2 Leptonic Interactions

2.1 The Current–Current Interaction (Charged Currents)

Let us first consider only the weak interactions between leptons. Today three leptonic hierarchies (e, μ, τ) are known; the experimental data are listed in Table 2.1. To recall, parity violation in nuclear β decay suggested an interaction of the form (see (1.30))

$$H_{\text{int}} = \frac{G}{\sqrt{2}} \int \mathrm{d}^3x \left[\bar{u}_{\mathrm{p}}(x)\gamma_\alpha(C_V + C_A\gamma_5)u_{\mathrm{n}}(x)\right]\left[\bar{u}_{\mathrm{e}}(x)\gamma^\alpha(1-\gamma_5)u_{\nu_{\mathrm{e}}}(x)\right] \tag{2.1}$$

where the leptonic contribution

$$\bar{u}_{\mathrm{e}}(x)\gamma^\alpha(1-\gamma_5)u_{\nu_{\mathrm{e}}}(x) \tag{2.2}$$

contains terms that resemble the electromagnetic current

$$j^\alpha(x) = \mathrm{e}\bar{\Psi}(x)\gamma^\alpha\Psi(x) \quad . \tag{2.3}$$

Table 2.1. Experimental data for leptons

Lepton	e	ν_{e}	μ	ν_μ	τ	ν_τ
mass (MeV)	0.511	$<17 \times 10^{-6}$	105.66	<0.27	1784 ± 4	<35
lifetime (s)	∞	∞	2.2×10^{-6}	∞?	3×10^{-13}	?

By analogy with the electromagnetic current, we therefore introduce the total *weak leptonic current* by adding the currents of the three leptonic families:

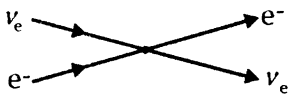
$$\begin{aligned} J_\alpha^{(\mathrm{L})}(x) &= \bar{u}_{\mathrm{e}}(x)\gamma_\alpha(1-\gamma_5)u_{\nu_{\mathrm{e}}}(x) + \bar{u}_\mu(x)\gamma_\alpha(1-\gamma_5)u_{\nu_\mu}(x) \\ &\quad + \bar{u}_\tau(x)\gamma_\alpha(1-\gamma_5)u_{\nu_\tau}(x) \\ &= J_\alpha^{(\mathrm{e})}(x) + J_\alpha^{(\mu)}(x) + J_\alpha^{(\tau)}(x) \quad . \end{aligned} \tag{2.4}$$

To describe the mutual weak interaction of leptons we generalize (2.1) by postulating that

$$H_{\text{int}}^{(\mathrm{L})} = \frac{G}{\sqrt{2}} \int \mathrm{d}^3x\, J_\alpha^{(\mathrm{L})\dagger}(x) J_{(\mathrm{L})}^\alpha(x) \quad . \tag{2.5}$$

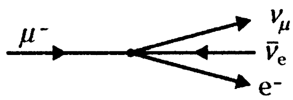
The consequences of this step are non-trivial. Since $H_{\text{int}}^{(\mathrm{L})}$ is quadratic in $J_\alpha^{(\mathrm{L})}$, each leptonic hierarchy interacts with itself as well as with each of the other two. The following diagrams are some examples for such possible processes (see also Exercise 2.1).

W. Greiner, B. Müller, *Gauge Theory of Weak Interactions*,
DOI 10.1007/978-3-540-87843-8_2, © Springer-Verlag Berlin Heidelberg 2009



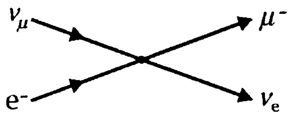
Neutrino–electron scattering:

$$J_\alpha^{(e)\dagger} J_{(e)}^\alpha = \left[\bar{u}_{\nu_e}\gamma_\alpha(1-\gamma_5)u_e\right]\left[\bar{u}_e\gamma^\alpha(1-\gamma_5)u_{\nu_e}\right] \quad .$$



Muon decay:

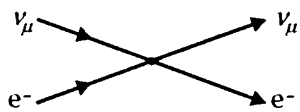
$$J_\alpha^{(\mu)\dagger} J_{(e)}^\alpha = \left[\bar{u}_{\nu_\mu}\gamma_\alpha(1-\gamma_5)u_\mu\right]\left[\bar{u}_e\gamma^\alpha(1-\gamma_5)u_{\nu_e}\right] \quad .$$



Muon production in muon-neutrino–electron scattering:

$$J_\alpha^{(e)\dagger} J_{(\mu)}^\alpha = \left[\bar{u}_{\nu_e}\gamma_\alpha(1-\gamma_5)u_e\right]\left[\bar{u}_\mu\gamma^\alpha(1-\gamma_5)u_{\nu_\mu}\right] \quad .$$

On the other hand, a process like



is *not* allowed. This means that ν_μ and e can interact *only* via the creation of a muon, which is an immediate consequence of the specific form of the currents $J_\mu^{(i)}$, allowing for a neutrino converting into a charged lepton (or vice versa!), but prohibiting an interaction without a conversion of particles. This property of the interaction is usually expressed by calling the currents (2.4) *charged currents* (more accurate by *charged transition currents*) since the charge of the particle of a particular leptonic hierarchy changes by one unit. In the electromagnetic current (2.3) the charge of the particle does not change, it is therefore called a *neutral current*. We shall later see that neutral currents also appear in the context of the gauge theory of weak interaction.

EXERCISE

2.1 Neutrino–Electron Exchange Current

Problem. Prove that $J_{(e)}^{\mu\,\dagger} = \bar{u}_{\nu_e}\gamma_\mu(1-\gamma_5)u_e \quad .$

Solution. With $\gamma_5{}^\dagger = \gamma_5$ we find

$$\begin{aligned} J_{(e)}^{\mu\,\dagger} &= \left[\bar{u}_e\gamma^\mu(1-\gamma_5)u_{\nu_e}\right]^\dagger \\ &= u_{\nu_e}{}^\dagger(1-\gamma_5)\gamma^{\mu\dagger}\bar{u}_e^\dagger \\ &= \bar{u}_{\nu_e}\gamma^0(1-\gamma_5)\gamma^{\mu\dagger}\gamma^{0\dagger}u_e \quad . \end{aligned} \tag{1}$$

Using the identity

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \quad , \tag{2}$$

that is, $\gamma^{i\,\dagger} = -\gamma^i$, $\gamma^{0\dagger} = \gamma^0$, yields the desired result:

$$J_{(e)}^{\mu\,\dagger} = \bar{u}_{\nu_e}\gamma^0(1-\gamma_5)\gamma^0\gamma^\mu u_e$$

$$
\begin{aligned}
&= \bar{u}_{\nu_e}(1+\gamma_5)\gamma^\mu u_e \\
&= \bar{u}_{\nu_e}\gamma^\mu(1-\gamma_5)u_e \quad ,
\end{aligned} \tag{3}
$$

where we have used the fact that γ_5 anticommutes with all other γ matrices.

2.2 The Decay of the Muon

Of all pure leptonic processes, muon decay was the first to be investigated with high accuracy. Muon decay occurs because of the general hypothesis (2.5) for the weak interaction of leptons. Its observation, therefore, is a very important check of the generalization (2.5) of the original Fermi theory of weak interactions. It is therefore appropriate to begin our study with this particular process. Since the decay implies a change in the state of the muon, and because the interaction that causes it is weak, it can be described in the framework of time-dependent perturbation theory.

The quantum mechanical wavefunction obeys a Schrödinger equation,

$$
\mathrm{i}\frac{\partial \Psi(\boldsymbol{x},t)}{\partial t} = \hat{H}(\boldsymbol{x},t)\Psi(\boldsymbol{x},t)
$$

which – after eliminating the space coordinates $\boldsymbol{x}$ – simply reads

$$
\mathrm{i}\frac{\partial \Psi(t)}{\partial t} = \hat{H}(t)\Psi(t) \quad . \tag{2.6}
$$

We now study the time development appropriate for our case (2.5) of weak interaction. Starting at t_0 with the initial wavefunction $\Psi_\mathrm{i} = \Psi(t_0)$, we obtain after a time step Δt_0

$$
\begin{aligned}
\Psi(t_1) &= \Psi(t_0+\Delta t_0) = \Psi(t_0) - \mathrm{i}\Delta t_0 \hat{H}(t_0)\Psi(t_0) \\
&= (1-\mathrm{i}\hat{H}(t_0)\Delta t_0)\Psi(t_0) \quad .
\end{aligned}
$$

After a next time step Δt_1 we get

$$
\begin{aligned}
\Psi(t_2) &= \Psi(t_0+\Delta t_0+\Delta t_1) = \Psi(t_1) - \mathrm{i}\Delta t_1 \hat{H}(t_1)\Psi(t_1) \\
&= \Psi(t_0) - \mathrm{i}\Delta t_0 \hat{H}(t_0)\Psi(t_0) \\
&\qquad - \mathrm{i}\Delta t_1 \hat{H}(t_1)\Psi(t_1) \quad ,
\end{aligned}
$$

and after N steps

$$
\begin{aligned}
\Psi(t) = \Psi(t_N) &= \Psi(t_0+\Delta t_0+\Delta t_1+\cdots+\Delta t_{N-1}) \\
&= \Psi(t_0) - \mathrm{i}\Delta t_0 \hat{H}(t_0)\Psi(t_0) \\
&\qquad - \mathrm{i}\Delta t_1 \hat{H}(t_1)\Psi(t_1) \\
&\qquad \vdots \\
&\qquad - \mathrm{i}\Delta t_{N-1} \hat{H}(t_{N-1})\Psi(t_{N-1})
\end{aligned}
$$

$$
\begin{aligned}
&= \Psi(t_0) - \mathrm{i} \sum_{i=0}^{N-1} \Delta t_i \hat{H}(t_i) \Psi(t_i) \\
&\approx \Psi(t_0) - \mathrm{i} \int_{t_0}^{t} \hat{H}(t') \Psi(t') \mathrm{d}t' \quad ,
\end{aligned} \tag{2.7}
$$

where higher-order terms in $\hat{H}$ were neglected.

Now, the S-matrix element S_{fi} for a transition from an initial state $|\Psi_{\mathrm{i}}\rangle$ to a final state $|\Psi_{\mathrm{f}}\rangle \neq |\Psi_{\mathrm{i}}\rangle$ is defined as[1]

$$
\begin{aligned}
S_{\mathrm{fi}} &= \lim_{t \to \infty} \langle \Psi_{\mathrm{f}} | \Psi_{\mathrm{i}}(t) \rangle \\
&= \lim_{\substack{t \to \infty \\ t_0 \to -\infty}} \left\langle \Psi_{\mathrm{f}} \middle| \Psi_{\mathrm{i}}(t_0) - \mathrm{i} \int_{t_0}^{t} \hat{H}(t') \Psi_{\mathrm{i}}(t') \mathrm{d}t' \right\rangle \\
&= \delta_{\mathrm{fi}} - \mathrm{i} \int_{-\infty}^{\infty} \langle \Psi_{\mathrm{f}} | \hat{H}(t') | \Psi_{\mathrm{i}}(t') \rangle \mathrm{d}t' \quad .
\end{aligned} \tag{2.8}
$$

Only $\hat{H}_{\mathrm{int}}$ of $\hat{H} = \hat{H}_0 + \hat{H}_{\mathrm{int}}$ contributes to the integral, because of the supposed orthogonality of initial and final state, $\langle \Psi_{\mathrm{f}} | \Psi_{\mathrm{i}} \rangle = 0$.

Specializing to the case of the muon decay, the lowest-order transition amplitude is

$$
S_{\mathrm{fi}} = -\mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}t \, H_{\mathrm{int}}^{(\mathrm{L})}(\mu^- \to \mathrm{e}^- \bar{\nu}_{\mathrm{e}} \nu_\mu) \quad . \tag{2.9}
$$

As discussed in Sect. 2.1, the relevant part of $H_{\mathrm{int}}^{(\mathrm{L})}$ contributing to this process is

$$
-\mathrm{i} \frac{G}{\sqrt{2}} \int \mathrm{d}^3x \left[\bar{u}_{\nu_\mu}(x) \gamma_\mu (1 - \gamma_5) u_\mu(x) \right] \left[\bar{u}_{\mathrm{e}}(x) \gamma^\mu (1 - \gamma_5) u_{\nu_{\mathrm{e}}}(x) \right] \quad . \tag{2.10}
$$

For this first-order approximation we may choose free wave functions to describe the four particles with four-momenta p, p', k, k' and spins s, s', t, t', respectively. According to the Feynman rules the (outgoing) antineutrino is represented by an (incoming) wave function with negative energy (see Fig. 2.1). Employing the form of the plane

[1] See W. Greiner: *Quantum Mechanics – An Introduction*, 4th ed. (Springer, Berlin, Heidelberg, 2001), and W. Greiner and J. Reinhardt: *Field Quantization*, 1st ed. (Springer, Berlin, Heidelberg, 1996).

waves of Appendix A.2 we have[2]

$$
\begin{aligned}
u_\mu(x) &= (2E_\mu V)^{-1/2} u_\mu(\boldsymbol{p}', s') \exp(-\mathrm{i} p'_\mu x^\mu) \quad , \\
u_\mathrm{e}(x) &= (2E_\mathrm{e} V)^{-1/2} u_\mathrm{e}(\boldsymbol{p}, s) \exp(-\mathrm{i} p_\mu x^\mu) \quad , \\
u_{\bar{\nu}_\mathrm{e}}(x) &= (2E_{\nu_\mathrm{e}} V)^{-1/2} v_{\nu_\mathrm{e}}(\boldsymbol{k}, t) \exp(+\mathrm{i} k_\mu x^\mu) \quad , \\
u_{\nu_\mu}(x) &= (2E_{\nu_\mu} V)^{-1/2} u_{\nu_\mu}(\boldsymbol{k}', t') \exp(-\mathrm{i} k'_\mu x^\mu) \quad ,
\end{aligned}
\tag{2.11}
$$

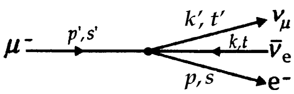


Fig. 2.1. Momenta and spins for the muon decay. The antineutrino $\bar{\nu}_\mathrm{e}$ is represented by an incoming wave with negative energy and negative momentum, i.e. negative four-momentum

where

$$
E_\mu = p'^0, \quad E_\mathrm{e} = p^0, \quad E_{\nu_\mathrm{e}} = k^0, \quad E_{\nu_\mu} = k'^0 \tag{2.12}
$$

and $u(\boldsymbol{p}, s)$, $v(\boldsymbol{p}, s)$ denote the spinor parts (E positive!)

$$
\begin{aligned}
u(\boldsymbol{p}, s) &= (E+m)^{\frac{1}{2}} \begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_s \end{pmatrix} \quad , \\
v(\boldsymbol{p}, s) &= (E+m)^{\frac{1}{2}} \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_s \\ \chi_s \end{pmatrix}
\end{aligned}
\tag{2.13}
$$

with the two-component unit spinors χ_s. Substituting this expression into the matrix element (2.9) yields

$$
\begin{aligned}
S(\mu^- \to \mathrm{e}^- \bar{\nu}_\mathrm{e} \nu_\mu) &= -\frac{\mathrm{i}G}{\sqrt{2}} \int \mathrm{d}^4 x \, \frac{\exp[\mathrm{i}(k'_\mu - p'_\mu + p_\mu + k_\mu) x^\mu]}{[16(k'^0 V)(p'^0 V)(p^0 V)(k^0 V)]^{\frac{1}{2}}} \\
&\quad \times \left[\bar{u}_{\nu_\mu}(\boldsymbol{k}', t') \gamma_\mu (1-\gamma_5) u_\mu(\boldsymbol{p}', s')\right] \\
&\quad \times \left[\bar{u}_\mathrm{e}(\boldsymbol{p}, s) \gamma^\mu (1-\gamma_5) v_{\nu_\mathrm{e}}(\boldsymbol{k}, t)\right] \\
&= -\mathrm{i}(2\pi)^4 \frac{G}{\sqrt{2}} \frac{\delta^4(p+k+k'-p')}{[16V^4 k'^0 k^0 p'^0 p^0]^{\frac{1}{2}}} \\
&\quad \times \left[\bar{u}_{\nu_\mu}(\boldsymbol{k}', t') \gamma_\mu (1-\gamma_5) u_\mu(\boldsymbol{p}', s')\right] \\
&\quad \times \left[\bar{u}_\mathrm{e}(\boldsymbol{p}', s) \gamma^\mu (1-\gamma_5) v_{\nu_\mathrm{e}}(\boldsymbol{k}, t)\right] \quad .
\end{aligned}
\tag{2.14}
$$

To obtain the transition probability, (2.14) must be multiplied with its Hermitian conjugate. This gives a factor

$$
\left[\delta^4(p+k+k'-p')\right]^2 = \delta^4(p+k+k'-p')\delta^4(0) \quad , \tag{2.15}
$$

which is replaced by

$$
\frac{VT}{(2\pi)^4} \delta^4(p+k+k'-p') \tag{2.16}
$$

according to the usual prescription, which can be derived heuristically (although mathematically oversimplified) as follows:

$$
\delta^4(0) = \lim_{q\to 0} \delta^4(q) = \lim_{q\to 0} \int \frac{\mathrm{d}^4 y}{(2\pi)^4} \mathrm{e}^{\mathrm{i} y_\mu q^\mu} = \int \frac{\mathrm{d}^4 y}{(2\pi)^4} = \frac{VT}{(2\pi)^4} \quad . \tag{2.17}
$$

---

[2] Note that we are using the index "μ" for two different purposes: it denotes the muon wave function u_μ and energy E_μ, and it occurs as a four-vector index, such as in p_μ, x_μ, γ_μ. Although this is somewhat unfortunate, we must get used to this double meaning.

V and T are understood to be macroscopic quantities so that the physical process takes place entirely within the finite space-time volume VT. In practice, the two neutrinos cannot be observed, that is, we need to sum or integrate over all possible final states. Furthermore, to obtain the transition probability within a small interval of momentum, we multiply by the density of the electron final states within an interval $V\,\mathrm{d}^3p/(2\pi)^3$. Finally we divide by T to get the *decay rate*, that is, the transition probability per unit time interval. Following these steps we find that

$$\begin{aligned}\mathrm{d}W &= \frac{1}{T}\frac{V\mathrm{d}^3p}{(2\pi)^3}V\int\frac{\mathrm{d}^3k}{(2\pi)^3}V\int\frac{\mathrm{d}^3k'}{(2\pi)^3}\sum_{t,t'}|S(\mu^-\to\mathrm{e}^-\bar{\nu}_\mathrm{e}\nu_\mu)|^2\\ &= \frac{G^2}{2}\frac{1}{(2\pi)^5}\frac{\mathrm{d}^3p}{2p'^0 2p^0}\int\frac{\mathrm{d}^3k}{2k^0}\int\frac{\mathrm{d}^3k'}{2k'^0}\delta^4(p+k+k'-p')\sum_{t,t'}|M|^2 \quad , \qquad (2.18)\end{aligned}$$

where

$$M = \left[\bar{u}_{\nu_\mu}\gamma^\mu(1-\gamma_5)u_\mu\right]\left[\bar{u}_\mathrm{e}\gamma_\mu(1-\gamma_5)v_{\nu_\mathrm{e}}\right] \quad . \qquad (2.19a)$$

The expression $|M|^2$ consists of two similar factors for the muonic and electronic transition currents. If we write $M = M^\mu E_\mu$ with $M^\mu = (\bar{u}_{\nu\mu}\gamma^\mu(1-\gamma_5)u_\mu)$ and $E_\mu = (\bar{u}_\mathrm{e}\gamma_\mu(1-\gamma_5)u_{\nu_\mathrm{e}})$, (2.19a) becomes

$$\sum_{t,t'}|M|^2 = \sum_{t,t'}(M^\mu E_\mu)(M^\nu E_\nu)^\dagger = \sum_{t,t'}(M^\mu M^{\nu\dagger})(E_\mu E_\nu^\dagger) \quad . \qquad (2.19b)$$

Let us first focus on the muonic factor, making use of Exercise 2.1:

$$\begin{aligned}X^{\mu\nu}(\mu) &= M^\mu M^{\nu\dagger}\\ &= \sum_{t'}\left[\bar{u}_{\nu_\mu}(\boldsymbol{k}',t')\gamma^\mu(1-\gamma_5)u_\mu(\boldsymbol{p}',s')\right]\\ &\quad\times\left[\bar{u}_{\nu_\mu}(\boldsymbol{k}',t')\gamma^\nu(1-\gamma_5)u_\mu(\boldsymbol{p}',s')\right]^\dagger\\ &= \sum_{t'}\bar{u}_{\nu_\mu}(\boldsymbol{k}',t')\gamma^\mu(1-\gamma_5)u_\mu(\boldsymbol{p}',s')\bar{u}_\mu(\boldsymbol{p}',s')\\ &\quad\times\gamma^\nu(1-\gamma_5)u_{\nu_\mu}(\boldsymbol{k}',t') \quad . \qquad (2.20)\end{aligned}$$

In order to evaluate this expression we make use of some helpful formulas for Dirac spinors and γ matrices (see Appendix A.2),[3]

$$\sum_{t'}u_{\nu_\mu}(k',t')_\alpha\bar{u}_{\nu_\mu}(k',t')_\beta = (\not{k}'+m_\nu)_{\alpha\beta} = \not{k}'_{\alpha\beta} \quad , \qquad (2.21)$$

where α,β denote the spinor indices and $m_\nu = 0$. Since the summation is not over the initial muon states, we have (see Appendix A.2)

$$u_\mu(p',s')_\alpha\bar{u}_\mu(p',s')_\beta = \left[(\not{p}'+m_\mu)\left(\frac{1+\gamma_5\not{s}'}{2}\right)\right]_{\alpha\beta} \quad , \qquad (2.22)$$

[3] See W. Greiner and J. Reinhardt: *Quantum Electrodynamics*, 4th ed. (Springer, Berlin, Heidelberg, 2009).

with the spin four-vector

$$s'_\mu = \left(\frac{\boldsymbol{p}' \cdot \boldsymbol{s}'}{m}, \boldsymbol{s}' + \frac{(\boldsymbol{p}' \cdot \boldsymbol{s}')\boldsymbol{p}'}{m(E'+m)} \right) \quad , \tag{2.23}$$

where $\boldsymbol{s}'$ is the spin vector with respect to the rest frame. Here $\boldsymbol{s}'$ is a unit vector so that $s'^\mu s'_\mu = -1$. Inserting the relations (2.21) and (2.22) into (2.20) we obtain the following expression for the muonic contribution to the transition currents $X^{\mu\nu}(\mu)$:

$$\begin{aligned} X^{\mu\nu}(\mu) &= \sum_{t'} \bar{u}_{\nu_\mu}(k',t')_\pi \gamma^\mu_{\pi\varrho}(1-\gamma_5)_{\varrho\alpha} \left[(\not{p}' + m_\mu) \left(\frac{1+\gamma_5 \not{s}'}{2} \right) \right]_{\alpha\beta} \\ &\quad \times \gamma^\nu_{\beta\sigma}(1-\gamma_5)_{\sigma\tau} u_{\nu_\mu}(k',t')_\tau \\ &= \gamma^\mu_{\pi\varrho}(1-\gamma_5)_{\varrho\alpha} \left[(\not{p}' + m_\mu) \left(\frac{1+\gamma_5 \not{s}'}{2} \right) \right]_{\alpha\beta} \gamma^\nu_{\beta\sigma}(1-\gamma_5)_{\sigma\tau} \not{k}'_{\tau\pi} \quad . \end{aligned} \tag{2.24}$$

Summing over the first and last index π means that we have to evaluate the trace of the (4 × 4) matrix:

$$X^{\mu\nu}(\mu) = \mathrm{Tr}\left\{ \gamma^\mu (1-\gamma_5)(\not{p}' + m_\mu) \left(\frac{1+\gamma_5 \not{s}'}{2} \right) \gamma^\nu (1-\gamma_5) \not{k}' \right\} \quad . \tag{2.25}$$

Since $\gamma^\alpha \gamma_5 = -\gamma_5 \gamma^\alpha$ and $\mathrm{Tr}\{AB\} = \mathrm{Tr}\{BA\}$ this yields

$$X^{\mu\nu}(\mu) = \frac{1}{2} \mathrm{Tr}\left\{ (\not{p}' + m_\mu)(1+\gamma_5 \not{s}') \gamma^\nu \not{k}' (1+\gamma_5) \gamma^\mu (1-\gamma_5) \right\} \quad . \tag{2.26}$$

Now we make use of the property that any trace of a product of an odd number of γ matrices vanishes (see Appendix A.2). Since $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ it consists of an even number of γ matrices. Furthermore it holds that $(1-\gamma_5)^2 = 2(1-\gamma_5)$, so that (2.26) becomes

$$\begin{aligned} X^{\mu\nu}(\mu) &= \mathrm{Tr}\left\{ (\not{p}' + m_\mu)(1+\gamma_5 \not{s}') \gamma^\nu \not{k}' \gamma^\mu (1-\gamma_5) \right\} \\ &= \mathrm{Tr}\left\{ \not{p}' \gamma^\nu \not{k}' \gamma^\mu (1-\gamma_5) + \not{p}' \gamma_5 \not{s}' \gamma^\nu \not{k}' \gamma^\mu (1-\gamma_5) \right. \\ &\quad \left. + m_\mu \gamma^\nu \not{k}' \gamma^\mu (1-\gamma_5) + m_\mu \gamma_5 \not{s}' \gamma^\nu \not{k}' \gamma^\mu (1-\gamma_5) \right\} \quad . \end{aligned} \tag{2.27}$$

Obviously the second and the third terms are “odd”; therefore they do not contribute. The remaining first and last terms are “even”. Taking into account that $\gamma_5(1-\gamma_5) = -(1-\gamma_5)$, we find that

$$\begin{aligned} X^{\mu\nu}(\mu) &= \mathrm{Tr}\left\{ \not{p}' \gamma^\nu \not{k}' \gamma^\mu (1-\gamma_5) - m_\mu \not{s}' \gamma^\nu \not{k}' \gamma^\mu (1-\gamma_5) \right\} \\ &= \mathrm{Tr}\left\{ (\not{p}' - m_\mu \not{s}') \gamma^\nu \not{k}' \gamma^\mu (1-\gamma_5) \right\} \quad . \end{aligned} \tag{2.28}$$

