

Pseudo-Anosov diffeomorphisms in pseudo-surfaces

There are diffeomorphisms on a compact surface S with uniformly hyperbolic 1 dimensional stable and unstable foliations if and only if S is a torus: the Anosov diffeomorphisms. What is happening on the other surfaces? This question leads to the study of pseudo-Anosov maps. Both Anosov and pseudo-Anosov maps appear as periodic points of the geodesic Teichmüller flow T_t on the unitary tangent bundle of the moduli space over S . We observe that the points of pseudo-Anosov maps are regular (the foliations are like the ones for the Anosov automorphisms) except for a finite set of points, called singularities, which are characterized by their number of prongs k . The stable and unstable foliations near the singularities are determined by the real and the imaginary parts of the quadratic differential $\sqrt{z^{k-2}(dz)^2}$. By a coordinate change $u(z) = z^{k/2}$ the quadratic differential $z^{k-2}(dz)^2$ gives rise to the quadratic differential $(du)^2$ and, in this new coordinates, the pseudo-Anosov maps are uniform contractions and expansions of the stable and unstable foliations. This fact inspired the construction of Pinto-Rand's pseudo-smooth structures, near the singularities, such that the pseudo-Anosov maps are smooth for this pseudo-smooth structures, and have the property that the stable and unstable foliations are uniformly contracted and expanded by the pseudo-Anosov dynamics. We define a pseudo-linear algebra, the first step in constructing the notion of the derivative of a map at a singularity. In this way, we obtain a pseudo-smooth structure at the singularity, leading to Pinto-Rand's pseudo-smooth manifolds, pseudo-smooth submanifolds, pseudo-smooth splittings and pseudo-smooth diffeomorphisms. The Stable Manifold Theorem, for pseudo-smooth manifolds, is presented giving the associated pseudo-Anosov diffeomorphisms.

14.1 Affine pseudo-Anosov maps

Let A_c be a conformal structure on a compact surface S . Two conformal structures A_c and B_c are equivalent if, and only if, there is a conformal map

h such that $A_c = h^*(B_c)$. The moduli space $M_S = \{[A_c]\}$ has a natural metric given by the minimal quasi-conformal distortion of the maps from the elements of a class $[A_c]$ to the elements of the other class $[B_c]$.

The geodesic (Teichmüller) flow T_t on the unitary tangent bundle of the moduli space has a dense set of periodic orbits. If the surface S is a torus, then the periodic points correspond to Anosov automorphisms. If the surface S is not a torus, then the periodic points correspond to pseudo-Anosov maps.

All the points of an Anosov automorphism are regular. The points of a pseudo-Anosov maps are regular, except for a finite set of points called singularities. A regular point is locally characterized by a quadratic differential $(dz)^2$. The stable and unstable foliations are determined by the real and the imaginary parts of $\sqrt{(dz)^2} = \pm dz$.

The singularities of pseudo-Anosov maps are characterized by their number of prongs k . A k -prong singularity is locally characterized by a quadratic differential $z^{k-2}(dz)^2$. The stable and unstable foliations are determined by the real and the imaginary parts of $\sqrt{z^{k-2}(dz)^2}$. If the pseudo-Anosov map has a singularity with an odd number of prongs, then the stable and unstable foliations are non-orientable.

By a coordinate change $u(z) = z^{k/2}$, the quadratic differential $z^{k-2}(dz)^2$ gives rise to the quadratic differential $(du)^2$. In this new coordinates, the pseudo-Anosov maps are locally affine contractions and expansions of the stable and unstable foliations by λ^{-1} and λ , respectively.

How can we regard the image of $u(z) = z^{k/2}$? The answer to this question leads us to the construction of Pinto-Rand's paper models, where the pseudo-Anosov maps constructed above are affine.

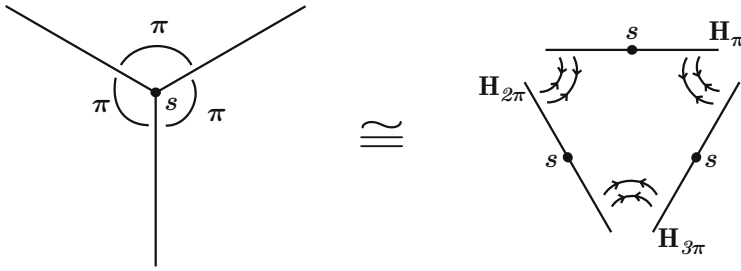
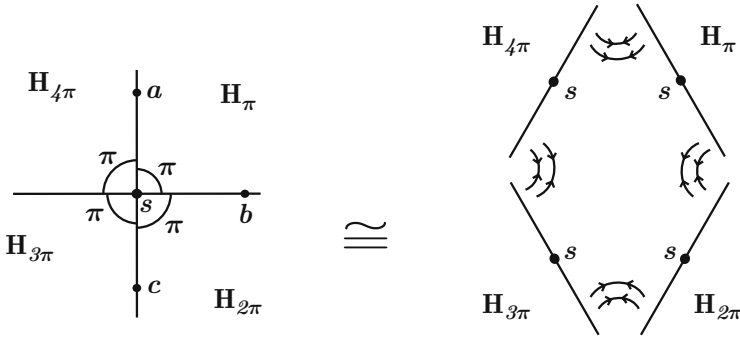
14.2 Paper models Σ_k

Let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ denote the upper half plane with the Euclidean metric d_E . Consider the space $\sqcup_{j \in \mathbb{Z}_k} \mathbb{H}_{j\pi}$ which is the disjoint union of k copies of \mathbb{H} , with $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$. Let the *paper models* Σ_k be the space obtained from $\sqcup_{j \in \mathbb{Z}_k} \mathbb{H}_{j\pi}$ by identifying $(x, 0) \in \mathbb{H}_{(j+1)\pi}$ with $(-x, 0) \in \mathbb{H}_{j\pi}$, for all $x \geq 0$. Let $s \in \Sigma_k$ be the point determined by $(0, 0) \in \mathbb{H}_{j\pi}$ for every $j \in \mathbb{Z}_k$. The Euclidean metric d_E on the upper half planes $\mathbb{H}_{j\pi}$ naturally define a flat metric on $\Sigma_k \setminus \{s\}$ which extends to a *continuous* metric d_k on Σ_k (see Figure 14.1).

The map $i : \mathbb{R} \rightarrow \Sigma_k$ is an *isometry* if, and only if, there is an isometry $i_{\mathbb{H}} : \mathbb{H} \rightarrow \Sigma_k$ such that $i_{\mathbb{H}}(x, 0) = i(x)$, for all $x \in \mathbb{R}$ (see Figure 14.2).

We say that:

- $l \subset \Sigma_k$ is a *straight line* in Σ_k if, and only if, there is an isometry $i : \mathbb{R} \rightarrow \Sigma_k$ such that $l = i(\mathbb{R})$;
- $l_{a \rightarrow b} \subset \Sigma_k$ is a *semi-straight line* in Σ_k , with origin at a and passing through b , if, and only if, there is an isometry $i : \mathbb{R} \rightarrow \Sigma_k$ such that $l_{a \rightarrow b} = i([a', +\infty))$ with $i(a') = a$ and $i(b') = b$, for some points $a' < b'$;


 Fig. 14.1. $k = 3$.

