Chapter 2
Deterministic and Statistical Approaches for Studying Rogue Waves

Depending on the objective in mind, two main approaches can be used for the water wave description, based on deterministic or statistical methods. Deterministic equations are very useful and powerful in understanding and describing the underlying physics of water waves; namely, they may be used in practice to estimate in detail wave impact upon structures and ships. Statistical equations are usually used to estimate typical wave motion and probability of this or that wave situation. When the sea surface elevation is such a complicated function of space and time, a statistical description is easier than a detailed description, but still may provide sufficient information about the waves.

In this chapter, we introduce first the basic equations governing the dynamics of water waves. The scales of the wavelength considered are long enough to neglect surface tension. Hence, the waves are called gravity waves since their main restoring force is gravity. Within the framework of water waves, we discuss and justify the different assumptions used to derive from the most complete system, the Navier-Stokes equations—a simplified set of equations describing realistic wave dynamics. In this way, the assumptions of incompressible and perfect fluid and irrotational motion are introduced successively to derive the simplified model. The simplified equations fall within the scope of the potential theory. Nevertheless, some of these assumptions may become questionable—for instance,, in shallow water where bottom friction can be important. Near the bottom a boundary layer of thickness of $O(2\nu/\Omega)$ develops, where ν and Ω are the molecular viscosity and the free surface wave frequency. So, for swells of 10 s, the boundary layer thickness is 0.17 cm with $\nu = 0.01\,\mathrm{cm}^2\,\mathrm{s}^{-1}$. The role of molecular viscosity in the formation of rogue waves can be considered as negligible. For turbulent boundary layers, the turbulent viscosity is much larger than the molecular viscosity ν and bottom friction may influence rogue wave dynamics. This aspect is discussed in Sect. 4.1.2. In the presence of breaking waves, the motion cannot be considered as irrotational and the dissipation of the waves is mainly due to turbulence (and not to molecular viscosity). Section 2.3 introduces concepts that will be used in subsequent chapters. Therefore, we focus attention on various physical mechanisms that contribute to the formation of extreme water wave events. Despite the complexity of the sea surface, we are aimed at describing quite simple realistic models that capture the essential features of rogue-wave phenomena.

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2.1 Deterministic Equations

2.1.1 Mass and Momentum Conservation Equations

An Eulerian description of the fluid motion is adopted. The motion is described by the velocity field $\mathbf{U} = (U,V,W)^t$ as a function of time T, horizontal coordinates (X,Y) and vertical coordinate Z. The illustration of the problem geometry is provided in Fig. 2.1. The unperturbed surface coincides with the plane OXY at $Z = 0$, and the horizontal bed is situated at $Z = -D$. Typically, the waves are supposedly propagating along the OX direction.

The mass conservation or continuity equation is

$$\frac{\partial \rho}{\partial T} + \nabla \cdot (\rho \mathbf{U}) = 0, \tag{2.1}$$

or

$$\frac{\mathrm{D}\rho}{\mathrm{D}T} + \rho \nabla \cdot \mathbf{U} = 0, \tag{2.2}$$

where ρ is the water density, $\nabla\cdot$ is the divergence operator, and $\mathrm{D}/\mathrm{D}T$ is the material derivative given by

$$\frac{\mathrm{D}}{\mathrm{D}T} = \frac{\partial}{\partial T} + (\mathbf{U} \cdot \nabla), \tag{2.3}$$

$\nabla = (\partial/\partial X, \partial/\partial Y, \partial/\partial Z)^t$ is the gradient operator and $(\bullet)^t$ indicates transposition.

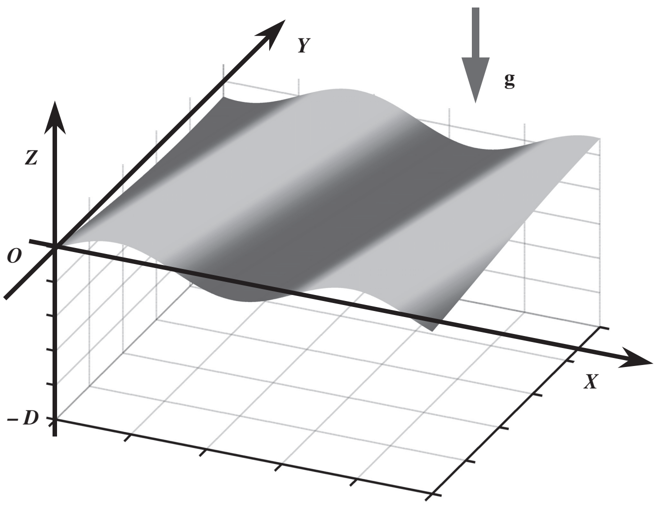

Fig. 2.1 Configuration of the problem

The incompressibility condition of water reads $\mathrm{D}\rho/\mathrm{D}T = 0$, hence from the continuity equation we have

$$\nabla \cdot \mathbf{U} = 0. \tag{2.4}$$

The momentum-conservation equation, based on Newton's second law, reduces to the Navier-Stokes equation when considering water as an incompressible Newtonian fluid. The vector form of this equation is

$$\rho \frac{\mathrm{D}\mathbf{U}}{\mathrm{D}T} = -\nabla P + \rho \mathbf{F} + \mu \Delta \mathbf{U} \tag{2.5}$$

where P is the pressure, μ is the dynamic viscosity of the fluid, and Δ is the Laplacian operator $\Delta = \nabla \cdot \nabla$. The first and last terms on the Right Hand Side (RHS) of this equation correspond to pressure forces and viscous forces, respectively, while $\mathbf{F}$ is the body force due to the gravitational acceleration: $\mathbf{F} = \mathbf{g}$.

The corresponding X–, Y– and Z– momentum equations are given by

$$\rho \frac{\mathrm{D}U}{\mathrm{D}T} = -\frac{\partial P}{\partial X} + \rho F_X + \mu \Delta U, \tag{2.6}$$

$$\rho \frac{\mathrm{D}V}{\mathrm{D}T} = -\frac{\partial P}{\partial Y} + \rho F_Y + \mu \Delta V, \tag{2.7}$$

$$\rho \frac{\mathrm{D}W}{\mathrm{D}T} = -\frac{\partial P}{\partial Z} + \rho F_Z + \mu \Delta W, \tag{2.8}$$

where F_X, F_Y and F_Z are the components of the body forces $\mathbf{F}$ experienced by the fluid. Hence Eq. (2.5) is rewritten as follows:

$$\frac{\mathrm{D}\mathbf{U}}{\mathrm{D}T} = -\frac{1}{\rho} \nabla P + \mathbf{g} + \nu \Delta \mathbf{U}, \tag{2.9}$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

Equation (2.9) may be written as follows:

$$\frac{\partial \mathbf{U}}{\partial T} + \frac{1}{2} \nabla \left(\mathbf{U}^2 \right) = \mathbf{U} \times \omega - \frac{1}{\rho} \nabla P + \mathbf{g} + \nu \Delta \mathbf{U}, \tag{2.10}$$

where $\omega = \nabla \times \mathbf{U}$ is the vorticity. The operator $\nabla \times$ is the curl operator. By taking the curl of Eq. (2.9) and using Eq. (2.4), we obtain the vorticity equation

$$\frac{\mathrm{D}\omega}{\mathrm{D}T} = (\omega \cdot \nabla) \mathbf{U} + \nu \Delta \omega. \tag{2.11}$$

For 3D motions, the nonlinear term on the RHS of Eq. (2.11) is responsible for the vortex stretching and tilting while the linear term corresponds to the diffusion of vorticity due to viscosity.

Generally, water is considered as a weakly viscous fluid. In the vicinity of free surfaces and solid boundaries (the sea bottom), the thickness of the vortical layer is $O(\nu^{1/2})$. Hence, it will be assumed that the vortical part of the flow is confined to a thin boundary layer of thickness that is small compared to the other scales of the problem, so viscous effects are dropped from the equations.