In Appendix A.2 it is shown that successive application of $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ yields the general relations

$$\begin{aligned} \mathrm{Tr}\{\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\tau\} &= 4(g^{\alpha\beta}g^{\sigma\tau} - g^{\alpha\sigma}g^{\beta\tau} + g^{\alpha\tau}g^{\beta\sigma}) \\ \mathrm{Tr}\{\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\tau\gamma_5\} &= -4\mathrm{i}\varepsilon^{\alpha\beta\sigma\tau} \quad . \end{aligned} \tag{2.29}$$

Using this for the trace, (2.28) gives the final result for the muonic part of the transition currents:

$$X^{\mu\nu}(\mu) = 4\big[(p' - m_\mu s')^\nu k'^\mu - (p' - m_\mu s')^\alpha k'_\alpha g^{\mu\nu} + (p' - m_\mu s')^\mu k'^\nu$$
$$+ \mathrm{i}\varepsilon^{\alpha\nu\beta\mu}(p' - m_\mu s')_\alpha k'_\beta\big] \quad . \tag{2.30}$$

The electronic contribution (2.19)–(2.20) is evaluated in a similar manner, which gives

$$\begin{aligned} X_{\mu\nu}(\mathrm{e}) &= E_\mu E_\nu^\dagger \\ &= \sum_t \big[\bar{u}_\mathrm{e}(p,s)\gamma_\mu(1-\gamma_5)v_{\nu_\mathrm{e}}(k,t)\big]\big[\bar{u}_\mathrm{e}(p,s)\gamma_\nu(1-\gamma_5)v_{\nu_\mathrm{e}}(k,t)\big]^\dagger \\ &= \mathrm{Tr}\left\{(\not{p} - m_\mathrm{e}\not{s})\gamma_\mu\not{k}\gamma_\nu(1-\gamma_5)\right\} \\ &= 4\big[(p - m_\mathrm{e}s)_\mu k_\nu - (p - m_\mathrm{e}s)^\alpha\, k_\alpha g_{\mu\nu} + (p - m_\mathrm{e}s)_\nu k_\mu \\ &\quad - \mathrm{i}\varepsilon_{\alpha\mu\beta\nu}(p - m_\mathrm{e}s)^\alpha k^\beta\big] \quad . \end{aligned} \tag{2.31}$$

The final result for the squared invariant matrix element (2.19b) is the product of the two expressions (2.30) and (2.31) which, after some work, is formed to be (see Exercise 2.3)

$$\sum_{t,t'} |M|^2 = X^{\mu\nu}(\mu)X_{\mu\nu}(\mathrm{e}) = 64(p' - m_\mu s')^\alpha k_\alpha (p - m_\mathrm{e}s)^\beta k'_\beta \quad . \tag{2.32}$$

EXERCISE

2.2 Proof of (2.31)

Problem. Prove the first part of (2.31)

$$X_{\mu\nu}(\mathrm{e}) = \mathrm{Tr}\big\{(\not{p} - m_\mathrm{e}\not{s})\gamma_\mu\not{k}\gamma_\nu(1-\gamma_5)\big\} \quad .$$

Solution. Starting from the expression (2.31) and performing the t summation, we arrive at

$$\begin{aligned} X_{\mu\nu}(\mathrm{e}) &= \sum_t E_\mu E_\nu^\dagger \\ &= \sum_t \big[\bar{u}_\mathrm{e}(\boldsymbol{p},s)\gamma_\mu(1-\gamma_5)v_{\nu_\mathrm{e}}(\boldsymbol{k},t)\big]\big[\bar{u}_\mathrm{e}(\boldsymbol{p},s)\gamma_\nu(1-\gamma_5)v_{\nu_\mathrm{e}}(\boldsymbol{k},t)\big]^\dagger \\ &= \sum_t \big[\bar{u}_\mathrm{e}(\boldsymbol{p},s)\gamma_\mu(1-\gamma_5)v_{\nu_\mathrm{e}}(\boldsymbol{k},t)\bar{v}_{\nu_\mathrm{e}}(\boldsymbol{k},t)\gamma_\nu(1-\gamma_5)u_\mathrm{e}(\boldsymbol{p},s)\big] \\ &= \bar{u}_\mathrm{e}(\boldsymbol{p},s)\gamma_\mu(1-\gamma_5)\underbrace{\left[\sum_t v_{\nu_\mathrm{e}}(\boldsymbol{k},t)\bar{v}_{\nu_\mathrm{e}}(\boldsymbol{k},t)\right]}_{=\not{k}-m_{\nu_\mathrm{e}}=\not{k}}\gamma_\nu(1-\gamma_5)u_\mathrm{e}(\boldsymbol{p},s) \\ &= \bar{u}_\mathrm{e}(\boldsymbol{p},s)_\pi(\gamma_\mu)_{\pi\varrho}(1-\gamma_5)_{\varrho\alpha}\not{k}_{\alpha\beta}(\gamma_\nu)_{\beta\sigma}(1-\gamma_5)_{\sigma\tau}u_\mathrm{e}(\boldsymbol{p},s)_\tau \quad . \end{aligned} \tag{1}$$

With the identity (see (2.22))

$$\bar{u}_\mathrm{e}(\boldsymbol{p},s)_\pi u_\mathrm{e}(\boldsymbol{p},s)_\tau = \left[(\not{p} + m_\mathrm{e})\frac{(1+\gamma_5\not{s})}{2}\right]_{\tau\pi} \tag{2}$$

we obtain

Exercise 2.2

$$X_{\mu\nu}(\mathrm{e}) = (\gamma_\mu)_{\pi\varrho}(1-\gamma_5)_{\varrho\alpha}\not{k}_{\alpha\beta}(\gamma_\nu)_{\beta\sigma}(1-\gamma_5)_{\sigma\tau}\left[(\not{p}+m_\mathrm{e})\left(\frac{1+\gamma_5\not{s}}{2}\right)\right]_{\tau\pi}$$
$$= \mathrm{Tr}\left\{\gamma_\mu(1-\gamma_5)\not{k}\gamma_\nu(1-\gamma_5)\left[(\not{p}+m_\mathrm{e})\left(\frac{1+\gamma_5\not{s}}{2}\right)\right]\right\}$$
$$= \mathrm{Tr}\left\{(\not{p}+m_\mathrm{e})(1+\gamma_5\not{s})\gamma_\mu\not{k}\gamma_\nu(1-\gamma_5)\right\} \tag{3}$$

where we have used the relation

$$\gamma_\mu(1-\gamma_5)\not{k}\gamma_\nu(1-\gamma_5) = \gamma_\mu\not{k}(1+\gamma_5)\gamma_\nu(1-\gamma_5)$$
$$= \gamma_\mu\not{k}\gamma_\nu(1-\gamma_5)^2$$
$$= \gamma_\mu\not{k}\gamma_\nu 2(1-\gamma_5)$$

and the trace identity $\mathrm{Tr}\{AB\} = \mathrm{Tr}\{BA\}$. This expression (3) transforms to (2.27) if we replace

$$\begin{aligned} \not{p} &\to \not{p}' \quad , \\ m_\mathrm{e} &\to m_\mu \quad , \\ \not{s} &\to \not{s}' \quad , \\ \gamma_\mu &\to \gamma^\nu \quad , \\ \not{k} &\to \not{k}' \quad , \\ \gamma_\nu &\to \gamma^\mu \quad . \end{aligned} \tag{4}$$

Therefore we may simply rewrite (2.28) by substituting for the muonic quantities the corresponding electron quantities:

$$X_{\mu\nu}(\mathrm{e}) = \mathrm{Tr}\left\{(\not{p}-m_\mathrm{e}\not{s})\gamma_\mu\not{k}\gamma_\nu(1-\gamma_5)\right\} \quad . \tag{5}$$

EXERCISE

2.3 Calculation of the Averaged Decay Matrix Element

Problem. Evaluate $\sum |M|^2$ in (2.32) by using the following relation for the antisymmetric Levi-Civita tensor,

$$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\bar{\alpha}\bar{\beta}\mu\nu} = 2(\delta^\alpha_{\bar{\beta}}\delta^\beta_{\bar{\alpha}} - \delta^\alpha_{\bar{\alpha}}\delta^\beta_{\bar{\beta}}) \quad ,$$

and the property that any product of $\varepsilon^{\alpha\beta\mu\nu}$ with a tensor that is symmetric in the indices μ, ν vanishes (see also Exercise 2.4).

Solution. Introducing the following abbreviations

$$(p' - m_\mu s')_\nu \equiv q'_\nu \quad , \tag{1}$$
$$(p - m_\mathrm{e} s)_\mu \equiv q_\mu \quad , \tag{2}$$

Exercise 2.3

$$q_\alpha k^\alpha \equiv (q \cdot k) \quad , \tag{3}$$

and using the relations (2.30)–(2.32) we rewrite $\sum_{t,t'} |M|^2$ as follows:

$$\begin{aligned}
\sum_{t,t'} |M|^2 &= 16\big[q'^{\nu} k'^{\mu} - (q' \cdot k') g^{\mu\nu} + q'^{\mu} k'^{\nu} - \mathrm{i}\varepsilon^{\alpha\nu\beta\mu} q'_\alpha k'_\beta\big] \\
&\quad \times \big[q_\mu k_\nu - (q \cdot k) g_{\mu\nu} + q_\nu k_\mu + \mathrm{i}\varepsilon_{\alpha\mu\beta\nu} q^\alpha k^\beta\big] \\
&= 16\big[(q' \cdot k)(q \cdot k') - (q \cdot k)(q' \cdot k') \\
&\quad + (q' \cdot q)(k' \cdot k) - \mathrm{i}\varepsilon_{\alpha\mu\beta\nu} q'^{\nu} k'^{\mu} q^\alpha k^\beta \\
&\quad - (q' \cdot k')\{(q \cdot k) - (q \cdot k) \cdot 4 + (q \cdot k)\} \\
&\quad + (q' \cdot q)(k' \cdot k) - (q \cdot k)(q' \cdot k') + (q' \cdot k)(k' \cdot q) \\
&\quad - \mathrm{i}\varepsilon_{\alpha\mu\beta\nu} q'^{\mu} k'^{\nu} q^\alpha k^\beta + \mathrm{i}\varepsilon^{\alpha\nu\beta\mu} q'_\alpha k'_\beta q_\mu k_\nu \\
&\quad + \mathrm{i}\varepsilon^{\alpha\nu\beta\mu} q'_\alpha k'_\beta q_\nu k_\mu - \varepsilon^{\alpha\beta\nu\mu} q'_\alpha k'_\beta \varepsilon_{\bar\alpha\bar\beta\mu\nu} q^{\bar\alpha} k^{\bar\beta}\big] \\
&= 16\big[2(q' \cdot k)(q \cdot k') + 2(q' \cdot q)(k' \cdot k) \\
&\quad \underbrace{-\mathrm{i}\varepsilon_{\alpha\mu\beta\nu}(q'^{\mu} k'^{\nu} + q'^{\nu} k'^{\mu}) q^\alpha k^\beta}_{=0} \underbrace{+\mathrm{i}\varepsilon^{\alpha\nu\beta\mu}(q_\mu k_\nu + q_\nu k_\mu) q'_\alpha k'_\beta}_{=0} \\
&\quad + \varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\bar\alpha\bar\beta\mu\nu} q'_\alpha k'_\beta q^{\bar\alpha} k^{\bar\beta}\big] \\
&= 32\big[(q' \cdot k)(q \cdot k') + (q' \cdot q)(k' \cdot k) + (\delta^\alpha_{\bar\beta} \delta^\beta_{\bar\alpha} - \delta^\alpha_{\bar\alpha} \delta^\beta_{\bar\beta}) q'_\alpha k'_\beta q^{\bar\alpha} k^{\bar\beta}\big] \\
&= 32\big[(q' \cdot k)(q \cdot k') + (q' \cdot q)(k' \cdot k) \\
&\quad + (q' \cdot k)(k' \cdot q) - (q' \cdot q)(k' \cdot k)\big] \\
&= 64 (q' \cdot k)(k' \cdot q) \quad .
\end{aligned} \tag{4}$$

Returning to the original notation (3) this result is equivalent to (2.32).

EXERCISE

2.4 A Useful Relation for the Levi-Civita Tensor

Problem. Prove the formula

$$\varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\bar\alpha\bar\beta\mu\nu} = 2(\delta^\alpha_{\bar\beta} \delta^\beta_{\bar\alpha} - \delta^\alpha_{\bar\alpha} \delta^\beta_{\bar\beta})$$

that was applied in the Exercise 2.3.

Solution. The totally antisymmetric Levi-Civita tensor $\varepsilon^{\alpha\beta\mu\nu}$ is defined as

$$\varepsilon^{\alpha\beta\mu\nu} = \begin{cases} \mathrm{sgn}(\hat P) & \text{if } (\alpha\beta\mu\nu) = \hat P(0123) \\ 0 & \text{otherwise} \end{cases} \quad , \tag{1}$$

where $\hat P$ denotes a permutation of the indices (0123). $\varepsilon^{\alpha\beta\mu\nu}$ vanishes if two of its indices are equal.

Exercise 2.4

Since $\varepsilon_{\bar\alpha\bar\beta\bar\mu\bar\nu} = g_{\bar\alpha\alpha} g_{\bar\beta\beta} g_{\bar\mu\mu} g_{\bar\nu\nu} \varepsilon^{\alpha\beta\mu\nu}$ is non-zero if and only if $(\bar\alpha\bar\beta\bar\mu\bar\nu)$ is a permutation of (0123), we then have for example for $(\bar\alpha\bar\beta\bar\mu\bar\nu) = (0123)$,

$$\begin{aligned}\varepsilon_{0123} &= g_{0\alpha} g_{1\beta} g_{2\mu} g_{3\nu} \varepsilon^{\alpha\beta\mu\nu} \\ &= 1 \cdot (-1)^3 \varepsilon^{0123} = 1 \times (-1)^3 = -1 \quad ,\end{aligned} \tag{2}$$

and a similar relation for all other non-vanishing components of the covariant Levi-Civita tensor, that is,

$$\varepsilon^{\alpha\beta\mu\nu} = -\varepsilon_{\alpha\beta\mu\nu} \quad . \tag{3}$$

Now consider the desired contraction with respect to the indices μ, ν,

$$\varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\bar\alpha\bar\beta\mu\nu} \quad . \tag{4}$$

For fixed values of $\alpha, \bar\alpha, \beta, \bar\beta$, only those terms contribute that contain tensor components with third and fourth indices different from $\alpha, \bar\alpha, \beta, \bar\beta$. Furthermore, since the third and fourth indices are the same for both the covariant and the contravariant tensor, an additional condition is that either

$$\alpha = \bar\alpha \quad , \quad \beta = \bar\beta \tag{5a}$$

or

$$\alpha = \bar\beta \quad , \quad \beta = \bar\alpha \quad . \tag{5b}$$

In each of the two cases (5a) and (5b) only two possible combinations for the values of the indices μ, ν remain, namely those of the two numbers (0123) that differ from α and β. We then have the following relations:

Case A:

$$\begin{aligned}\alpha = \bar\alpha \quad , \quad \beta = \bar\beta \quad : \quad \varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\bar\alpha\bar\beta\mu\nu} &= \sum_{\mu,\nu} \varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\bar\alpha\bar\beta\mu\nu} \\ &= \sum_{\mu,\nu} \varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\alpha\beta\mu\nu} \\ &= 2 \cdot 1 \cdot (-1) \\ &= -2 \quad .\end{aligned} \tag{6a}$$

Case B:

$$\begin{aligned}\alpha = \bar\beta \quad , \quad \beta = \bar\alpha \quad : \quad \varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\bar\alpha\bar\beta\mu\nu} &= \sum_{\mu,\nu} \varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\beta\alpha\mu\nu} \\ &= -\sum_{\mu\nu} \varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\alpha\beta\mu\nu} \\ &= -(-2) \\ &= +2 \quad .\end{aligned} \tag{6b}$$

Exercise 2.4

Therefore the final result is

$$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\bar{\alpha}\bar{\beta}\mu\nu} = \begin{cases} -2 \quad , & \alpha=\bar{\alpha}, \beta=\bar{\beta} \\ +2 \quad , & \alpha=\bar{\beta}, \beta=\bar{\alpha} \\ 0 \quad , & \text{otherwise} \end{cases}$$

$$= 2\{\delta^{\alpha}_{\bar{\beta}}\delta^{\beta}_{\bar{\alpha}} - \delta^{\alpha}_{\bar{\alpha}}\delta^{\beta}_{\bar{\beta}}\} \quad , \tag{7}$$

which was to be shown.

For calculating the decay rate $\mathrm{d}W$ we proceed by inserting the result of (2.32) into the expression for $\mathrm{d}W$ (2.18). In order to perform the required integration, we need to evaluate the integral

$$I_{\alpha\beta} = \int \frac{\mathrm{d}^3 k}{2k^0} \int \frac{\mathrm{d}^3 k'}{2k'^0} k_\alpha k'_\beta \delta^4(p+k+k'-p') \quad . \tag{2.33}$$

$I_{\alpha\beta}$ is manifestly Lorentz covariant. This is obvious because $\delta^4(p+k+k'-p')$ and $\mathrm{d}^3p/2p_0 = \int_{-\infty} \mathrm{d}^4 p\,\delta(p^2-m_0^2)\theta(p_0)$ are Lorentz invariant. The latter has been shown in *Quantum Electrodynamics*.[4] Since the variables k and k' are integrated over only the two second-rank tensors $g_{\alpha\beta}$ and $(p'-p)_\alpha(p'-p)_\beta = q_\alpha q_\beta$ can occur in the result. Note that the vector $q=(p'-p)$ is different from that defined in Exercise 2.3! We keep this in mind and proceed with the ansatz

$$I_{\alpha\beta} = Aq^2 g_{\alpha\beta} + Bq_\alpha q_\beta \quad , \tag{2.34}$$

where $q^2 = q^\alpha q_\alpha$ was split off in order to have A and B dimensionless. From (2.34) we construct the following invariants:

$$g^{\alpha\beta} I_{\alpha\beta} = (4A+B)q^2 \quad , \tag{2.35a}$$

$$q^\alpha q^\beta I_{\alpha\beta} = (A+B)q^4 \quad . \tag{2.35b}$$

To proceed, we now distinguish two cases:

(i) The vector $q = p'-p$ is time-like, that is $q^2 > 0$. With this condition we can always perform a proper Lorentz transformation, such that

$$\tilde{q}^\nu := a^\nu{}_\mu q^\mu = (\tilde{q}^0, 0) \tag{2.36}$$

defines the reference system. With respect to this reference frame we have

$$g^{\alpha\beta} I_{\alpha\beta} = \int \frac{\mathrm{d}^3 k}{2k^0} \int \frac{\mathrm{d}^3 k'}{2k'^0} k_\alpha k'^\alpha \delta^3(\boldsymbol{k}+\boldsymbol{k}')\delta(k^0+k'^0-\tilde{q}^0)$$

$$= \int \frac{\mathrm{d}^3 k}{2k^0} \int \frac{\mathrm{d}^3 k'}{2k'^0} \big[(k^0)^2 - (\boldsymbol{k}'\cdot\boldsymbol{k})\big]\delta^3(\boldsymbol{k}+\boldsymbol{k}')\delta(k^0+k'^0-\tilde{q}^0)$$

---

[4] See W. Greiner and J. Reinhardt: *Quantum Electrodynamics*, 4th ed. (Springer, Berlin, Heidelberg 2009), equation (3.72).

$$= \int \frac{d^3k}{4(k^0)^2} 2(k^0)^2 \delta(2k^0 - \tilde{q}^0) \quad , \tag{2.37}$$

since $\boldsymbol{k}' = -\boldsymbol{k}$ and consequently $k'^0 = k^0 = |\boldsymbol{k}| = |\boldsymbol{k}'|$. The integral can further be simplified by substituting $x = 2k^0$,

$$g^{\alpha\beta} I_{\alpha\beta} = 2\pi \int\limits_0^\infty (k^0)^2 \, dk_0 \delta(2k^0 - \tilde{q}^0) = \frac{\pi}{4} \int\limits_0^\infty x^2 \, dx \delta(x - \tilde{q}^0) \quad . \tag{2.38}$$

For positive $\tilde{q}^0$ the argument of the δ function has its zero value within the integration interval. By means of the Θ function

$$\Theta(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \quad , \tag{2.39}$$

the above result can be expressed as

$$g^{\alpha\beta} I_{\alpha\beta} = \frac{\pi}{4} (\tilde{q}^0)^2 \Theta(\tilde{q}^0) \quad . \tag{2.40}$$

In order to rewrite this in a Lorentz invariant form we remark that for time-like four-vectors q^μ the sign of the zeroth component q^0 remains unchanged under proper Lorentz transformations, that is, $\Theta(\tilde{q}^0) = \Theta(q^0)$. Furthermore, with respect to our chosen reference frame we have $\tilde{q}^2 = (\tilde{q}^0)^2 = q^2$. Hence the result (2.40) can be stated in the Lorentz invariant form

$$g^{\alpha\beta} I_{\alpha\beta} = \frac{\pi}{4} q^2 \Theta(q^0) \quad \text{for} \quad q^2 > 0 \quad . \tag{2.41}$$

Similarly we obtain

$$\begin{aligned} q^\alpha q^\beta I_{\alpha\beta} &= (\tilde{q}^0)^2 I_{00} \\ &= (\tilde{q}^0)^2 \int \frac{d^3k}{2} \int \frac{d^3k'}{2} \delta^3(\boldsymbol{k} + \boldsymbol{k}') \delta(k^0 + k'^0 - \tilde{q}^0) \\ &= \frac{1}{4} (\tilde{q}^0)^2 \int d^3k \delta(2k^0 - \tilde{q}^0) \\ &= \pi (\tilde{q}^0)^2 \int (k^0)^2 \, dk^0 \delta(2k^0 - \tilde{q}^0) = \frac{\pi}{8} (\tilde{q}^0)^4 \Theta(\tilde{q}^0) \\ &= \frac{\pi}{8} q^4 \Theta(q^0) \quad \text{for} \quad q^2 > 0 \quad . \end{aligned} \tag{2.42}$$

(ii) The vector q_μ is space-like, that is $q^2 < 0$. In this case the argument of the δ function, $(k + k' - q)$, is non-zero everywhere. This property can be understood by recalling that, owing to the vanishing mass of the neutrinos, it holds that

$$\begin{aligned} k^2 = k'^2 &= 0 \quad , \\ \boldsymbol{k} \cdot \boldsymbol{k}' &= k^0 k'^0 \cos\theta \quad , \end{aligned} \tag{2.43}$$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$. Consequently we have

$$(k + k')^2 = 2(k^0 k'^0 - \boldsymbol{k} \cdot \boldsymbol{k}') = 2k^0 k'^0 (1 - \cos\theta) \geq 0 \quad , \tag{2.44}$$

which implies that $q^\mu = k^\mu + k'^\mu$ cannot be satisfied. Therefore

$$I_{\alpha\beta} = 0 \quad \text{for} \quad q^2 < 0 \quad . \tag{2.45}$$

The results of (2.41), (2.42) and (2.45) may be summarized as follows:

$$g^{\alpha\beta} I_{\alpha\beta} = \frac{\pi}{4} q^2 \Theta(q^0)\Theta(q^2) \tag{2.46a}$$

$$q^\alpha q^\beta I_{\alpha\beta} = \frac{\pi}{8} q^4 \Theta(q^0)\Theta(q^2) \quad . \tag{2.46b}$$

Equating these expressions with (2.35) gives

$$4A + B = \frac{\pi}{4}\Theta(q^0)\Theta(q^2) \quad , \tag{2.47a}$$

$$A + B = \frac{\pi}{8}\Theta(q^0)\Theta(q^2) \quad , \tag{2.47b}$$

yielding the solution

$$A = \frac{\pi}{24}\Theta(q^0)\Theta(q^2) \quad , \tag{2.48a}$$

$$B = \frac{\pi}{12}\Theta(q^0)\Theta(q^2) \quad . \tag{2.48b}$$

Substituting in (2.34) we finally obtain

$$I_{\alpha\beta} = \frac{\pi}{24}(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta)\Theta(q^0)\Theta(q^2) \quad . \tag{2.49}$$

The decay rate of a muon with polarization s' into an electron with polarization s is given in terms of (2.18), (2.32) and (2.49); thus we find that

$$\begin{aligned} \mathrm{d}W &= \frac{G^2}{2}\frac{1}{(2\pi)^5}\frac{\mathrm{d}^3 p}{2p'^0 2p^0}\int \frac{\mathrm{d}^3 k}{2k^0}\int \frac{\mathrm{d}^3 k'}{2k'^0}\sum_{t,t'}\delta^4(p+k+k'-p')|M|^2 \\ &= \frac{G^2}{2}\frac{1}{(2\pi)^5}\frac{64\mathrm{d}^3 p}{2p'^0 2p^0} I_{\alpha\beta}(p' - m_\mu s')^\alpha (p - m_\mathrm{e} s)^\beta \\ &= \frac{G^2}{3}\frac{\pi \mathrm{d}^3 p}{(2\pi)^5 p'^0 p^0}\big[(p'-p)^2 (p' - m_\mu s')^\alpha (p - m_\mathrm{e} s)_\alpha \\ &\quad + 2(p'-p)_\alpha (p' - m_\mu s')^\alpha (p'-p)_\beta (p - m_\mathrm{e} s)^\beta\big] \\ &\quad \times \Theta(p'^0 - p^0)\Theta((p'-p)^2) \quad . \end{aligned} \tag{2.50}$$

Note that the effect of time dilatation, which accompanies the observation of the muon lifetime, becomes obvious from (2.50). For a moving muon we have $p'^0 = \gamma m_\mu$ with the Lorentz factor $\gamma = (1 - v^2/c^2)^{-1/2}$. As can be seen from the expression for $\mathrm{d}W$ (2.50), $\mathrm{d}W \propto 1/\gamma$, implying that the decay rate decreases considerably for fast-moving muons, that is, the life-time $\tau_\mu \propto \gamma$ is prolonged. To proceed we switch to the *rest frame of the muon*, which is characterized by $p^{\alpha\prime} - p^\alpha = (m_\mu - p^0, -\boldsymbol{p})$. Since

$$\begin{aligned} (p'-p)^2 &= (m_\mu - p^0)^2 - \boldsymbol{p}^2 = (m_\mu - p^0)^2 - (p^{0^2} - m_\mathrm{e}^2) \\ &= -2p^0 m_\mu + m_\mu^2 + m_\mathrm{e}^2 \quad , \end{aligned} \tag{2.51}$$

the condition $(p'-p)^2>0$ for a non-vanishing $\mathrm{d}W$ yields the restriction

$$p^0 < p^0_{\max} = ({m_\mu}^2 + {m_\mathrm{e}}^2)/2m_\mu \quad , \tag{2.52}$$

which consequently requires $p'^0 - p^0 > 0$, since

$$p'^0 - p^0 = m_\mu - p^0 > m_\mu - p^0_{\max} = ({m_\mu}^2 - {m_\mathrm{e}}^2)/2m_\mu > 0 \quad . \tag{2.53}$$

The condition $p^0 < p^0_{\max}$ in (2.52) and (2.53) assures that the first $\Theta(p'_0 - p_0)$ function in (2.50) is automatically fulfilled. Therefore we may replace the product of the two Θ functions in (2.50) by $\Theta(p^0_{\max} - p_0)$. Furthermore, with respect to the rest frame of the muon, it holds that $s^{\mu\prime} = (0, \boldsymbol{s}')$, so that the final result is

$$\begin{aligned} \mathrm{d}W(\boldsymbol{s}') = \frac{G^2}{3}\frac{\pi\, \mathrm{d}^3p}{(2\pi)^5 p^0}\Big\{&\big[(m_\mu - p^0)^2 - \boldsymbol{p}^2\big]\big[(p_0 - m_\mathrm{e}s^0) + \boldsymbol{s}'\cdot(\boldsymbol{p} - m_\mathrm{e}\tilde{\boldsymbol{s}})\big] \\ &+ 2[m_\mu - p^0 - \boldsymbol{s}'\cdot\boldsymbol{p}]\big[(m_\mu - p^0)(p^0 - m_\mathrm{e}s^0) + \boldsymbol{p}\cdot(\boldsymbol{p} - m_\mathrm{e}\tilde{\boldsymbol{s}})\big]\Big\} \\ &\times \Theta(p^0_{\max} - p^0) \quad . \end{aligned} \tag{2.54}$$

Here $\tilde{\boldsymbol{s}} = \boldsymbol{s} + \frac{(ps)\boldsymbol{p}}{m_\mathrm{e}(p^0 + m_\mathrm{e})}$ is the space component of the electron spin vector (2.23).