 Fig. 14.2. There is a straight line passing through a and b . There is no straight line passing through a and c .

- $l_{a,b} \subset \Sigma_k$ is a *segment straight line* in Σ_k , with endpoints a and b , if, and only if, there is an isometry $i : \mathbb{R} \rightarrow \Sigma_k$ such that $l_{a,b} = i([a', b'])$ with $i(a') = a$ and $i(b') = b$, for some points $a' < b'$. The interior $\text{int}l_{a,b}$ of $l_{a,b}$ is equal to $l_{a,b} \setminus \{a, b\}$.

Let $l_{s \rightarrow a}$ and $l_{s \rightarrow b}$ be two semi-straight lines in Σ_k . To fix ideas, let us suppose that $l_{s \rightarrow a} \subset \mathbb{H}_{j\pi}$ and $l_{s \rightarrow b} \subset \mathbb{H}_{(j+n)\pi}$, with $j, j+n \in \mathbb{Z}_k$. Let $l_{s \rightarrow c}$ be the semi-straight line formed by the points of $\mathbb{H}_{j\pi}$ and $\mathbb{H}_{(j+1)\pi}$ that were identified at the construction of Σ_k . Analogously, let $l_{s \rightarrow d}$ be the semi-straight line formed by the points of $\mathbb{H}_{(j+n-1)\pi}$ and $\mathbb{H}_{(j+n)\pi}$ that were identified at the construction of Σ_k . Let $\alpha \in [0, \pi]$ be the angle $\angle(l_{s \rightarrow a}, l_{s \rightarrow c})$ between the semi-straight lines $l_{s \rightarrow a}$ and $l_{s \rightarrow c}$, and let $\beta \in [0, \pi]$ be the angle $\angle(l_{s \rightarrow d}, l_{s \rightarrow b})$ between the semi-straight lines $l_{s \rightarrow d}$ and $l_{s \rightarrow b}$. We say that the angle $\angle(l_{s \rightarrow a}, l_{s \rightarrow b})$ between the semi-straight lines $l_{s \rightarrow a}$ and $l_{s \rightarrow b}$ is given by

$$\angle(l_{s \rightarrow a}, l_{s \rightarrow b}) = \alpha + (n-1)\pi + \beta.$$

Given $\alpha \in \mathbb{R}/k\pi\mathbb{R}$ and two points $x, y \in \Sigma_k$, we say that they are in an α -angular region, if $\angle(l_{s \rightarrow x}, l_{s \rightarrow y}) \leq \alpha$ (see Figure 14.3).

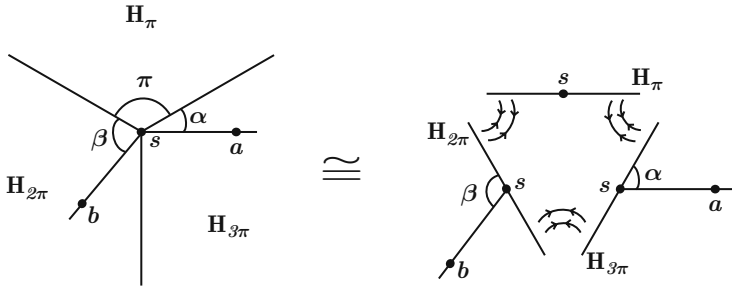


Fig. 14.3. The angle $\angle(l_{s \rightarrow a}, l_{s \rightarrow b}) = \alpha + \pi + \beta$.

14.3 Pseudo-linear algebra

Given two points $x, y \in \Sigma_k$, we say that $\mathbf{y} = y - x$ is a *vector* if, and only if, there is a segment straight line $l_{x,y} \subset \Sigma_k$ with endpoints x and y ; we call x the *origin* and y the *endpoint* of the vector $y - x$. The *norm* $\|y - x\|$ of the vector $y - x$ is given by $d_k(x, y)$.

Given a vector $\mathbf{y} = y - x$ and a constant $\lambda \in \mathbb{R}$, the vector $w - x = \lambda(y - x)$ is well-defined if, and only if, there is an isometry $i_{\mathbb{H}} : \mathbb{H} \rightarrow \Sigma_k$ with the following property: there are points $x_{\mathbb{H}}, y_{\mathbb{H}}, w_{\mathbb{H}} \in \mathbb{H}$ such that

- (i) $x = i_{\mathbb{H}}(x_{\mathbb{H}})$, $y = i_{\mathbb{H}}(y_{\mathbb{H}})$ and $w = i_{\mathbb{H}}(w_{\mathbb{H}})$;
- (ii) $w_{\mathbb{H}} - x_{\mathbb{H}} = \lambda(y_{\mathbb{H}} - x_{\mathbb{H}})$;
- (iii) if $s \in \text{int}l_{x,w}$, then $s \in \text{int}l_{x,y}$;
- (iv) if $s = x$, then $\lambda \geq 0$.

We note that the vector $\lambda(y - x)$ is well-defined, for all $0 \leq \lambda \leq 1$. The above conditions (iii) and (iv) imply that the vector $w - x$ does not depend upon the isometry considered, and so $w - x$ is uniquely determined.

Given two vectors $\mathbf{y} = y - x$ and $\mathbf{z} = z - x$ with the same origin, the vector $\mathbf{w} = w - x$, with $\mathbf{w} = \mathbf{y} + \mathbf{z}$, is equal to the *sum* of the vectors $y - x$ with $z - x$ if, and only if, there is an isometry $i_{\mathbb{H}} : \mathbb{H} \rightarrow \Sigma_k$ with the following property: there exists a constant $\lambda > 0$ and there are points $x_{\mathbb{H}}, y_{\mathbb{H}}, z_{\mathbb{H}}, w_{\mathbb{H}} \in \mathbb{H}$ such that (see Figure 14.4)

- (i) the vectors $y' - x = \lambda(y - x)$, $z' - x = \lambda(z - x)$ and $w' - x = \lambda(w - x)$ are well-defined;
- (ii) $x = i_{\mathbb{H}}(x_{\mathbb{H}})$, $y' = i_{\mathbb{H}}(y_{\mathbb{H}})$, $z' = i_{\mathbb{H}}(z_{\mathbb{H}})$ and $w' = i_{\mathbb{H}}(w_{\mathbb{H}})$;
- (iii) $w_{\mathbb{H}} = y_{\mathbb{H}} + z_{\mathbb{H}} - x_{\mathbb{H}}$;
- (iv) if $s \in \text{int}l_{x,w}$, then $s \in \text{int}l_{x,y} \cup \text{int}l_{x,z}$.

The above condition (iv) implies that the vector $\mathbf{w} = w - x$ does not depend upon the isometry considered. If s is a singularity, with order k , then there are k distinct vectors $x_1 - s, \dots, x_k - s$, all with norm equal to one, such that $x_i - s + x_{i+1} - s = s - s$, for all $i \in \mathbb{Z}_k$ (see Figure 14.5).