We can consider that water waves have been generated from a fluid that was initially at rest—that is, from an irrotational motion. When the fluid is incompressible and inviscid, and external forces derive from a potential, the Kelvin-Lagrange theorem states that the motion remains irrotational. Hence, $|\omega| = 0$ and the velocity $\mathbf{U}$ derives from a potential $\phi(X,Y,Z,T)$ such that

$$\mathbf{U} = \nabla\phi. \tag{2.12}$$

Under the hypotheses of irrotational motion and inviscid fluid, Eqs. (2.4) and (2.10) become, respectively

$$\Delta\phi = 0 \tag{2.13}$$

and

$$\frac{\partial \mathbf{U}}{\partial T} + \frac{1}{2}\nabla\left(\mathbf{U}^2\right) = -\frac{1}{\rho}\nabla P + \mathbf{g}. \tag{2.14}$$

Substituting $\nabla\phi$ for $\mathbf{U}$ in Eq. (2.14) gives

$$\nabla\left(\frac{\partial\phi}{\partial T} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{P}{\rho}\right) - \mathbf{g} = 0 \tag{2.15}$$

Noting that $\mathbf{g} = (0,0,-g)^t$ so that $\mathbf{g} = \nabla(-g\mathrm{Z})$, the previous equation takes the following form

$$\nabla\left(\frac{\partial\phi}{\partial T} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{P}{\rho} + gZ\right) = 0. \tag{2.16}$$

Integration with respect to space variables yields the Bernoulli equation

$$\frac{\partial\phi}{\partial T} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{P}{\rho} + gZ = C(T). \tag{2.17}$$

The time dependent function $C(T)$ can be incorporated into the potential ϕ by the following transformation

$$\phi \to \phi + \int_0^T C(\xi)\,d\xi. \tag{2.18}$$

Thus, Eq. (2.17) is rewritten as follows:

$$\frac{\partial\phi}{\partial T} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{P}{\rho} + gZ = 0. \tag{2.19}$$

To solve the Laplace equation (2.13), conditions on boundaries are needed.

2.1.2 Boundary Conditions

The fluid domain that is considered is bounded by two kinds of boundaries: the interface, which separates the air from the water; and the wetted surface of an impenetrable solid (the sea bottom, for instance). The air-sea interface is assumed to be a free surface whose equation is given by

$$S(X,Y,Z,T)=0. \tag{2.20}$$

The kinematic boundary condition states that the normal velocity of the surface is equal to the normal velocity of the fluid at the surface. The normal velocity of the surface is

$$V_n=-\frac{1}{|\nabla S|}\frac{\partial S}{\partial T}, \tag{2.21}$$

and the normal velocity of the fluid is

$$U_n=\mathbf{n}\cdot\mathbf{U}. \tag{2.22}$$

where $\mathbf{n}=\nabla S/|\nabla S|$ is the unit vector normal to the surface.

The mathematical expression of the kinematic boundary condition is therefore

$$V_n=U_n. \tag{2.23}$$

Hence,

$$\frac{\partial S}{\partial T}+\mathbf{U}\cdot\nabla S=0, \tag{2.24}$$

$$\frac{\mathrm{D}S}{\mathrm{D}T}=0. \tag{2.25}$$

Equation (2.25) means that a fluid particle located on the free surface will remain on it.

An alternative form of the surface equation is

$$S(X,Y,Z,T)=\eta(X,Y,T)-Z=0, \tag{2.26}$$

where $\eta(X,Y,T)$ represents the free surface elevation measured from $Z=0$. Thus, Eq. (2.25) takes the form

$$\frac{\partial\eta}{\partial T}+U\frac{\partial\eta}{\partial X}+V\frac{\partial\eta}{\partial Y}-W=0 \quad \text{on} \quad Z=\eta \tag{2.27}$$

or, equivalently

$$\frac{\partial\eta}{\partial T}+\frac{\partial\phi}{\partial X}\frac{\partial\eta}{\partial X}+\frac{\partial\phi}{\partial Y}\frac{\partial\eta}{\partial Y}-\frac{\partial\phi}{\partial Z}=0 \quad \text{on} \quad Z=\eta. \tag{2.28}$$

Equations (2.23), (2.25) and (2.28) correspond to different forms of the kinematic boundary condition.

Since η and ϕ are both unknown on the free surface, a second boundary condition is needed: the dynamic boundary condition. This condition is derived from the Bernoulli equation (2.19). When surface tension is neglected, the pressure P in the fluid on the free surface is equal to the atmospheric pressure P_a. Hence, the Bernoulli equation (2.19) on the free surface takes the form

$$\frac{\partial \phi}{\partial T} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{P_a}{\rho} + gZ = 0 \quad \text{on} \quad Z = \eta. \tag{2.29}$$

The atmospheric pressure P_a is chosen as reference and we can set P_a equal to zero without loss of generality. Hence,

$$\frac{\partial \phi}{\partial T} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + gZ = 0 \quad \text{on} \quad Z = \eta. \tag{2.30}$$

For the rigid boundary, we have $S(X,Y,Z) = Z + D(X,Y) = 0$, thus $Z = -D(X,Y)$ is the equation of the sea bottom and Eq. (2.28) takes the form

$$\frac{\partial \phi}{\partial X}\frac{\partial D}{\partial X} + \frac{\partial \phi}{\partial Y}\frac{\partial D}{\partial Y} + \frac{\partial \phi}{\partial Z} = 0 \quad \text{on} \quad Z = -D(X,Y). \tag{2.31}$$

Although the Laplace equation is a linear partial differential equation, the difficulty in solving water wave problems arises from the nonlinearity of kinematic and dynamic boundary conditions. Furthermore, these equations apply on a surface that is unknown *a priori*. To summarize, the water wave problem reduces to solve the system of equations consisting of the Laplace equation (2.13), kinematic boundary condition (2.28), dynamic boundary condition (2.30) and sea bottom condition (2.31), with initial and boundary values for ϕ and η.

2.1.3 Linearization: Equations for Small Amplitude Waves

As emphasized in the previous section, we need values of the partial derivatives of the potential ϕ on a surface η that is unknown *a priori*. To solve the water wave equations, a free surface known *a priori* will be introduced through the linearization of the problem, which corresponds to an approximation of the nonlinear problem.

The nonlinearity of Eq. (2.30) is due to the presence of the convective term of the momentum equation, namely $(\mathbf{U} \cdot \nabla)\mathbf{U}$. Let us consider the simple example of a two-dimensional (2D) fluid motion. For waves propagating along the X direction, we consider the X–momentum equation and thus the corresponding nonlinear term is $U\partial U/\partial X + V\partial U/\partial Y$. Let us compare the first term to the linear term $\partial U/\partial T$ of the momentum equation. Let A, T_p and λ be the characteristic amplitude, period and wavelength of waves on the free surface, respectively. During a specific period, the fluid particles suffer displacements of order A. The corresponding fluid velocity and horizontal velocity gradient are then approximately A/T_p and $A/T_p\lambda$. Hence,

$$U\frac{\partial U}{\partial X}=O\left(\frac{A^2}{\lambda T_p^2}\right)$$

and

$$\frac{\partial U}{\partial T}=O\left(\frac{A}{T_p^2}\right).$$

The linearization condition can, therefore, be written as

$$\left|U\frac{\partial U}{\partial X}\right|<<\left|\frac{\partial U}{\partial T}\right|\rightarrow A<<\lambda.$$

The condition for linearization of the equations is that the amplitude is small against the wavelength. Using $\lambda=2\pi/K$, where K is the wavenumber, the previous equation yields to the condition

$$\varepsilon=AK<<1, \tag{2.32}$$

where ε is the linearization parameter. Physically, this parameter is the wave steepness.

The water wave equations, which are nonlinear, can be transformed into a sequence of linear problems by using a perturbation procedure. Let us assume the following perturbation expansions in the parameter ε for the unknowns ϕ and η (i.e., Mei 1983 or Johnson 1997)

$$\phi\left(X,Y,Z,T\right)=\sum_{n=1}^{\infty}\varepsilon^n\phi_n\left(X,Y,Z,T\right), \tag{2.33}$$

$$\eta\left(X,Y,T\right)=\sum_{n=1}^{\infty}\varepsilon^n\eta_n\left(X,Y,T\right). \tag{2.34}$$

The temporal and spatial derivatives of the velocity potential ϕ, which occur in the free surface conditions (2.28) and (2.30), are expanded in the Taylor series about the still water level $Z=0$:

$$\frac{\partial\phi}{\partial r}\left(X,Y,Z=\eta,T\right)=\sum\frac{\eta^n}{n!}\frac{\partial^n}{\partial Z^n}\left(\frac{\partial\phi}{\partial r}\right)\left(X,Y,Z=0,T\right), \tag{2.35}$$

where r may represent temporal or spatial variables.