EXERCISE

2.5 The Endpoint of the Electron Energy Spectrum in Muon Decay

Problem. Show that the highest electron energy is given in terms of (2.52) by energy and momentum conservation.

Solution. The highest energy of the electron corresponds to the largest value of its momentum. The latter is obtained if both neutrinos are emitted in one direction while the electron is scattered in the other direction (Fig. 2.2), that is,

$$\boldsymbol{p} = -(\boldsymbol{k} + \boldsymbol{k}') \quad . \tag{1}$$

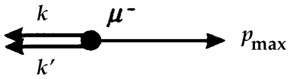


Fig. 2.2. Configuration for which the electron reaches its maximum value of momentum

Because $k^0 = |\boldsymbol{k}|$ and $k'^0 = |\boldsymbol{k}'|$ it holds that

$$\begin{aligned} m_\mu &= p^0_{\max} + k^0 + k'^0 = p^0_{\max} + |\boldsymbol{k}| + |\boldsymbol{k}'| \\ &= p^0_{\max} + |\boldsymbol{p}_{\max}| \\ &= p^0_{\max} + \big[(p^0_{\max})^2 - m_\mathrm{e}^2\big]^{\frac{1}{2}} \quad . \end{aligned} \tag{2}$$

Inverting this relation gives

$$\begin{aligned} p^0_{\max} &= \frac{m_\mu^2 + m_\mathrm{e}^2}{2m_\mu} = 52.83\ \mathrm{MeV} \quad , \\ |\boldsymbol{p}_{\max}| &= \frac{m_\mu^2 - m_\mathrm{e}^2}{2m_\mu} \quad , \end{aligned} \tag{3}$$

which agrees exactly with the conditions (2.52) and (2.53).

2.3 The Lifetime of the Muon

To determine the muon lifetime τ_μ we sum over the electron spin orientations s, average over the spin orientation of the muon s', and integrate over the electron momentum $\boldsymbol{p}$:

$$\begin{aligned}\frac{1}{\tau_\mu} &= W_\mu = \frac{1}{2}\sum_{s,s'}\int \mathrm{d}W \\ &= 2\frac{G^2}{3}\frac{\pi}{(2\pi)^5}\int \frac{\mathrm{d}^3p}{p^0}\big\{\big[(m_\mu - p^0)^2 - \boldsymbol{p}^2\big]p^0 + 2(m_\mu - p^0) \\ &\quad \times \big[(m_\mu - p^0)p^0 + \boldsymbol{p}^2\big]\big\}\Theta(p^0_{\max} - p^0) \\ &= 2\frac{G^2}{3}\int \frac{\pi\,\mathrm{d}^3p}{(2\pi)^5 p^0}[-4m_\mu(p^0)^2 + 3p^0(m_\mu^2 + m_\mathrm{e}^2) - 2m_\mu m_\mathrm{e}^2] \\ &\quad \times \Theta(p^0_{\max} - p^0) \quad . \end{aligned} \tag{2.55}$$

In deriving (2.55) we used the fact that the averaging over $\boldsymbol{s}$ gives $\langle \boldsymbol{s}\rangle = 0$ so that also $\langle \boldsymbol{s}^0\rangle = \frac{1}{m}\langle \boldsymbol{p}\cdot\boldsymbol{s}\rangle = 0$ (cf. (2.23)). If we employ the following identity:

$$\int \mathrm{d}^3\boldsymbol{p}[\ldots]\Theta(p^0_{\max} - p^0) = 4\pi \int_0^{|\boldsymbol{p}_{\max}|} |\boldsymbol{p}|^2\,\mathrm{d}|\boldsymbol{p}|[\ldots] \quad , \tag{2.56}$$

and take into account that $\boldsymbol{p}^2 = (p_0^2 - m_\mathrm{e}^2)$ and therefore that

$$\mathrm{d}|\boldsymbol{p}|/\mathrm{d}p^0 = p^0/|\boldsymbol{p}| \quad ,$$

we can rewrite W_μ in the form

$$\begin{aligned} W_\mu &= \frac{2G^2}{3(2\pi)^3}\int_{m_\mathrm{e}}^{p^0_{\max}} \mathrm{d}p^0\sqrt{(p^0)^2 - m_\mathrm{e}^2}\big[-4m_\mu(p^0)^2 + 3p^0(m_\mu^2 + m_\mathrm{e}^2) - 2m_\mu m_\mathrm{e}^2\big] \\ &= \frac{G^2 {m_\mu}^5}{192\pi^3}[1 - 8y + 8y^3 - y^4 - 12y^2\ln y] \quad , \end{aligned} \tag{2.57}$$

with the abbreviation $y \equiv m_\mathrm{e}^2/m_\mu^2$. The contributions involving y lead only to small corrections, namely

$$W_\mu = \frac{G^2 m_\mu^5}{192\pi^3}(1 - 1.87\times 10^{-4}) \quad . \tag{2.58}$$

From (2.57) it is obvious that the decay rate would vanish if $y = 1$. This reflects the fact that in this (academic) case the muon would be stable since $|\boldsymbol{p}_{\max}| = 0$, so that there would be no phase space available for the final-state electron.

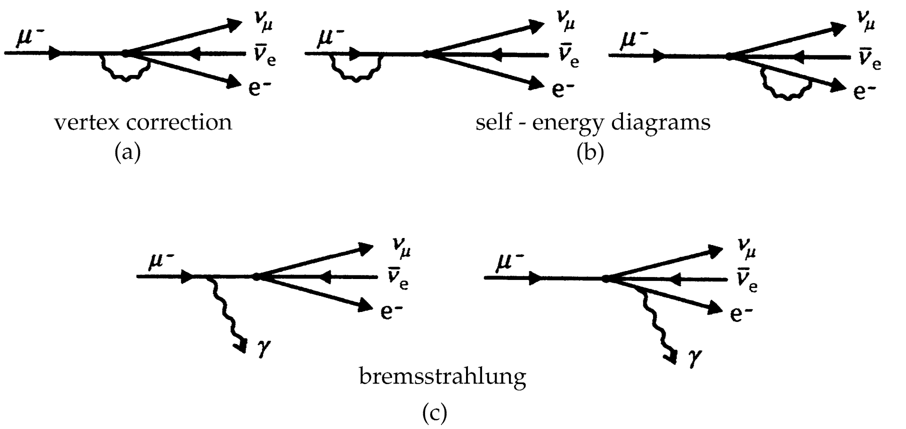


Fig. 2.3. Vertex correction (**a**), self-energy (**b**), and bremsstrahlung (**c**) contributions

The result (2.58) does not include the so-called radiative corrections, which also need to be considered. These effects are represented by diagrams in which one of the charged particles interacts with the electromagnetic field (see Fig. 2.3). The bremsstrahlung diagram has to be included since, owing to the vanishing photon mass, photons with arbitrary small energies may be emitted. On the other hand, because of the limited experimental resolution, it is impossible to distinguish the muon decay accompanied by emission of an extremely "soft" photon from a decay without radiation. This contribution exactly cancels the divergent terms in the self-energy diagrams for very soft photons (infrared divergence).[5] The calculation of these contributions leads to a modification of the decay rate W by a factor[6]

$$1-\frac{\alpha}{2\pi}\left(\pi^2-\frac{25}{4}\right)=0.9958\ldots \quad . \tag{2.59}$$

Hence, the radiative corrections are of greater importance than the influence of the finite mass of the electron. The final result for the muon decay rate is now given by

$$W_\mu=\frac{1}{\tau_\mu}=\frac{G^2m_\mu^5}{192\pi^3}\left(1-\frac{\alpha}{2\pi}\left(\pi^2-\frac{25}{4}\right)-8\frac{m_e^2}{m_\mu^2}\quad\ldots\right) \quad . \tag{2.60}$$

Using this formula we may calculate the value of the Fermi coupling constant G by taking into account the experimental value for the average life time of the muon

$$\tau_\mu=(2.19703\pm0.00004)\times10^{-6}\ \mathrm{s} \quad ,$$

i.e.

$$W_\mu=\tau_\mu^{-1}=2.996\times10^{-16}\ \mathrm{MeV} \quad . \tag{2.61}$$

With the most accurate value for the muon mass

$$m_\mu=(105.658387\pm0.000034)\ \mathrm{MeV} \tag{2.62}$$

---

[5] See W. Greiner and J. Reinhardt: *Quantum Electrodynamics*, 4th ed. (Springer, Berlin, Heidelberg, 2009).

[6] S.M. Berman: Phys. Rev. **112**, 267 (1958); M. Roos and A. Sirling: Nucl. Phys. **B 29**, 296 (1971); L.D. Landau, E.M. Lifschitz: *Theoretical Physics* (Pergamon, Oxford, 1974), Vol. IVb, p. 147.

we obtain

$$G = (1.166\,37 \pm 0.000\,02) \times 10^{-11}\ \text{MeV}^{-2} \quad . \tag{2.63}$$

Since G is not a dimensionless quantity, a direct comparison with the electromagnetic coupling constant is not possible. The effective strength of the weak interaction obviously increases with growing mass of the particles. This is evident from the inverse lifetime of the muon which, because of the uncertainty relation, corresponds to the uncertainty of its rest mass. The ratio of its value to the rest mass itself,

$$\frac{\tau_\mu^{-1}}{m_\mu} = \frac{W_\mu}{m_\mu} \approx \frac{1}{192\pi^3}(Gm_\mu^2)^2 \tag{2.64}$$

manifests the role of Gm_μ^2 as an effective coupling strength. For curiosity's sake we may now evaluate the mass M for which the effective coupling constant equals the fine-structure constant α:

$$GM^2 = \alpha \quad \to \quad M = \sqrt{\alpha/G} = 25\ \text{GeV} \quad . \tag{2.65}$$

The experimental investigation of this energy region has become possible with the large particle accelerators of DESY (Hamburg), SLAC (Stanford), CERN (Geneva), and Fermilab (Chicago). As we will soon see, these investigations have revealed new information concerning the nature of the weak interaction.

EXERCISE

2.6 Myon Decay for Finite Neutrino Masses

Problem. Generalize the relation (2.49),

$$\begin{aligned} I_{\alpha\beta} &= \int \frac{\mathrm{d}^3k}{2k^0} \int \frac{\mathrm{d}^3k'}{2k'^0} k_\alpha k'_\beta \delta^4(k+k'-q) \\ &= \frac{\pi}{24}\left(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta\right) \quad , \end{aligned}$$

which is valid for $q^2, q^0 > 0$, to the case of non-vanishing rest masses m, m', of the two decay products with the four-momenta k_α, k'_β.

Solution. As in (2.34) we make the ansatz

$$I_{\alpha\beta} = Aq^2 g_{\alpha\beta} + Bq_\alpha q_\beta \quad , \tag{1}$$

which implies the relations (2.35),

$$g^{\alpha\beta} I_{\alpha\beta} = (4A+B)q^2 \quad , \tag{2a}$$

$$q^\alpha q^\beta I_{\alpha\beta} = (A+B)q^4 \quad . \tag{2b}$$

For the calculation of these two Lorentz invariants, we take the frame of reference in which q^α consists of a time-like component only,

$$\tilde{q}^\alpha = \left(\tilde{q}^0 = \sqrt{q^2}, \boldsymbol{O}\right) \quad .$$

Thus, with *Exercise 2.6*

$$k^0 = \left[\boldsymbol{k}^2 + m^2\right]^{1/2} \quad , \quad k'^0 = \left[\boldsymbol{k}'^2 + m'^2\right]^{1/2} \quad , \tag{3}$$

and after performing the k' integration, we get

$$\begin{aligned} g^{\alpha\beta} I_{\alpha\beta} &= \int \frac{\mathrm{d}^3 k}{2k^0} \int \frac{\mathrm{d}^3 k'}{2k'^0} k_\alpha k'^\alpha \delta^3(\boldsymbol{k}+\boldsymbol{k}')\delta(k^0 + k'^0 - \tilde{q}^0) \\ &= \frac{1}{4} \int \mathrm{d}^3 k \left[\sqrt{(\boldsymbol{k}^2 + m^2)}\sqrt{(\boldsymbol{k}^2 + m'^2)}\right]^{-1} \\ &\quad \times \left[\sqrt{(\boldsymbol{k}^2 + m^2)}\sqrt{(\boldsymbol{k}^2 + m'^2)} + \boldsymbol{k}^2\right] \\ &\quad \times \delta\left(\sqrt{(\boldsymbol{k}^2 + m^2)} + \sqrt{(\boldsymbol{k}^2 + m'^2)} - \tilde{q}^0\right) \quad . \end{aligned} \tag{4}$$

We substitute the sum of the two square roots in the argument of the δ function by x,

$$x = \left[\boldsymbol{k}^2 + m^2\right]^{1/2} + \left[\boldsymbol{k}^2 + m'^2\right]^{1/2} \quad , \tag{5}$$

and transform to polar coordinatesmentum space. The volume element transforms into

$$\frac{|\boldsymbol{k}|\,\mathrm{d}|\boldsymbol{k}|}{\sqrt{(\boldsymbol{k}^2 + m^2)}\sqrt{(\boldsymbol{k}^2 + m'^2)}} = \frac{\mathrm{d}x}{x} \quad , \tag{6}$$

and by squaring (5), we have

$$\begin{aligned} m^2 - m'^2 - x^2 &= -2x\left[\boldsymbol{k}^2 + m'^2\right]^{1/2} \quad , \\ \boldsymbol{k}^2 &= \frac{(x^2 - m^2 + m'^2)^2}{4x^2} - m'^2 \\ &= \frac{(x^2 - m^2 - m'^2)^2}{4x^2} - \frac{m^2 m'^2}{x^2} \quad . \end{aligned} \tag{7}$$

Squaring (5) also yields the relation

$$\boldsymbol{k}^2 + \left[\boldsymbol{k}^2 + m^2\right]^{1/2}\left[\boldsymbol{k}^2 + m'^2\right]^{1/2} = \frac{(x^2 - m^2 - m'^2)}{2} \quad . \tag{8}$$

Equations (5)–(8) now give

$$\begin{aligned} g^{\alpha\beta} I_{\alpha\beta} &= \pi \int_0^\infty \frac{\mathrm{d}x}{x} |\boldsymbol{k}| \frac{1}{2}(x^2 - m^2 - m'^2)\delta(x - \tilde{q}^0) \\ &= \frac{\pi}{4(\tilde{q}^0)^2}\left((\tilde{q}^0)^2 - m^2 - m'^2\right)\left[\left((\tilde{q}^0)^2 - m^2 - m'^2\right)^2 - 4m^2 m'^2\right]^{1/2} \\ &= \frac{\pi}{4q^2}(q^2 - m^2 - m'^2)\left[(q^2 - m^2 - m'^2)^2 - 4m^2 m'^2\right]^{1/2} \quad , \end{aligned} \tag{9}$$

where the expression in its last form again is written in a manifestly Lorentz invariant form. In the same way we get for the second invariant

$$q^\alpha q^\beta I_{\alpha\beta} = \frac{(\tilde{q}^0)^2}{4} \int \mathrm{d}^3 k \int \mathrm{d}^3 k' \delta(\boldsymbol{k}+\boldsymbol{k}')\delta(k^0 + k'^0 - \tilde{q}^0)$$

Exercise 2.6

$$= \frac{(\tilde{q}^0)^2}{4} \int \mathrm{d}^3 k \delta\Big(\sqrt{(\boldsymbol{k}^2+m^2)} + \sqrt{(\boldsymbol{k}^2+m'^2)} - \tilde{q}^0\Big)$$

$$= \pi (\tilde{q}^0)^2 \int\limits_0^\infty \frac{\mathrm{d}x}{x} |\boldsymbol{k}| \big[\boldsymbol{k}^2+m^2\big]^{1/2} \big[\boldsymbol{k}^2+m'^2\big]^{1/2} \delta(x-\tilde{q}^0) \quad . \tag{10}$$

Combining (7) and (8) gives

$$\begin{aligned}
&\big[\boldsymbol{k}^2+m^2\big]^{1/2}\big[\boldsymbol{k}^2+m'^2\big]^{1/2} \\
&\quad = \frac{x^2-m^2-m'^2}{2} - \frac{(x^2-m^2-m'^2)^2}{4x^2} + \frac{m^2 m'^2}{x^2} \\
&\quad = (4x^2)^{-1}\big[x^4-(m^2+m'^2)^2+4m^2m'^2\big] \\
&\quad = (4x^2)^{-1}\big[x^4-(m^2-m'^2)^2\big] \quad ,
\end{aligned} \tag{11}$$

which facilitates the final calculation,

$$\begin{aligned}
q^\alpha q^\beta I_{\alpha\beta} &= \frac{\pi}{8(\tilde{q}^0)^2}\big[(\tilde{q}^0)^4-(m^2-m'^2)^2\big] \\
&\qquad \times \Big[\big((\tilde{q}^0)^2-m^2-m'^2\big)^2-4m^2m'^2\Big]^{1/2} \\
&= \frac{\pi}{8q^2}\big[q^4-(m^2-m'^2)^2\big] \\
&\qquad \times \big[(q^2-m^2-m'^2)^2-4m^2m'^2\big]^{1/2} \quad .
\end{aligned} \tag{12}$$

In addition, the δ function of (4) tells us that the results (9) and (12) are valid only for $\tilde{q}^0 = \sqrt{q^2} > m + m'$. This is expressed by the fact, that the expression under the square root in (9) and (12) may be written as follows:

$$\begin{aligned}
&(q^2-m^2-m'^2)^2-4m^2m'^2 \\
&\quad = \big[q^2-(m+m')^2\big]\big[q^2-(m-m')^2\big] \quad ,
\end{aligned} \tag{13}$$

which is easily checked. The radicand in (12) becomes negative for $q^2 < (m+m')^2$.

With the aid of definition (2) the quantities A and B can be determined:

$$\begin{aligned}
A &= (3q^2)^{-1}\left(g^{\alpha\beta} I_{\alpha\beta} - \frac{q^\alpha q^\beta}{q^2} I_{\alpha\beta}\right) \\
&= \frac{\pi}{24q^6}\big[q^2-(m+m')^2\big]^{1/2}\big[q^2-(m-m')^2\big]^{1/2} \\
&\quad \times \big[2q^2(q^2-m^2-m'^2)-q^4+(m^2-m'^2)^2\big]
\end{aligned}$$

Exercise 2.6

$$
\begin{aligned}
&= \frac{\pi}{24q^6}\left[q^2-(m+m')^2\right]^{3/2}\left[q^2-(m-m')^2\right]^{3/2} \\
&= \frac{\pi}{24}\left[1-\frac{(m+m')^2}{q^2}\right]^{3/2}\left[1-\frac{(m-m')^2}{q^2}\right]^{3/2} \quad , \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
B &= (3q^2)^{-1}\left(4\frac{q^\alpha q^\beta}{q^2}I_{\alpha\beta}-g^{\alpha\beta}I_{\alpha\beta}\right) \\
&= \frac{\pi}{12q^6}\left[q^2-(m+m')^2\right]^{1/2}\left[q^2-(m-m')^2\right]^{1/2} \\
&\quad \times\left[2q^4-2(m^2-m'^2)^2-q^2(q^2-m^2-m'^2)\right] \\
&= \frac{\pi}{12}\left[1-\frac{(m+m')^2}{q^2}\right]^{1/2}\left[1-\frac{(m-m')^2}{q^2}\right]^{1/2} \\
&\quad \times\left[1+\frac{m^2+m'^2}{q^2}-2\frac{(m^2-m'^2)^2}{q^4}\right] \quad . \qquad (15)
\end{aligned}
$$

The final result is thus

$$
\begin{aligned}
I_{\alpha\beta} &= \frac{\pi}{24}\left[1-\frac{(m+m')^2}{q^2}\right]^{1/2}\left[1-\frac{(m-m')^2}{q^2}\right]^{1/2} \\
&\quad \times\left[g_{\alpha\beta}q^2\left(1-\frac{(m+m')^2}{q^2}\right)\left(1-\frac{(m-m')^2}{q^2}\right)\right. \\
&\quad \left.+2q_\alpha q_\beta\left(1+\frac{m^2+m'^2}{q^2}-2\frac{(m^2-m'^2)^2}{q^4}\right)\right] \\
&\quad \times\Theta\left(q^2-(m+m')^2\right) \quad . \qquad (16)
\end{aligned}
$$

In the limit $m=m'=0$ one again gets (2.49) as is to be expected. For later use we note the special case $m'=0$ (that is, one of the two particles is a neutrino),

$$
\begin{aligned}
I_{\alpha\beta}(m) &= \frac{\pi}{24}\left(1-\frac{m^2}{q^2}\right)^2\left[q^2\left(1-\frac{m^2}{q^2}\right)g_{\alpha\beta}\right. \\
&\quad \left.+2\left(1+\frac{2m^2}{q^2}\right)q_\alpha q_\beta\right]\Theta(q^2-m^2) \quad . \qquad (17)
\end{aligned}
$$

2.4 Parity Violation in the Muon Decay

We now want to discuss two experiments which prove the violation of reflection invariance in muon decay. The first experiment observes the *decay of unpolarized muons* and measures the average helicity of the emitted electrons. In the second experiment one starts with *polarized muons*, which are produced by pion decay (see (1.32) and the subsequent equations), and measures only the angular distribution of the electron momenta with respect to the spin direction of the decaying muons.

Let us start with the first experiment. For unpolarized muons the expression (2.54) has to be averaged over $\boldsymbol{s}'$. Therefore all terms containing $\boldsymbol{s}'$ vanish:

$$\begin{aligned} \mathrm{d}\tilde{W} &= \frac{1}{2}\sum_{s'} \mathrm{d}W(\boldsymbol{s}') \\ &= \frac{G^2}{3}\frac{\pi\,\mathrm{d}^3p}{(2\pi)^5 p^0}\Theta(p^0_{\max}-p^0)\big\{\big[3(m_\mu-p^0)^2-\boldsymbol{p}^2\big] \\ &\quad\times(p^0-m_\mathrm{e}s^0)+2(m_\mu-p^0)(\boldsymbol{p}^2-m_\mathrm{e}\boldsymbol{p}\cdot\tilde{\boldsymbol{s}})\big\} \quad , \end{aligned} \tag{2.66}$$

where $\tilde{\boldsymbol{s}}$ is the space-like component of the spin four-vector s^α. The four-vector of the electron spin also satisfies (2.23), and we get

$$\begin{aligned} m_\mathrm{e}s^0 &= \boldsymbol{p}\cdot\boldsymbol{s} \quad , \\ \boldsymbol{p}\cdot\tilde{\boldsymbol{s}} &= (\boldsymbol{p}\cdot\boldsymbol{s})\left[1+\frac{|\boldsymbol{p}|^2}{m_\mathrm{e}(p^0+m_\mathrm{e})}\right]=\frac{p^0}{m_\mathrm{e}}(\boldsymbol{p}\cdot\boldsymbol{s}) \quad , \end{aligned} \tag{2.67}$$

where $\boldsymbol{s}$ is the spin vector defined in the muon rest frame. The two possible eigenstates of the helicity operator $\hat{\Lambda}=\boldsymbol{\sigma}\cdot\boldsymbol{p}/|\boldsymbol{p}|$ correspond to the values

$$\boldsymbol{p}\cdot\boldsymbol{s}=\pm|\boldsymbol{p}| \quad . \tag{2.68}$$

Remember that the spin vector $\boldsymbol{s}$ within relativistic quantum mechanics (2.23) is normalized to 1, i.e. $\boldsymbol{s}\cdot\boldsymbol{s}=1$, so that $s_\mu s^\mu=-1$.[7] In the first case, electron spin and direction of motion are parallel and in the second case antiparallel; the corresponding helicities are $(+1)$ and (-1), respectively. Because of (2.66) we obtain the following average value of the helicity operator:

$$\begin{aligned} \langle\Lambda\rangle &= \frac{\mathrm{d}\tilde{W}(\boldsymbol{p}\cdot\boldsymbol{s}=|\boldsymbol{p}|)-\mathrm{d}\tilde{W}(\boldsymbol{p}\cdot\boldsymbol{s}=-|\boldsymbol{p}|)}{\mathrm{d}\tilde{W}(\boldsymbol{p}\cdot\boldsymbol{s}=|\boldsymbol{p}|)+\mathrm{d}\tilde{W}(\boldsymbol{p}\cdot\boldsymbol{s}=-|\boldsymbol{p}|)} \\ &= \frac{-2|\boldsymbol{p}|[3(m_\mu-p^0)^2-\boldsymbol{p}^2+2(m_\mu-p^0)p^0]}{2[3(m_\mu-p^0)^2p^0-\boldsymbol{p}^2p^0+2(m_\mu-p^0)\boldsymbol{p}^2]} \quad . \end{aligned} \tag{2.69}$$

We demonstrate this simply for the nominator only:

$$\begin{aligned} &\big[3(m_\mu-p^0)^2-|\boldsymbol{p}|^2\big](p^0-|\boldsymbol{p}|)+2(m_\mu-p^0)(\boldsymbol{p}^2-p^0|\boldsymbol{p}|) \\ &\quad-\Big[\big[3(m_\mu-p^0)^2-|\boldsymbol{p}|^2\big](p^0+|\boldsymbol{p}|)+2(m_\mu-p^0)(\boldsymbol{p}^2+p^0|\boldsymbol{p}|)\Big] \\ &=\big[3(m_\mu-p^0)^2-|\boldsymbol{p}|^2\big](-2|\boldsymbol{p}|)+2(m_\mu-p^0)(-2p^0|\boldsymbol{p}|) \\ &=-2|\boldsymbol{p}|\big[3(m_\mu-p^0)^2-|\boldsymbol{p}|^2+2(m_\mu-p^0)p^0\big] \quad . \end{aligned}$$

[7] See W. Greiner: *Relativistic Quantum Mechanics – Wave Equations*, 3rd ed. (Springer, Berlin, Heidelberg, 2000).