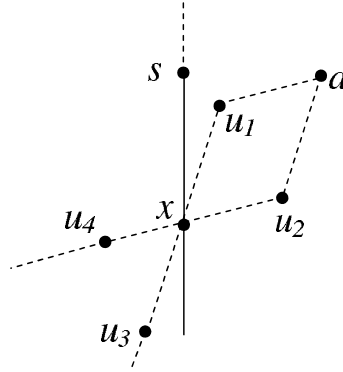


Fig. 14.4. $u_1 + u_2 = a$ and $((u_1, u_3), (u_2, u_4))$ is a basis of \mathbb{V}_x .

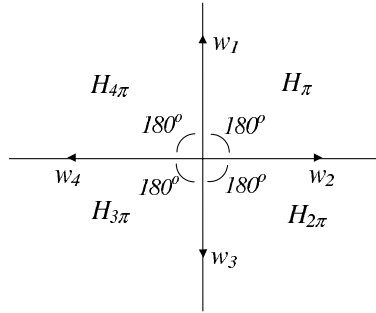


Fig. 14.5. $+$ is not associative: $(\mathbf{w}_1 + \mathbf{w}_2) + \mathbf{w}_3 = \mathbf{w}_3$; $\mathbf{w}_1 + (\mathbf{w}_2 + \mathbf{w}_3) = \mathbf{w}_1$. There is not a unique "inverse": $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{0}$; $\mathbf{w}_1 + \mathbf{w}_4 = \mathbf{0}$, where $\mathbf{0} = s - s$. $\mathbf{w}_2 + \mathbf{w}_4$ is not well-defined.

The *pseudo-linear space* \mathbb{V}_x at x is the set of all vectors with origin at x , together with the operations of addition of vectors and of multiplication of a vector by a constant, as constructed above. Let l_x be either (i) the empty set or (ii) a semi-straight line contained in a semi-straight line with origin at x . The *branched linear space* \mathbb{V}_{l_x} is given by $\mathbb{V}_x \setminus \text{int} l_x$ (see Figure 14.6).

A *pseudo-linear subspace* \mathbb{S}_x of a pseudo-linear space \mathbb{V}_x (see Figure 14.7) is a subset of \mathbb{V}_x with the following properties:

- (i) For all $\mathbf{u}, \mathbf{v} \in \mathbb{S}_x$ such that $\mathbf{u} + \mathbf{v}$ is well-defined, we have that $\mathbf{u} + \mathbf{v} \in \mathbb{S}_x$;
- (ii) For all $\lambda \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{S}_x$ such that $\lambda \mathbf{u}$ is well-defined, we have that $\lambda \mathbf{u} \in \mathbb{S}_x$.

A *full pseudo-linear space* \mathbb{S}_x is a pseudo-linear subspace \mathbb{S}_x with the following property: If $\mathbf{u} \in \mathbb{S}_x$ and $\mathbf{v} \in \mathbb{V}_x$ are such that $\mathbf{u} + \mathbf{v} = \mathbf{0}$, then $\mathbf{v} \in \mathbb{S}_x$. Hence, a full pseudo-linear subspace \mathbb{S}_s , $\mathbb{S}_s \neq \mathbb{V}_s$, at the singularity s , with order k , is the image of an isometry $i : \Sigma_k^1 \rightarrow \mathbb{V}_s$.

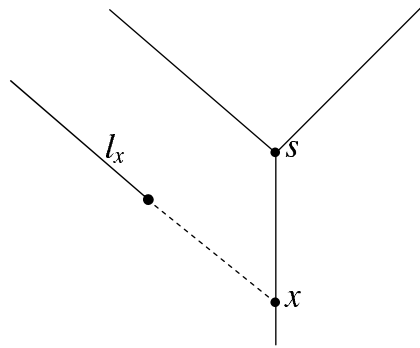


Fig. 14.6. The branched linear space \mathbb{V}_{l_x} .

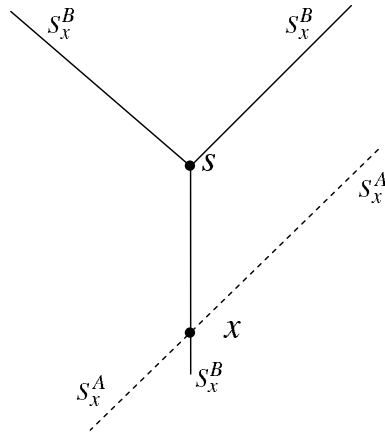


Fig. 14.7. Pseudo-linear subspaces \mathbb{S}_x^A and \mathbb{S}_x^B at x .

A pseudo-affine subspace \mathbb{S} at a point $x \in \Sigma_k \setminus \{s\}$, with $\mathbb{S}_x \neq \mathbb{V}_x$, is the image of an isometry $i : A \rightarrow \mathbb{V}_x$ with A equal either \mathbb{R} or Σ_k^1 .

A map $L : \mathbb{V}_{l_x} \rightarrow \mathbb{V}_y$ is *linear* (see Figures 14.8 and 14.9), if the set \mathbb{V}_{l_x} is a branched linear space in Σ_k , \mathbb{V}_y is a pseudo-linear space in $\Sigma_{k'}$ and L satisfies the following properties:

- (i) For every $\mathbf{v}, \mathbf{w} \in \mathbb{V}_{x,l}$ such that the vectors $\mathbf{v} + \mathbf{w}$ and $L(\mathbf{v}) + L(\mathbf{w})$ are well-defined, we have $L(\mathbf{v} + \mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w})$;
- (ii) For every $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{V}_{x,l}$ such that the vectors $\lambda \mathbf{v}$ and $L(\lambda \mathbf{v})$ are well-defined, we have $L(\lambda \mathbf{v}) = \lambda L(\mathbf{v})$;
- (iii) $L(a - x) = s - y$, where a is the origin of l_x , $a - x \in \mathbb{V}_x$ is the vector with origin at x and $s - y \in \mathbb{V}_y$ is the vector with origin at y .

Given two linear maps $L_1 : \mathbb{V}_{l_x} \rightarrow \mathbb{V}_y$ and $L_2 : \mathbb{V}_{l_y} \rightarrow \mathbb{V}_z$, there is a unique linear map $L_3 : \mathbb{V}_{l'_x} \rightarrow \mathbb{V}_z$ such that $L_3|_{\mathbb{V}_{l_x} \cap \mathbb{V}_{l'_x}} = L_2 \circ L_1$, where l'_x might be

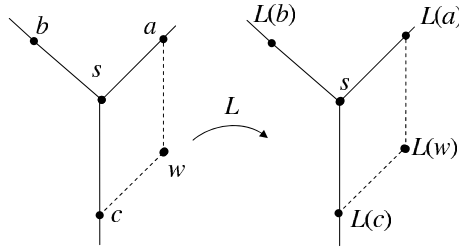


Fig. 14.8. Linear map at the singularity s .

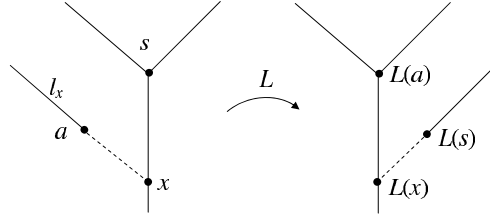


Fig. 14.9. Linear map at the point x .

distinct of l_x (see Figure 14.10). Hence, the *composition* $L_2 \circ L_1$ of two linear maps is well-defined by $L_3 = L_2 \circ L_1$, and so it is a linear map.