Substituting expansions (2.33), (2.34) and (2.35) into Eqs. (2.13), (2.28), (2.30), (2.31), and collecting the coefficients of the first power of ε, one finds

$$\Delta\phi_1=0,-D<Z<0, \tag{2.36}$$

$$\frac{\partial\eta_1}{\partial T}-\frac{\partial\phi_1}{\partial Z}=0\quad\text{on}\quad Z=0, \tag{2.37}$$

$$\frac{\partial \phi_1}{\partial T} + g\eta_1 = 0 \quad \text{on} \quad Z = 0, \tag{2.38}$$

$$\frac{\partial \phi_1}{\partial X}\frac{\partial D}{\partial X} + \frac{\partial \phi_1}{\partial Y}\frac{\partial D}{\partial Y} + \frac{\partial \phi_1}{\partial Z} = 0 \quad \text{on} \quad Z = -D. \tag{2.39}$$

For small amplitude water waves $\varepsilon << 1$, we can ignore the terms of order $O(\varepsilon^n)$ with $n > 1$ in the expansions of (2.33), (2.34). Hence, the velocity potential and free surface elevation are approximated as

$$\phi\left(X,Y,Z,T\right) = \varepsilon\phi_1, \tag{2.40}$$

$$\eta\left(X,Y,T\right) = \varepsilon\eta_1. \tag{2.41}$$

The corresponding linear system of equations to be solved is

$$\Delta\phi = 0, -D < Z < 0, \tag{2.42}$$

$$\frac{\partial \eta}{\partial T} - \frac{\partial \phi}{\partial Z} = 0 \quad \text{on} \quad Z = 0, \tag{2.43}$$

$$\frac{\partial \phi}{\partial T} + g\eta = 0 \quad \text{on} \quad Z = 0, \tag{2.44}$$

$$\frac{\partial \phi}{\partial X}\frac{\partial D}{\partial X} + \frac{\partial \phi}{\partial Y}\frac{\partial D}{\partial Y} + \frac{\partial \phi}{\partial Z} = 0 \quad \text{on} \quad Z = -D. \tag{2.45}$$

2.1.4 Dispersion Relation

For the sake of simplicity, the bottom elevation, D, is considered to be constant. Hence, Eq. (2.45) becomes:

$$\frac{\partial \phi}{\partial Z} = 0 \quad \text{on} \quad Z = -D. \tag{2.46}$$

We look for a 2D periodic solution of the linear system of Eqs. (2.42), (2.43), (2.44) and (2.46) that admits the following velocity potential:

$$\phi\left(X,Z,T\right) = B\cosh\left[K\left(Z+D\right)\right]\sin\left(KX-\Omega T\right), \tag{2.47}$$

where B is a constant and Ω and K are the cyclic frequency and wave number, respectively. This form automatically satisfies the Laplace equation (2.42) and the bottom condition (2.46). Substituting (2.47) into the dynamical condition (2.44), one obtains

$$\eta\left(X,T\right) = \frac{B\Omega}{g}\cosh\left(KD\right)\cos\left(KX-\Omega T\right). \tag{2.48}$$

Let

$$A = \frac{B\Omega}{g}\cosh\left(KD\right). \tag{2.49}$$

Hence,

$$\eta(X,T) = A\cos(KX - \Omega T), \tag{2.50}$$

and the potential can be rewritten as follows:

$$\phi(X,Z,T) = \frac{Ag}{\Omega}\frac{\cosh[K(Z+D)]}{\cosh(KD)}\sin(KX - \Omega T). \tag{2.51}$$

The linear dispersion relation is obtained by stating that the velocity potential (2.51) and the free surface elevation (2.50) correspond to nontrivial solutions satisfying the kinematic boundary condition (2.43),

$$\Omega^2 = gK\tanh(KD). \tag{2.52}$$

The frequency Ω is given as a function of K in Fig. 2.2. Equations (2.50) and (2.51) represent 2D gravity waves of permanent form propagating with a constant phase velocity on water of uniform depth.

Equation (2.52) links the frequency Ω to the wave number, K. The phase velocity is given by

$$C_{ph} = \frac{\Omega}{K} = \sqrt{\frac{g}{K}\tanh(KD)}. \tag{2.53}$$

Since ${C_{ph}}'(K) \neq 0, \forall K \neq 0$, the gravity water waves are dispersive. This is an important property of water waves, which means that waves of different wave numbers propagate at different phase velocities. Nevertheless, a stronger condition introduced by Whitham (1974) to define dispersive waves is $\forall K : \Omega''(K) \neq 0$.

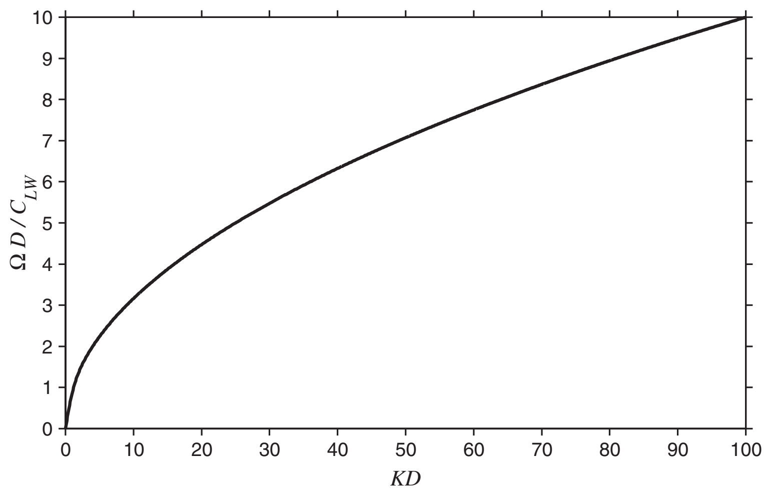

Fig. 2.2 Water wave dispersion relation curve as normalized frequency versus dimensionless water depth. The long wave velocity is defined as $C_{LW} = (gD)^{1/2}$

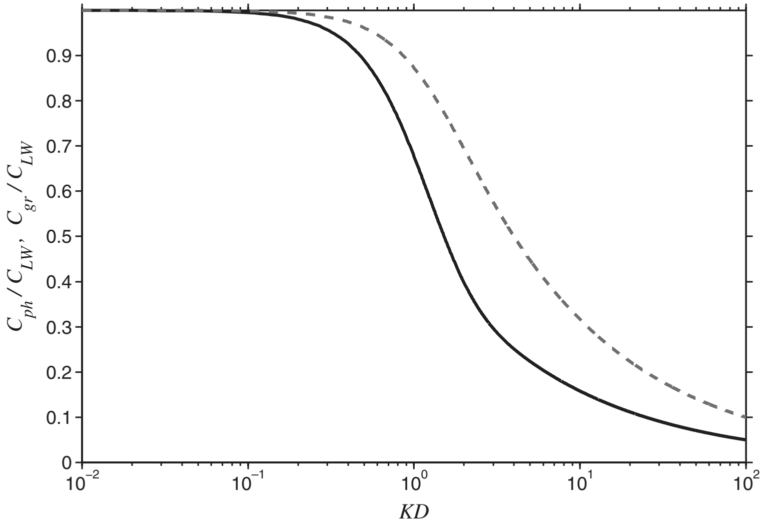

Fig. 2.3 Phase (*dotted line*) and group (*solid line*) velocity dependencies (C_{ph} and C_{gr} are normalized by C_{LW}) versus the dimensionless depth. Note logarithmic scale of the abscissa

The group velocity is defined as

$$C_{gr} = \frac{\partial \Omega}{\partial K} = \frac{g}{2\Omega}\left[\tanh\left(KD\right) + KD\left(1 - \tanh^2 KD\right)\right]. \tag{2.54}$$

In the shallow water limit $KD \to 0$, the group and phase velocities become equal and $C_{ph} \approx C_{gr} \to C_{LW}, C_{LW} = (gD)^{1/2}$; this means that the waves become nondispersive. The velocities C_{ph} and C_{gr} are given in Fig. 2.3 as functions of the dimensionless depth KD.