Applying the value (2.52) for the maximum electron energy, the expression (2.69) can be written in the form

$$\frac{-|\boldsymbol{p}|m_\mu(3p^0_{\max}-2p^0-m_\mathrm{e}^2/m_\mu)}{m_\mu p^0(3p^0_{\max}-2p^0-m_\mathrm{e}^2/p^0)}=-1+O\left(\frac{m_\mathrm{e}^2}{(p^0)^2}\right)\quad . \tag{2.70}$$

Let us quickly verify this result by inserting

$$\boldsymbol{p}^2=(p^0)^2-m_\mathrm{e}^2$$

and

$$p^0_{\max}=\frac{m_\mu^2+m_\mathrm{e}^2}{2m_\mu}\quad\text{or}\quad m_\mu=2p^0_{\max}-\frac{m_\mathrm{e}^2}{m_\mu},$$

which yields for nominator $\mathcal{N}$:

$$\begin{aligned}
\mathcal{N}&=-2|\boldsymbol{p}|\big(3m_\mu^2-6m_\mu p^0+3(p^0)^2-(p^0)^2+m_\mathrm{e}^2+2m_\mu p^0-2(p^0)^2\big)\\
&=-2|\boldsymbol{p}|\left(3m_\mu^2-4m_\mu p^0+\frac{m_\mathrm{e}^2}{m_\mu}m_\mu\right)\\
&=-2|\boldsymbol{p}|m_\mu\left(3\left(2p^0_{\max}-\frac{m_\mathrm{e}^2}{m_\mu}\right)-4p^0+\frac{m_\mathrm{e}^2}{m_\mu}\right)\\
&=-4|\boldsymbol{p}|m_\mu\left(3p^0_{\max}-2p^0-\frac{m_\mathrm{e}^2}{m_\mu}\right),
\end{aligned}$$

and for the denominator $\mathcal{D}$:

$$\begin{aligned}
\mathcal{D}=2\Big(&3m_\mu^2p^0-6m_\mu(p^0)^2+3(p^0)^3-(p^0)^3+p^0m_\mathrm{e}^2\\
&+2m_\mu(p^0)^2-2m_\mu m_\mathrm{e}^2-2(p^0)^3+2p^0m_\mathrm{e}^2\Big)\\
=\;&2m_\mu p^0\left(3m_\mu-4p^0+3\frac{m_\mathrm{e}^2}{m_\mu}-2\frac{m_\mathrm{e}^2}{p^0}\right)\\
=\;&2m_\mu p^0\left(3\cdot 2p^0_{\max}-4p^0-2\frac{m_\mathrm{e}^2}{p^0}\right)\\
=\;&4m_\mu p^0\left(3p^0_{\max}-2p^0-\frac{m_\mathrm{e}^2}{p^0}\right)
\end{aligned}$$

and, therefore,

$$\langle\Lambda\rangle=\frac{-|\boldsymbol{p}|}{p^0}\frac{3p^0_{\max}-2p^0-\frac{m_\mathrm{e}^2}{m_\mu}}{3p^0_{\max}-2p^0-\frac{m_\mathrm{e}^2}{p^0}}.$$

The result (2.70) is most interesting. We notice that for energies $p^0\gg m_\mathrm{e}$ the electron is predicted to be in an almost completely left-handed state. For the average electron helicity in the kinematically allowed energy interval $[m_e, p^0_{max}]$, with $p^0_{max} \approx 100 m_e$, the experimentally observed value is[8]

$$\overline{\langle \Lambda \rangle} = -1.00 \pm 0.13 \quad . \tag{2.71}$$

The fact that the electron is limited to a left-handed state follows directly from the interaction (2.10), because the electronic transition current can be written as $\bar{U}_e \gamma^\mu (1-\gamma_5) U_{\nu_e} = \frac{1}{2}\overline{U_2(1-\gamma_5)}\gamma^\mu U_{\nu_e}$. Thus, the electron, like the electron neutrino, has negative helicity. High-energy electrons are thus negatively polarized.

Next we consider the experiment in which the angular distribution of the electrons emitted in the decay of polarized muons is measured. Since the electron helicity is not observed, we must sum over the electron spin in (2.54). The value of the muon spin is assumed to be fixed. Let us begin with the expression (2.50), which we denote once more

$$\begin{aligned} \mathrm{d}W = \frac{G^2}{3}\frac{\pi \mathrm{d}^3 p}{(2\pi)^5 p'^0 p^0}\big[&(p'-p)^2 (p'-m_\mu s')^\alpha (p-m_e s)_\alpha \\ &+ 2(p'-p)_\alpha (p'-m_\mu s')^\alpha (p'-p)_\beta (p-m_e s)^\beta\big]\Theta(p^0_{max}-p^0) \quad . \end{aligned}$$

We remember that due to the discussion following (2.51)–(2.53) the two step functions in (2.50) can be abbreviated by $\Theta(p^0_{max}-p^0)$. Now the summation of the expression in the bracket $[\ldots]$ over the electron spins yields

$$\sum_{\pm s}[\ldots] = 2(p'-p)^2 (p'-m_\mu s')^\alpha p_\alpha + 4(p'-p)_\alpha (p'-m_\mu s')^\alpha (p'-p)_\beta p^\beta \quad .$$

It is easier to continue the calculation in the rest frame of the muon, for which $p'^\nu = (m_\mu, 0)$, $s'^\nu = (0, \boldsymbol{s}')$ holds. Then

$$\begin{aligned} \sum_{\pm s}[\ldots] = {} & 2\big[(m_\mu - p^0)^2 - \boldsymbol{p}^2\big]\big[m_\mu p^0 + m_\mu \boldsymbol{s}'\cdot\boldsymbol{p}\big] \\ & + 4\big[(m_\mu - p^0)m_\mu - m_\mu \boldsymbol{s}'\cdot\boldsymbol{p}\big]\big[(m_\mu - p^0)p^0 + \boldsymbol{p}^2\big] \quad . \end{aligned}$$

Inserting $\boldsymbol{p}^2 = p_0^2 - m_e^2$, $p^0_{max} = (m_\mu^2 + m_e^2)/2m_\mu$, $\boldsymbol{s}'\cdot\boldsymbol{p} = 1\cdot|\boldsymbol{p}|\cos\theta$ and separating terms proportional to $\cos\theta$ yields

$$\begin{aligned} \sum_{\pm s}[\ldots] &= 4m_\mu^2\left[\frac{m_\mu^2 + m_e^2}{2m_\mu} - p^0\right]\big[p^0 + \boldsymbol{s}'\cdot\boldsymbol{p}\big] \\ &\quad + 4m_\mu^2\big[(m_\mu - p^0) - \boldsymbol{s}'\cdot\boldsymbol{p}\big]\left[p^0 - \frac{m_e^2}{m_\mu}\right] \\ &= 4m_\mu^2\left[p^0\left(p^0_{max} - p^0 + m_\mu - p^0 - \frac{m_e^2}{p^0} + \frac{m_e^2}{m_\mu}\right)\right. \\ &\quad \left. + |\boldsymbol{p}|\cos\theta\left(p^0_{max} - p^0 - p^0 + \frac{m_e^2}{m_\mu}\right)\right] \end{aligned}$$

[8] Review of Particle Properties in: Review of Modern Physics (April 1988); J. Duclos, J. Heintze, A. de Rujula, V. Soergel: Phys. Lett. **9**, 62 (1964).

$$= 4m_\mu^2 p^0 \left[\left(3p^0_{\max} - 2p^0 - \frac{m_e^2}{p^0} \right) + \frac{|\boldsymbol{p}|}{p^0} \cos\theta \left(p^0_{\max} - 2p^0 + \frac{m_e^2}{m_\mu} \right) \right] \quad .$$

Therefore, the decay rate summed over the electron spin is given by

$$\begin{aligned} \mathrm{d}\bar{W} &= \sum_s \mathrm{d}W(s) \\ &= \frac{2G^2}{3(2\pi)^3} m_\mu |\boldsymbol{p}| p^0 \mathrm{d}p^0 \sin\theta \mathrm{d}\theta \left[3p^0_{\max} - 2p^0 - \frac{m_e^2}{p^0} \right. \\ &\quad \left. + \frac{|\boldsymbol{p}|}{p^0} \cos\theta \left(p^0_{\max} - 2p^0 + \frac{m_e^2}{m_\mu} \right) \right] \Theta(p^0_{\max} - p^0) \quad . \end{aligned} \tag{2.72}$$

Here θ denotes the angle between the muon spin $\boldsymbol{s}'$ and the electron momentum $\boldsymbol{p}$. The volume element of the electron momentum space has been used according to (2.56) in the form

$$\mathrm{d}^3 p = 2\pi |\boldsymbol{p}|^2 \mathrm{d}|\boldsymbol{p}| \sin\theta \mathrm{d}\theta = 2\pi |\boldsymbol{p}| p^0 \mathrm{d}p^0 \sin\theta \mathrm{d}\theta \quad . \tag{2.73}$$

Equation (2.72) does not yet contain the electromagnetic corrections. If one considers the corrections of the order $\alpha = e^2/\hbar c \simeq 1/137$, some terms are added to $\mathrm{d}\bar{W}$. But the parity-violating structure, which is expressed in the factor $\cos\theta$ in (2.72), is not changed. The agreement between the predicted angular distribution $\mathrm{d}\bar{W}$ and the experimentally measured one is better than 0.5%.

EXERCISE

2.7 Average Helicity and Parity Violation

Problem. Calculate the helicity expectation value averaged over the whole energy region and show that the result $\overline{\langle \Lambda \rangle} = -1$ is evidence for the violation of parity invariance.

Solution. (a) We set $|\boldsymbol{p}| = p$ and

$$\mathrm{d}\tilde{W}(\boldsymbol{p} \cdot \boldsymbol{s} = \pm|\boldsymbol{p}|) = \mathrm{d}\tilde{W}^{\pm}(p) \quad . \tag{1}$$

The probability of an electron being emitted with momentum $\boldsymbol{p}$ is

$$\mathrm{d}\tilde{W}^{+}(p) + \mathrm{d}\tilde{W}^{-}(p) \quad . \tag{2}$$

The average of the expectation values is therefore

$$\begin{aligned} \langle \Lambda \rangle &= \frac{\int [\overline{\langle \Lambda \rangle}(\mathrm{d}\tilde{W}^{+}(p)) + \overline{\langle \Lambda \rangle}(\mathrm{d}\tilde{W}^{-}(p))]}{\int [\mathrm{d}\tilde{W}^{+}(p) + \mathrm{d}\tilde{W}^{-}(p)]} \\ &= \frac{\int [\mathrm{d}\tilde{W}^{+}(p) - \mathrm{d}\tilde{W}^{-}(p)]}{W_\mu} \quad . \end{aligned} \tag{3}$$

Exercise 2.7

W_μ is already known from (2.57); we only need to calculate the numerator. With the help of (2.66)–(2.68) we get

$$\begin{aligned}
&\int \left[\mathrm{d}\tilde{W}^+(p) - \mathrm{d}\tilde{W}^-(p)\right] \\
&= -\frac{2}{3}G^2\pi(2\pi)^{-5}\int \frac{\mathrm{d}^3 p}{p^0}\Theta(p^0_{\max} - p^0)p\left[3(m_\mu - p^0)^2 - p^2 + 2(m_\mu - p^0)p^0\right] \\
&= -\frac{2}{3}G^2\pi(2\pi)^{-5}4\pi \int\limits_{m_\mathrm{e}}^{p^0_{\max}} p^2\mathrm{d}p^0\left[3(m_\mu - p^0)^2 - p^2 + 2(m_\mu - p^0)p^0\right] \\
&= -\frac{2}{3}G^2(2\pi)^{-3}m_\mu \int\limits_{m_\mathrm{e}}^{p^0_{\max}} \mathrm{d}p^0\left[(p^0)^2 - m_\mathrm{e}^2\right]\left[3m_\mu - 4p^0 + \frac{m_\mathrm{e}^2}{m_\mu}\right] \quad . \qquad (4)
\end{aligned}$$

Performing the integral yields

$$\begin{aligned}
&\int \left[\mathrm{d}\tilde{W}^+(p) - \mathrm{d}\tilde{W}^-(p)\right] \\
&= -\frac{2}{3}G^2(2\pi)^{-3}m_\mu\left\{\left(m_\mu + \frac{m_\mathrm{e}^2}{3m_\mu}\right)\left[(p^0_{\max})^3 - 3m_\mathrm{e}^2 p^0_{\max} + 2m_\mathrm{e}^3\right]\right. \\
&\quad \left. - (p^0_{\max})^4 + 2m_\mathrm{e}^2(p^0_{\max})^2 - m_\mathrm{e}^4\right\} \\
&= -\frac{G^2 m_\mu^5}{24(2\pi)^3}\left(1 - \frac{40}{3}y + 2\sqrt{y^3} - 30y^2 + \frac{32}{3}y\sqrt{y^5} - \frac{1}{3}y^4\right) \quad , \qquad (5)
\end{aligned}$$

where again $y = (m_\mathrm{e}/m_\mu)^2$. Applying (2.57) we obtain in lowest order in y

$$\begin{aligned}
\langle\Lambda\rangle &\simeq -\frac{1 - \frac{40}{3}y}{1 - 8y} \\
&\simeq -1 + \frac{16}{3} \times \frac{m_\mathrm{e}^2}{m_\mu^2} + \cdots \\
&\simeq -0.999\,88 \quad . \qquad (6)
\end{aligned}$$

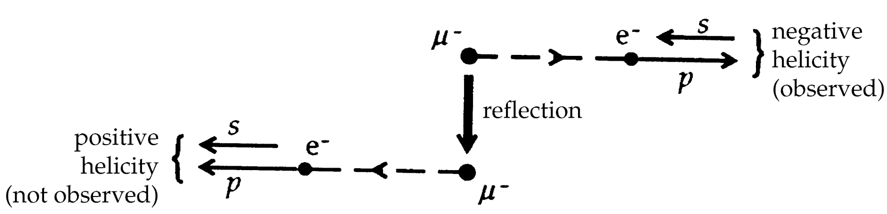


Fig. 2.4. Parity violation in muon decay

(b) Obviously (almost) all the electrons emitted in muon decay have negative helicity ($\lambda = -1$). A space reflection (see Fig. 2.4) would give the electrons positive helicity ($\lambda = +1$). In the case of parity invariance of the process, one would therefore measure equal numbers of electrons with positive and negative helicity. This is not the case; thus parity invariance must be broken.

EXERCISE

2.8 Angular Distribution and Parity Violation

Problem. Show that the violation of parity is due to the appearance of $\cos\theta$ in (2.72), describing the angular distribution.

Solution. We can write the angular distribution as

$$\frac{\mathrm{d}\bar{W}}{\mathrm{d}p^0\mathrm{d}\theta} = \sin\theta\big[A(p^0) + B(p^0)\cos\theta\big] \quad , \tag{1}$$

where $A(p^0)$ and $B(p^0)$ are given by comparison with (2.72). The geometry is displayed in Fig. 2.5. If we perform a space reflection, θ changes to $\theta_s = \pi - \theta$, and

$$\begin{aligned} \sin\theta &\to \sin\theta_s = \sin\theta \quad , \\ \cos\theta &\to \cos\theta_s = -\cos\theta \end{aligned} \tag{2}$$

thus the angular distribution becomes

$$\frac{\mathrm{d}\bar{W}}{\mathrm{d}p^0\mathrm{d}\theta_s} = \sin\theta\{A(p^0) - B(p^0)\cos\theta\} \quad . \tag{3}$$

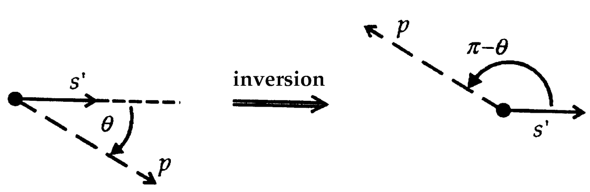


Fig. 2.5. Parity violation in an angular distribution

$\mathrm{d}\bar{W}/\mathrm{d}p^0\mathrm{d}\theta_s$ and $\mathrm{d}\bar{W}/\mathrm{d}p^0\mathrm{d}\theta$ differ from each other in the sign of the term proportional to $\cos\theta$: the angular distribution is not parity invariant. This argument is supported by geometrical considerations. The figure shows the intensity of the emitted electrons for $B(p^0) = 0$: no electrons are emitted in the direction of $\boldsymbol{s}'$. If $B(p^0) > 0$ and $B(p^0) <$

$A(p^0)$, the distribution is deformed and not reflection invariant, because $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$.

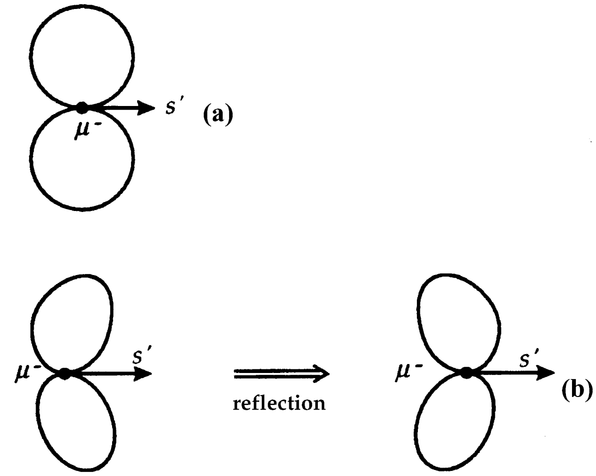


Fig. 2.6. Angular distribution of electrons with respect to the muon spin: (**a**) without violation of parity; (**b**) with violation of parity

EXERCISE

2.9 Electron Helicity in Muon Decay

Problem. Show, for the limit $p^0 \gg m_e$, that for the decay of a muon with spin $\boldsymbol{s}'$ in its rest system, the emission of an electron with spin $\boldsymbol{s}$ is given by $\mathrm{d}W \sim \sin^2(\theta/2)$, where $\mathrm{d}W$ is calculated in the limit $p^0 \gg m_e$ and θ denotes the angle between the electron spin $\boldsymbol{s}$ and momentum $\boldsymbol{p}$ of the electron.

Solution. We start from (2.23) inserted in (2.54) ($m = m_e$) and neglect systematically all terms with m_e. It is important to recognize that terms like $m_e s^0$ or $m_e \boldsymbol{p}\cdot\tilde{\boldsymbol{s}}$ do not contain the effective electron mass. Remember, $\tilde{\boldsymbol{s}}$ is the space component of the electron spin vector (2.23)! In this spirit we have

$$|\boldsymbol{p}| = \sqrt{(p^0)^2 + m_e^2} \approx p^0 \quad , \tag{1}$$

so we can write

$$\begin{aligned}
\mathrm{d}W \approx\; & \frac{G^2}{3}\frac{\pi\,\mathrm{d}^3p}{(2\pi)^5 p^0 m_\mu}\Theta(p^0_{\max} - p^0) \\
& \times\Bigg[m_\mu(m_\mu - 2p^0)\left(p^0 - \boldsymbol{p}\cdot\boldsymbol{s} + \boldsymbol{s}'\cdot\boldsymbol{p} - \frac{(\boldsymbol{p}\cdot\boldsymbol{s})(\boldsymbol{p}\cdot\boldsymbol{s}')}{p^0}\right) \\
& + 2(m_\mu - p^0 - \boldsymbol{p}\cdot\boldsymbol{s}')\big[(m_\mu - p^0)(p^0 - \boldsymbol{p}\cdot\boldsymbol{s}) + p^0(p^0 - \boldsymbol{p}\cdot\boldsymbol{s})\big]\Bigg] \\
=\; & \frac{G^2}{3}\frac{\pi\,\mathrm{d}^3p}{(2\pi)^5 m_\mu}\Theta(p^0_{\max} - p^0)\left(1 - \frac{\boldsymbol{p}\cdot\boldsymbol{s}}{p^0}\right)m_\mu
\end{aligned}$$

$$\times\left[(m_\mu - 2p^0)\left(1-\frac{\boldsymbol{p}\cdot\boldsymbol{s}'}{p^0}\right)+2(m_\mu - p^0 - \boldsymbol{p}\cdot\boldsymbol{s}')\right]$$

$$=\frac{G^2}{3}\frac{\mathrm{d}^3 p}{(2\pi)^5}\Theta(p^0{}_{\max}-p^0)\left(1-\frac{\boldsymbol{p}\cdot\boldsymbol{s}}{p^0}\right)$$

$$\times\left[3p^0_{\max}-2p^0-\frac{\boldsymbol{p}\cdot\boldsymbol{s}'}{p^0}p^0_{\max}\right] \quad . \tag{2}$$

In the last step we have applied (2.52) for the matrix elements of the electron

$$p^0_{\max}=\frac{m_\mu^2+m_\mathrm{e}^2}{2m_\mu}\simeq\frac{1}{2}m_\mu \quad . \tag{3}$$

The coefficient in (2) which contains the spin $\boldsymbol{s}$ of the electron gives the desired angular dependence:

$$1-\frac{\boldsymbol{p}\cdot\boldsymbol{s}}{p^0}=1-\frac{|\boldsymbol{p}|}{p^0}\cos\theta\simeq 2\sin^2\frac{\theta}{2} \quad . \tag{4}$$

The maximum of the distribution is at $\theta=\pi$, that is the electrons are preferentially polarized against their momentum (negative helicity). The result (4) is in accordance with the angular distribution in (1.12), which we calculated from the β decay of cobalt, if we take $\theta=\pi-\theta$ (here the z axis points downwards!). This is another confirmation of the heuristic consideration in relation to the experiment of Wu et al. (see Sect. 1.2).

In the limit $p^0\to p^0_{\max}$ the last factor takes the form

$$3p^0_{\max}-2p^0-p^0_{\max}\frac{\boldsymbol{p}\cdot\boldsymbol{s}'}{p^0}\to p^0_{\max}\left(1-\frac{\boldsymbol{p}\cdot\boldsymbol{s}'}{p^0}\right) \quad , \tag{5}$$

that is, the preferential emission of the electron is opposite to the polarization of the muons. This is easy to see for the case $p^0=p^0_{\max}$ in Fig. 2.2, where the two neutrinos are emitted in the same direction while the electron goes in the opposite direction. Because $\bar{\nu}_\mathrm{e}$ and ν_μ have opposite helicities, the sum of their angular momenta is equal to zero. The result is that the electron must acquire the spin of the decaying muon. Because of its negative helicity the electron is preferentially emitted opposite to the muon spin.

EXERCISE

2.10 CP Invariance in Muon Decay

Problem. The term $J^{\alpha\dagger}_{(\mathrm{e})}\,J^{(\mu)}_{\alpha}$ in the current–current coupling is responsible for the decay of the positive muon, μ^+. Show that this leads to a change of the sign of the spin-dependent terms in the squared transition amplitude (2.32). On the basis of these results discuss the connection between violation of the invariance under spatial reflection and the invariance under charge conjugation.

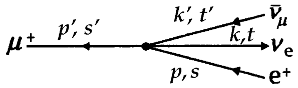


Fig. 2.7. Decay of a μ^+ particle[7]

Solution. (a) The S-matrix element for the μ^+ decay is given by[9]

$$S(\mu^+ \to e^+\nu_e\bar{\nu}_\mu) = -i\int d^4x \frac{G}{\sqrt{2}}\left[\bar{u}_\mu(x)\gamma^\mu(1-\gamma_5)u_{\nu_\mu}(x)\right]\left[\bar{u}_{\nu_e}(x)\gamma_\mu(1-\gamma_5)u_e(x)\right] \quad . \tag{1}$$

As now all particles are antiparticles, except the ν_e, the spinors are given in analogy to (2.11) by

$$\begin{aligned} u_\mu(x) &= v_\mu(\boldsymbol{p}', s')\exp(ip'_\mu x^\mu)(2p'^0V)^{-1/2} \quad , \\ u_e(x) &= v_e(\boldsymbol{p}, s)\exp(ip_\mu x^\mu)(2p^0V)^{-1/2} \quad , \\ u_{\nu_e}(x) &= u_{\nu_e}(\boldsymbol{k}, t)\exp(-ik_\mu x^\mu)(2k^0V)^{-1/2} \quad , \\ u_{\nu_\mu}(x) &= v_{\nu_\mu}(\boldsymbol{k}', t')\exp(ik'_\mu x^\mu)(2k'^0V)^{-1/2} \quad . \end{aligned} \tag{2}$$

The calculation proceeds exactly as before up to (2.18), (2.19), because the δ function does not change when the sign of its argument is inverted. Thus we obtain

$$dW(\mu^+) = \frac{G^2}{2}\frac{1}{(2\pi)^5}\frac{d^3p}{2p'^0 2p^0} \times \int\frac{d^3k}{2k^0}\int\frac{d^3k'}{2k'^0}\delta^4(p+k+k'-p')\sum_{t,t'}|M^2| \quad , \tag{3}$$

with

$$M = \left[\bar{v}_\mu\gamma^\mu(1-\gamma_5)v_{\nu_\mu}\right]\left[\bar{u}_{\nu_e}\gamma_\mu(1-\gamma_5)v_e\right] \quad . \tag{4}$$

The only difference compared to μ^- decay is in the spinors which enter into the transition amplitude M, where all particles are replaced by antiparticle spinors and vice versa. $\sum_{t,t'}|M|^2$ separates again into two similar contributions for the muonic and electronic particles. First we repeat the calculation from (2.20) to (2.30) for the muonic part. Here we need the analogous relation to (2.21) (see Appendix A.2),

$$\sum_{t'} v_{\nu_\mu}(k', t')_\alpha \bar{v}_{\nu_\mu}(k', t')_\beta = \not{k}'_{\alpha\beta} \quad , \tag{5}$$

and to (2.22),

$$\bar{v}_\mu(p', s')_\pi v_\mu(p', s')_\tau = \left[(\not{p}' - m_\mu)\frac{1+\gamma_5\not{s}'}{2}\right]_{\tau\pi} \quad . \tag{6}$$

With these expressions we find that

$$\begin{aligned} X^{\mu\nu}(\mu) &= \sum_{t'}\bar{v}_\mu(p', s')_\pi\gamma^\mu_{\pi\rho}(1-\gamma_5)_{\rho\alpha} \\ &\quad \times v_{\nu_\mu}(k', t')_\alpha\bar{v}_{\nu_\mu}(k', t')_\beta\gamma^\nu_{\beta\sigma}(1-\gamma_5)_{\sigma\tau}v_\mu(p', s')_\tau \\ &= \mathrm{Tr}\left\{\gamma^\mu(1-\gamma_5)\not{k}'\gamma^\nu(1-\gamma_5)(\not{p}'-m_\mu)\left(\frac{1+\gamma_5\not{s}'}{2}\right)\right\} \quad . \end{aligned} \tag{7}$$

---

[9] M.L. Perl: Ann. Rev. Nucl. Part. Science **30**, 299 (1980).