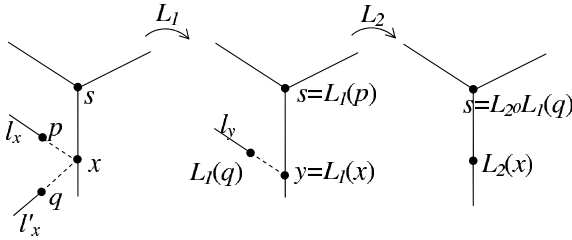


Fig. 14.10. The composition $L_3 = L_2 \circ L_1$ is well-defined.

A map $L_1 : \mathbb{V}_{l_x} \rightarrow \mathbb{V}_y$ is an *isomorphism* if, and only if, there is a linear map $L_2 : \mathbb{V}_{l_y} \rightarrow \mathbb{V}_x$ such that $L_2 \circ L_1|_{\mathbb{V}_{l_x} \cap L_1^{-1}(\mathbb{V}_{l_y})}$ and $L_1 \circ L_2|_{\mathbb{V}_{l_y} \cap L_2^{-1}(\mathbb{V}_{l_x})}$ are the identity maps. We note that if the linear map L_2 exists, then it is unique. Hence, the *inverse map* L_1^{-1} of L_1 is well-defined by $L_1^{-1} = L_2$. The kernel of a linear map $L : \mathbb{V}_{l_x} \rightarrow \mathbb{V}_y$ is equal to the intersection $\mathbb{V}_{l_x} \cap \mathbb{S}_x$ of a pseudo-linear subspace \mathbb{S}_x with \mathbb{V}_{l_x} .

We say that a vector $y - x$ has a *parallel transport* from x to z (see Figure 14.11), if there are a vector $w - z$, a constant λ , with $|\lambda| \leq 1$, and an isometry $i_{\mathbb{H}} : \mathbb{H} \rightarrow \Sigma_k$ with the following property: there are points $x_{\mathbb{H}}, y_{\mathbb{H}}, z_{\mathbb{H}}, w_{\mathbb{H}} \in \mathbb{H}$ such that

- (i) $w' - z = \lambda(w - z)$ and $y' - x = \lambda(y - x)$ are well-defined;

- (ii) $x = i_{\mathbb{H}}(x_{\mathbb{H}})$, $z = i_{\mathbb{H}}(z_{\mathbb{H}})$, $y' = i_{\mathbb{H}}(y_{\mathbb{H}})$ and $w' = i_{\mathbb{H}}(w_{\mathbb{H}})$;
- (iii) $w_{\mathbb{H}} - z_{\mathbb{H}} = y_{\mathbb{H}} - x_{\mathbb{H}}$;
- (iv) if $s \in l_{z,w} \setminus \{w\}$, then $s \in \text{int}l_{x,y}$.

The parallel transport is uniquely determined, if $s \notin l_{z,w} \setminus \{w\}$ or if $s \in l_{z,w} \setminus \{w\} \cap \text{int}l_{x,y}$. Let $\mathbb{V}_{x \rightarrow z}$ be the set of all vectors that have a parallel transport from x to z . The *parallel transport map* $\mathbb{P}_{x \rightarrow z} : \mathbb{V}_{x \rightarrow z} \rightarrow \mathbb{V}_z$ is well-defined by $\mathbb{P}_{x \rightarrow z}(u) = v$, where the vector v is the parallel transport of the vector u from x to z , when $\mathbb{V}_{x \rightarrow z}$ is non-empty.

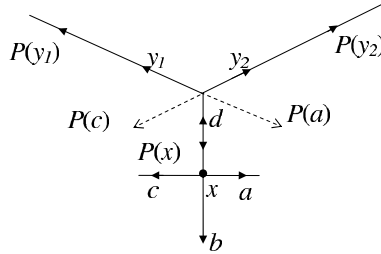


Fig. 14.11. Parallel transport from x to s .

We note that the parallel transport map $\mathbb{P}_{x \rightarrow z}$ is a linear map, except in the case where $x = s$ and $z \neq s$, because $\mathbb{P}_{s \rightarrow z}$ is just defined in an open 2π -angular region. However, $\mathbb{P}_{z \rightarrow s} : \mathbb{V}_{l_z} \rightarrow \mathbb{V}_s$ is a linear map and $\mathbb{P}_{s \rightarrow z} \circ \mathbb{P}_{z \rightarrow s}|_{\mathbb{V}_{l_z}}$ is the identity.

We say that a map $G : \mathbb{V}_{x_1}^m \rightarrow \mathbb{V}_y$ is an m -multilinear map, if, for every $(\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{0}, \mathbf{a}_{i+1}, \dots, \mathbf{a}_m)$, there is \mathbb{V}_{l_i} , where l_i depends upon $(\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{0}, \mathbf{a}_{i+1}, \dots, \mathbf{a}_m)$, such that the map $g : \mathbb{V}_{l_i} \rightarrow \mathbb{V}_y$ defined by $g(\mathbf{a}_i) = G(\mathbf{a}_1, \dots, \mathbf{a}_i, \dots, \mathbf{a}_m)$ is a linear map.

Lemma 14.1. *Let $L : \mathbb{V}_{x_1}^m \rightarrow \mathbb{V}_{y_1}$ be an m -multilinear map. Let x_2 and y_2 be such that the parallel transport maps $\mathbb{P}_{x_1 \rightarrow x_2}$ and $\mathbb{P}_{y_1 \rightarrow y_2}$ are well-defined. Suppose that if x_1 is a singularity with order k , then x_2 is a singularity with order $2nk$, for some $n \geq 1$. Then, there is an m -multilinear map $L_P : \mathbb{V}_{x_2}^m \rightarrow \mathbb{V}_{y_2}$ such that*

$$L_P(P_{x_1 \rightarrow x_2}(\mathbf{v}_1), \dots, P_{x_1 \rightarrow x_2}(\mathbf{v}_m)) = P_{y_1 \rightarrow y_2}(L(\mathbf{v}_1, \dots, \mathbf{v}_m)),$$

whenever both sides are well-defined.

We call the above linear map L_P the *parallel transport of L from (x_1, y_1) to (x_2, y_2)* . We note that the parallel transport L_P of L is an isomorphism.

Proof. The map $P_{y_1 \rightarrow y_2} \circ L_1 \circ P_{x_1 \rightarrow x_2}^{-1}$ has a unique extension to a linear map.