The 3D plane wave solution is given by the following formulas:

$$\eta\left(\mathbf{X}, T\right) = A\cos\left(\mathbf{K}\cdot\mathbf{X} - \Omega T\right) \tag{2.55}$$

and

$$\phi\left(\mathbf{X}, Z, T\right) = \frac{Ag}{\Omega}\frac{\cosh\left[\left|\mathbf{K}\right|\left(Z + D\right)\right]}{\cosh\left(\left|\mathbf{K}\right|D\right)}\sin\left(\mathbf{K}\cdot\mathbf{X} - \Omega T\right), \tag{2.56}$$

where $\mathbf{K}$ is the wave vector and $\mathbf{X} = (X, Y)^t$. The corresponding linear dispersion relation is

$$\Omega^2 = g\left|\mathbf{K}\right|\tanh\left(\left|\mathbf{K}\right|D\right). \tag{2.57}$$

Once the velocity potential ϕ is known, it is easy to calculate the velocity field $\mathbf{U} = \nabla\phi(\mathbf{X}, Z, T)$. The velocity components are

$$U = \frac{AgK_X}{\Omega}\frac{\cosh\left[\left|\mathbf{K}\right|\left(Z + D\right)\right]}{\cosh\left(\left|\mathbf{K}\right|D\right)}\cos\left(\mathbf{K}\cdot\mathbf{X} - \Omega T\right), \tag{2.58}$$

$$V = \frac{AgK_Y}{\Omega} \frac{\cosh\left[\left|\mathbf{K}\right|(Z+D)\right]}{\cosh\left(\left|\mathbf{K}\right|D\right)} \cos\left(\mathbf{K}\cdot\mathbf{X} - \Omega T\right), \tag{2.59}$$

$$W = \frac{Ag\left|\mathbf{K}\right|}{\Omega} \frac{\sinh\left[\left|\mathbf{K}\right|(Z+D)\right]}{\cosh\left(\left|\mathbf{K}\right|D\right)} \sin\left(\mathbf{K}\cdot\mathbf{X} - \Omega T\right), \tag{2.60}$$

where K_X and K_Y are the $X-$ and $Y-$ components of $\mathbf{K}$, respectively. The pressure $P(\mathbf{X};Z;T)$ is obtained from the Bernoulli equation (2.19).

For infinite depth $D \to \infty$, the bottom condition becomes

$$\nabla\phi \to 0 \text{ as } Z \to -\infty, \tag{2.61}$$

and the corresponding 2D gravity waves of permanent form propagating with a constant phase velocity are given by Eq. (2.50) and

$$\phi\left(X,Z,T\right) = \frac{Ag}{\Omega} \exp\left(KZ\right) \sin\left(KX - \Omega T\right) \tag{2.62}$$

with

$$\Omega^2 = gK. \tag{2.63}$$

2.2 Statistical Description

The second approach to studying waves is statistical. Water waves, of course, obey physical laws. They all, in principle, may be taken into account in a deterministic model, and therefore this model will be able (theoretically) to describe wave dynamics. In practice, this approach fails due to incomplete information about the initial state of the fluid, complexity of the physics, and growing fluctuations (this means that small perturbations with time may result in very different dynamics). Generally, the system of equations suffers from sensitive dependence on initial conditions. This feature is met in chaotic and turbulent systems. We know from our everyday experience that sea waves behave irregularly and unpredictably in even rather short time scales, although they show some periodicity. So, the dynamic system *Ocean* manifests *random* wave dynamics. Therefore, at certain sea conditions (significant wave height, wave age, winds, currents, etc.), different *realizations* (concerning the wave elevation – they are functions $\eta(\mathbf{X}, T)$) of sea waves are equally possible and may be considered as the object of investigation. The collection of realizations $\{\eta_j(\mathbf{X}, T)\}$ (integer subscript j counts them) builds an *ensemble*. In that way, the sea surface at one moment of time T_0 in one point $\mathbf{X}_0$ is represented by random functions numbered by the index j: $\eta_j(\mathbf{X}_0, T_0)$ with some statistical properties. This approach is referred to as *stochastic* and is aimed at a statistical description of sea wave dynamics.

The ultimate goal here is to describe and foresee the dynamics of certain realizations on the basis of the dynamics of averaged statistical characteristics. This approach is currently the center of attention of both theorists and experts, especially in the fields of ocean and atmospheric research; it is widely used in ocean engeneering. To obtain the time-dependence of statistical properties, one may perform stochastic simulations—i.e., to use deterministic models to compute a number of randomly chosen realizations (Monte Carlo simulations). Thus, one takes the position that the simulation of a sufficiently large but finite number of realizations represents the evolution of the whole ensemble in a statistical sense. The other approach is to compose and study models for direct computation of the evolution of statistical wave parameters. This is aimed at the theories that deal with spectral kinetic equations.

2.2.1 The Rayleigh Probability

Let us consider the surface displacement $\eta(\mathbf{X}, T)$—a function of space and time. Its autocorrelation function is defined as

$$R(\mathbf{X},T,\mathbf{r},\tau) = E\left[\eta(\mathbf{X},T)\cdot\eta(\mathbf{X}+\mathbf{r},T+\tau)\right], \tag{2.64}$$

where $E[\cdot]$ denotes statistical averaging over the ensemble of realizations $\eta_j(\mathbf{X}, T)$:

$$E\left[\eta(\mathbf{X},T)\cdot\eta(\mathbf{X}+\mathbf{r},T+\tau)\right] = \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\eta_j(\mathbf{X},T)\cdot\eta_j(\mathbf{X}+\mathbf{r},T+\tau). \tag{2.65}$$

In practice, N is finite, but it should be sufficiently large to provide a good estimate of the limit (2.65). Averaging over an ensemble is convenient for reproducible laboratory experimental conditions, but not the real ocean, where waves do not repeat themselves. For natural observations, a long time series is split into many shorter samples—“realizations”—that are used for averaging. This approach needs the random process to be *stationary* (i.e., its statistical properties do not depend on time). If these two ways of averaging result in the same statistics, the process is called ergodic. Although it is impossible to prove the ergodicity property for water waves via direct natural experiments, it is commonly invoked for the study of waves on the sea surface.

The statistical stationarity and statistical homogeneity in space imply that the autocorrelation function does not depend on $\mathbf{X}$ and T: $R = R(\mathbf{r}, \tau)$.

Averaging (2.64) may be also rewritten in terms of the probability function as

$$R(\mathbf{r},\tau) = \int_{-\infty}^{\infty}\eta_1\eta_2 f(\eta_1,\eta_2,\mathbf{r},\tau)\,d\eta_1 d\eta_2, \tag{2.66}$$

where f is the two-point probability density function defined as

$$f(\eta_1, \eta_2, \mathbf{r}, \tau) = \frac{\partial^2 F(\eta_1, \eta_2, \mathbf{r}, \tau)}{\partial \eta_1 \partial \eta_2}, \tag{2.67}$$

and the distribution function F measures probability such that $\eta(\mathbf{X}, T)$ and $\eta(\mathbf{X} + \mathbf{r}, T + \tau)$ do not exceed η_1 and η_2, respectively.

$$F(\eta_1, \eta_2, \mathbf{r}, \tau) = P(\eta(\mathbf{X}, T) \le \eta_1 \,|\, \eta(\mathbf{X} + \mathbf{r}, T + \tau) \le \eta_2). \tag{2.68}$$

Functions F and f do not depend on $\mathbf{X}$ and T if the field is both statistically homogeneous in space and stationary.

The probability distribution function or probability density function defines the statistical properties of the random field. To simplify the analysis of the statistics, integral parameters are often used. The nth statistical moment is defined as

$$\mu_n = E[\eta^n] = \int_{-\infty}^{\infty} \eta^n f(\eta)\, d\eta, \tag{2.69}$$

where f is the probability density function for η. Due to the normalization of the probability density function,

$$\mu_0 = 1. \tag{2.70}$$

The centered moments are defined as

$$\mu_n^c = E[(\eta - \mu)^n] = \int_{-\infty}^{\infty} (\eta - \mu)^n f(\eta)\, d\eta, \quad \mu \equiv \mu_1. \tag{2.71}$$

Only few low-order statistical moments have specific names due to their great importance in statistics. The first statistical moment μ in this instance is the mean water level. The variance σ^2 is equal to the second central moment

$$\sigma^2 = \mu_2^c = E\left[(\eta - \mu)^2\right], \tag{2.72}$$

and σ is the standard deviation. The skewness γ and kurtosis κ are defined through

$$\gamma = \frac{\mu_3^c}{\sigma^3} \tag{2.73}$$

and

$$\kappa = \frac{\mu_4^c}{\sigma^4}. \tag{2.74}$$

The skewness is usually used to estimate the vertical asymmetry of the sea surface elevation, whereas the kurtosis corresponds to the peakedness of the distribution when compared with the normal distribution (see Massel 1996).

The Central Limit Theorem proves that a superposition

$$\eta = \sum_j \eta_j \tag{2.75}$$

of statistically independent[1] variables η_j with mean values μ_j and variances σ_j^2 results in the Gaussian probability density

$$f(\eta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\eta-\mu)^2}{2\sigma^2}\right] \tag{2.76}$$

with mean

$$\mu = \sum_j \mu_j \tag{2.77}$$

and variance

$$\sigma^2 = \sum_j \sigma_j^2. \tag{2.78}$$

For Gaussian statistics, the skewness and kurtosis are $\gamma = 0$ and $\kappa = 3$, respectively.

