Preface

The principal aim of the book is to give a comprehensive account of the variety of approaches to such an important and complex concept as Integrability. Developing mathematical models, physicists often raise the following questions: whether the model obtained is integrable or close in some sense to an integrable one and whether it can be studied in depth analytically. In this book we have tried to create a mathematical framework to address these issues, and we give descriptions of methods and review results.

In the Introduction we give a historical account of the birth and development of the theory of integrable equations, focusing on the main issue of the book – the concept of integrability itself. A universal definition of *Integrability* is proving to be elusive despite more than 40 years of its development. Often such notions as "exact solvability" or "regular behaviour" of solutions are associated with integrable systems. Unfortunately these notions do not lead to any rigorous mathematical definition. A constructive approach could be based upon the study of hidden and rich algebraic or analytic structures associated with integrable equations. The requirement of existence of elements of these structures could, in principle, be taken as a definition for integrability. It is astonishing that the final result is not sensitive to the choice of the structure taken; eventually we arrive at the same pattern of equations. The relationship between the different approaches is often far from obvious and needs to be understood better.

Integrable equations possess hidden symmetries and actually possess infinite hierarchies of local symmetries. This property is taken as a definition of integrability in the symmetry approach. A detailed introduction and review of the modern state of the symmetry approach is given in Chap. 1, written by A.V. Mikhailov and V. Sokolov. The symmetry approach provides powerful necessary conditions for the existence of local higher symmetries and/or conservation laws for systems of differential equations. For a given system of equations these conditions are easily verifiable and eventually can serve as a criterion of integrability. Chapter 1 also contains an account of classification results obtained and an extensive bibliography.

For evolutionary equations whose right-hand side is a homogeneous differential polynomial, the symbolic representation and powerful results of number theory allow us to achieve global classification results (Chap. 2, written by J. Sanders and J.P. Wang). One of the most spectacular results of this theory can be formulated as follows: any scalar integrable evolutionary equation whose right-hand side is a homogeneous differential polynomial (with a positive weight) belongs to one of the infinite hierarchies of equations of order $2, 3$ or 5 and all these integrable hierarchies are explicitly listed. It is shown that for a scalar evolutionary equation the existence of one higher symmetry implies the existence of an infinite hierarchy of hidden symmetries and therefore the integrability of the equation. For systems of equations a similar statement is not valid: there are examples of systems which have only a finite number of higher local symmetries. Chapter 2 is an excellent introduction to the symbolic method and contains relevant number theory results in applications to the theory of integrable equations.

In Chap. 3, written by S.P. Novikov, the phenomenon of integrability is associated with hidden symmetries of linear spectral problems. Darboux and Laplas transformations for one- and two-dimensional Schrödinger operators are famous examples of the spectral symmetries. The proper discretisation of these operators, the corresponding discrete Darboux and Laplas transformations and their relation to integrable equations and finite gap solutions are discussed. Chapter 4, written by A. Shabat is devoted to a detailed study of continuous and discrete spectral symmetries in the one-dimensional case. A connection of these symmetries with the famous list of Painlevé equations and with dressing chains is discussed.

Chapters 5 and 6 explore perturbative and asymptotic aspects of integrable equations. The concept of approximate integrability, approximate symmetries and conservation laws are discussed in Chap. 5, written by Y. Hiraoka and Y. Kodama. It is an attempt to extend the classical theory of normal forms to the case of partial differential equations. If the main approximation is given by an integrable equation, the higher order corrections often violate integrability and give rise to new effects, such as inelasticity in soliton interaction, creation of new solitons as a result of soliton collisions, etc. Chapter 6, written by A. Degasperis, addresses multiscaling expansion and universal equations, i.e. nonlinear equations which determine the leading term in the asymptotic expansion. Francesco Calogero gave a simple explanation for why integrable equations, which are rather exceptional, are widely applicable. Universal equations have a good chance to be integrable, since the multiscaling expansion preserves the main attributes of integrability, such as symmetries, local conservation laws, etc. The analysis of higher order corrections in a multiscale expansion of a given system provides necessary conditions for integrability of the system.

In the analytical theory of differential equations we study the structure of singularities of the solutions. The absence of movable critical singularities can be taken as a criteria for isolation of integrable systems. This is at the heart of the Painlevé approach and its generalisations described in Chap. 7 written by A. Hone.

Chapter 8, written by J. Hietarinta, describes the modern development of the Hirota approach and bi-linear representation of integrable equations. This kind of representation proved to be very useful for construction and analysis of explicit multi-soliton solutions. It can also be used for a classification of integrable equations of special form.

Quantum integrability is a separate and well-developed subject. It deserves a separate volume. We include lectures of T. Miwa (Chap. 9) in order to give a flavour of quantum integrability and to highlight the symmetry aspects of quantum integrable systems in the example of XXZ model.

This book is a unique collection of articles which could serve as the core material for a number of graduate lecture courses. The chapters in the book are independent and self-contained. They can be read in any order. Chapter 1 is probably more pedagogical than others and can be recommended for those wishing to become acquainted with the subject. The book was specifically designed to be accessible to graduate students and post-docs.

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