□

Let $L_1 : \mathbb{V}_{x_1}^m \rightarrow \mathbb{V}_{y_1}$ and $L_2 : \mathbb{V}_{x_1}^m \rightarrow \mathbb{V}_{y_2}$ be two m -multilinear maps. Let $0 \leq h \leq 1$ be such that $L_1(\mathbf{v})$ and $L_2(\mathbf{v})$ are well-defined, for all \mathbf{v} with $\|\mathbf{v}\| = h$, and such that there is $\mathbf{w}(\mathbf{v})$ with the property that $\mathbf{w}(\mathbf{v}) + L_1(\mathbf{v}) = L_2(\mathbf{v})$. We define the *distance* $d(L_1, L_2)$ between the m -multilinear maps L_1 and L_2 as follows:

$$d(L_1, L_2) = \begin{cases} +\infty, & \text{if } h = 0 \\ \max_{\mathbf{v}} \frac{\|\mathbf{w}(\mathbf{v})\|}{h}, & \text{otherwise} \end{cases}$$

Let $L_1 : \mathbb{V}_{x_1}^m \rightarrow \mathbb{V}_{y_1}$ and $L_2 : \mathbb{V}_{x_2}^m \rightarrow \mathbb{V}_{y_2}$ be two m -multilinear maps. Let \mathbb{L} be the set of all parallel transport L_P of L_2 from (x_2, y_2) to (x_1, y_1) . We define the *distance* $d(L_1, L_2)$ between the m -multilinear maps L_1 and L_2 as follows:

$$d(L_1, L_2) = \begin{cases} +\infty, & \text{if } \mathbb{L} = \emptyset \\ \min_{L_P \in \mathbb{L}} d(L_1, L_P), & \text{otherwise} \end{cases}$$

We note that $d(L_1, L_2) = d(L_2, L_1)$.

14.4 Pseudo-differentiable maps

Let $f : A \subset \Sigma_k \rightarrow \Sigma_{k'}$ be a map defined on an open neighbourhood A of x in Σ_k . We say that the map f is *pseudo-differentiable at x* , if there is a linear map $D_x f : \mathbb{V}_{l_x} \rightarrow \mathbb{V}_{f(x)}$ with the following property: For all $\mathbf{v} \in \mathbb{V}_{l_x}$, there exists a constant $h_0 > 0$ such that there is a unique vector $\mathbf{w}(h, \mathbf{v})$ satisfying

$$\mathbf{w}(h, \mathbf{v}) + f(x) = f(x + h\mathbf{v}),$$

for all $0 < h < h_0$, and

$$D_x f(\mathbf{v}) = \lim_{h \rightarrow 0} \frac{\mathbf{w}(h, \mathbf{v})}{h}.$$

By induction, let us suppose that the $(m-1)^{\text{th}}$ -derivative $D_x^{m-1} f : \mathbb{V}_x^m \rightarrow \mathbb{V}_{f(x)}$ of f is well-defined in an open set A containing x . We say that f is *m -pseudo-differentiable at x* , if there is an m -multilinear map

$$D_x^m f : \mathbb{V}_x^m \rightarrow \mathbb{V}_{f(x)}$$

with the following property: For all $\mathbf{v} \in \mathbb{V}_x^m$, there exists a constant $h_0(\mathbf{v}) > 0$ such that there is a unique vector $\mathbf{w}(h, \mathbf{v})$ satisfying

$$\mathbf{w}(h, \mathbf{v}_1, \dots, \mathbf{v}_m) + D_x^{m-1} f(\mathbf{v}_1, \dots, \mathbf{v}_m) = D_{x+h\mathbf{v}_1}^{m-1} f(\mathbf{v}_2, \dots, \mathbf{v}_m),$$

for all $0 < h < h_0(\mathbf{v})$, and

$$D_x^m f(\mathbf{v}_1, \dots, \mathbf{v}_m) = \lim_{h \rightarrow 0} \frac{1}{h} \mathbf{w}(h, \mathbf{v}_1, \dots, \mathbf{v}_m).$$

A map $f : A \rightarrow \Sigma_{k'}$ is C^m , with $m \in \mathbb{N}$, in the open set $A \subset \Sigma_k$, if f is m -differentiable for all $x \in A$, and the m -derivative $D_x f$ varies continuously

with x . We say that f is a $C^{m+\alpha}$, with $m \in \mathbb{N}$ and $0 < \alpha \leq 1$, if f is C^m and there exists $c > 0$ such that

$$\|D_x f - D_y f\| \leq c\|x - y\|^\alpha,$$

for all $x, y \in A$ with the property that there is a parallel transport L_p from x to y .

We say that $B_\varepsilon = B_\varepsilon(x, s) \subset A$ is an *avoid singularity cone*, if $d(x, y) = \varepsilon d(x, s)$ and $\alpha = d(x, s)/\varepsilon$ (see Figure 14.12).

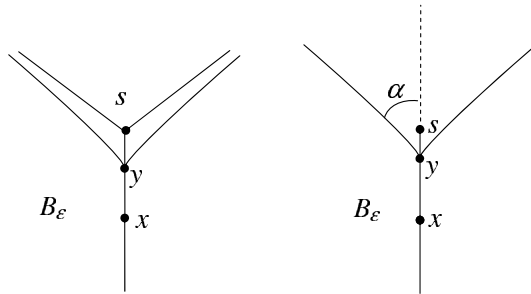


Fig. 14.12. Avoid singularity cone.

Theorem 14.2. (*Taylor's Theorem*) Let $f : A \subset \Sigma_k \rightarrow \Sigma_k$ be a C^m pseudo-map defined on an open set A . Let $B_\varepsilon \subset A$ be an avoid singularity cone and $0 < \varepsilon < 1$ small. Then, for all $x, y \in B_\varepsilon$ with $\|y - x\| \leq \varepsilon$, the vectors $\mathbf{z}_m(x, y)$ and $\mathbf{w}_m(x, y)$ are well-defined by

$$\begin{aligned} \mathbf{z}_m(x, y) &= (\dots (D_x f(y - x) + D_x^2 f(y - x, y - x)) + \dots) + \\ &\quad + \frac{1}{m!} D_x^m f(y - x, \dots, y - x) \\ f(y) - f(x) &= \mathbf{z}_m(x, y) + \mathbf{w}_m(x, y). \end{aligned}$$

Furthermore,

$$\|\mathbf{w}_m(x, y)\| \leq \chi(\|y - x\|)\|y - x\|^m,$$

where $\chi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is a continuous map with $\chi(0) = 0$.

Let l_1, \dots, l_{2k} be semi-straight lines with origin at s such that $0 < \angle(l_i, l_{i+1}) < \pi$ and $\angle(l_i, l_{i+2}) = \pi$ for every $i \in \mathbb{Z}_{2k}$. Then, $S_s^1 = \cup_{i \in \mathbb{Z}_{2k}} l_{2i}$ and $S_s^2 = \cup_{i \in \mathbb{Z}_{2k}} l_{2i+1}$ are pseudo-linear subspaces at the singularity s . We call the direct sum $S_s^1 \oplus S_s^2$ of S_s^1 and S_s^2 to the set of all pairs (\mathbf{u}, \mathbf{v}) of vectors with the property that if $\mathbf{u}_i \in l_i$, then $\mathbf{u}_{i+1} \in l_{i+1}$, for all $i \in \mathbb{Z}_{2k}$. By construction, there are one-to-one maps

$$\begin{aligned} \Theta_1 : \mathbb{V}_s &\rightarrow S_s^1 \oplus S_s^2 \\ \Theta_2 : \Sigma_k &\rightarrow S_s^1 \oplus S_s^2 \end{aligned}$$

given by $\Theta_1^{-1}(\mathbf{u}, \mathbf{v}) = \mathbf{u} + \mathbf{v}$ and $\Theta_2^{-1}(\mathbf{u}, \mathbf{v}) = (\mathbf{u} + \mathbf{v}) + s$. We say that $\langle (\mathbf{u}_1, \dots, \mathbf{u}_{2k-1}), (\mathbf{u}_2, \dots, \mathbf{u}_{2k}) \rangle$ is a *basis* of \mathbb{V}_s , if $\mathbf{u}_i \in l_i$ and $\mathbf{u}_i + \mathbf{u}_{i+2} = \mathbf{0}$, for every $i \in \mathbb{Z}_{2k}$ (see Figures 14.13 and 14.14).