With a cyclical permutation we take the last two factors to the front; we also permute $\not{k}'$ with $(1-\gamma_5)$, changing the sign in $(1-\gamma_5)$. We thus obtain

$$X^{\mu\nu}(\mu) = \frac{1}{2}\,\mathrm{Tr}\big\{(\not{p}' - m_\mu)(1+\gamma_5\not{s}')\gamma^\mu \not{k}'(1+\gamma_5)\gamma^\nu(1-\gamma_5)\big\} \quad . \tag{8}$$

This result is distinct from (2.26) by the sign of m_μ and also by permutation of the Lorentz indices μ and ν. Hence we can skip all subsequent calculations and write down directly the analogue of (2.30):

$$\begin{aligned} X^{\mu\nu}(\mu) = 4\big[&(p' + m_\mu s')^\mu k'^\nu - (p' + m_\mu s')^\alpha k'_\alpha g^{\nu\mu} \\ &+ (p' + m_\mu s')^\nu k'^\mu + \mathrm{i}\epsilon^{\alpha\mu\beta\nu}(p' + m_\mu s')_\alpha k'_\beta\big] \quad . \end{aligned} \tag{9}$$

For the electronic part the same relation holds; compared to (2.31) it changes the sign of m_e, and μ and ν have to be permuted:

$$\begin{aligned} X_{\mu\nu}(\mathrm{e}) &= \sum_t \bar{u}_{\nu_\mathrm{e}}(k,t)\gamma_\mu(1-\gamma_5)v_\mathrm{e}(p,s)\bar{v}_\mathrm{e}(p,s)\gamma_\nu(1-\gamma_5)u_{\nu_\mathrm{e}}(k,t) \\ &= 4\big[(p + m_\mathrm{e}s)_\nu k_\mu - (p + m_\mathrm{e}s)^\alpha k_\alpha g_{\nu\mu} \\ &\quad + (p + m_\mathrm{e}s)_\mu k_\nu + \mathrm{i}\epsilon_{\alpha\nu\beta\mu}(p + m_\mathrm{e}s)^\alpha k^\beta\big] \quad . \end{aligned} \tag{10}$$

The permutation can be reversed, because by constructing $|M|^2$ we sum over the indices μ and ν. What remains is just the change of the sign of m_μ and m_e in (2.32):

$$\begin{aligned} \sum_{t,t'} |M|^2 &= X^{\mu\nu}(\mu)X_{\mu\nu}(\mathrm{e}) \\ &= 64(p' + m_\mu s')^\alpha k_\alpha (p + m_\mathrm{e}s)^\beta k'_\beta \quad . \end{aligned} \tag{11}$$

As the spin vectors s^μ and s'^μ enter only in the combination $m_\mu s'$ or $m_\mathrm{e}s$, we can also easily get the result (11) from (2.32) by inversion of the spin vectors: $s, s' \to -s, -s'$.

(b) Equation (11) follows from (2.32) if we invert the sign of the charge of the decaying muon, that is, it follows from the operation of charge conjugation. We thus see that *the β decay of the muon is not invariant against charge conjugation.*

An interesting point is that (11) could also be obtained by space reflection. On being reflected, the momentum vector $\boldsymbol{p}'$ changes its sign, whereas the axial spin vector (in the rest system) $\boldsymbol{s}'$ does not change:

$$p'^\alpha = (p'^0, \boldsymbol{p}') \to (p'^0, -\boldsymbol{p}') \quad , \tag{12}$$

$$\begin{aligned} s'^\alpha &= \left(\frac{\boldsymbol{p}'\cdot\boldsymbol{s}'}{m_\mu}, \boldsymbol{s}' + \frac{(\boldsymbol{p}'\cdot\boldsymbol{s}')}{m_\mu(p'^0 + m_\mu)}\boldsymbol{p}'\right) \\ &\to \left(-\frac{\boldsymbol{p}'\cdot\boldsymbol{s}'}{m_\mu}, \boldsymbol{s}' + \frac{(\boldsymbol{p}'\cdot\boldsymbol{s}')}{m_\mu(p'^0 + m_\mu)}\boldsymbol{p}'\right) \quad , \end{aligned} \tag{13}$$

and

$$k^\alpha = (k^0, \boldsymbol{k}) \to (k^0, -\boldsymbol{k})$$

or

$$k_\alpha = (k^0, -\boldsymbol{k}) \to (k^0, \boldsymbol{k}) \quad . \tag{14}$$

With these we obtain the following change to (2.32):

$$
\begin{aligned}
&(p' - m_\mu s')^\alpha k_\alpha \\
&\quad = \left(p'^0 - \frac{\boldsymbol{p}' \cdot \boldsymbol{s}'}{m_\mu}\right)k^0 - \left(\boldsymbol{p}' - m_\mu \boldsymbol{s}' - \frac{\boldsymbol{p}' \cdot \boldsymbol{s}'}{p^0 + m_\mu}\boldsymbol{p}'\right) \cdot \boldsymbol{k} \\
&\quad \to \left(p'^0 + \frac{\boldsymbol{p}' \cdot \boldsymbol{s}'}{m_\mu}\right)k^0 + \left(-\boldsymbol{p}' - m_\mu \boldsymbol{s}' - \frac{(\boldsymbol{p}' \cdot \boldsymbol{s}')}{p'^0 + m_\mu}\boldsymbol{p}'\right) \cdot \boldsymbol{k} \\
&\quad = (p' + m_\mu s')^\alpha k_\alpha
\end{aligned} \tag{15}
$$

and analogously

$$
(p - m_\mathrm{e} s)^\beta k'_\beta \to (p + m_\mathrm{e} s)^\beta k'_\beta \quad . \tag{16}
$$

We can thus conclude that the weak interaction (of the leptons) behaves under charge conjugation $\hat{C}$ in the same way as under space reflection $\hat{P}$. Since according to this the simultaneous application of $\hat{C}$ and $\hat{P}$ yields the identity, that is, everything remains invariant, it means that the weak interaction is invariant under the product $\hat{C}\hat{P}$. (We shall see later, in Chap. 8, how the weak interaction among quarks can lead to a slight violation of CP invariance.)

2.5 The Michel Parameters

We now ask how far the muon decay confirms the V–A theory. For this purpose we write down the most general form of the coupling matrix element,

$$
\begin{aligned}
\tilde{H}_{\mu^- \to \mathrm{e}^- \bar{\nu}_\mathrm{e} \nu_\mu} = \frac{G}{\sqrt{2}} \int \mathrm{d}^3 x \; \sum_i \left[\bar{u}_{\nu_\mu}(x) \hat{O}_i u_\mu(x)\right] \\
\times \left[\bar{u}_\mathrm{e}(x) \hat{O}^i (A_i + A'_i \gamma_5) u_{\nu_\mathrm{e}}(x)\right] \quad ,
\end{aligned} \tag{2.74}
$$

and allow this time every type of coupling $\hat{O}_i = (\mathrm{S}, \mathrm{V}, \mathrm{T}, \mathrm{A}, \mathrm{P})$. It is customary to use other constants C_i, C'_i instead of A_i, A'_i. The two sets of constants are related to each other through the transformation (the so-called Fierz transformation, see Supplement 2.12):

$$
\begin{aligned}
&C_i = \sum_j \Lambda_{ij} A_j \quad , \quad C'_i = \sum_j \Lambda_{ij} A'_j \quad , \\
&(\Lambda_{ij}) = \frac{1}{4} \begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} \quad .
\end{aligned} \tag{2.75}
$$

With C_i and C'_i, we can write the coupling in the form

$$
\tilde{H} = \frac{G}{\sqrt{2}} \int \mathrm{d}^3 x \; \sum_i \left[\bar{u}_\mathrm{e}(x) \hat{O}_i u_\mu(x)\right]\left[\bar{u}_{\nu_\mu}(x) \hat{O}^i (C_i + C'_i \gamma_5) u_{\nu_\mathrm{e}}(x)\right] \quad . \tag{2.76}
$$

One easily checks that pure vector coupling minus axial vector coupling in (2.74) results also in a vector minus axial vector coupling in (2.76) up to the minus sign, while the other couplings in (2.74) result in complicated superpositions in (2.76). This circumstance may seem to endow the V–A law with particular significance. However, four other combinations have comparable properties, as can be found by diagonalizing the matrix Λ_{ij}. The possibility of using the invariance under transpositions among the fields as a basis for singling out the correct coupling was explored extensively without decisive results.[10]

The advantage of the notation (2.76) is that the wave function of the observable particles – the electron and the muon – are connected in one matrix element, whereas the wave functions of the two neutrinos are separated in the second spinor matrix element. In the interaction (2.74) or (2.76) only the conservation of electron and muon number, and Lorentz invariance, is assumed. Let us introduce the abbreviation

$$a_i = |C_i|^2 + |C_i'|^2 \quad . \tag{2.77}$$

Since the factor G stands in front of the expression (2.76), the proper coupling constants are given by GC_i or GC_i', respectively. It is obvious that a variation of the value of G can be compensated by a multiplication of all constants C_i, C_i' with a common factor. If we determine G by experiment, the C_i, C_i' are no longer independent, that is, they must satisfy a normalization condition. We choose this condition to be

$$a_{\mathrm{S}} + 4a_{\mathrm{V}} + 6a_{\mathrm{T}} + 4a_{\mathrm{A}} + a_{\mathrm{P}} = 16 \quad . \tag{2.78}$$

It is necessary to calculate the muon decay once more, but now with all types of coupling allowed. We assume that the μ^- is polarized before the decay, but we do not observe the polarizations of the three decay products (see (2.54) and also Exercise 2.9, but remember that those results were valid for V–A coupling only). With the abbreviation

$$x = \frac{p^0}{p^0_{\max}} = \frac{2m_\mu p^0}{m_\mu^2 + m_\mathrm{e}^2} \tag{2.79}$$

and the emission angle θ of the electron with respect to the muon spin

$$\cos\theta = \frac{\boldsymbol{p}\cdot\boldsymbol{s}}{|\boldsymbol{p}|} \quad , \tag{2.80}$$

we get after a lengthy calculation the following electron spectrum:

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}\Omega\,\mathrm{d}p^0} = \frac{G^2 m_\mu}{12\pi^4}|\boldsymbol{p}|p^0\Bigg\{&3(p^0_{\max}-p^0) + \frac{2}{3}\rho\left(4p^0 - 3p^0_{\max} - \frac{1}{3}\frac{m_\mathrm{e}^2}{m_\mu}\right)\\ &+3\frac{m_\mathrm{e}}{p^0}\eta(p^0_{\max}-p^0) - \xi\frac{|\boldsymbol{p}|}{p^0}\cos\theta\Bigg[(p^0_{\max}-p^0)\\ &+\frac{2}{3}\delta(4p^0 - 3p^0_{\max} - m_\mathrm{e}^2/m_\mu)\Bigg]\Bigg\}\theta(p^0_{\max}-p^0) \quad .\end{aligned} \tag{2.81a}$$

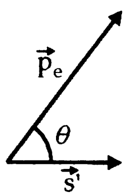


Fig. 2.8. The angle of electron emission relative to the spin $\boldsymbol{s}'$ of the muon

[10] See many papers beginning with C. Gitchfield: Phys. Rev. **63**, 417 (1943) through to E. Caianello: Nuovo Cimento **8**, 749 (1952), in which references to earlier work can be found. See also E.J. Konopinski: *The Theory of Beta Radioactivity* (Oxford University Press, London, 1966).

The details of this calculation are layed down in Exercise 2.11. If we neglect the mass of the electron and make use of definition (2.79) this becomes

$$\frac{\mathrm{d}W}{\mathrm{d}\Omega\,\mathrm{d}x} = \frac{G^2 m_\mu{}^5}{192\pi^4} x^2 \left\{ \frac{1}{1+4\eta\frac{m_\mathrm{e}}{m_\mu}} \left[4(x-1) + \frac{2}{3}\rho(4x-3) + 6\frac{m_\mathrm{e}}{m_\mu}\frac{1-x}{x}\eta \right] \right.$$
$$\left. - \xi\cos\theta \left[(1-x) + \frac{2}{3}\delta(4x-3) \right] \right\} \quad . \tag{2.81b}$$

In this formula ρ, η, ξ, and δ are the so-called **Michel** parameters

$$\begin{aligned} \rho &= \frac{1}{16}(3a_\mathrm{V} + 6a_\mathrm{T} + 3a_\mathrm{A}) \quad , \quad \xi = \frac{-1}{16}(4b' + 3a' - 14c') \quad , \\ \eta &= \frac{1}{16}(a_\mathrm{S} - 2a_\mathrm{V} + 2a_\mathrm{A} - a_\mathrm{P}) \quad , \quad \delta = \frac{-1}{16\xi}(3b' - 6c') \quad , \end{aligned} \tag{2.82}$$

where

$$\begin{aligned} a' &= 2\,\mathrm{Re}\{\mathrm{C_S C_P'}^* + \mathrm{C_S' C_P^*}\} \quad , \\ b' &= 2\,\mathrm{Re}\{\mathrm{C_V C_A'}^* + \mathrm{C_V' C_A^*}\} \quad , \\ c' &= 2\,\mathrm{Re}\{\mathrm{C_T C_T'}^*\} \quad . \end{aligned} \tag{2.83}$$

The parameters are chosen in such a way that if one integrates over x from 0 to 1 then ρ and δ disappear. Therefore the lifetime of the muon is independent of ρ and δ. For a pure V–A coupling, which was assumed during the discussion in Sects. 2.2, 2.3, and 2.4, we get

$$\begin{aligned} C_\mathrm{S} &= C_\mathrm{S}' = C_\mathrm{T} = C_\mathrm{T}' = C_\mathrm{P} = C_\mathrm{P}' = 0 \quad , \\ C_\mathrm{V} &= C_\mathrm{V}' = -C_\mathrm{A} = -C_\mathrm{A}' = 1 \quad . \end{aligned} \tag{2.84}$$

Considering Supplement 2.12, (23), one gets

$$\begin{aligned} M &= [\bar{u}_\mathrm{e}\gamma_\mu u_\mu]\big[\bar{u}_{\nu_\mu}\gamma^\mu(1-\gamma_5)u_{\nu_\mathrm{e}}\big] + [u_\mathrm{e}\gamma_5\gamma_\mu u_\mu]\big[u_{\nu_\mu}\gamma_5\gamma^\mu(1-\gamma_5)u_{\nu_\mathrm{e}}\big] \\ &= \big[\bar{u}_\mathrm{e}\gamma_\mu(1-\gamma_5)u_\mu\big]\big[\bar{u}_{\nu_\mu}\gamma^\mu(1-\gamma_5)u_{\nu_\mathrm{e}}\big] \quad . \end{aligned} \tag{2.85}$$

By inserting this value into (2.82) we obtain the prediction of the V–A theory for the Michel parameters:

$$\rho = \frac{3}{4} \quad , \quad \xi = 1 \quad , \quad \eta = 0 \quad , \quad \delta = \frac{3}{4} \quad . \tag{2.86a}$$

The experimental values are derived from a careful measurement of the electron spectrum (or the positron spectrum in the case of the μ^+ decay) and of the angular distribution. Equation (2.81) tells us that ρ must be fitted to the whole spectrum, whereas η is mainly sensitive to low energies ($x \to 0$). It is not surprising, therefore, that η is the most uncertain of the parameters. ξ can be obtained by integrating the angular distribution over the energy, whereas δ can be determined by measuring the energy dependence of this distribution. The best experimental values are

$$\begin{aligned} \rho &= 0.7517 \pm 0.0026 \quad , \\ \eta &= -0.12 \pm 0.21 \quad , \end{aligned}$$

$$\xi = 0.972 \pm 0.013 \quad ,$$
$$\delta = 0.7551 \pm 0.0085 \quad . \tag{2.86b}$$

These values are in very good agreement with the predictions (2.85) of the V–A theory; see also Fig. 2.9.

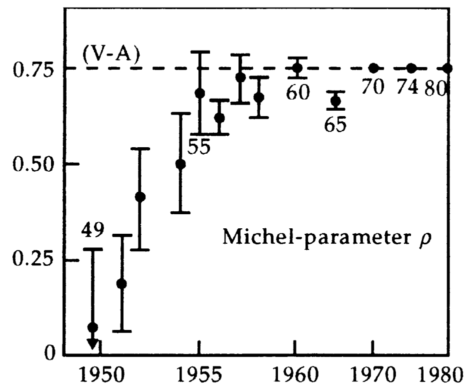


Fig. 2.9. Experimental determination of the Michel parameter ϱ since 1950. The curve shows the improvement of the experiments, but perhaps also the prejudice of the experimentalists

EXERCISE

2.11 Muon Decay and the Michel Parameters

Problem. Calculate the muon decay with the general interaction (2.76) in the same manner as in Sect. 2.2 and derive (2.81) by summing over the polarizations of the outgoing particles.

Solution. To derive (2.81) we repeat the steps which lead us from (2.10) to (2.18). The normalization and the phase-space factor are obtained in the same manner. The only difference occurs in the matrix element M. With (2.76) this is given by

$$M = \sum_i [\bar{u}_e \hat{O}_i u_\mu][\bar{u}_{\nu_\mu} \hat{O}^i (C_i - C_i' \gamma_5) u_{\nu_e}] \quad . \tag{1}$$

First we calculate the part of $|M|^2$ which stems from the neutrinos. We sum over the unobservable neutrino spins and get

$$\begin{aligned} X(\nu) &= \sum_{t,t'} \left[\bar{u}_{\nu_\mu}(\boldsymbol{k}',t') \hat{O}^i (C_i - C_i'\gamma_5) u_{\nu_e}(\boldsymbol{k},t)\right] \\ &\quad \times \left[\bar{u}_{\nu_\mu}(\boldsymbol{k}',t') \hat{O}^k (C_k - C_k'\gamma_5) u_{\nu_e}(\boldsymbol{k},t)\right] \\ &= \sum_{t,t'} \left[\bar{u}_{\nu_\mu}(\boldsymbol{k}',t') \hat{O}^i (C_i - C_i'\gamma_5) u_{\nu_e}(\boldsymbol{k},t)\right] \\ &\quad \times \left[\bar{u}_{\nu_e}(\boldsymbol{k},t) \gamma_0 (C_k^* - C_k'^* \gamma_5^\dagger)(\hat{O}^k)^\dagger \gamma_0 u_{\nu_\mu}(\boldsymbol{k}',t')\right] \\ &= \sum_{t,t'} \left[\bar{u}_{\nu_\mu}(\boldsymbol{k}',t') \hat{O}^i (C_i - C_i'\gamma_5) u_{\nu_e}(\boldsymbol{k},t)\right] \\ &\quad \times \left[\bar{u}_{\nu_e}(\boldsymbol{k},t) (C_k^* + C_k'^* \gamma_5)(\hat{O}^k) u_{\nu_\mu}(\boldsymbol{k}',t')\right] \end{aligned} \tag{2}$$

Exercise 2.11

where we have inserted $\gamma_0^2 = 1$ in front of $(\hat{O}^k)^\dagger$ and used Supplement 2.12, (3), which yields

$$\gamma_0 \gamma_5^\dagger \gamma_0 = -\gamma_5 \quad . \tag{3}$$

With (2.21) we get

$$X(\nu) = \mathrm{Tr}\left\{\hat{O}^i (C_i - C_i' \gamma_5) \not{k} (C_k^* + C_k'^* \gamma_5) \hat{O}^k \not{k}'\right\} \quad . \tag{4}$$

Because

$$\gamma_5 \gamma_\mu = -\gamma_\mu \gamma_5 \tag{5}$$

and the trace is invariant under cyclic permutations, this form may be transcribed to

$$\begin{aligned} X(\nu) &= \mathrm{Tr}\left\{(C_i + C_i'\gamma_5)(C_k^* + C_k'^*\gamma_5)\hat{O}^k \not{k}' \hat{O}^i \not{k}\right\} \\ &= \mathrm{Tr}\left\{\{C_i C_k^* + C_i' C_k'^* + (C_i C_k'^* + C_i' C_k^*)\gamma_5\}\hat{O}^k \not{k}' \hat{O}^i \not{k}\right\} \\ &= \mathrm{Tr}\left\{(A_{ik} + B_{ik}\gamma_5)\hat{O}^k \not{k}' \hat{O}^i \not{k}\right\} \\ &= \mathrm{Tr}\left\{(A_{ik} \pm B_{ik}\gamma_5)\hat{O}^i \not{k} \hat{O}^k \not{k}'\right\} \quad , \end{aligned} \tag{6}$$

where we have $+$ for $\hat{O} = \hat{S}, \hat{P}, \hat{T}$ and $-$ for $\hat{O} = V, A$, and the following abbreviations have been introduced:

$$\begin{aligned} A_{ik} &= C_i C_k^* + C_i' C_k'^* \quad , \\ B_{ik} &= C_i C_k'^* + C_i' C_k^* \quad . \end{aligned} \tag{7}$$

Notice that $X(\nu)$ is non-zero only if both $\hat{O}^i$ and $\hat{O}^k$ contain either an even or an odd number of γ matrices. Otherwise the trace in (6) vanishes. This property will be useful for the evaluation of $X(\mu, \mathrm{e})$, since we can then restrict our consideration to the corresponding combinations of $\hat{O}^i$ and $\hat{O}^k$. In determining $X(\mu, \mathrm{e})$ we assume that the electron spin is not observed, and we therefore sum over the spin orientations. Furthermore we make the approximation of neglecting the electron mass.

We then find

$$\begin{aligned} X(\mu, \mathrm{e}) &= \sum_s \left[\bar{u}_\mathrm{e}(\boldsymbol{p}, s)\hat{O}_i u_\mu(\boldsymbol{p}', s')\right]\left[\bar{u}_\mathrm{e}(\boldsymbol{p}, s)\hat{O}_k u_\mu(\boldsymbol{p}', s')\right]^\dagger \\ &= \sum_s \left[\bar{u}_\mathrm{e}(\boldsymbol{p}, s)\hat{O}_i u_\mu(\boldsymbol{p}', s')\right]\left[\bar{u}_\mu(\boldsymbol{p}', s')\hat{O}_k u_\mathrm{e}(\boldsymbol{p}, s)\right] \\ &= \mathrm{Tr}\left[\hat{O}_i(\not{p}' + m_\mu)\frac{1 + \gamma_5 \not{s}'}{2}\hat{O}_k \not{p}\right] \quad . \end{aligned} \tag{8}$$

If both $\hat{O}_i$ and $\hat{O}_k$ contain an even or an odd number of γ matrices, $X(\mu, \mathrm{e})$ reduces to

$$X(\mu, \mathrm{e}) = \frac{1}{2}\mathrm{Tr}\left\{\hat{O}_i \not{p}' \hat{O}_k \not{p}\right\} + \frac{1}{2} m_\mu \mathrm{Tr}\left\{\hat{O}_i \gamma_5 \not{s}' \hat{O}_k \not{p}\right\} \quad . \tag{9}$$

All other terms in (8) do not contribute, since a trace consisting of an uneven number of γ matrices vanishes.

Let us now consider the particular combinations of $\hat{O}_i$ and $\hat{O}_k$ in detail. For this purpose we again employ the formulas listed in Appendix A.2, especially *Exercise 2.11*

$$(\gamma_5)^2 = 1 \quad \text{and} \quad \gamma_5\gamma_\mu = -\gamma_\mu\gamma_5 \quad .$$

- $\hat{O}^i = \hat{O}^k = 1$:

$$X(\mu, \mathrm{e}) = \frac{1}{2}\,\mathrm{Tr}\{\not{p}\not{p}'\} = 2(p \cdot p') \quad ,$$
$$X(\nu) = \mathrm{Tr}\{(A_{\mathrm{SS}} - B_{\mathrm{SS}}\gamma_5)\not{k}\not{k}'\} = 4A_{\mathrm{SS}}(k \cdot k') \quad . \tag{10}$$

- $\hat{O}^i = \mathrm{i}\gamma_5$, $\hat{O}^k = 1$:

$$X(\mu, \mathrm{e}) = \frac{1}{2}m_\mu\,\mathrm{Tr}\{\mathrm{i}\gamma_5\gamma_5\not{s}'\not{p}\} = 2\mathrm{i}m_\mu(p \cdot s') \quad ,$$
$$\begin{aligned} X(\nu) &= \mathrm{Tr}\{(A_{\mathrm{PS}} - B_{\mathrm{PS}}\gamma_5)\mathrm{i}\gamma_5\not{k}\not{k}'\} \\ &= -4\mathrm{i}B_{\mathrm{PS}}(k \cdot k') \quad . \end{aligned} \tag{11}$$

- $\hat{O}^i = 1$, $\hat{O}^k = \mathrm{i}\gamma_5$:

$$X(\mu, \mathrm{e}) = \frac{1}{2}m_\mu\,\mathrm{Tr}\{\gamma_5\not{s}' \cdot \mathrm{i}\gamma_5\not{p}\} = -2\mathrm{i}m_\mu(p \cdot s') \quad ,$$
$$\begin{aligned} X(\nu) &= \mathrm{Tr}\{(A_{\mathrm{SP}} - B_{\mathrm{SP}}\gamma_5)\not{k}\mathrm{i}\gamma_5\not{k}'\} \\ &= +4\mathrm{i}B_{\mathrm{SP}}(k \cdot k') \quad . \end{aligned} \tag{12}$$

- $\hat{O}^i = \mathrm{i}\gamma_5$, $\hat{O}^k = \mathrm{i}\gamma_5$:

$$X(\mu, \mathrm{e}) = \frac{1}{2}\,\mathrm{Tr}\{\mathrm{i}\gamma_5\not{p}' \cdot \mathrm{i}\gamma_5\not{p}\} = 2(p \cdot p') \quad ,$$
$$\begin{aligned} X(\nu) &= \mathrm{Tr}\{(A_{\mathrm{PP}} - B_{\mathrm{PP}}\gamma_5)\mathrm{i}\gamma_5\not{k}\mathrm{i}\gamma_5\not{k}'\} \\ &= 4A_{\mathrm{PP}}(k \cdot k') \quad . \end{aligned} \tag{13}$$

Collecting together (10)–(13), we obtain

$$\begin{aligned} \sum_{i,k=\mathrm{S,P}} X(\mu, \mathrm{e})X(\nu) = 8(k \cdot k')\big[&(A_{\mathrm{SS}} + A_{\mathrm{PP}})(p \cdot p') \\ &+ (B_{\mathrm{PS}} + B_{\mathrm{SP}})m_\mu(p \cdot s')\big] \quad , \end{aligned} \tag{14}$$

or, adopting the abbreviations (2.77) and (2.83),

$$\sum_{i,k=\mathrm{S,P}} X(\mu, \mathrm{e})X(\nu) = 8(k \cdot k')\big[(a_{\mathrm{S}} + a_{\mathrm{P}})(p \cdot p') + a' m_\mu(p \cdot s')\big] \quad . \tag{15}$$

(Note that according to the convention (23) of Supplement 2.12 it holds that $B_{\mathrm{PS}} + B_{\mathrm{SP}} = -a'$.)

