksaul, a certain tree in the Gobi Desert which can best resist the drought there. Freidlin and Svetlosanov [Fr-Sv 76] then calculate the mean exit time from D_2 by the exit time law. Such an exit and hence an approach of $K_1 = 0$ means the extinction of the tree population. The parameters a, l and κ parametrize the sprout rate, the extinction rate and the influence of neighboring trees, respectively. □

Example 2.6.9 (Evolution). Newman et al [Nm-Co-Ki 85] use (2.2) to describe Neo-darwinian evolution: X^ε depicts the population mean of some genetically determined character; the function $-U$ is then interpreted as the mean fitness of the population. Natural selection pushes the population mean X^ε towards higher values of this fitness landscape. The parameter $\sqrt{\varepsilon}$ measures the magnitude of the random genetic variations relative to that of natural selection. □

Example 2.6.10 (The Stochastic Disk Dynamo Model for the polarity of the earth's magnetic field). The *stochastic disk dynamo* is given by the SDE (2.1) in $\mathbb{R}^2$ for the drift function

$$b(x_1, x_2) \;:=\; \begin{pmatrix} -\,\nu_1\, x_1 \;+\; x_1\, x_2 \\ -\,\nu_2\, x_2 \;+\; 1 \;-\; x_1^2 \end{pmatrix} ,$$

where $\nu_1, \nu_2 > 0$ are constants such that $\nu_1\nu_2 < 1$. This drift has two stable equilibria at

$$K_{1,2} := \left(\pm\sqrt{1 - \nu_1\nu_2}, \nu_1\right)$$

and an unstable fixed point at $K_3 := (0, 1/\nu_2)$. Ito and Mikami, [Ito-Mik 96] and [Ito 88], use this system as a model for the earth's magnetic field: The stable equilibria K_1 and K_2 then correspond to the two respective polarities. □

The case of a compact state space

Instead of considering a diffusion on $\mathbb{R}^d$ one could also consider a diffusion X^ε on a d-dimensional compact manifold. In this case **(V)** is not necessary. Furthermore, a density satisfying the Fokker-Planck equation is always normalizable in the compact case.

The following SDE provides an example of a system on the circle for which the stationary density can be calculated as in remark 2.2.1 due to the one-dimensionality of its state space:

Example 2.6.11 (Noisy north-south-flow). The *noisy north-south-flow* is the solution $X^\varepsilon := \alpha^\varepsilon$ for the SDE

$$d\,\alpha_t^\varepsilon \;=\; -\,\cos\alpha_t^\varepsilon\,dt \;+\; \sqrt{\varepsilon}\;dW_t$$

on the compact state space $S^1 \;\equiv\; \mathbb{R}/2\pi\mathbb{Z} \;\simeq\; [0, 2\pi)$. The state space being one-dimensional, the system can be thought of as the random perturbation of the *north-south-flow*

$$d\,\alpha_t \;=\; -\,U'(\alpha_t)\,dt \qquad \text{with the potential} \qquad U(\alpha) \;:=\; \sin\alpha$$

on S^1; this deterministic dynamical system has two equilibria: $\alpha_N := \frac{\pi}{2}$ which is unstable and $\alpha_S := \frac{3\pi}{2}$ which is stable. Calling α_N and α_S the north pole and the south pole of the "planet" S^1, respectively, one gets an explanation of the name of this system.

The assumptions **(S)** and **(E)** are clearly met; **(K)** holds true with $l = 1$, $K_1 := \alpha_S \equiv \frac{3\pi}{2}$; additionally, $l' = 2$ and $K_2 := \alpha_N \equiv \frac{\pi}{2}$. Furthermore, also **(G)** is satisfied.

The invariant measure of $(X_t^\varepsilon)_{t\geq 0} \equiv (\alpha_t^\varepsilon)_{t\geq 0}$ can be also calculated explicitly: As in remark 2.2.1 on stationary measures, the invariant measure has a density $p^\varepsilon(\alpha)$ with respect to the Lebesgue measure on the circle, which needs to solve the *Fokker-Planck-equation*

$$\frac{\varepsilon}{2}\,\frac{d^2}{d\alpha^2}\,p^\varepsilon(\alpha) \;-\; \frac{d}{d\alpha}\big(-\,\cos(\alpha)\,p^\varepsilon(\alpha)\big) \;=\; 0$$

for $\alpha \in (0, 2\pi)$, fulfill the normalization constraint

$$\int_0^{2\pi} p^\varepsilon(\alpha)\,d\alpha = 1$$

and additionally the continuity (periodicity) requirement

$$p^\varepsilon(0) = p^\varepsilon(2\pi)\;.$$

The solution to this problem is given by Khasminskii, [Kh 67] and [Kh 80, Sec.VI.8], as

$$p^{\varepsilon}(\alpha) \;=\; c_{\varepsilon}\,\frac{1}{\varepsilon}\,\left[1 \;+\; \frac{\mathcal{W}(2\pi)-1}{\int_0^{2\pi}\mathcal{W}(s)ds}\,\int_0^{\alpha}\mathcal{W}(u)du\right]\mathcal{W}(\alpha)^{-1}\;,$$

where

$$\mathcal{W}(\alpha) \;:=\; \exp\left\{+\,\frac{2}{\varepsilon}\,U(\alpha)\right\}$$

and c_{ε} is the normalization constant; this can be seen by a direct calculation. Furthermore, the one peak of the stationary density p^{ε} is located at the south pole α_S and in the small noise limit $\varepsilon \to 0$ the invariant measure converges weakly to δ_{α_S}. The latter property can be either directly verified from the above formula or be deduced from corresponding results by Freidlin and Wentzell, [We-Fr 69] and [Fr-We 98, Th.6.4.2], since the state space is compact.

This noisy north-south-flow system has been considered by Carverhill [Cv 85b, p.290ff.]; note that Carverhill [Cv 85b, p.290] defines the noise free north-south-flow as the stereographic projection of the flow $\eta_t(x) := x\mathrm{e}^{-t}$ in $\mathbb{R}$ to the unit circle; it can be calculated that this is equivalent to the above potential function $U(\alpha) := \sin\alpha$ on S^1. A rotated version (a noisy west-east-flow so to say) has been investigated and simulated by Carverhill et al. [Cv-Cl-Ew 86, p.54ff.]. Carverhill [Cv 85b] calculates the (one) Lyapunov exponent λ^{ε} for this system α_t^{ε}, obtains a Furstenberg-Khasminskii-type formula for it and notes that in the small noise limit $\varepsilon \to 0$, λ^{ε} converges to $-U''(\alpha_S) = \sin(-\frac{\pi}{2}) = -1$.

A more general investigation concerning the invariant measures of diffusion processes on a circle with small diffusion is given by Nevel'son [Nev 64].

The impetus for us mentioning this system on the circle here is twofold: On the one hand it illustrates how the results from the Freidlin-Wentzell theory also hold in the setting of compact state spaces. On the other hand — what is more — elliptic SDEs of the above type typically arise in the study of the angle (and hence of the Lyapunov exponent) of linear, two-dimensional stochastic systems with *white* noise. The crucial property when studying the SDE (1) is, however, that its angle satisfies a differential equation with *real* noise; see (1.6). Therefore, when considering the angle system of (1), the Freidlin-Wentzell results only hold for the noise process X^{ε}, but not for the angle process; in particular, when examining (1) for (local) Lyapunov exponents, the Freidlin-Wentzell results cannot be applied to the angle process.

For this reason we study exit probabilities of *degenerate* stochastic systems in the following chapter.

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