Linear superposition of random periodic waves

$$\eta(\mathbf{X}, T) = \sum_n A_n \cos(\mathbf{K}_n\mathbf{X} - \Omega_n T + \theta_n) \tag{2.79}$$

is a natural representation of sea waves. Here, amplitudes A_n obey some probability distribution, and frequencies Ω_n and wave vectors $\mathbf{K}_n$ are dependent according to the dispersion relation; the wave phases θ_n are supposed to be uniformly distributed on the interval $[0, 2\pi]$. In this approximation, the surface elevation is described by the Gaussian statistics (2.76).

Let us now consider the linear superposition (2.79) of statistically independent Gaussian processes with variances σ_n^2. In the narrow-band assumption, the field may be represented in the following form

$$\eta = |B| \cos(\mathbf{K}_c\mathbf{X} - \Omega_c T + \varphi) \tag{2.80}$$

where $B = |B| \exp(\mathrm{i}\varphi)$ is a slowly varying function of $\mathbf{X}$ and T, and σ_n^2 is rapidly decaying when values $\mathbf{K}_n$ (or Ω_n) are not close to $\mathbf{K}_c$ (or Ω_c, respectively). In this limit, the distribution for the linear wave amplitude $|B|$ is described by the Rayleigh function (Massel 1996)

$$f(|B|) = \frac{|B|}{\sigma^2} \exp\left(-\frac{|B|^2}{2\sigma^2}\right). \tag{2.81}$$

In the limit of small bandwidth, the wave height is twice the envelope, $H = 2|B|$, and therefore

$$f(H) = \frac{H}{4\sigma^2} \exp\left(-\frac{H^2}{8\sigma^2}\right), \tag{2.82}$$

---

[1] Two random variables are statistically independent if their joint probability density function may be factorized: $f(x,y) = f_x(x) \cdot f_y(y)$.

and the probability that the wave height exceeds the value H (the exceedance probability) is

$$P(H) = 1 - F(H) = \exp\left(-\frac{H^2}{8\sigma^2}\right). \tag{2.83}$$

In Chap. 1, we introduced the significant wave height, which is the mean value of one-third of the highest waves. According to this definition and formula (2.82), the significant wave height is defined as (Massel 1996)

$$H_s = \frac{3}{4}\int\limits_{\sigma\sqrt{8\ln 3}}^{\infty} \frac{H^2}{4\sigma^2}\exp\left(-\frac{H^2}{8\sigma^2}\right)dH \approx 4.004\sigma. \tag{2.84}$$

Integral (2.84) may be expressed through the error function (see Massel 1996). Usually a simplified relation is used, $H_s = 4\sigma$. Hence, formula (2.83) may be written in the convenient form

$$P(H) \approx \exp\left(-2\frac{H^2}{H_s^2}\right) \tag{2.85}$$

that helps to easily estimate the probability of high waves. For instance, a freak wave ($H > 2H_s$) should appear once among about 3,000 waves. For a typical sea wave period of 10 s, this gives the estimation that one should meet a freak wave every 8–9 h. In a Gaussian sea, a wave exceeding three times the significant height may occur once in 20 years.

Study of rogue waves in the framework of Gaussian statistics is already a tricky task. But waves (especially extreme waves) in the real ocean are obviously non-Gaussian due to various reasons: dissipation including wave breaking, insufficiently narrow spectrum, and nonlinear effects. Because rogue waves are rare events, and the sea state is persistently changing, the statistical stationarity condition also breaks down.

Nonlinear effects contribute to bound corrections to the wave shape as well as to the interaction between different harmonics, so periodic waves in superposition (2.79) become correlated. Due to the nonlinearity, waves become asymmetric: the crests are sharper and higher, while the troughs are flatter and shallower. The approximate bound nonlinear corrections may be taken into account with the help of the perturbation technique. In the deep-water case, the second-order small steepness ($KA << 1$) approximation gives

$$\eta(X,T) = A\cos(KX - \Omega T + \theta) + \frac{1}{2}KA^2\cos[2(KX - \Omega T + \theta)]. \tag{2.86}$$

for a regular monochromatic (Stokes) wave.

Assuming that the linear wave amplitude preserves the Rayleigh distribution, Formula (2.85) can be used to estimate the probability exceedance for wave crests (η_{cr}) and troughs (η_{tr}) by

$$P(\eta_{cr} > \eta) = \exp\left(-\frac{8}{H_s^2}\frac{\left(\sqrt{1+2K\eta}-1\right)^2}{K^2}\right) \tag{2.87}$$

and

$$P(\eta_{tr} > \eta) = \exp\left(-\frac{8}{H_s^2}\frac{\left(\sqrt{1-2K\eta}-1\right)^2}{K^2}\right). \tag{2.88}$$

Formulas (2.87) and (2.88) predict that extreme waves have larger crests than troughs. We should note, however, that representation (2.86) does not lead to a change of the crest-to-trough wave height at this level of accuracy.

Different types of modified distribution functions, taking into account weak nonlinear bound corrections, were developed in Tayfun (1980), Tung and Huang (1985), and Mori and Yasuda (2002) and many others (see survey by Prevosto 2001); other modifications of the Rayleigh distribution are being developed, as are empirical formulas. Apparently, second-order statistical models turn out to be insufficient for the adequate description of rogue waves (Bitner-Gregersen and Magnusson 2005, Rosenthal 2005, Petrova et al. 2007). Nonlinear corrections of higher orders should be taken into account (Creamer et al. 1989, Huang et al. 1983, Zhang et al. 1999); these corrections may enhance the probability of high waves by ten (Prevosto and Bouffandeau 2002) or even one hundred (Stansell 2004, Forristall 2005) times! Since nonlinear properties of surface waves depend on depth, the depth is one more parameter in the statistical model (see Massel 1996).

The considered theory is applied to one-point observations. Real needs and recent 3D observations require development of a statistical model describing wave probability over a specific area. Reduction from the point statistics is not trivial for this purpose (Forristall 2005, Socquet-Juglard et al. 2005), and may be very important. Thus, Forristall (2005) estimates that for the air gap under a fixed structure with a deck $50\,\text{m} \times 50\,\text{m}$, the maximum wave crest is almost 20% higher than the one expected at a single point.

The sea state is rather changeable; this results in failure of the condition of statistical stationarity. Donelan and Magnusson (2005) and Müller et al. (2005) show how the probability of high waves grows in a mixed sea constituted by two wave trains. Baxevani and Rychlik (2006) considered a Gaussian sea evolving in time and also studied the effects of wave spreading. They report that the neglect of these effects leads to an underestimation of the high wave probability.

2.2.2 Wave Spectra

The Fourier transform of the autocorrelation function R gives the *wave spectrum*

$$\hat{S}(\mathbf{K},\Omega) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} R(\mathbf{r},\tau)\cdot\exp\left[i\left(\mathbf{K}\mathbf{r}-\Omega\tau\right)\right]d\mathbf{r}d\tau. \tag{2.89}$$

Here, $\mathbf{K} = (K_X, K_Y)$ is the wave vector and Ω is the frequency. The *frequency spectrum* and *wave vector spectrum* (or *two-dimensional wavenumber spectrum* or *spatial spectrum*) are defined, respectively, by

$$\hat{S}(\Omega) = \int_{-\infty}^{\infty} \hat{S}(\mathbf{K}, \Omega)\, d\mathbf{K}, \tag{2.90}$$

and

$$\hat{S}(\mathbf{K}) = \int_{-\infty}^{\infty} \hat{S}(\mathbf{K}, \Omega)\, d\Omega. \tag{2.91}$$

The wavenumber spectrum is defined as

$$\hat{S}(K) = \int_{-\pi}^{\pi} K\hat{S}(\mathbf{K})\, d\alpha, \text{ where } \mathbf{K} = (K\cos\alpha, K\sin\alpha), \tag{2.92}$$

and $K = |\mathbf{K}| > 0$ is the wavenumber. The *directional spectrum* is

$$\hat{S}(\alpha) = \int_{0}^{\infty} dK \int_{-\infty}^{\infty} d\Omega K\hat{S}(\mathbf{K}, \Omega). \tag{2.93}$$

The frequency and wave vector (wavenumber) spectra can be related to one another; this can be achieved with the help of the dispersion relation. For instance, for the deep-water case, the dispersion relation is as follows

$$KdK = 2\frac{\Omega^3}{g^2}d\Omega, \tag{2.94}$$

and hence

$$\hat{S}(\Omega) = \frac{2\Omega^3}{g^2}\hat{S}(K). \tag{2.95}$$

For a real process—statistically stationary and statistically homogeneous in space—the correlation function possesses the symmetry property $R(-\mathbf{r}, -\tau) = R(\mathbf{r}, \tau)$. Then the spectrum is real, and $\hat{S}(-\mathbf{K}, -\Omega) = \hat{S}(\mathbf{K}, \Omega)$. That is why only one half of the spectrum is commonly used in the analysis: $\Omega > 0$ for the frequency spectrum, and $K > 0$ for the wavenumber spectrum.