- $\hat{O}^i = \gamma^\mu$, $\hat{O}^k = \gamma^\nu$:

$$\begin{aligned} X(\mu, \mathrm{e}) &= \frac{1}{2}\,\mathrm{Tr}\{\gamma_\mu\not{p}'\gamma_\nu\not{p}\} + \frac{1}{2}m_\mu\,\mathrm{Tr}\{\gamma_\mu\gamma_5\not{s}'\gamma_\nu\not{p}\} \\ &= 2(p'_\mu p_\nu + p_\mu p'_\nu - g_{\mu\nu}\, p \cdot p') + 2\mathrm{i}m_\mu\varepsilon_{\sigma\nu\tau\mu}s'^\sigma p^\tau \quad , \end{aligned} \tag{16}$$

Exercise 2.11

$$
\begin{aligned}
X(\nu) &= \mathrm{Tr}\{(A_{\mathrm{VV}} + B_{\mathrm{VV}}\gamma_5)\gamma^\mu \not{k}\gamma^\nu \not{k}'\} \\
&= 4A_{\mathrm{VV}}\big[k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(k\cdot k')\big] + 4\mathrm{i}B_{\mathrm{VV}}\varepsilon^{\mu\alpha\nu\beta}k_\alpha k'_\beta \quad .
\end{aligned} \tag{17}
$$

Evaluating the products leads to

$$
\begin{aligned}
X(\mu,\mathrm{e})X(\nu) = 8A_{\mathrm{VV}}\big[&(p'\cdot k)(p\cdot k') + (p'\cdot k')(p\cdot k) \\
&- (p\cdot p')(k\cdot k') + (p\cdot k)(p'\cdot k') \\
&+ (p\cdot k')(p'\cdot k) - (p\cdot p')(k\cdot k') \\
&- (p\cdot p')(k\cdot k') - (p\cdot p')(k\cdot k') \\
&+ 4(p\cdot p')(k\cdot k')\big] \\
&- 8m_\mu B_{\mathrm{VV}}\varepsilon_{\sigma\nu\tau\mu}\varepsilon^{\mu\alpha\nu\beta}s'^\sigma p^\tau k_\alpha k'_\beta \\
= 16A_{\mathrm{VV}}\big[&(p'\cdot k)(p\cdot k') + (p'\cdot k')(p\cdot k)\big] \\
&- 8m_\mu B_{\mathrm{VV}}\varepsilon_{\sigma\nu\tau\mu}\varepsilon^{\mu\alpha\nu\beta}s'^\sigma p^\tau k_\alpha k'_\beta \quad .
\end{aligned} \tag{18}
$$

The last term does not contribute, since, in the course of the further evaluation, $k_\alpha k'_\beta$ yields the symmetric tensor $I_{\alpha\beta}$ which is contracted with $\varepsilon^{\mu\alpha\nu\beta}$; thus there is no need to evaluate this term further. The next three cases may be treated in just the same way.

- $\hat{O}^i = \gamma_5\gamma^\mu$, $\hat{O}^k = \gamma^\nu$:

$$
\begin{aligned}
X(\mu,\mathrm{e}) &= \frac{1}{2}\mathrm{Tr}\big\{\gamma_5\gamma_\mu \not{p}'\gamma_\nu \not{p}\big\} + \frac{1}{2}m_\mu \mathrm{Tr}\big\{\gamma_5\gamma_\mu\gamma_5 \not{s}'\gamma_\nu \not{p}\big\} \\
&= -2\mathrm{i}\varepsilon_{\mu\sigma\nu\tau}p'^\sigma p^\tau - 2m_\mu\big[s'_\mu p_\nu + s'_\nu p_\mu - g_{\mu\nu}(s'\cdot p)\big] \quad , \\
X(\nu) &= \mathrm{Tr}\big\{(A_{\mathrm{AV}} + B_{\mathrm{AV}}\gamma_5)\gamma_5\gamma^\mu \not{k}\gamma^\nu \not{k}'\big\} \\
&= -4\mathrm{i}A_{\mathrm{AV}}\varepsilon^{\mu\alpha\nu\beta}k_\alpha k'_\beta + 4B_{\mathrm{AV}}\big[k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(k\cdot k')\big] \quad , \\
X(\mu,\mathrm{e})X(\nu) &= -16m_\mu B_{\mathrm{AV}}\big[(k\cdot p)(k'\cdot s') + (k\cdot s')(k'\cdot p)\big] \\
&\quad - 8A_{\mathrm{AV}}\varepsilon_{\mu\sigma\nu\tau}\varepsilon^{\mu\alpha\nu\beta}p'^\sigma p^\tau k_\alpha k'_\beta \quad .
\end{aligned} \tag{19}
$$

- $\hat{O}^i = \gamma^\mu$, $\hat{O}^k = \gamma_5\gamma^\nu$:

$$
\begin{aligned}
X(\mu,\mathrm{e}) &= \frac{1}{2}\mathrm{Tr}\big\{\gamma_\mu \not{p}'\gamma_5\gamma_\nu \not{p}\big\} + \frac{1}{2}m_\mu \mathrm{Tr}\big\{\gamma_\mu\gamma_5 \not{s}'\gamma_5\gamma_\nu \not{p}\big\} \\
&= 2\mathrm{i}\varepsilon_{\nu\sigma\mu\tau}p'^\sigma p^\tau - 2m_\mu\big[s'_\mu p_\nu + s'_\nu p_\mu - g_{\mu\nu}(s'\cdot p)\big] \quad , \\
X(\nu) &= \mathrm{Tr}\big\{(A_{\mathrm{VA}} + B_{\mathrm{VA}}\gamma_5)\gamma^\mu \not{k}\gamma_5\gamma^\nu \not{k}'\big\} \\
&= -4\mathrm{i}A_{\mathrm{VA}}\varepsilon^{\nu\beta\mu\alpha}k_\alpha k'_\beta + 4B_{\mathrm{VA}}\big[k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(k\cdot k')\big] \quad , \\
X(\mu,\mathrm{e})X(\nu) &= -16m_\mu B_{\mathrm{VA}}\big[(k\cdot p)(k'\cdot s') + (k\cdot s')(k'\cdot p)\big] \\
&\quad + 8A_{\mathrm{VA}}\varepsilon_{\nu\sigma\mu\tau}\varepsilon^{\nu\beta\mu\alpha}p'^\sigma p^\tau k_\alpha k'_\beta \quad .
\end{aligned} \tag{20}
$$

- $\hat{O}^i = \gamma_5\gamma^\mu$, $\hat{O}^k = \gamma_5\gamma^\nu$:

$$
\begin{aligned}
X(\mu,\mathrm{e}) &= \frac{1}{2}\mathrm{Tr}\big\{\gamma_5\gamma_\mu \not{p}'\gamma_5\gamma_\nu \not{p}\big\} + \frac{1}{2}m_\mu \mathrm{Tr}\big\{\gamma_5\gamma_\mu\gamma_5 \not{s}'\gamma_5\gamma_\nu \not{p}\big\} \\
&= 2\big[p'_\mu p_\nu + p'_\nu p_\mu - g_{\mu\nu}(p\cdot p')\big] + 2\mathrm{i}m_\mu\varepsilon_{\mu\sigma\nu\tau}s'^\sigma p^\tau \quad ,
\end{aligned}
$$

Exercise 2.11

$$X(\nu) = \mathrm{Tr}\{(A_{\mathrm{AA}} + B_{\mathrm{AA}}\gamma_5)\gamma_5\gamma^\mu \not{k}\gamma_5\gamma^\nu \not{k}'\}$$
$$= 4A_{\mathrm{AA}}\big[k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(k\cdot k')\big] - 4\mathrm{i}B_{\mathrm{AA}}\varepsilon^{\nu\beta\mu\alpha}k_\alpha k'_\beta \quad , \tag{21}$$
$$X(\mu,\mathrm{e})X(\nu) = 16A_{\mathrm{AA}}\big[(k\cdot p)(k'\cdot p') + (k'\cdot p)(k\cdot p')\big]$$
$$- 8m_\mu B_{\mathrm{AA}}\varepsilon_{\mu\sigma\nu\tau}\varepsilon^{\nu\beta\mu\alpha}s'^\sigma p^\tau k_\alpha k'_\beta \quad .$$

Combining the last four results, we find that

$$\sum_{i,k=\mathrm{V},\mathrm{A}} X(\mu,e)X(\nu)$$
$$= 16(A_{\mathrm{VV}} + A_{\mathrm{AA}})\big[(k\cdot p)(k'\cdot p') + (k\cdot p')(k'\cdot p)\big]$$
$$- 16m_\mu(B_{\mathrm{AV}} + B_{\mathrm{VA}})\big[(k\cdot p)(k'\cdot s') + (k\cdot s')(k'\cdot p)\big] + X^{\alpha\beta}k_\alpha k'_\beta$$
$$= 16(a_{\mathrm{V}} + a_{\mathrm{A}})\big[(k\cdot p)(k'\cdot p') + (k\cdot p')(k'\cdot p)\big]$$
$$+ 16m_\mu b'\big[(k\cdot p)(k'\cdot s') + (k\cdot s')(k'\cdot p)\big] + X^{\alpha\beta}k_\alpha k'_\beta \quad . \tag{22}$$

Here $X^{\alpha\beta}$ contains all terms which are antisymmetric in the indices α and β. In the course of further evaluation $k_\alpha k'_\beta$ yields the symmetric tensor $I_{\alpha\beta}$ and therefore the term containing $X^{\alpha\beta}$ will vanish. Again, with respect to the convention (23) of Supplement 2.12, we have $b' = -(B_{\mathrm{VA}} + B_{\mathrm{AV}})$.

• $\hat{O}^i = \sigma^{\mu\nu}$, $\hat{O}^k = \sigma^{\bar{\mu}\bar{\nu}}$:
This case requires the evaluation of

$$\mathrm{Tr}\{\sigma^{\mu\nu}\gamma^\alpha\sigma^{\bar{\mu}\bar{\nu}}\gamma^\beta\} \quad . \tag{23}$$

For this purpose we first consider

$$\mathrm{Tr}\{\mathrm{i}\gamma^\mu\gamma^\nu\gamma^\alpha i\gamma^{\bar{\mu}}\gamma^{\bar{\nu}}\gamma^\beta\} \quad . \tag{24}$$

We use

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \tag{25}$$

and antisymmetrize (24) with respect to the indices μ and ν (that is, exchange μ and ν, subtract the result from the original term, and finally divide by 2) and then with respect to the indices $\bar{\mu}$ and $\bar{\nu}$. Finally, by repeated application of (A.33), we obtain

$$-\mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^{\bar{\mu}}\gamma^{\bar{\nu}}\gamma^\beta\}$$
$$= -\big(g^{\mu\nu}\,\mathrm{Tr}\{\gamma^\alpha\gamma^{\bar{\mu}}\gamma^{\bar{\nu}}\gamma^\beta\} - g^{\mu\alpha}\,\mathrm{Tr}\{\gamma^\nu\gamma^{\bar{\mu}}\gamma^{\bar{\nu}}\gamma^\beta\}$$
$$+ g^{\mu\bar{\mu}}\,\mathrm{Tr}\{\gamma^\nu\gamma^\alpha\gamma^{\bar{\nu}}\gamma^\beta\} - g^{\mu\bar{\nu}}\,\mathrm{Tr}\{\gamma^\nu\gamma^\alpha\gamma^{\bar{\mu}}\gamma^\beta\} + g^{\mu\beta}\,\mathrm{Tr}\{\gamma^\nu\gamma^\alpha\gamma^{\bar{\mu}}\gamma^{\bar{\nu}}\}\big)$$
$$= -4\big[g^{\mu\nu}(g^{\alpha\bar{\mu}}g^{\bar{\nu}\beta} - g^{\alpha\bar{\nu}}g^{\mu\bar{\beta}} + g^{\alpha\beta}g^{\bar{\mu}\bar{\nu}})$$
$$- g^{\mu\alpha}(g^{\nu\bar{\mu}}g^{\bar{\nu}\beta} - g^{\nu\bar{\nu}}g^{\bar{\mu}\beta} + g^{\nu\beta}g^{\bar{\mu}\bar{\nu}})$$
$$+ g^{\mu\bar{\mu}}(g^{\nu\alpha}g^{\bar{\nu}\beta} - g^{\nu\bar{\nu}}g^{\alpha\beta} + g^{\nu\beta}g^{\alpha\bar{\nu}})$$
$$- g^{\mu\bar{\nu}}(g^{\nu\alpha}g^{\bar{\mu}\beta} - g^{\nu\bar{\mu}}g^{\alpha\beta} + g^{\nu\beta}g^{\alpha\bar{\mu}})$$
$$+ g^{\mu\beta}(g^{\nu\alpha}g^{\bar{\mu}\beta} - g^{\nu\bar{\mu}}g^{\alpha\bar{\nu}} + g^{\nu\bar{\nu}}g^{\alpha\bar{\mu}})\big] \quad . \tag{26}$$

Exercise 2.11

Owing to the procedure of antisymmetrization with respect to μ and ν, as well as to $\bar{\mu}$ and $\bar{\nu}$, all terms proportional to $g^{\mu\nu}$ and $g^{\bar{\mu}\bar{\nu}}$ vanish, so that we are left with

$$\begin{aligned}
&-\mathrm{Tr}\{\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar{\mu}\bar{\nu}}\gamma^{\beta}\}\\
&\quad=4\big[(g^{\mu\bar{\mu}}g^{\nu\bar{\nu}}-g^{\mu\bar{\nu}}g^{\nu\bar{\mu}})g^{\alpha\beta}-g^{\mu\bar{\mu}}(g^{\nu\alpha}g^{\bar{\nu}\beta}+g^{\bar{\nu}\alpha}g^{\mu\beta})\\
&\qquad+g^{\mu\bar{\nu}}(g^{\nu\alpha}g^{\bar{\mu}\beta}+g^{\bar{\mu}\alpha}g^{\nu\beta})+g^{\bar{\mu}\nu}(g^{\mu\alpha}g^{\bar{\nu}\beta}+g^{\bar{\nu}\alpha}g^{\mu\beta})\\
&\qquad-g^{\bar{\nu}\nu}(g^{\mu\alpha}g^{\bar{\mu}\beta}+g^{\bar{\mu}\alpha}g^{\mu\beta})\big]\quad .
\end{aligned}\tag{27}$$

For later purposes it is worth mentioning that this term is simply the antisymmetrized form of

$$8g^{\mu\bar{\mu}}\{g^{\nu\bar{\nu}}g^{\alpha\beta}-2g^{\nu\alpha}g^{\bar{\nu}\beta}-2g^{\bar{\nu}\alpha}g^{\nu\beta}\}\quad .\tag{28}$$

This is easily checked by multiplying (27) or (28) by a term which itself is antisymmetric with respect to μ, ν and $\bar{\mu}$, $\bar{\nu}$; the two corresponding results are identical.

In order to evaluate the quantity $X(\mu,\mathrm{e})X(\nu)$ we need to consider

$$\begin{aligned}
&\mathrm{Tr}\{\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar{\mu}\bar{\nu}}\gamma^{\beta}\}\,\mathrm{Tr}\{\sigma_{\mu\nu}\gamma_{\varrho}\sigma_{\bar{\mu}\bar{\nu}}\gamma_{\sigma}\}\\
&\quad=8g^{\bar{\mu}\bar{\mu}}(g^{\nu\bar{\nu}}g^{\alpha\beta}-2g^{\nu\alpha}g^{\bar{\nu}\beta}-2g^{\bar{\nu}\alpha}g^{\nu\beta})\\
&\qquad\times 4\big[(g_{\mu\bar{\mu}}g_{\nu\bar{\nu}}-g_{\mu\bar{\nu}}g_{\nu\bar{\mu}})g_{\varrho\sigma}\\
&\qquad-g_{\mu\bar{\mu}}(g_{\nu\varrho}g_{\bar{\nu}\sigma}+g_{\bar{\nu}\varrho}g_{\nu\sigma})+g_{\mu\bar{\nu}}(g_{\nu\varrho}g_{\bar{\mu}\sigma}+g_{\bar{\mu}\varrho}g_{\nu\sigma})\\
&\qquad+g_{\bar{\mu}\nu}(g_{\mu\varrho}g_{\bar{\nu}\sigma}+g_{\bar{\nu}\varrho}g_{\mu\sigma})-g_{\bar{\nu}\nu}(g_{\mu\varrho}g_{\bar{\mu}\sigma}+g_{\bar{\mu}\varrho}g_{\mu\sigma})\big]\\
&\quad=32\big(g^{\nu\bar{\nu}}g^{\alpha\beta}-2g^{\nu\alpha}g^{\bar{\nu}\beta}-2g^{\bar{\nu}\alpha}g^{\nu\beta}\big)\\
&\qquad\times\big[3g_{\nu\bar{\nu}}g_{\varrho\sigma}-4(g_{\nu\varrho}g_{\bar{\nu}\sigma}+g_{\bar{\nu}\varrho}g_{\nu\sigma})\\
&\qquad+g_{\nu\varrho}g_{\bar{\nu}\sigma}+g_{\bar{\nu}\varrho}g_{\nu\sigma}+g_{\nu\varrho}g_{\bar{\nu}\sigma}+g_{\bar{\nu}\varrho}g_{\nu\sigma}-2g_{\nu\bar{\nu}}g_{\varrho\sigma}\big]\\
&\quad=32(g^{\nu\bar{\nu}}g^{\alpha\beta}-2g^{\nu\alpha}g^{\bar{\nu}\beta}-2g^{\bar{\nu}\alpha}g^{\nu\beta})(g_{\nu\bar{\nu}}g_{\varrho\sigma}-2g_{\nu\varrho}g_{\bar{\nu}\sigma}-2g_{\bar{\nu}\varrho}g_{\nu\sigma})\\
&\quad=32\big(4g^{\alpha\beta}g_{\varrho\sigma}-2g^{\alpha\beta}g_{\varrho\sigma}-2g^{\alpha\beta}g_{\varrho\sigma}-2g^{\alpha\beta}g_{\varrho\sigma}\\
&\qquad+4\delta^{\alpha}_{\varrho}\delta^{\beta}_{\sigma}+4\delta^{\alpha}_{\sigma}\delta^{\beta}_{\varrho}-2g^{\alpha\beta}g_{\varrho\sigma}+4\delta^{\alpha}_{\sigma}\delta^{\beta}_{\varrho}+4\delta^{\alpha}_{\varrho}\delta^{\beta}_{\sigma}\big)\\
&\quad=128\big(-g^{\alpha\beta}g_{\varrho\sigma}+2\delta^{\alpha}_{\varrho}\delta^{\beta}_{\sigma}+2\delta^{\alpha}_{\varrho}\delta^{\beta}_{\sigma}\big)\quad .
\end{aligned}\tag{29}$$

The evaluation of (6), or of (9), furthermore contains terms of the form

$$\mathrm{Tr}\{\gamma_5\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar{\mu}\bar{\nu}}\gamma^{\beta}\}\quad .\tag{30}$$

However, these can be reduced to (27) by employing the relation

$$\gamma_5\sigma^{\mu\nu}=\frac{\mathrm{i}}{2}\varepsilon^{\mu\nu\varrho\tau}\sigma_{\varrho\tau}\quad ,\tag{31}$$

so that we obtain

$$\begin{aligned}
&\mathrm{Tr}\{\gamma_5\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar{\mu}\bar{\nu}}\gamma^{\beta}\}=\frac{\mathrm{i}}{2}\varepsilon^{\mu\nu\varrho\tau}\,\mathrm{Tr}\{\sigma_{\varrho\tau}\gamma^{\alpha}\sigma^{\bar{\mu}\bar{\nu}}\gamma^{\beta}\}\\
&\quad=\frac{\mathrm{i}}{2}\varepsilon^{\mu\nu\varrho\tau}\times 4\big[(\delta^{\bar{\mu}}_{\varrho}\delta^{\bar{\nu}}_{\tau}-\delta^{\bar{\mu}}_{\tau}\delta^{\bar{\nu}}_{\varrho})g^{\alpha\beta}-\delta^{\bar{\mu}}_{\varrho}(\delta^{\alpha}_{\tau}g^{\bar{\nu}\beta}+\delta^{\beta}_{\tau}g^{\bar{\nu}\alpha})
\end{aligned}$$

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$$
\begin{aligned}
&+\delta^{\bar\nu}_{\varrho}(\delta^{\alpha}_{\tau}g^{\bar\mu\beta}+\delta^{\beta}_{\tau}g^{\bar\mu\alpha})+\delta^{\bar\mu}_{\tau}(\delta^{\alpha}_{\varrho}g^{\bar\nu\beta}+\delta^{\beta}_{\varrho}g^{\bar\nu\alpha})\\
&-\delta^{\bar\nu}_{\tau}(\delta^{\alpha}_{\sigma}g^{\bar\mu\beta}+\delta^{\beta}_{\sigma}g^{\bar\mu\alpha})\big]\\
&=4\mathrm{i}(\varepsilon^{\mu\nu\bar\mu\bar\nu}g^{\alpha\beta}-\varepsilon^{\mu\nu\bar\mu\alpha}g^{\bar\nu\beta}-\varepsilon^{\mu\nu\bar\mu\beta}g^{\bar\nu\alpha}+\varepsilon^{\mu\nu\bar\nu\alpha}g^{\bar\mu\beta}+\varepsilon^{\mu\nu\bar\nu\beta}g^{\bar\mu\alpha}) \quad . \qquad (32)
\end{aligned}
$$

Another typical term that occurs in $X(\mu,\mathrm{e})X(\nu)$ is

$$
\mathrm{Tr}\{\gamma_5\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\}\,\mathrm{Tr}\{\sigma_{\mu\nu}\gamma_{\varrho}\sigma_{\bar\mu\bar\nu}\gamma_{\sigma}\} \quad . \qquad (33)
$$

However, it is easily verified that this contribution vanishes. The first factor is again antisymmetric with respect to μ and ν and also to $\bar\mu$ and $\bar\nu$. Thus, for the second trace we may substitute the expression (28), which leads to

$$
\begin{aligned}
&\mathrm{Tr}\{\gamma_5\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\}\cdot\mathrm{Tr}\{\sigma_{\mu\nu}\gamma_{\varrho}\sigma_{\bar\mu\bar\nu}\gamma_{\sigma}\}\\
&=4\mathrm{i}(\varepsilon^{\mu\nu\bar\mu\bar\nu}g^{\alpha\beta}-\varepsilon^{\mu\nu\bar\mu\alpha}g^{\bar\nu\beta}-\varepsilon^{\mu\nu\bar\mu\beta}g^{\bar\nu\alpha}+\varepsilon^{\mu\nu\bar\nu\alpha}g^{\bar\mu\beta}+\varepsilon^{\mu\nu\bar\nu\beta}g^{\bar\mu\alpha})\\
&\quad\times 8g_{\mu\bar\mu}(g_{\nu\bar\nu}g_{\varrho\sigma}-2g_{\nu\varrho}g_{\bar\nu\sigma}-2g_{\nu\sigma}g_{\bar\nu\varrho})\\
&=-32\mathrm{i}(\varepsilon^{\beta\nu\bar\nu\alpha}+\varepsilon^{\alpha\nu\bar\nu\beta})(g_{\nu\bar\nu}g_{\varrho\sigma}-2g_{\nu\varrho}g_{\bar\nu\sigma}-2g_{\nu\sigma}g_{\bar\nu\varrho})=0 \quad , \qquad (34)
\end{aligned}
$$

Using (28) and the relation

$$
\varepsilon_{\mu\nu\gamma\omega}\varepsilon^{\mu\nu\lambda\tau}=2(\delta^{\lambda}_{\omega}\delta^{\tau}_{\gamma}-\delta^{\lambda}_{\gamma}\delta^{\tau}_{\omega}) \quad , \qquad (35)
$$

we finally evaluate the following expression

$$
\begin{aligned}
&\mathrm{Tr}\{\gamma_5\sigma_{\mu\nu}\gamma_{\varrho}\sigma_{\bar\mu\bar\nu}\gamma_{\sigma}\}\,\mathrm{Tr}\{\gamma_5\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\}\\
&=-\frac{1}{4}\varepsilon_{\mu\nu\gamma\omega}\varepsilon^{\mu\nu\lambda\tau}\,\mathrm{Tr}\{\sigma^{\gamma\omega}\gamma_{\varrho}\sigma_{\bar\mu\bar\nu}\gamma_{\varrho}\}\,\mathrm{Tr}\{\sigma_{\lambda\tau}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\}\\
&=\frac{1}{2}(\delta^{\lambda}_{\gamma}\delta^{\tau}_{\omega}-\delta^{\lambda}_{\omega}\delta^{\tau}_{\gamma})\,\mathrm{Tr}\{\sigma^{\gamma\omega}\gamma_{\varrho}\sigma_{\bar\mu\bar\nu}\gamma_{\sigma}\}\,\mathrm{Tr}\{\sigma_{\lambda\tau}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\}\\
&=\mathrm{Tr}\{\sigma^{\gamma\omega}\gamma_{\varrho}\sigma_{\bar\mu\bar\nu}\gamma_{\sigma}\}\,\mathrm{Tr}\{\sigma_{\gamma\omega}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\}\\
&=\mathrm{Tr}\{\sigma_{\mu\nu}\gamma_{\varrho}\sigma_{\bar\mu\bar\nu}\gamma_{\sigma}\}\,\mathrm{Tr}\{\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\} \quad . \qquad (36)
\end{aligned}
$$

This result exactly coincides with the one we previously obtained in (29). Now we have all the ingredients necessary to consider the contribution of tensor coupling $\hat O^i=\sigma^{\mu\nu}$, $\hat O^k=\sigma^{\bar\mu\bar\nu}$.
With respect to (6) we obtain

$$
\begin{aligned}
X(\nu)&=A_{\mathrm{TT}}\,\mathrm{Tr}\{\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\}k_{\alpha}k'_{\beta}-B_{\mathrm{TT}}\,\mathrm{Tr}\{\gamma_5\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\}k_{\alpha}k'_{\beta}\\
&=a_{\mathrm{T}}\,\mathrm{Tr}\{\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\}k_{\alpha}k'_{\beta}-c'\,\mathrm{Tr}\{\gamma_5\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\}k_{\alpha}k'_{\beta} \quad , \qquad (37)
\end{aligned}
$$

where we have again adopted the abbreviations (2.77) and (2.83). The contribution of the massive leptons is given by

$$
X(\mu,\mathrm{e})=\frac{1}{2}\,\mathrm{Tr}\{\sigma_{\mu\nu}\gamma_{\varrho}\sigma_{\bar\mu\bar\nu}\gamma_{\sigma}\}p'^{\varrho}p^{\sigma}+\frac{1}{2}m_{\mu}\,\mathrm{Tr}\{\gamma_5\sigma_{\mu\nu}\gamma_{\varrho}\sigma_{\bar\mu\bar\nu}\gamma_{\sigma}\}s'^{\varrho}p^{\sigma} \quad . \qquad (38)
$$

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All other terms vanish, since they contain an uneven number of γ matrices. We recall that the expression (33) does not contribute, so that we obtain

$$\sum_{\mathrm{T}} X(\nu)X(\mu,\mathrm{e}) = \frac{1}{4}\Bigg[\frac{1}{2}a_{\mathrm{T}}k_\alpha k'_\beta p'^{\varrho} p^{\sigma}\,\mathrm{Tr}\{\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\}\,\mathrm{Tr}\{\sigma_{\mu\nu}\gamma_{\varrho}\sigma_{\bar\mu\bar\nu}\gamma_{\sigma}\}$$
$$-\frac{1}{2}m_\mu c' k_\alpha k'_\beta s'^{\varrho} p^{\sigma}\,\mathrm{Tr}\{\gamma_5\sigma^{\mu\nu}\gamma^{\alpha}\sigma^{\bar\mu\bar\nu}\gamma^{\beta}\}$$
$$\times\,\mathrm{Tr}\{\gamma_5\sigma_{\mu\nu}\gamma_{\varrho}\sigma_{\bar\mu\bar\nu}\gamma_{\sigma}\}\Bigg] \quad . \tag{39}$$

Here we have introduced a factor $\frac{1}{4}$ in order to avoid double counting of $\sigma^{\mu\nu}$, or $\sigma^{\bar\mu\bar\nu}$, since the sum includes $\sigma^{\mu\nu}$ as well as $\sigma^{\nu\mu} = -\sigma^{\mu\nu}$!