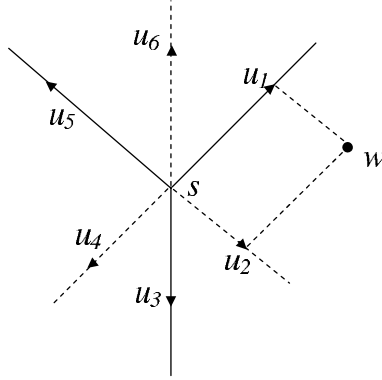


Fig. 14.13. $\mathbf{u}_1 + \mathbf{u}_2 = \mathbf{w}$ and $\langle (\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_5), (\mathbf{u}_2, \mathbf{u}_4, \mathbf{u}_6) \rangle$ is a basis of \mathbb{V}_s .

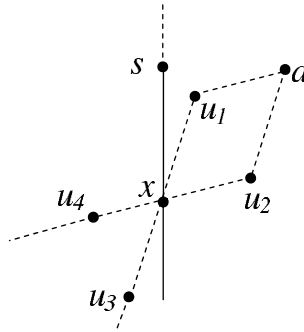


Fig. 14.14. $\mathbf{u}_1 + \mathbf{u}_2 = \mathbf{a}$ and $\langle (\mathbf{u}_1, \mathbf{u}_3), (\mathbf{u}_2, \mathbf{u}_4) \rangle$ is a basis of \mathbb{V}_x .

For every $i \in \mathbb{Z}_{2k}$, let $\mathbf{u}_i \in l_i$ be such that $\|\mathbf{u}_i\| = 1$. Let $D_{K_i} = \mathbb{R}^2 \setminus ((-\infty, 0) \times \{0\})$. We define the map $K_i : D_{K_i} \rightarrow \mathbb{V}_s$ at the singularity by

$$K_i(a, b) = \begin{cases} a\mathbf{u}_i + b\mathbf{u}_{i+1}, & \text{if } a, b \geq 0 \\ a\mathbf{u}_i + b\mathbf{u}_{i-1}, & \text{if } a \geq 0, b \leq 0 \\ a\mathbf{u}_{i+2} + b\mathbf{u}_{i+1}, & \text{if } a \leq 0, b > 0 \\ a\mathbf{u}_{i-2} + b\mathbf{u}_{i-1}, & \text{if } a \leq 0, b < 0 \end{cases}$$

The set of maps K_1, \dots, K_{2k} is called a *coordinate system* for \mathbb{V}_s ($\cong \Sigma_k$) given by $S_1 \oplus S_2$.

Lemma 14.3. *Let K_1, \dots, K_{2k} be a coordinate system for Σ_k .*

- (i) Let $L : V_{l_s} \rightarrow V_{l'_s}$ be a linear map at the singularity. Then, there is a unique linear map $L' : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $L'(a, b) = K_j^{-1} \circ L \circ K_i = (a, b)$, where $j = j(i, a, b)$ has the property that $L \circ K_i(a, b) \in D_{K_j}$.
- (ii) A map $f : A \rightarrow \Sigma_{k'}$ is C^r on $A \subset \Sigma_k$ if, and only if, $K_j^{-1} \circ f \circ K_i$ is C^r , where $j = j(i, a, b)$ has the property that $f \circ K_i(a, b) \in D_{K_j}$.

14.4.1 C^r pseudo-manifolds

Let M be a topological space. A *chart* $c : U \rightarrow \Sigma_k$ is a homeomorphism onto its image defined on an open set U of M (recall that $\Sigma_2 = \mathbb{R}^2$). If $k \neq 2$, then we call $c : U \rightarrow \Sigma_k$ a *singular chart*. A *topological atlas* \mathcal{A} of M is a collection of charts

$$c_x : U_x \rightarrow \Sigma_{k_x}$$

such that the union $\cup_{x \in M} U_x$ of the open sets cover M . A C^r *pseudo-atlas* \mathcal{A} of M is a topological atlas \mathcal{A} of M with the following properties: (i) \mathcal{A} has just a finite set of singular charts; (ii) the overlap maps

$$c_x \circ c_y^{-1} : c_y(U_x \cap U_y) \rightarrow c_x(U_x \cap U_y)$$

are C^r diffeomorphisms. A topological space M with a C^r pseudo-atlas \mathcal{A} is called a C^r *pseudo-manifold*, that we will denote by the pair (M, \mathcal{A}) . A topological space N contained in a C^r manifold (M, \mathcal{A}) is a *pseudo-submanifold* of M , if there is a collection \mathcal{B} of charts

$$e_x : V_x \rightarrow \Sigma_{k_x}$$

with the following properties (see Figure 14.15):

- (i) The set N is contained in the union $\cup_{x \in N} V_x$;
- (ii) For all $x \in N$, $e_x(N \cap V_x)$ is the intersection of a pseudo-linear subspace $S_{e_x(x)}$ at $e_x(x)$ with an open set of M ;
- (iii) The dimension of $S_{e_x(x)}$ is 1;
- (iv) The overlap maps

$$e_x \circ c_x^{-1} : c_x(U_x \cap V_x) \rightarrow e_x(U_x \cap V_x)$$

between the charts $c_x \in \mathcal{A}$ and $e_x \in \mathcal{B}$ are C^r diffeomorphisms.

Hence, the first derivative at every point is locally a bijection over a corresponding pseudo-linear subspace with dimension 1. We call the above charts e_x the *submanifold charts* of N .

Definition 14.4. Let (M, \mathcal{A}) and (M', \mathcal{A}') be C^r manifolds. The map $f : M \rightarrow M'$ is *pseudo C^r* if, and only if, the maps $c_x \circ f \circ e_y^{-1}$ are C^r with respect to charts $c_x \in \mathcal{A}$ and $e_y \in \mathcal{A}'$. The map $f : M \rightarrow M'$ is C^r pseudo-diffeomorphism if, and only if, $f : M \rightarrow M'$ is a homeomorphism and the maps $c_x \circ f \circ c_y^{-1}$ are C^r with respect to charts $c_x \in \mathcal{A}$ and $c_y \in \mathcal{A}'$.

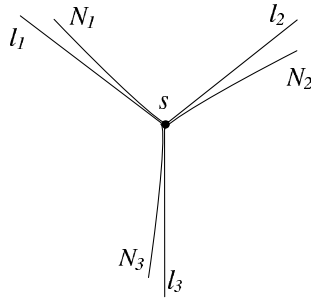


Fig. 14.15. The full subspace $\mathbb{S}_s = \cup_{i=1}^3 l_i$ at the singularity, and the pseudo-submanifold $N = \cup_{i=1}^3 N_i$.