In the first approximation, the wave field may be represented as a linear superposition of periodic waves (2.79). To see this, let us consider a single cosine wave

$$\eta(\mathbf{X}, T) = A_0 \cos(\mathbf{K}_0\mathbf{X} - \Omega_0 T + \theta), \tag{2.96}$$

where A_0, $\mathbf{K}_0$ and Ω_0 are defined, but θ is a random value uniformly distributed within the interval $[-\pi, \pi]$. Then, the corresponding correlation function is

$$
\begin{aligned}
& R(\mathbf{r},\tau) \\
& = \int_{-\pi}^{\pi} A_0 \cos(\mathbf{K}_0\mathbf{X} - \Omega_0 T + \theta) A_0 \cos[\mathbf{K}_0(\mathbf{X}+\mathbf{r}) - \Omega_0(T+\tau) + \theta] \frac{d\theta}{2\pi} \\
& = \frac{A_0^2}{2} \cos(\mathbf{K}_0\mathbf{r} - \Omega_0 \tau).
\end{aligned}
\tag{2.97}
$$

Furthermore, the wave spectrum for a single cosine wave reads

$$
\begin{aligned}
\hat{S}(\mathbf{K},\Omega) &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} R(\mathbf{r},\tau) \cdot \exp[i(\mathbf{K}\mathbf{r} - \Omega\tau)] d\mathbf{r} d\tau \\
&= \frac{A_0^2}{4} (\delta(\mathbf{K}+\mathbf{K}_0)\delta(\Omega+\Omega_0) + \delta(\mathbf{K}-\mathbf{K}_0)\delta(\Omega-\Omega_0))
\end{aligned}
. \tag{2.98}
$$

Hence, for the linear superposition of periodic waves (2.79), the spectrum has the form

$$
\hat{S}(\mathbf{K},\Omega) = \sum_n \frac{A_n^2}{4} (\delta(\mathbf{K}+\mathbf{K}_n)\delta(\Omega+\Omega_n) + \delta(\mathbf{K}-\mathbf{K}_n)\delta(\Omega-\Omega_n)). \tag{2.99}
$$

Thus, the wave spectrum is represented by Dirac delta functions and has non-zero values in the ($\mathbf{K}$, Ω) space only at points corresponding to the waves represented in the superposition (2.79).

It is well known that the total energy of a linear plane progressive wave (2.96) is defined as

$$
En = \rho g \frac{A_0^2}{2}. \tag{2.100}
$$

Alternatively, Formula (2.98) gives

$$
En = \rho g \int_{-\infty}^{\infty} \hat{S}(\mathbf{K},\Omega) d\mathbf{K} d\Omega. \tag{2.101}
$$

Therefore, the wave spectrum has the meaning of the wave energy distribution in the space of wave vectors and frequencies; the quantity $\rho g \hat{S}$ is called the *energetic spectrum*.

The wave amplitudes may be expressed through the relation

$$
A_n^2 = 2 \int \hat{S}(\mathbf{K}_n,\Omega_n) d\mathbf{K} d\Omega, \tag{2.102}
$$

where the integration is effective only in closed intervals around $\pm\mathbf{K}_n$ and $\pm\Omega_n$. Relations (2.99) and (2.102) allow us to define the spectrum as the squared absolute value of the Fourier transform of the process. The relationship between the spectrum and the autocorrelation function (2.89) is then called the Wiener-Khintchine Theorem. The spectrum concept is a powerful tool for investigating time series, since it displays the distribution of wave energy among frequencies and scales represented by harmonics. Data processing in the spectral space (such as filtering) may be very powerful. Although the wave height, peak period, and main wave direction are sufficient to describe sea states for most practical purposes, Olagnon and Magnusson (2004) pointed out at the same time that it is likely that no spectral parameter alone can provide useful information on the risk and potentially abnormal wave events when it is estimated. Thus, this approach needs improvement to be applied to the needs of rogue wave research.

2.2.2.1 Frequency Spectrum

For most of the numerous measurements of sea waves represented by time series at one spatial point, only the frequency spectrum may be obtained. Since it is an even function of frequency, instead of the symmetric function $\hat{S}(\Omega), \Omega \in (\infty;\infty)$, only one part is used in experimental practice (so-called nonsymmetric spectrum): $S(\Omega) = 2\hat{S}(\Omega), \Omega \in [0;\infty)$. The longer the record is, the more statistical material it provides; on the other hand, sea conditions may change if the realization takes too long. Usually wave record samples of duration 10–30 min are retrieved for analysis to fulfill these contradictory requirements. The relationship between the spectrum and the wave amplitudes persists

$$A_n = \sqrt{2S(\Omega_n)\Delta\Omega}, \tag{2.103}$$

where $\Delta\Omega$ is the frequency discretization interval. As an estimator for frequency spectrum $S(\Omega)$, the Fourier transform of the wave field is usually employed in practice:

$$S(\Omega) \cong 2|\eta_\Omega|^2, \quad \eta_\Omega = \frac{1}{T}\int_0^T \eta(t)\exp(i\Omega t)\,dt. \tag{2.104}$$

When analyzing the wave spectrum, the spectral moments are frequently used; they are, in general, defined as

$$m_n = \int_0^\infty \Omega^n S(\Omega)d\Omega. \tag{2.105}$$

Inversing the Fourier transform (2.89), one obtains

$$R(\tau=0) = \int_{-\infty}^{\infty} \hat{S}(\Omega)\,d\Omega = \int_0^\infty S(\Omega)\,d\Omega, \tag{2.106}$$

therefore the zero spectral moment is expressed through the second statistical moment

$$m_0 = \mu_2, \tag{2.107}$$

or for the case of a field with zero mean,

$$m_0 = \mu_2^c = \sigma^2. \tag{2.108}$$

The mean wave frequency and wave period are defined as

$$\Omega_p = \frac{m_1}{m_0}, \quad T_p = \frac{2\pi}{\Omega_p} = 2\pi\frac{m_1}{m_0}, \tag{2.109}$$

although other possible ways to define the mean frequency exist.

Central moments

$$m_n^c = \int_0^\infty (\Omega - \Omega_p)^n S(\Omega) d\Omega \tag{2.110}$$

are also used. The central moment m_2^c is a measure of the concentration of the spectral wave energy around the frequency Ω_p, which characterizes the spectral width through the dimensionless parameter

$$\delta_\Omega = \frac{1}{\Omega_p}\sqrt{\frac{m_2^c}{m_0}}. \tag{2.111}$$

The spectral shape displays the distribution of energy between scales and thus contains information about the physical mechanisms supporting and generating the waves. Concerning wind-generated waves, the energy growth due to the wind action is balanced by the wave interactions—which transfer energy between frequencies—and energy dissipation. Following the hypothesis of similarity for ocean waves, the energy spectrum should be represented by a function of the form (Massel 1996)

$$S(\Omega) = S\left(\Omega, X_f, T, U_w, g\right), \tag{2.112}$$

where X_f is the fetch, T is related to the wave age, and U_w is the wind velocity, or alternatively,

$$S(\Omega) = S(\Omega, g, \sigma, \Omega_p). \tag{2.113}$$

The suggested spectral shapes usually have the general form

$$S(\Omega) = C_1\Omega^{-p}\exp\left(-C_2\Omega^{-q}\right). \tag{2.114}$$

One of the most popular spectrums was suggested by Pierson and Moskowitz (1964) on the basis of theoretical discoveries and field data analysis:

$$\begin{aligned} S(\Omega) &= \alpha g^2\Omega^{-5}\exp\left(-B\left(\frac{\Omega U_X}{g}\right)^{-4}\right) \\ &= \alpha g^2\Omega^{-5}\exp\left(-\frac{5}{4}\left(\frac{\Omega}{\Omega_p}\right)^{-4}\right) \end{aligned}, \tag{2.115}$$

where $\alpha = 8.1 \times 10^{-3}$, and $B = 0.74$. It was proposed for a fully developed sea, when the wave phase speed is equal to the wind speed. It is controlled by a single parameter, which is the wind speed.