Equation (39) may be further reduced by using (29) and (36):

$$\begin{aligned}
\sum_{\mathrm{T}} X(\nu)X(\mu,\mathrm{e})
&= 16a_{\mathrm{T}}k_\alpha k'_\beta p'^{\varrho} p^{\sigma}(-g^{\alpha\beta}g_{\varrho\sigma} + 2\delta^{\alpha}_{\varrho}\delta^{\beta}_{\sigma} + 2\delta^{\alpha}_{\sigma}\delta^{\beta}_{\varrho})\\
&\quad - 16m_\mu c' k_\alpha k'_\beta s'^{\varrho} p^{\sigma}(-g^{\alpha\beta}g_{\varrho\sigma} + 2\delta^{\alpha}_{\varrho}\delta^{\beta}_{\sigma} + 2\delta^{\alpha}_{\sigma}\delta^{\beta}_{\varrho})\\
&= 16a_{\mathrm{T}}\big[-(k\cdot k')(p\cdot p') + 2(k\cdot p)(k'\cdot p') + 2(k\cdot p')(k'\cdot p)\big]\\
&\quad - 16m_\mu c'\big[-(k\cdot k')(s'\cdot p') + 2(k\cdot s')(k'\cdot p) + 2(k\cdot p)(k'\cdot s')\big] \quad .
\end{aligned} \tag{40}$$

With the following argument we can conclude that all other combinations of $\hat{O}^i$ and $\hat{O}^k$ do not contribute: if for example we identify $\hat{O}^i$ with V or A, then $\hat{O}^k$ can neither be S nor P nor T, since otherwise $X(\nu)$ in (6) would contain an uneven number of γ matrices. On the other hand, all remaining combinations lead to an $X(\nu)$ which is antisymmetric with respect to the exchange of k and k', for example, for the combination "ST",

$$\begin{aligned}
&\mathrm{Tr}\big\{(A_{\mathrm{ST}} - B_{\mathrm{ST}}\gamma_5)\cdot 1\cdot \not{k}\sigma^{\mu\nu}\not{k}'\big\}\\
&\quad = 4\mathrm{i}A_{\mathrm{ST}}(k^{\mu}k'^{\nu} - k^{\nu}k'^{\mu}) + 4B_{\mathrm{ST}}\varepsilon^{\alpha\mu\nu\beta}k_\alpha k'_\beta \quad .
\end{aligned} \tag{41}$$

As we have already mentioned in connection with (18), such terms do not contribute to the decay rate.

Combining the previous results (15), (22) and (40) as well as the terms of (41), we find that

$$\begin{aligned}
\sum_{i,k} X(\nu)X(\mu,\mathrm{e})
&= \big\{8g^{\alpha\beta}\big[(a_{\mathrm{S}} + a_{\mathrm{P}})(p\cdot p') - a' m_\mu (p\cdot s')\big]\\
&\quad + 16(a_{\mathrm{V}} + a_{\mathrm{A}})\big[p^{\alpha}p'^{\beta} + p'^{\alpha}p^{\beta}\big] + 16m_\mu b'\big[p^{\alpha}s'^{\beta} + s'^{\alpha}p^{\beta}\big]\\
&\quad + 16a_{\mathrm{T}}\big[-g^{\alpha\beta}(p\cdot p') + 2p^{\alpha}p'^{\beta} + 2p'^{\alpha}p^{\beta}\big]\\
&\quad - 16m_\mu c'\big[-g^{\alpha\beta}(s'\cdot p) + 2s'^{\alpha}p^{\beta} + 2p^{\alpha}s'^{\beta}\big]\\
&\quad + Y^{\alpha\beta}\big\}k_\alpha k'_\beta \quad ,
\end{aligned} \tag{42}$$

where $Y^{\alpha\beta}$ is an antisymmetric tensor which contains terms like $X^{\alpha\beta}$ of (22) as well as the contribution corresponding to (41). *Exercise 2.11*

In (2.18) we now replace

$$\sum_{t,t'} |M|^2$$

by $X(\mu, \mathrm{e})X(\nu)$ of (42) thereby abbreviating the last term by $Z^{\alpha\beta}k_\alpha k_{\beta'}$. This results in

$$\mathrm{d}W = \frac{G^2}{2}\frac{1}{(2\pi)^5}\frac{d^3p}{2p'^0 2p^0}\int\frac{\mathrm{d}^3k}{2k^0}\int\frac{\mathrm{d}^3k'}{2k'^0}\delta^4(p+k+k'-p')Z^{\alpha\beta}k_\alpha k'_\beta \quad . \tag{43}$$

Now, utilizing (2.51)–(2.53), we employ (2.49), according to which it holds that

$$\begin{aligned} I_{\alpha\beta} &\equiv \int\frac{\mathrm{d}^3k}{2k^0}\int\frac{\mathrm{d}^3k'}{2k'^0}k_\alpha k'_\beta\delta^4(k+k'-q) \\ &= \frac{\pi}{24}\left(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta\right)\Theta\left(p^0_{\max} - p^0\right) \quad . \end{aligned} \tag{44}$$

The quantity

$$\mathrm{d}W = \frac{G^2}{2(2\pi)^5}\frac{d^3p}{(2p'^0 2p^0)}I_{\alpha\beta}Z^{\alpha\beta} \tag{45}$$

is now easily evaluated. Neglecting the electron rest mass, the rest frame of the muon is again characterized by

$$\begin{aligned} p'^0 &= m_\mu \quad , \quad \boldsymbol{p}' = 0 \quad , \\ (p\cdot p') &= m_\mu p^0 \quad , \quad (p'\cdot s') = 0 \quad , \\ (p\cdot s') &= -\boldsymbol{p}\cdot\boldsymbol{s}' = -|\boldsymbol{p}|\cos\theta = -p^0\cos\theta \quad . \end{aligned} \tag{46}$$

Introducing $q = p' - p$, from these relations we obtain

$$\begin{aligned} \frac{\pi}{24}\left(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta\right)g^{\alpha\beta} &= \frac{\pi}{24}6m_\mu\left(m_\mu - 2p^0\right) \quad , \\ \frac{\pi}{24}\left(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta\right)p^\alpha p'^\beta &= \frac{\pi}{24}\left(3m_\mu^3 p^0 - 4m_\mu^2(p^0)^2\right) \quad , \\ \frac{\pi}{24}\left(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta\right)p^\alpha s'^\beta &= -\frac{\pi}{24}m_\mu(m_\mu - 4p^0)(p^0)\cos\theta \end{aligned} \tag{47}$$

and finally

$$\begin{aligned} \mathrm{d}W = \frac{G^2}{2}&\frac{1}{(2\pi)^5}\frac{d^3p}{2p'^0 2p^0}\frac{\pi}{24}\Theta\left(p^0 - p^0_{\max}\right) \\ &\times\left\{\left[48(a_\mathrm{S}+a_\mathrm{P}) + 96a_\mathrm{T} + 96(a_\mathrm{V}+a_\mathrm{A})\right]m_\mu^3 p^0\right. \\ &- \left[96(a_\mathrm{S}+a_\mathrm{P}) + 64a_\mathrm{T} + 128(a_\mathrm{V}+a_\mathrm{A})\right]m_\mu^2(p^0)^2 \\ &+ (48a' - 32b' - 32c')m_\mu^3 p^0\cos\theta \\ &\left.- (96a' - 128b' + 64c')m_\mu^2(p^0)^2\cos\theta\right\} \quad . \end{aligned} \tag{48}$$

Neglecting the electron mass implies that

$$p^0_{\mathrm{max}} = \frac{m_\mu}{2} \quad , \tag{49}$$

so that

$$p^0 = x p^0_{\mathrm{max}} = x \frac{m_\mu}{2} \tag{50}$$

and

$$\mathrm{d}^3 p = |\boldsymbol{p}|^2 \mathrm{d}|\boldsymbol{p}| \mathrm{d}\Omega = (p^0)^2 \,\mathrm{d}p^0 \mathrm{d}\Omega = \frac{m_\mu^3}{8} x^2 \mathrm{d}x \,\mathrm{d}\Omega \quad . \tag{51}$$

We collect all these expressions and substitute them into (48). The final result is

$$\begin{aligned} \mathrm{d}W = \frac{G^2 m_\mu^5}{192\pi^4} x^2 \mathrm{d}x \,\mathrm{d}\Omega \frac{1}{16} \Big\{ & \big[3(a_\mathrm{S} + a_\mathrm{P}) + 6a_\mathrm{T} + 6(a_\mathrm{V} + a_\mathrm{A})\big] \\ & - \big[3(a_\mathrm{S} + a_\mathrm{P}) + 2a_\mathrm{T} + 4(a_\mathrm{V} + a_\mathrm{A})\big] \cdot x + [3a' - 2b' - 2c'] \cos\theta \\ & - [3a' - 4b' + 2c'] x \cos\theta \Big\} \Theta(1 - x) \quad . \end{aligned} \tag{52}$$

This agrees with (2.81b), as is easily verified by inserting the Michel parameters and using (2.78).

MATHEMATICAL SUPPLEMENT

2.12 The Fierz Transformation

Within the framework of the Fermi theory there are two different but equivalent ways of describing a reaction $\psi_1 + \psi_2 \to \psi_3 + \psi_4$, namely

$$(\bar{\psi}_3 \hat{\Gamma} \psi_1)(\bar{\psi}_4 \hat{\Gamma} \psi_2) \quad \text{and} \quad (\bar{\psi}_4 \hat{\Gamma} \psi_1)(\bar{\psi}_3 \hat{\Gamma} \psi_2) \quad . \tag{1}$$

The properties of the Clifford algebra[11] allow us to form 16 matrices

$$\{1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_5\gamma_\mu, \mathrm{i}\gamma_5\} =: \{\hat{O}_1, \dots, \hat{O}_{16}\} \quad . \tag{2a}$$

$$\{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma_5\gamma^\mu, \mathrm{i}\gamma_5\} =: \{\hat{O}^1, \dots, \hat{O}^{16}\} \quad . \tag{2b}$$

which form a basis for any 4×4 matrix. Furthermore it holds that

$$\gamma_0 \hat{O}_i^\dagger \gamma_0 = \hat{O}_i \quad . \tag{3}$$

Hence we may expand $\hat{\Gamma}$ in terms of the $\hat{O}^i$.

$$\sum_{i=1}^{16} C_i \bar{\psi}_3 \hat{O}_i \psi_1 \bar{\psi}_4 \hat{O}^i \psi_2$$

[11] See W. Greiner: *Relativistic Quantum Mechanics – Wave Equations*, 3rd ed. (Springer, Berlin, Heidelberg, 2000).

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$$\sum_{i=1}^{16} C_i' \bar{\psi}_4 \hat{O}_i \psi_1 \bar{\psi}_3 \hat{O}^i \psi_2 \quad . \tag{4a}$$

The requirement for Lorentz invariance demands that

$$C_2 = \cdots = C_5 \quad , \quad C_6 = C_7 = \cdots = C_{11} \quad , \quad C_{12} = \cdots = C_{15} \quad . \tag{4b}$$

Since the two representations (4a) are equivalent, these expressions must be identical for arbitrary values of ψ_1, ψ_2 ,ψ_3, and ψ_4. In terms of the particular components this implies that

$$\sum_{i=1}^{16} C_i (\hat{O}_i)_{\gamma\alpha} (\hat{O}^i)_{\delta\beta} = \sum_{i=1}^{16} C_i' (\hat{O}_i)_{\delta\alpha} (\hat{O}^i)_{\gamma\beta} \quad . \tag{5}$$

In the following steps we will solve this equation for C_i, which requires the determination of the transformation matrix Λ_{ij} connecting the two representations, that is,

$$C_i = \sum_j \Lambda_{ij} C_j' \quad . \tag{6}$$

The transformation from the C_j' to the C_i (or vice versa) is called the **Fierz** transformation.

Multiplying (5) by $(O^l)^{\alpha\gamma} (O_l)^{\beta\delta}$ and summing over α, β, γ, and δ yields

$$\sum_{i=1}^{16} C_i \operatorname{Tr}\{\hat{O}_i \hat{O}^l\} \operatorname{Tr}\{\hat{O}^i \hat{O}_l\} = \sum_{j=1}^{16} \operatorname{Tr}\{\hat{O}_j \hat{O}^l \hat{O}^j \hat{O}_l\} C_j' \quad . \tag{7}$$

We employ the following formulas (see Appendix A.2):

$$\begin{aligned}
\operatorname{Tr}\{1\} &= 4 \quad , \\
\operatorname{Tr}\{\sigma_{\mu\nu}\} &= \operatorname{Tr}\{\gamma_\mu\} = \operatorname{Tr}\{\mathrm{i}\gamma_5\} \\
&= \operatorname{Tr}\{\gamma_5\gamma_\nu\} = 0 \quad , \\
\operatorname{Tr}\{\gamma_\mu \gamma^\nu\} &= 4 g_\mu{}^\nu \quad , \\
\operatorname{Tr}\{\gamma^\mu \sigma_{\mu\nu}\} &= \operatorname{Tr}\{\gamma^\mu \mathrm{i}\gamma_5\} = \operatorname{Tr}\{\gamma^\mu \gamma_5 \gamma_\nu\} = 0 \quad , \\
\operatorname{Tr}\{\sigma_{\mu\nu}\sigma^{\lambda\varrho}\} &= -\frac{1}{4} \operatorname{Tr}\{[\gamma_\mu, \gamma_\nu]\gamma^\lambda \gamma^\varrho - [\gamma_\mu, \gamma_\nu]\gamma^\varrho \gamma^\lambda\} \\
&= -\{g_{\mu\nu} g^{\lambda\varrho} + g_\mu{}^\varrho g_\nu{}^\lambda - g_\mu{}^\lambda g_\nu{}^\varrho \\
&\quad - g_{\mu\nu} g^{\lambda\varrho} - g_\nu{}^\varrho g_\mu{}^\lambda + g_\nu{}^\lambda g_\mu{}^\varrho \\
&\quad - g_{\mu\nu} g^{\varrho\lambda} - g_\mu{}^\lambda g_\nu{}^\varrho + g_\mu{}^\varrho g_\nu{}^\lambda \\
&\quad + g_{\mu\nu} g^{\varrho\lambda} + g_\nu{}^\lambda g_\mu{}^\varrho - g_\nu{}^\varrho g_\mu{}^\lambda\} \\
&= 4\{g_\mu{}^\lambda g_\nu{}^\varrho - g_\nu{}^\lambda g_\mu{}^\varrho\} \quad , \\
\operatorname{Tr}\{\sigma_{\mu\nu}\gamma_5\gamma^\lambda\} &= \operatorname{Tr}\{\sigma_{\mu\nu}\mathrm{i}\gamma_5\} = 0 \quad ,
\end{aligned}$$

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$$\mathrm{Tr}\{\mathrm{i}\gamma_5\gamma_\mu\mathrm{i}\gamma_5\gamma^\nu\} = \mathrm{Tr}\{\gamma_\mu\gamma^\nu\} = 4g_\mu{}^\nu = 4\delta_{\mu\nu} \quad ,$$
$$\mathrm{Tr}\{\mathrm{i}\gamma_5\gamma_\mu\gamma_5\} = 0 \quad ,$$
$$\mathrm{Tr}\{\gamma_5\gamma_5\} = 4 \quad .$$

All these relations may be combined to give

$$\mathrm{Tr}\{\hat{O}^i\hat{O}_l\} = 4\delta^i{}_l\varepsilon_l \quad , \quad \varepsilon_l = \begin{cases} +1 & \text{for} \quad l = 1, \ldots, 11 \\ -1 & \text{for} \quad l = 12, \ldots, 16 \end{cases} \quad . \tag{8}$$

Inserting (8) into (7), we then have

$$C_l = \frac{1}{16}\sum_{j=1}^{16} C'_j \,\mathrm{Tr}\{\hat{O}_j\hat{O}^l\hat{O}^j\hat{O}_l\} \quad . \tag{9}$$

There remains the evaluation of

$$\tilde{\Lambda}_{lj} = \frac{1}{16}\mathrm{Tr}\{\hat{O}_j\hat{O}^l\hat{O}^j\hat{O}_l\} = \tilde{\Lambda}_{jl} \quad . \tag{10}$$

In order to solve for $\tilde{\Lambda}_{jl}$ we consider the particular cases separately.

- $j = 1$:

$$\tilde{\Lambda}_{1l} = 4\varepsilon_l\frac{1}{16} = \frac{1}{4}\varepsilon_l \quad , \tag{11}$$

according to (8).

- $j = 2, \ldots, 5$; $l = 2, \ldots, 5$:

$$\begin{aligned} \tilde{\Lambda}_{jl} &= \frac{1}{16}\mathrm{Tr}\{\gamma_{j-2}\gamma^{l-2}\gamma^{j-2}\gamma_{l-2}\} = \frac{1}{4}\{2\delta_{jl} - 1\} \quad , \\ (\tilde{\Lambda}_{jl}) &= \frac{1}{4}\begin{pmatrix} +1 & -1 & -1 & -1 \\ -1 & +1 & -1 & -1 \\ -1 & -1 & +1 & -1 \\ -1 & -1 & -1 & +1 \end{pmatrix} \quad , \end{aligned} \tag{12}$$

where the index j labels the rows and l the columns.

- $l = 6, \ldots, 11$:

$$\begin{aligned} \hat{O}_l &= \mathrm{i}\gamma_\nu\gamma_\mu \quad (\nu \neq \mu) \quad , \\ \hat{O}_6 &= \mathrm{i}\gamma_0\gamma_1 \quad , \quad \hat{O}_7 = \mathrm{i}\gamma_0\gamma_2 \quad , \\ \hat{O}_8 &= \mathrm{i}\gamma_0\gamma_3 \quad , \quad \hat{O}_9 = \mathrm{i}\gamma_1\gamma_2 \quad , \\ \hat{O}_{10} &= \mathrm{i}\gamma_1\gamma_3 \quad , \quad \hat{O}_{11} = \mathrm{i}\gamma_2\gamma_3 \quad , \\ \tilde{\Lambda}_{jl} &= -\frac{1}{16}\mathrm{Tr}\{\gamma_{j-2}\gamma^\nu\gamma^\mu\gamma^{j-2}\gamma_\nu\gamma_\mu\} \\ &= -\frac{1}{8}\delta^\nu_{j-2}\,\mathrm{Tr}(\gamma^\mu\gamma^{j-2}\gamma_\nu\gamma_\mu) + \frac{1}{16}\mathrm{Tr}(\gamma^\nu\gamma_{j-2}\gamma^\mu\gamma^{j-2}\gamma_\nu\gamma_\mu) \\ &= -\frac{1}{8}\delta^\nu_{j-2}\,\mathrm{Tr}(\gamma^{j-2}\gamma_\nu\gamma_\mu\gamma^\mu) + \frac{1}{8}\delta^\mu_{j-2}\,\mathrm{Tr}(\gamma^\nu\gamma^{j-2}\gamma_\nu\gamma_\mu) \end{aligned} \tag{13}$$

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$$
\begin{aligned}
&-\frac{1}{16}\operatorname{Tr}(\gamma^{\nu}\gamma^{\mu}\gamma_{j-2}\gamma^{j-2}\gamma_{\nu}\gamma_{\mu})\\
&=-\frac{1}{8}\delta^{\nu}_{j-2}\operatorname{Tr}(\gamma^{j-2}\gamma_{\nu})+\frac{1}{8}\delta^{\mu}_{j-2}\operatorname{Tr}(\gamma^{\nu}\gamma^{j-2}\gamma_{\nu}\gamma_{\mu})-\frac{1}{16}\operatorname{Tr}(\gamma^{\nu}\gamma^{\mu}\gamma_{\nu}\gamma_{\mu}) \quad .
\end{aligned}
$$

For the first term we get

$$
-\frac{1}{8}\delta^{\nu}_{j-2}\operatorname{Tr}\big(\gamma^{j-2}\gamma_{\nu}\big)=-\frac{1}{8}\delta^{\nu}_{j-2}\cdot 4\delta^{j-2}_{\nu}=-\frac{1}{2}\delta^{\nu}_{j-2} \quad .
$$

To evaluate the last two terms we take into consideration that $\mu\neq\nu$ and therefore $\gamma_{\mu}\gamma_{\nu}=-\gamma_{\nu}\gamma_{\mu}$, yielding

$$
\begin{aligned}
\frac{1}{8}\delta^{\mu}_{j-2}\operatorname{Tr}(\gamma^{\nu}\gamma^{j-2}\gamma_{\nu}\gamma_{\mu}) &= -\frac{1}{8}\delta^{\mu}_{j-2}\operatorname{Tr}(\gamma^{\nu}\gamma^{j-2}\gamma_{\mu}\gamma_{\nu})\\
&= -\frac{1}{8}\delta^{\mu}_{j-2}\operatorname{Tr}(\gamma_{\nu}\gamma^{\nu}\gamma^{j-2}\gamma_{\mu})\\
&= -\frac{1}{8}\delta^{\mu}_{j-2}\operatorname{Tr}(\gamma^{j-2}\gamma_{\mu})\\
&= -\frac{1}{8}\delta^{\mu}_{j-2}\cdot 4\delta^{j-2}_{\mu}\\
&= -\frac{1}{2}\delta^{\mu}_{j-2} \quad ,
\end{aligned}
$$

in which we have used the fact that the trace is constant under cyclic permutation. For the third term we then obtain

$$
\begin{aligned}
-\frac{1}{16}\operatorname{Tr}(\gamma^{\nu}\gamma^{\mu}\gamma_{\nu}\gamma_{\mu}) &= \frac{1}{16}\operatorname{Tr}(\gamma^{\nu}\gamma_{\nu}\gamma^{\mu}\gamma_{\mu})\\
&= \frac{1}{16}\operatorname{Tr}(1)=\frac{1}{4} \quad ,
\end{aligned}
$$

and, in summary, we finally have

$$
\begin{aligned}
&\tilde{\Lambda}_{jl}=\frac{1}{4}(1-2\delta_{j-2,\nu}-2\delta_{j-2,\mu}) \quad ,\\
&(\tilde{\Lambda}_{jl})=\frac{1}{4}\begin{pmatrix}-1&-1&-1&+1&+1&+1\\-1&+1&+1&-1&-1&+1\\+1&-1&+1&-1&+1&-1\\+1&+1&-1&+1&-1&-1\end{pmatrix} \quad .
\end{aligned}
\tag{14}
$$

- $l=12,\ldots,15$:

$$
\begin{aligned}
&\tilde{\Lambda}_{jl}=\frac{1}{16}\operatorname{Tr}\{\gamma_{j-2}\gamma_5\gamma^{l-12}\gamma^{j-2}\gamma_5\gamma_{l-12}\} \quad ,\\
&(\tilde{\Lambda}_{jl})=\frac{1}{4}\begin{pmatrix}+1&-1&-1&-1\\-1&+1&-1&-1\\-1&-1&+1&-1\\-1&-1&-1&+1\end{pmatrix} \quad .
\end{aligned}
\tag{15}
$$