14.4.2 Pseudo-tangent spaces

The *pseudo-tangent fiber bundle* $T\Sigma_k$ of Σ_k is the set $\cup_{x \in \Sigma_k} \mathbb{V}_x$, with the natural induced topology by Σ_k . We also call the pseudo-linear space \mathbb{V}_x at x the *pseudo-tangent space* $T_x \Sigma_k$ at x ($T_x \Sigma_k \cong \mathbb{V}_x$).

The *pseudo-tangent space* $T_x M$ at $x \in M$ of a C^r pseudo-manifold (M, \mathcal{A}) is a pseudo-linear space isomorphic to $T_{c_x(x)} \Sigma_{k_x}$, where $c_x : U_x \rightarrow \Sigma_{k_x}$ is a chart in \mathcal{A} with $x \in U_x$. The *tangent fiber bundle* TM of a C^r manifold (M, \mathcal{A}) is the topological set $\cup_{x \in M} T_x M$, with the induced topology by the topological sets

$$\cup_{x \in U_x} T_{c_x(x)} \Sigma_{k_x}.$$

The *tangent space* $T_x N$ at $x \in N$ of a C^r submanifold N of M is a pseudo-linear subspace $T_x N \subset T_x M$ isomorphic to the pseudo-linear subspace $S_{e_x(x)}$ at $e_x(x)$. The *tangent fiber subbundle* $TN \subset TM$ of a C^r submanifold N of M is the topological set $\cup_{x \in N} T_x N$.

14.4.3 Pseudo-inner product on Σ_k

Let $I_x \subset \mathbb{V}_x \times \mathbb{V}_x$ be the set of all pairs $(\mathbf{u}, \mathbf{v}) \in I_x$ such that $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \pi$. A *pseudo-inner product*

$$i : I_x \rightarrow \mathbb{R}$$

at a point $x \in \Sigma_k$ is a bi-linear map with the following properties:

- $i(\mathbf{u}, \mathbf{v}) = i(\mathbf{v}, \mathbf{u})$, for all $(\mathbf{u}, \mathbf{v}) \in I_x$;
- $i(\mathbf{u}, \mathbf{u}) \geq 0$, for all $(\mathbf{u}, \mathbf{u}) \in I_x$;
- $i(\mathbf{u}, \mathbf{u}) = 0$ if, and only if, $\mathbf{u} = 0$ ($= x - x$).

A C^r *pseudo-Riemannian metric* in an open set $U \subset \Sigma_k$ is a map

$$\langle, \rangle : \cup_{x \in U} I_x \rightarrow \mathbb{R}$$

with the following properties:

- $\langle, \rangle_x = \langle, \rangle |I_x$ is an inner product;
- For every isometry $i_{\mathbb{H}} : \mathbb{H} \rightarrow \Sigma_k$, the pullback by $i_{\mathbb{H}}$

$$\langle y - x, z - x \rangle_{x, \mathbb{H}} = \langle i_{\mathbb{H}}(y) - i_{\mathbb{H}}(x), i_{\mathbb{H}}(z) - i_{\mathbb{H}}(x) \rangle_{i_{\mathbb{H}}(x)}$$

of the inner products $\langle, \rangle_{i_{\mathbb{H}}(x)}$ in U induces a C^r Riemannian metric in $i_{\mathbb{H}}^{-1}(U)$.

Let (M, \mathcal{A}) be a C^r manifold. Let $J_x \subset T_x M \times T_x M$ be the pull-back by the derivative of the chart $c_i : U_i \rightarrow \Sigma_{k_i}$ in \mathcal{A} of $I_{c_i(x)}$. A C^r pseudo-Riemannian metric in a C^r manifold (M, \mathcal{A}) is a map

$$\langle, \rangle : \cup_{x \in M} J_x \rightarrow \mathbb{R}$$

such that, for every chart $c_i : U_i \rightarrow \Sigma_{k_i}$ in \mathcal{A} , the push-forward of \langle, \rangle is a C^r Riemannian metric $\langle, \rangle_{c_i(U_i)}$ in $c_i(U_i)$.

We say that (x, \mathbf{u}_x) and (x, \mathbf{v}_x) are *direction equivalent* $(x, \mathbf{u}_x) \sim (x, \mathbf{v}_x)$ if, and only if, \mathbf{u}_x and \mathbf{v}_x belong to a same dimension 1 full subspace \mathbb{S}_x . $T\Sigma_k / \sim$ is the [direction set. A C^r *direction field* is a continuous map

$$\phi : \Sigma_k \rightarrow T\Sigma_k / \sim$$

such that for every isometry $i_{\mathbb{H}} : \mathbb{H} \rightarrow \Sigma_k$, the map $\hat{\phi} : \mathbb{H} \rightarrow T\mathbb{H} / \sim$ given by $\hat{\phi} = di_{\mathbb{H}}^{-1} \circ \phi \circ i_{\mathbb{H}}$ is C^r .

A C^r *splitting* is a pair (ϕ_s, ϕ_u) of C^r direction fields such that, for every $x \in \Sigma_k$, we have

$$\mathbb{V}_x = \mathbb{S}_{\phi_s(x)} \oplus \mathbb{S}_{\phi_u(x)},$$

where $\mathbb{S}_{\phi_{\iota}(x)}$ is a dimension 1 full subspace containing $\phi_{\iota}(x)$.

Definition 14.5. Let (M, \mathcal{A}) be a C^r pseudo-manifold with a pseudo-Riemannian metric. A C^r pseudo-diffeomorphism $f : M \rightarrow M$ is a C^r pseudo-Anosov diffeomorphism, if M has a 1 dimensional smooth splitting $E^s \oplus E^u$ of the tangent bundle, with the following properties: (i) the splitting is invariant under Tf , and (ii) Tf expands uniformly E^u and contracts uniformly E^s .

The set of all C^r pseudo-Anosov diffeomorphisms on M is an open set.

Theorem 14.6. (Stable Manifold Theorem) If $f : M \rightarrow M$ is a C^r pseudo-Anosov diffeomorphism, then the stable and unstable sets at the points of Λ are C^r pseudo-submanifolds with dimension 1.

Proof. First, we prove that the stable and unstable sets through the singularities are C^r pseudo-submanifolds. Then, we prove that the stable and unstable sets through the other points are also C^r pseudo-submanifolds. The singularities are periodic points, because f is a pseudo-diffeomorphism and so the image of a singularity is a singularity with the same order. Let us construct the unstable manifold at the singularity s (for simplicity $f(s) = s$). Let

$E_{1,cut}, \dots, E_{k,cut}$ at a singularity s be the *cut sets* represented in Figure 14.16. By the Whitney's extension theorem, there is a C^r diffeomorphism F_1 on the plane such that $F_1|_{E_{1,cut}} = f$. By the Hirsch and Pugh [48] Stable Manifold Theorem, the unstable set passing through $(0,0)$ of F_1 is a C^r submanifold $W^u = W_1^u \cup W_2^u$. Doing the same with respect to $E_{i,cut}$, we get that the unstable set

$$W^u(s) = \bigcup_{i=1}^k W_i^u$$

at the singularity is a C^r submanifold tangent to the unstable subspace (see Figure 14.17).