The Joint North Sea Wave Project (JONSWAP) Spectrum extends the Pierson-Moskowitz Formula (2.115) to include fetch-limited seas through inclusion of one more governing parameter manifesting peakedness, γ

$$S(\Omega) = \alpha g^2 \Omega^{-5} \exp\left(-\frac{5}{4}\left(\frac{\Omega}{\Omega_p}\right)^{-4}\right)\gamma^{\delta} \tag{2.116}$$

$$\delta = \exp\left[-\frac{(\Omega-\Omega_p)^2}{2\sigma_0^2\Omega_p^2}\right]$$

where $\gamma = 3.3$; $\sigma_0 = 0.07$, if $\Omega \leq \Omega_\mathrm{p}$; and $\sigma_0 = 0.09$, if $\Omega > \Omega_\mathrm{p}$.

$$\alpha = 0.076\left(\frac{gX_f}{U_w^2}\right)^{-0.22}, \tag{2.117}$$

$$\Omega_p = 7\pi\frac{g}{U_w}\left(\frac{gX_f}{U_w^2}\right)^{-0.33}. \tag{2.118}$$

The JONSWAP Spectrum was built on the basis of an extensive wave measurement in the North Sea. This area is very popular in recent studies, owing to its great economical importance and large number of instrumental observations. With an extra free parameter, this shape is a convenient model spectrum. Other spectral shapes have been suggested and may be found in Massel (1996), but will not be considered in the present book.

2.2.3 Kinetic Models

So far, only statistically stationary and spatially homogeneous processes have been considered. This approach does not describe a realistic sea state where the wave field is evolving and changing from one area to another. The variability of waves may be computed when local (in time and space) statistics, that are represented by the wave spectrum, are considered. So, the spectrum function $\hat{S}$ depends on slow variables $\mathbf{X}$ and T, and all the wave conditions and statistical wave parameters may vary slowly in space and time.

Energy conservation results in the energy balance equation if there are no currents. Generally, it is given by the balance equation for the wave action, N (Whitham 1974)

$$\frac{dN}{dT} = G, \tag{2.119}$$

where

$$N = \frac{\hat{S}}{\Omega_i}. \tag{2.120}$$

Value Ω_i is the intrinsic frequency (in absence of current) that is related to the wavenumber through Eq. (2.57),

$$\Omega_i^2 = g\,|\mathbf{K}|\tanh\left(|\mathbf{K}|D\right), \tag{2.121}$$

while the observed (apparent) frequency in the presence of the current with velocity $\mathbf{U}_c$ is given by

$$\Omega = \Omega_i + \mathbf{K}\mathbf{U}_c. \tag{2.122}$$

The term G on the RHS of Eq. (2.119) defines the external action and losses, is called the *collision integral*, and takes into account different physical mechanisms: income of energy from the wind (pressure fluctuations, wave-flow linear and nonlinear interactions); interaction with the atmosphere and sea turbulence; dissipation due to bottom friction; wave breaking; and spectral nonlinear exchange, etc.

$$G = \sum_n G_n \tag{2.123}$$

The wave action is a function of wave vector, apparent frequency, and slow variables $\mathbf{X}$ and T, thus the conservation of volume in space ($\mathbf{X}$, $\mathbf{K}$, Ω) results in

$$\frac{dN}{dT} = \frac{\partial N}{\partial T} + \frac{\partial N}{\partial \mathbf{X}}\frac{d\mathbf{X}}{dT} + \frac{\partial N}{\partial \mathbf{K}}\frac{d\mathbf{K}}{dT} + \frac{\partial N}{\partial \Omega}\frac{d\Omega}{dT} = G. \tag{2.124}$$

The assumption that all changes happen much slower (in time and space) than the period and length of the waves allows us to use the ray theory. Hence the wave filed may be represented as

$$\eta = A\left(\mathbf{X}, T\right)\exp\left(i\theta\right), \tag{2.125}$$

where θ is the phase. It is natural to define the local wave vector and frequency as

$$\mathbf{K}\left(\mathbf{X}, T\right) = \frac{\partial \theta}{\partial \mathbf{X}}, \text{ and } \Omega\left(\mathbf{K}, \mathbf{X}, T\right) = -\frac{\partial \theta}{\partial T}. \tag{2.126}$$

These relations then give the kinematic conservation equations

$$\frac{\partial \mathbf{K}}{\partial T} + \nabla\Omega = 0, \quad \frac{\partial K_X}{\partial Y} = \frac{\partial K_Y}{\partial X}, \tag{2.127}$$

where $\mathbf{K} = (K_X, K_Y)$, $\nabla = (\partial/\partial X, \partial/\partial Y)$. The second equation means that the wave vector field is irrotational. It follows then that

$$\frac{d\mathbf{K}}{dT} = -\frac{\partial \Omega}{\partial \mathbf{X}}, \quad \frac{d\mathbf{X}}{dT} = \frac{\partial \Omega}{\partial \mathbf{K}}, \text{ and } \frac{d\Omega}{dT} = \frac{\partial \Omega}{\partial T}, \tag{2.128}$$

which means that the quantity Ω may be understood as the Hamiltonian, while $\mathbf{X}$ is the position and $\mathbf{K}$ is the momentum. With the help of relations (2.128), the balance equation (2.124) transforms into

$$\frac{\partial N}{\partial T} + \frac{\partial N}{\partial \mathbf{X}}\frac{\partial \Omega}{\partial \mathbf{K}} - \frac{\partial N}{\partial \mathbf{K}}\frac{\partial \Omega}{\partial \mathbf{X}} + \frac{\partial N}{\partial \Omega}\frac{\partial \Omega}{\partial T} = G. \tag{2.129}$$

The spectral energy balance equation (2.129) is called *the radiative transfer equation*, or *the transport equation*, or *the kinetic equation* and is used for forecasting spectral changes of sea waves. The first step for describing the evolution of the wave spectrum was done by Gelci et al. (1956, 1957) who introduced the concept of the spectral transport equation.

The first term at the Left Hand Side (LHS) of (2.129) expresses the local time evolution of the spectrum, and the second one represents the evolution of the spectrum for the horizontally inhomogeneous wave field and provides the energy transport with the group velocity $\partial\Omega/\partial\mathbf{K}$. The third term in (2.129) reflects the effects of refraction and shoaling due to the spatial change of the dispersion relation (because of variable bathymetry or currents), while the fourth term describes the temporal evolution of the dispersion relation due to changing conditions.

The basic difficulty in solving Eq. (2.129) is an evaluation of the source function G. The theory of weak nonlinear interactions for wind-induced waves was first formulated by Hasselmann (1962, 1968); it involves nine terms in the sum (2.123). The terms representing the wave-wave interactions are quite bulky and make Eq. (2.129) an integro-differential type. The nonlinear interaction coefficients have been obtained through tedious computations for low-order nonlinear interactions (up to five-wave interactions) (see Zakharov 1974, 1999; Krasitskii 1994; and Davidan et al. 1985; Lavrenov 2003; Janssen 2004; Polnikov 2007). In this application, the Hamiltonian approach is very convenient when the theory is expressed in terms of specially defined canonical variables (Zakharov 1968, 1974, 1999; Zakharov and Kuznetsov 1997; Polnikov 2007). The kinetic equation may be obtained rigorously starting from the primitive hydrodynamic equations, or from weakly nonlinear dynamical models such as the Zakharov equations (Zakharov 1974, 1999; Krasitskii 1994). However, the still open question is whether—and, if so, how much and under which conditions—the numerical evolution of a spectrum evaluated with the kinetic equation corresponds to the spectrum obtained with the full integration of the dynamical model starting from the actual surface distribution (Cavaleri 2005).

It has already been pointed out that in addition to the bound corrections to the wave shapes, the wave nonlinearity results in interactions between Fourier harmonics (that are independent in the linear limit). Due to this interaction, the energy in the spectral space may focus on one scale (uniform waves), or spread over many frequencies, and under certain conditions form very steep intensive waves. On the surface of deep water, the main part of the wave-wave nonlinear interactions in G is represented by the four-wave interaction. The spectral energy balance equation was used by Janssen (2003) to find the corrections to the Gaussian statistics of high sea waves when four-wave interactions are taken into account. The results were compared against the stochastic Monte Carlo simulations of dynamical models. The nonlinear effects in wave dynamics causing significant wave enhancement will be considered further in Chaps. 4 and 5.

Both considered approaches—deterministic and statistical—have strong and weak points that indicate their successful application for different purposes. In Sect. 2.2.1, we describe the one-point approach imposed by the instrumental data that is presently available. Other restrictions are due to the hypotheses employed by the approach (such as statistical homogeneity and stationarity). Realistic statistical models are still developing, and they should be verified versus observations. Some statistical aspects of nonlinear waves over deep, shallow, and coastal waters are discussed further in Sects. 4.4, 4.7, 5.3, and 5.5.