- $l=16$:

$$
\tilde{\Lambda}_{jl}=-\frac{1}{16}\operatorname{Tr}\{\gamma_{j-2}\gamma_5\gamma^{j-2}\gamma_5\}=\frac{1}{4} \quad .
\tag{16}
$$

Mathematical Supplement 2.12

- $j = 6, \ldots, 11;\ l = 6, \ldots, 11$:

$$\begin{aligned}
\hat{O}_j &= \mathrm{i}\gamma_\nu\gamma_\mu \quad \text{with} \quad \mu > \nu \quad , \\
O_l &= \mathrm{i}\gamma_\varrho\gamma_\lambda \quad \text{with} \quad \lambda > \varrho \quad , \\
\tilde{\Lambda}_{jl} &= \frac{1}{16}\,\mathrm{Tr}\{\gamma_\nu\gamma_\mu\gamma^\varrho\gamma^\lambda\gamma^\nu\gamma^\mu\gamma_\varrho\gamma_\lambda\} \\
&= \frac{1}{16}\big[-2\delta^\nu{}_\mu\,\mathrm{Tr}\{\gamma_\mu\gamma^\lambda\gamma^\nu\gamma^\mu\gamma_\varrho\gamma_\lambda\} + 2\delta^\lambda{}_\nu\,\mathrm{Tr}\{\gamma_\mu\gamma^\varrho\gamma^\nu\gamma^\mu\gamma_\varrho\gamma_\lambda\} \\
&\quad - 2\delta_\mu{}^\varrho\,\mathrm{Tr}\{\gamma^\lambda\gamma^\mu\gamma_\varrho\gamma_\lambda\} + 2\delta_\mu{}^\lambda\,\mathrm{Tr}\{\gamma^\varrho\gamma^\mu\gamma_\varrho\gamma_\lambda\} - \mathrm{Tr}\{\gamma^\varrho\gamma^\lambda\gamma_\varrho\gamma_\lambda\}\big] \\
&= \frac{1}{16}\big[+2\delta_{\mu\varrho}\cdot 4(2\delta_{\nu\lambda}-1) + 2\delta_{\nu\lambda}\cdot 4(2\delta_{\mu\varrho}-1) - 8\delta_{\mu\varrho} - 8\delta_{\mu\lambda} + 4\big] \\
&= \frac{1}{4}\big[1 - 2(\delta_{\nu\varrho}+\delta_{\nu\lambda}+\delta_{\mu\varrho}+\delta_{\mu\lambda}) + 4\delta_{\nu\varrho}\delta_{\mu\lambda} + 4\delta_{\nu\lambda}\delta_{\mu\varrho}\big] \quad , \\
\tilde{\Lambda}_{jl} &= \frac{1}{4}\begin{pmatrix} +1 & -1 & -1 & -1 & -1 & +1 \\ -1 & +1 & -1 & -1 & +1 & -1 \\ -1 & -1 & +1 & +1 & -1 & -1 \\ -1 & -1 & +1 & +1 & -1 & -1 \\ -1 & +1 & -1 & -1 & +1 & -1 \\ +1 & -1 & -1 & -1 & -1 & +1 \end{pmatrix} \quad .
\end{aligned} \tag{17}$$

- $l = 12, \ldots, 15$:

$$\tilde{\Lambda}_{jl} = -\frac{1}{16}\,\mathrm{Tr}\{\gamma_\nu\gamma_\mu\gamma_5\gamma^{l-12}\gamma^\nu\gamma^\mu\gamma_5\gamma_{l-12}\} \quad .$$

Together with (14) it follows that

$$\tilde{\Lambda}_{jl} = \frac{1}{4}\begin{pmatrix} +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & -1 & -1 & +1 \\ -1 & +1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ -1 & -1 & +1 & +1 \end{pmatrix} \quad . \tag{18}$$

- $l = 16$:

From (8) it follows that

$$\tilde{\Lambda}_{j,16} = -\frac{1}{16}\,\mathrm{Tr}(\hat{O}_j\hat{O}^j) = -\frac{1}{4} \quad . \tag{19}$$

- $j = 12, \ldots, 15;\ l = 12, \ldots, 15$:

$$\begin{aligned}
\tilde{\Lambda}_{jl} &= \frac{1}{16}\,\mathrm{Tr}\{\gamma_{j-12}\gamma^{l-12}\gamma^{j-12}\gamma_{l-12}\} \\
&= \frac{1}{4}\{2\delta_{jl} - 1\} \quad , \\
\tilde{\Lambda}_{jl} &= \frac{1}{4}\begin{pmatrix} +1 & -1 & -1 & -1 \\ -1 & +1 & -1 & -1 \\ -1 & -1 & +1 & -1 \\ -1 & -1 & -1 & +1 \end{pmatrix} \quad .
\end{aligned} \tag{20}$$

• $l = 16$: *Mathematical Supplement 2.12*

$$\tilde{\Lambda}_{j,16} = -\frac{1}{16}\,\mathrm{Tr}\{\gamma_5\gamma_{j-12}\gamma_5\gamma_5\gamma^{j-12}\gamma_5\} = -\frac{1}{4} \quad . \tag{21}$$

• $j = 16; l = 16$:

$$\tilde{\Lambda}_{16,16} = \frac{1}{16}4 = \frac{1}{4} \quad . \tag{22}$$

According to (4a) we may combine the C_i , C_j as follows:

$$\begin{aligned}
C_\mathrm{S} &= C_1 \quad ,\\
C_\mathrm{V} &= C_2 = C_3 = C_4 = C_5 \quad ,\\
C_\mathrm{T} &= C_6 = \cdots = C_{11} \quad ,\\
C_\mathrm{A} &= -C_{12} = \cdots = -C_{15} \quad ,\\
C_\mathrm{P} &= -C_{16} \quad ,
\end{aligned} \tag{23}$$

where the negative signs correspond to the convention. Similarly (6) now reads as follows:

$$\begin{aligned}
C_i &= \sum_j C'_j \tilde{\Lambda}_{ij}\\
&= \tilde{\Lambda}_{i1}C'_1 + \sum_{j=2}^{5}\tilde{\Lambda}_{ij}C'_j + \sum_{j=6}^{11}\tilde{\Lambda}_{ij}C'_j + \sum_{j=12}^{15}\tilde{\Lambda}_{ij}C'_j + \tilde{\Lambda}_{i,16}C'_{16}\\
&= \tilde{\Lambda}_{i1}C_\mathrm{S} + \left(\sum_{j=2}^{5}\tilde{\Lambda}_{ij}\right)C_\mathrm{V} + \left(\sum_{j=6}^{11}\tilde{\Lambda}_{ij}\right)C_\mathrm{T} - \left(\sum_{j=12}^{15}\tilde{\Lambda}_{ij}\right)C_\mathrm{A} - \tilde{\Lambda}_{i,16}C_\mathrm{P} \quad .
\end{aligned}$$

Thus it follows that

$$\Lambda_{IJ} = \sum_{j \text{ in } J} \varepsilon^I \varepsilon^J \tilde{\Lambda}_{ij} \quad \text{with } i \text{ in } I \quad , \tag{24}$$

where

$$\varepsilon^I = \begin{cases} +1 & \text{for} \quad \mathrm{S, V, T} \\ -1 & \text{for} \quad \mathrm{A, P} \end{cases} \quad . \tag{25}$$

From (11)–(22) it follows that

$$\Lambda_{IJ} = \frac{1}{4}\begin{pmatrix} 1 & 4 & 6 & 4 & 1\\ 1 & -2 & 0 & 2 & -1\\ 1 & 0 & -2 & 0 & 1\\ 1 & 2 & 0 & -2 & -1\\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} \quad . \tag{26}$$

This is the standard representation of the Fierz transformation and the one most commonly used. It is easily checked that $\Lambda^2 = 1$, and therefore $\Lambda = \Lambda^{-1}$, i.e. Λ is its own inverse.

Mathematical Supplement 2.12 An alternative form is obtained from (22) and (23) by introducing

$$
\begin{aligned}
\mathrm{S}(3,1;4,2) &:= (\bar{\psi}_3 1 \psi_1)(\bar{\psi}_4 1 \psi_2) \quad , \\
\mathrm{V}(3,1;4,2) &:= (\bar{\psi}_3 \gamma_\mu \psi_1)(\bar{\psi}_4 \gamma^\mu \psi_2) \quad , \\
\mathrm{T}(3,1;4,2) &:= \frac{1}{2}(\bar{\psi}_3 \sigma_{\mu\nu} \psi_1)(\bar{\psi}_4 \sigma^{\mu\nu} \psi_2) \quad , \\
\mathrm{A}(3,1;4,2) &:= (\bar{\psi}_3 \gamma_5 \gamma_\mu \psi_1)(\bar{\psi}_4 \gamma^\mu \gamma_5 \psi_2) \quad , \\
\mathrm{P}(3,1;4,2) &:= (\bar{\psi}_3 \gamma_5 \psi_1)(\bar{\psi}_4 \gamma_5 \psi_2) \quad ,
\end{aligned}
\tag{27}
$$

and replacing (5) by

$$
\sum_{I=\mathrm{S,V,T,A,P}} C_I I(3,1;4,2) = \sum_{J=\mathrm{S,V,T,A,P}} C'_J \, J(4,1;3,2) \quad . \tag{28}
$$

The transformation of the matrices I and J is then given by

$$
I(3,1;4,2) = \sum_J \Lambda_{JI} J(4,1;3,2) \quad ,
$$

and because Λ is self-inversive it also follows that

$$
J(4,1;32) = \sum \Lambda_{IJ} I(3,1;42) \quad .
$$

2.6 The Tau Lepton

In the year 1975 a further lepton was discovered at Stanford (SLAC) by **Perl**, which has been named the τ lepton.[12] With a mass of 1784 ± 3 MeV it is almost 20 times heavier than the muon. Its lifetime is

$$
T_\tau = (3.4 \pm 0.5) \times 10^{-13}\ \mathrm{s} \quad . \tag{2.87}
$$

The scheme of τ lepton decay is completely analogous to muon decay, which we have discussed in detail. Since both the electron and the muon have smaller masses than the τ lepton, both decay processes are possible:

$$
\tau^- \to \mu^- + \bar{\nu}_\mu + \nu_\tau \quad , \tag{2.88a}
$$

$$
\tau^- \to \mathrm{e}^- + \bar{\nu}_\mathrm{e} + \nu_\tau \quad . \tag{2.88b}
$$

In addition, the τ lepton may also decay into strongly interacting particles, especially into three or more pions together with a τ neutrino. These hadronic processes contribute about 65% to the total decay probability of the τ lepton (see Table 2.2); however, we will not consider them here but will rather focus on the leptonic processes.

[12] M.L. Perl et al.: Phys. Rev. Lett. **35**, 148 (1975); M.L. Perl: Ann. Rev. Nucl. Part. Science **30**, 299 (1980); G.S. Abrams, M.L. Perl et al.: Phys. Rev. Lett **43**, 1555 (1979).

Table 2.2. Decay probabilities of the τ lepton

Decay	%
$\tau^- \to e^- \bar{\nu}_e \nu_\tau$	16.4 ± 1.8
$\mu^- \bar{\nu}_\mu \nu_\tau$	16.0 ± 1.7
$\pi^- \nu_\tau$	10.3 ± 1.2
$\rho^- \nu_\tau$	22.1 ± 2.4
$K^- \nu_\tau$	1.3 ± 0.5
$\pi^- \rho^0 \nu_\tau$	5.4 ± 1.7
further hadronic decays	26.0 ± 1.3

Besides the properties that result from its rather large mass, the τ lepton behaves just like an electron or muon. This fact is sometimes termed e–μ–τ *universality*. For example, the τ lepton is observed with large accuracy to be point-like. Its internal extension amounts to less than 0.004 fm. Furthermore the electron and muon spectra observed in the decay processes (2.88) may be analyzed in terms of Michel parameters, in analogy to the case of muon decay, which we have already treated. The result is[13]

$$\rho_\tau = 0.742 \pm 0.035 \pm 0.020 \quad , \tag{2.89}$$

which is a strong argument for V–A coupling ($\rho = 0.75$) and unambiguously excludes V+A coupling ($\rho = 0$), as well as pure V or A coupling ($\rho = 0.375$). This behavior also becomes obvious from Fig. 2.10, which compares the observed electron spectrum with the predictions of the V–A and V+A theory.

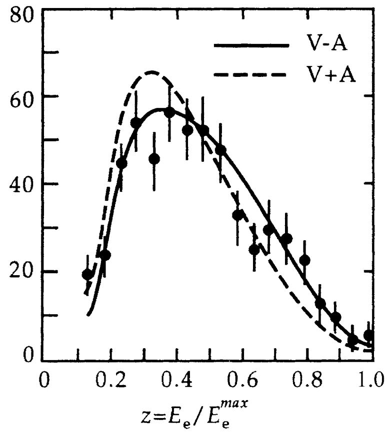

Fig. 2.10. Electron spectrum of the τ decay

From a detailed analysis of the shape of the high-energy end of the muon spectrum, an upper limit for the mass of the τ neutrino can be inferred (see Fig. 2.11). The most accurate value today is

$$m_{\nu_\tau} \leq 70\,\mathrm{MeV} \tag{2.90}$$

[13] H. Albrecht et al. [ARGUS Collaboration]: Phys. Lett. **B246** (1990) 278–284.

However, it is not unlikely that its rest mass vanishes. If we assume that the weak current of the τ particles is of the familiar form

$$J_\mu^{(\tau)}(x) = \bar{u}_\tau(x)\gamma_\mu(1-\gamma_5)u_{\nu_\tau}(x) \quad , \tag{2.91}$$

we immediately obtain the decay rates into the leptonic channels (2.88) by simply adopting the formula (2.57) for the muonic decay. Again, we set $m_e = 0$, but do not neglect the muon mass:

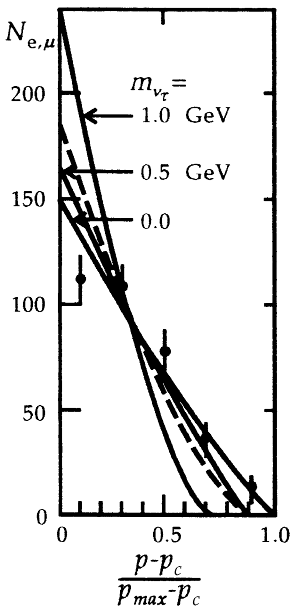

Fig. 2.11. The number of the observed electrons and muons is depicted as a function of the momentum.[14] Here $p_c = 0.65$ GeV is the lower limit of the momentum observed in the experiment. Each curve stands for one value of the mass of the τ neutrino. The dashed curve is for V+A coupling and $m_{\nu_\tau} = 0$

$$W_{\tau^-\to e^-\nu_e\nu_\tau} = \frac{G^2 m_\tau^5}{192\pi^3} = W_{\mu^-\to e^-\nu_e\nu_\mu}\left(\frac{m_\tau}{m_\mu}\right)^5 \quad , \tag{2.92a}$$

$$\begin{aligned} W_{\tau^-\to\mu^-\nu_\mu\nu_\tau} &= \frac{G^2 m_\tau^5}{192\pi^3}\left(1-8\frac{m_\mu^2}{m_\tau^2}\right) \\ &= W_{\mu^-\to e^-\nu_e\nu_\mu}\left(\frac{m_\tau}{m_\mu}\right)^5\left(1-8\frac{m_\mu^2}{m_\tau^2}\right) \quad . \end{aligned} \tag{2.92b}$$

Inserting the value $m_\tau/m_\mu = 16.86$, we obtain

$$W_{\tau\to e} = 0.620\times 10^{+12}\ \mathrm{s}^{-1} \quad , \tag{2.93a}$$

$$W_{\tau\to\mu} = 0.603\times 10^{+12}\ \mathrm{s}^{-1} \quad . \tag{2.93b}$$

The ratio of these quantities is

$$\frac{W_{\tau\to\mu}}{W_{\tau\to e}} = \left(1-8\frac{m_\mu^2}{m_\tau^2}\right) = 0.972 \quad . \tag{2.94}$$

On the other hand, the experimental determination of the relative probability for these two decay processes, compared with the total decay rate, yields the following values:[15]

$$\begin{aligned} B_{\tau\to e} &= W_{\tau\to e}/W_\tau = (17.7\pm 0.4) \quad , \\ B_{\tau\to\mu} &= W_{\tau\to\mu}/W_\tau = (17.8\pm 0.4) \quad . \end{aligned} \tag{2.95}$$

These values yield the experimental ratio

$$\frac{B_{\tau\to\mu}}{B_{\tau\to e}} = 0.9\pm 0.1 \quad , \tag{2.96}$$

which agrees with the theoretical prediction (2.94) within the accuracy of the experiment. By inserting (2.95) in (2.92a) we can give a theoretical prediction for the lifetime of the τ lepton:

$$T_\tau = \frac{B_{\tau\to e}}{W_{\tau\to e}} = (2.6\pm 0.2)\times 10^{-13}\mathrm{s} \quad , \tag{2.97}$$

[14] M.L. Perl: Ann. Rev. Nucl. Part. Science **30**, 299 (1980).

[15] Review of particle properties in M. Aguilar-Benitez et al.: Phys. Rev. D **45**, Part II (June 1992).

which at least does not contradict the experimental value (2.87). From this it follows that the coupling constant G occurring in (2.92a) cannot differ significantly from the coupling constant G of muon decay.

To summarize, we conclude that according to the actual data the τ lepton fits perfectly into the family of leptons (e, μ, τ). The only differences between these leptons are their masses and a quantum number that guarantees the separate conservation of the electronic, muonic, and τ-leptonic particle numbers. In particular, the leptons exhibit a completely universal behavior in electromagnetic and weak interactions.

EXAMPLE

2.13 The Discovery of the Tau Lepton

The τ lepton was discovered at the SPEAR storage ring by the magnetic detector of the SLAC–LBL collaboration (Stanford Linear Accelerator Center – Lawrence Berkeley Laboratory).[16] The principle of a storage ring is that particle and antiparticle beams circulate within the ring in opposite directions and are forced to overlap in the region of the detector (see Fig. 2.12). The detector was constructed in such a way that electrons, muons, and photons, as well as hadrons, could be detected and identified within a large solid angle. In addition, the trajectories of the charged particles in the magnetic field allowed for a determination of their momentum.

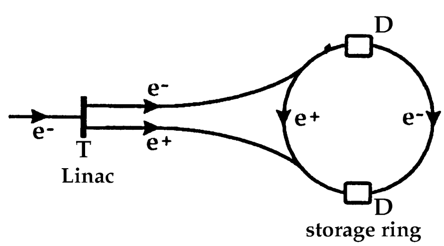

Fig. 2.12. Schematic picture of the storage ring facility SPEAR at SLAC

Through investigations of electron–positron collisions a number of events of the form

$$e^+ + e^- \to \left\{ \begin{matrix} e^+ + \mu^- \\ e^- + \mu^+ \end{matrix} \right\} + \text{ at least 2 unobserved particles}$$

were observed – until 1975 a total number of 105 events. These processes could not be understood in terms of a conventional interpretation, especially since the possible uncertainty in particle identification by the detector had already been taken into account, that is to say, the most unfavorable assumption was made, namely that all processes with three observed charged particles implied the production of hadrons only. Thus every "electron" or "muon" was claimed to be a misinterpretation of the detector. This allowed an estimate to be made of how reliable particle identification was. It was therefore possible to evaluate, from the number of observed events in which a lepton and a hadron, or two hadrons, occurred, the number of misinterpreted e–μ events. It followed that of the 139 events originally observed, 34 were spurious and had to

[16] G.J. Feldman and M.L. Perl: Phys. Rep. **19**, 233 (1975).

Example 2.13

be subtracted. The immediate conclusion was that the uncertainty in particle identification could not explain all these events. One might argue that at least one of the observed particles was a charged particle, a photon, or a neutral pion decaying into two photons, but one not reaching the effective region of the detector. On the other hand, such processes would imply the occurrence of corresponding events in which the particle is actually detected. However, this was not the case.

The sole remaining explanation was the interpretation of the e–μ events in terms of the production of hardly detectable particles such as neutrons, K_L^0 (see Chap. 8), or neutrinos. However, the K_L^0 is ruled out by the reasonable assumption that the production rates for K_L^0 and K_S^0 are equal. The latter could easily be identified by its decay products π^+ and π^-. However, up until 1976 only a single event had been observed.

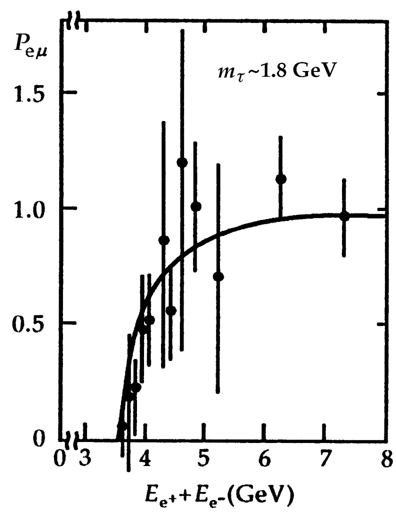

Fig. 2.13. Cross section for electron–muon events in storage ring experiments

A characteristic feature of the e–μ events is their threshold energy of about 3.6–4 GeV, that is, they do not occur at lower energies (see Fig 2.13). Another significant property is that with increasing energy the electron and muon are preferentially emitted collinearly in opposite directions. This strongly suggests the production of a particle–antiparticle pair,

$$e^+ + e^- \to \tau^+ + \tau^- \quad .$$

Owing to momentum conservation, the two particles should be emitted in exactly opposite directions. Hence, a higher energy implies that the particles have a larger momentum. Subsequently, the two particles decay into an electron (positron) or a muon which is emitted isotropically with respect to the rest frame of the corresponding τ particle. However, the larger the velocity of the τ particle, the less the direction of emission with respect to the τ particle's rest frame contributes to the emission actually observed within the lab system, whereby the latter is then essentially determined by the direction of emission of the τ particle.

The observed threshold energy leads to the conclusion that the mass of the τ particle lies in the range 1.6–2 GeV. In order to characterize the nature of the τ particle, there were in practice two options: either it is a lepton that decays according to

$$\tau^- \to \nu_\tau + e^- + \bar{\nu}_e \quad , \quad \tau^- \to \nu_\tau + \mu^- + \bar{\nu}_\mu \quad ,$$

and similarly for the antiparticle τ^+, or it is a boson with the following decay channels: *Example 2.13*

$$\tau^- \to e^- + \bar{\nu}_e \quad , \quad \tau^- \to \mu^- + \bar{\nu}_\mu \quad ,$$

as are observed for example, in the decay of negatively charged pions. The latter interpretation, however, could certainly be ruled out by the analysis of the momentum distribution of produced electrons (positrons) and muons (the specific form of this distribution also contradicts the interpretation of the observed particle in terms of a neutron).

Therefore the sole explanation that remained was the classification of the τ particle as a new, heavy lepton. Figure 2.14 illustrates how the total process results in the observed e–μ events. Since 1975 the properties of the τ lepton have been extensively studied, its mass has been accurately determined to be 1784 ± 3 MeV, its Michel parameters were obtained[17] as $\rho = 0.731 \pm 0.031$, $\xi = 1.03 \pm 0.11$, $\xi\delta = 0.63 \pm 0.09$, and thus the V–A coupling of its decay has been verified in detail.

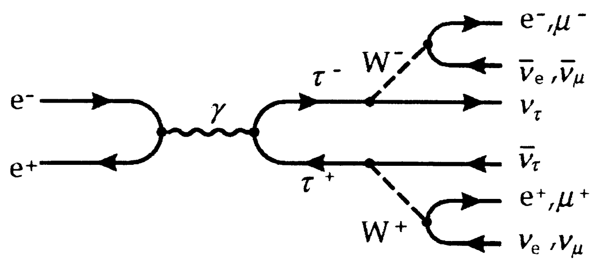

Fig. 2.14. Production and decay of the τ lepton

2.7 Biographical Notes

FIERZ, Markus, *20.6.1912 in Basel (Switzerland), †20.6.2006 in Küsnacht (Switzerland), professor at the University of Basel 1944–1960, since 1960 successor of W. Pauli at the ETH Zürich, in 1969 appointed director of the Theoretical Division at CERN, Geneva.

LEVI-CIVITA, Tullio, mathematician, *29.3.1873 in Padua (Italy), †29.12.1941 in Rome. In 1898 he became professor of mechanics in Padua, since 1918 at the University of Rome. He developed differential and tensor calculus, which laid the basis for Einstein's general theory of relativity. He introduced the idea of parallel transport and developed the theory of curved spaces.

MICHEL, Louis, *4.5.1923 in Roanne (France), †30.12.1999 in Bures-sur-Yvette (France), professor at the Ecole Polytechnique in Paris, since 1962.

PERL, Martin, L. *1927 in New York. Attended New York city schools. After military services in World War II, he received a Bachelor in Chemical Engineering degree from the Polytechnic Institute of Brooklyn in 1948. After several years working for the General Electric Co. as a chemical engineer, he went to graduate school in physics at Columbia University, studied under

[17] H. Albrecht et al. (51 authors): The ARGUS Collaboration, DESY-preprint 97-194.

I. I. Rabi, and received his Ph.D. in 1955. From 1955 to 1963 he did research and taught at the University of Michigan. Since 1963 he has been at the Stanford Linear Accelerator Center at Stanford University where he is a Professor of Physics and Group Leader. In 1990–1992 he was a Distinguished Visiting Professor at the University of Michigan.

His major research interest is experimental elementary particle physics. Other research interests are optical and electronic devices, and the application of small drop technology. He is also interested in applying these technologies to industry, biology, and medicine.

He received the 1995 Nobel Prize in Physics in recognition of his discovery of the tau lepton, the heaviest known member of the electron–muon–tau sequence of charged leptons. Finding the tau lepton subsequently led to the discovery of the three generations of elementary particles, an essential ingredient in what has now become the Standard Model of fundamental particles and interactions. He has published 250 papers in physics and science editions and edited or authored five books including Reflectors on Experimental Science which he wrote in 1996.

In addition to the 1995 Nobel Prize in Physics, he received the 1982 Wolf Prize in Physics. He is a fellow of the American Physical Society and a member of the U.S. National Academy of Science and American Academy of Arts and Science. He holds honorary degrees from the University of Chicago in 1990 and Polytechnic University in 1996.

Gauge Theory of Weak Interactions
Greiner, W.; Müller, B.
2009, XVI, 404 p. With 75 Worked Examples and Problems., Softcover
ISBN: 978-3-540-87842-1