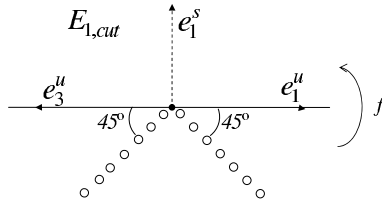


Fig. 14.16. A $E_{1,cut}$ cut set at a singularity.

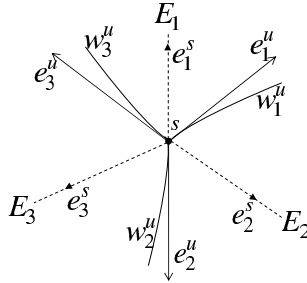


Fig. 14.17. The unstable set at a singularity $s \in \Sigma_3$.

Away from the singularities, let $(x_n)_{n \in \mathbb{Z}}$ be an orbit of f . If $x_n \in E_{i_n,cut}$, then we take the C^r diffeomorphism F_{i_n} such that $F_{i_n}|_{E_{i_n,cut}} = f$ in a neighbourhood of x_n . Applying the Hirsch and Pugh [48] Stable Manifold Theorem to this orbit, we get that the unstable set at every point of the orbit is a C^r submanifold tangent to the unstable subspace. \square

14.5 C^r foliations

A C^{1+} pseudo-foliation satisfies the properties of a C^r foliation with the extra *turntable condition* that we now describe. If s is a singularity, with order $k = k(s)$, then a singular leaf W^ι on M , containing s , is such that $W^\iota \setminus \{s\}$ is the union of k disjoint leaves ℓ_j^ι , $j \in \mathbb{Z}_k$, whose closures intersect in s . The components $\ell_1^\iota, \dots, \ell_k^\iota$ of $W^\iota(s, \varepsilon) \setminus s$ are called *separatrices* of s . We call W^ι a *singular spinal set* and call the sets ℓ_j^ι *emph* the separatrices of s .

A C^{1+} foliation satisfies the *turntable condition*: if for all singular spinal sets W^ι with separatrices ℓ_j^ι , $j \in \mathbb{Z}_k$, there are leaf charts (i_j, ℓ_j^ι) , such that the maps defined by $i_{j,l}|\ell_j^\iota = -i_j$ and $i_{j,l}|\ell_l^\iota = i_l$ are smooth. A C^{1+} foliation induced by a C^{1+} pseudo-Anosov diffeomorphism satisfies the turntable condition (see Pinto and Rand [160]).

The HR structures and the solenoid functions also apply to C^r pseudo-Anosov diffeomorphisms with the extra turntable condition that we now describe.

For any triple (v_1, v_2, v_3) of points v_1, v_2 and v_3 contained in same ι -leaf, we define the *solenoid limit* $s_\iota^z(v_1, v_2, v_3)$ as follows. For all $i \geq 0$, let

$$(z_1^i, z_2^i, z_3^i), (z_2^i, z_3^i, z_4^i), \dots, (z_{n_i-2}^i, z_{n_i-1}^i, z_{n_i}^i) \in \text{sol}^\iota$$

be a sequence of triples such that for some $1 < j_i < n_i$

$$v_1 = \lim_{i \rightarrow \infty} f_\iota^i(z_1^i) \quad , \quad v_2 = \lim_{i \rightarrow \infty} f_\iota^i(z_{j_i}^i) \quad \text{and} \quad v_3 = \lim_{i \rightarrow \infty} f_\iota^i(z_{n_i}^i).$$

The solenoid *limit* $s_\iota^z(v_1, v_2, v_3)$ is equal to

$$s_\iota^z(v_1, v_2, v_3) = \frac{\sum_{j=j_i-1}^{n_i-2} (s_\iota(z_1, z_2, z_3) \dots s_\iota(z_j, z_{j+1}, z_{j+2}))}{\sum_{j=1}^{j_i-2} (s_\iota(z_1, z_2, z_3) \dots s_\iota(z_j, z_{j+1}, z_{j+2}))}.$$

For all singularities s , with order $k = k(s)$, and for all $i \in \mathbb{Z}_k$, let $a_i = (v_i, s, v_{i+1})$ be a triple contained in a leaf ℓ_i^ι which intersects an ι' boundary of a Markov rectangle just in the points v_i and v_{i+1} or in the points v_i, s and v_{i+1} . The limit solenoids $s_\iota^{z_i}(a_i)$ satisfy the following *turntable condition*:

$$\prod_{i=1}^k s_\iota^{z_i}(a_i) = 1.$$

If $k(s) = 1$ and $v_1 = v_2$, then $s_\iota^z(v_1, s, v_2) = 1$.

The solenoid functions determined by C^r pseudo-Anosov diffeomorphisms satisfy the turntable condition (see Pinto and Rand [160]).

The train-tracks and the self-renormalizable structures also apply to C^r pseudo-Anosov diffeomorphisms with the extra turntable condition that we now describe.

A C^{1+} atlas \mathcal{B} satisfies the *turntable condition* at a singularity s , with order $k = k(s)$: if for all singular spinal sets on the train-track with separatrices ℓ_j^ℓ , $j \in \mathbb{Z}_k$, there are leaf charts (i_j, ℓ_j^ℓ) , such that the maps defined by $i_{j,l}|_{\ell_j^\ell} = -i_j$ and $i_{j,l}|_{\ell_l^\ell} = i_l$ are smooth.

A C^{1+} foliation induced by a C^{1+} pseudo-Anosov determines a C^{1+} train-track atlas satisfying the turntable condition that comes from the turntable condition of a C^{1+} foliation. For example, let s be a singularity with order 3, as in Figure 14.1. The Markov partition determines a singular spinal set S^ℓ with separatrices ℓ_j^ℓ , $j \in \mathbb{Z}_3$, such that there are train-track charts (i_j, ℓ_j^ℓ) , whose maps defined by $i_{j,l}|_{\ell_j^\ell} = -i_j$ and $i_{j,l}|_{\ell_l^\ell} = i_l$ are smooth.

14.6 Further literature

The theory developed in this book has a natural extension to C^r pseudo-Anosov diffeomorphisms using the turntable conditions (see Pinto and Rand [160]). Pinto and Pujals [155] relate the pseudo-Anosov diffeomorphisms with the Pujals and Sambarino [181, 184] non-uniformly hyperbolic diffeomorphisms. The symplectic forms are defined similarly to the Riemannian metric. Let (M, \mathcal{A}) be a C^r pseudo-manifold with a pseudo-volume form ω . Pinto and Viana [176] proved that there is a residual set \mathcal{R} contained in the set of all C^1 pseudo-diffeomorphisms, preserving the volume form, such that if $f \in \mathcal{R}$, then either f is a C^1 pseudo-diffeomorphism or has almost everywhere both Lyapunov exponents zero. In that way we recover the duality given by Mañé-Bochi Theorem in the torus to the other surfaces. This chapter is based on Pinto [152] and Pinto and Rand [160].

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