Unlike field observations and laboratory experiments, which usually give either temporal data at a few locations in space or spatial data at a few instants, the direct numerical simulations may provide both temporal and spatial data of a large-scale wave field. At the same time, the complexity and nonreproducibility of sea wave dynamics make the phase-resolving, long-time dynamic simulations practically useless. Draper in 1964 remarked that it is probably not possible to predict rogue-wave occurrence at a given time and space, although their probability might be estimated exceeding the framework of the stationary Gaussian process.

To obtain realistic rogue statistics, wave data collection is probably not the most adequate approach. Besides the problem of instrumental measurement briefly discussed in Chap. 1, one will face the following question: is the observed extreme wave a very rare realization from the typical slightly non-Gaussian sea surface population, or is it a typical realization of a very rare and strongly non-Gaussian sea surface population (Haver and Andersen 2000, Haver 2005)? The ensemble technique widely used in meteorology is promising. Each simulation is obtained by perturbing the conditions and/or initial sea wave field and letting the system evolve. Given the spectrum at a certain instant of time and location, one can choose a possible realization or a number of realizations. This would provide robust statistics of the sea surface, inclusive of all the nonlinear processes. Rather than acting only on the phases, one could act on the spectrum, both as amplitude and directions. In addition, the perturbations could be done not at random, but acting, for example, on specific groups of components chosen according to the situation (Cavaleri 2005).

2.3 Possible Physical Mechanisms of Rogue Wave Generation

Freak waves have been observed in basins of arbitrary depth (in deep as well as shallow water) with or without current and with or without wind. To resume, they may potentially occur everywhere on the ocean surface under any sea state conditions (see Chap. 1).

Before briefly presenting the main physical mechanisms leading to huge waves, let us introduce the critical depth parameter that allows separation between deep water and shallow water. Wave properties depend crucially on the water depth. This evident feature follows from the dispersion relation. While very long waves are not dispersive, dispersion becomes essential for shorter waves (see Figs. 2.2, 2.3). Furthermore, the dependence of the dispersion on water depth results in different manifestations of nonlinear wave-wave interactions. In shallow water, three-wave interactions play a major role, whereas in deep or moderately deep water, the main contribution to nonlinear wave interactions comes generally from four-wave interactions. The variety of nonlinear properties of sea waves over finite depths provides a rich and complex picture of nonlinear instabilities that could spawn rogue waves.

The natural parameter used to define deep water or shallow water conditions is the dimensionless depth, KD, where K is the wave number and D is the water depth. Following Fenton (1979), shallow water conditions correspond to $KD < \pi/4$, otherwise waves are propagating on finite depth or deep water. For $KD > \pi/4$, the Stokes-like expansion is relevant to calculating accurately nonlinear wave fields, whereas in shallow water it is the cnoidal-like expansion that prevails. In deep water and finite depth, the small parameter used in the Stokes expansion is the wave steepness AK (A denotes the wave amplitude), while for shallower water this parameter becomes A/D. In this book, we use the critical value $\pi/4$ of the normalized depth KD to separate deep water from shallow water. Note that in Chap. 3, the distinction between deep and shallow water is not used, whereas Chaps. 4 and 5 consider physical mechanisms that act in deep and shallow seas, respectively.

As it was noted previously, in addition to the dispersive parameter KD, there exist nonlinear parameters AK and A/D for deep water and shallow water, respectively. The parameter AK was already introduced in Sect. 2.1.3 for linearization of the equations. Chapter 3 will focus on linear aspects of rogue occurrence, while Chaps. 4 and 5 will consider nonlinear aspects. The main efforts are focused on the nonlinear and strongly nonlinear dynamics of the rogue-wave phenomenon based on recent research, because such waves are more dangerous. We also collect the results of statistical processing of natural registrations in Sect. 4.7.2; they, in part, support theoretical trends, although in the present state they are, in fact, often contradictory.

There are various physical mechanisms generating rogue waves on the sea surface. They can be due to geometrical focusing of directionally spread waves, refraction phenomena (presence of variable current or bottom topography), frequency modulation (dispersive focusing or modulational instability of Benjamin-Feir type), or soliton interactions that may produce wave energy that focuses in a small area. It was recently suggested that wave fields resulting from the nonlinear interaction of two wave systems (crossing seas) could be unstable to modulational instability and therefore produce rogue-wave occurrence (see Sect. 4.6). In this section, the different mechanisms are briefly listed and presented, they will be investigated and discussed deeply in the subsequent chapters. These effects have been previously reviewed by Kharif and Pelinovsky (2003) and Dysthe et al. (2008).

2.3.1 Wave-Current Interaction

Freak-wave occurrence on currents is a well-understood problem (see Sect. 3.4) that can explain the formation of rogue waves when wind waves or swells are propagating against a current. Besides more sophisticated models, the use of basic equations describing conservation of kinematical and dynamical properties of water-wave fields can be very convenient in determining the transformation of water waves by currents.

2.3.2 Geometrical or Spatial Focusing

Meanwhile, freak waves are seemingly observed throughout the world's oceans without significant currents as well. Underwater topography modifies the wave propagation. The result is spatial variations of the kinematic and dynamic variables of the problem that can be solved by the use of ray theory. Hence, rogue wave occurrence corresponds to caustics (see Sect. 3.1).

2.3.3 Focusing Due to Dispersion: The Spatio-Temporal Focusing

The spatio-temporal wave focusing due to the dispersive nature of water waves is a classic mechanism yielding wave-energy concentration in a small area (see Sect. 3.2). This effect, which can occur at the sea surface, can be reproduced easily in a laboratory experiment. Interactions with sea currents and wind flows represent specific features of sea waves. The effect of wind action is taken into account within the linear approximation in Sect. 3.3.

It is evident that once the wave steepness becomes finite, nonlinearity needs to be included. Effects of water wave nonlinearity on the above processes are discussed in Chap. 3 and further in Chaps. 4 and 5. Both weak and strong nonlinear approaches are presented. The achieved conclusions are verified against available results of laboratory experiments.

2.3.4 Focusing Due to Modulational Instability

This phenomenon is essentially nonlinear. Nonlinear uniform wave trains suffer an instability known as the Benjamin-Feir instability, which produces growing modulations of the envelope. These modulations that evolve into short groups of steep waves correspond to a nonlinear focusing of the wave energy. At the maximum of modulation, rogue waves can occur followed by the demodulation of the envelope. Rogue waves resulting from the modulational instability are considered in Chap. 4.

2.3.5 Soliton Collision

Uniform wave trains under modulational instability transform into a system of envelope solitons that may collide to give rise to huge wave events. Instability of quasi-solitons of large amplitude followed by collapse has been suggested as a proper scenario of rogue wave occurrence as well. These mechanisms that are working on finite and infinite water depths are presented in Chap. 4. Rogue waves can also occur in shallow water due to soliton interaction. The latter aspect is discussed in Chap. 5.

List of Notations

C_{gr}	group velocity
C_{LW}	long wave velocity
C_{ph}	phase velocity
D	water depth
$\mathrm{D}\,/\,\mathrm{D}T$	material derivative
$E[\cdot]$	statistical averaging
f	probability density function
F	probability distribution function
$\mathbf{g} = (0,0,-g)^t$	acceleration vector due to gravity
H	wave height
H_s	significant wave height
$\mathbf{K} = (K_X, K_Y)$	wave vector
K	wavenumber
K_p	mean wavenumber
m_n	n^{th} spectral moment
m_n^c	n^{th} central spectral moment
$\mathbf{n}$	unit vector normal to the surface
N	wave action in the kinetic equation
P	pressure
P_a	atmosphere pressure
R	autocorrelation function
S	non-symmetric spectrum
$\hat{S}$	wave spectrum
T	time
T_p	mean wave period
$\mathbf{U} = (U,V,W)^t$	the velocity field
$\mathbf{U}_c$	current velocity
U_n	normal to the surface velocity
U_w	wind velocity
$\mathbf{X} = (X,Y)^t$	horizontal plane coordinate
(X,Y,Z)	coordinates
X_f	fetch
δ_Ω	spectral width
ε	linearization parameter
$\phi(X,Y,Z,T)$	velocity potential
γ	peakedness in the JONSWAP spectrum
γ	skewness
$\eta(X,Y,T)$	surface elevation
κ	kurtosis
λ	wavelength
μ	dynamic viscosity
$\mu \equiv \mu_1$	first statistical moment, the expected value
μ_n	n^{th} statistical moment

μ_n^c — n^{th} central statistical moment
ν — kinematic viscosity
ρ — water density
σ — standard deviation, σ^2 is the variance
ω — vorticity
Ω — cyclic wave frequency
Ω_i — intrinsic frequency
Ω_p — mean wave frequency
∇ — gradient operator

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