

## Chapter 2

### Delay Model, SSA and Brownian Motion

This chapter presents three other tools to approach complex, nonlinear and chaotic dynamics. We will consider the *Delay-model*, the *Singular Spectrum analysis* and the *Brownian motions* (fractional or non-fractional). Firstly, we present the delay-model which is applied to the logistic equation. According to the Medio's work, a discrete-delay is integrated into the construction of an economic model by means of a convolution. The lengths of lags are distributed in a random way in the population. The delay is in fact modelled by means of a random variable which is characterized by its probability distribution. We will notice that in this way, the system built rocks more tardily to the chaos. We will observe a shift of bifurcation points, but also an *unhooking* in the trajectory.

*Delay-model applied to the logistic equation.* We will use the equation with the first-order differences used by Robert May. The central element of the model is the concept of "delay". For a macroeconomic consumption model for example, if we postulate that there is an (unspecified) great number of agents, and that all these agents answer to a certain stimulation with given discrete-lags, the lengths of lags are different for various agents and are distributed in a random way in the population. In a global model, in the whole population, the reaction times are aggregate. In the described case, we can model the reaction-time by means of a random variable that will represent the global length of the lag.

The Singular Spectrum Analysis is the second investigation tool of complex dynamics presented in this chapter. The method associates the *Takens reconstruction* technique and a technique known in *matrix algebra* which is the *Singular Value Decomposition*. In such a framework, the purpose is to project a time series on a basis of *eigenvectors extracted* from this same time-series. We project a *trajectory matrix* on the space described by the *eigenvectors of the covariance matrix of the time series*. The *eigenvalues obtained* can be ordered and be the subject of a filtering with the aim to extract the *deterministic* part of the signal cleaned of its background noise. The SSA was used in signal theory, and its applications to dynamical system theory have been introduced by Broomhead and King in 1986, in connection with their version of the Takens theorem (see Chap. 1). The method is presented in the framework of the delay-model behavior applied to the logistic equation, but also on the French stock index.

The last concept described in this chapter is the Brownian motion, which is a formidable tool to study chaotic behaviors. We will describe their constructions and experiment different types of Brownian motions, fractional ( $H \neq 1/2$ ) or non-fractional ( $H = 1/2$ ). The concepts associated with Brownian motions, such as *persistence*, *memory*, *Levy distribution*, *fractal dimension* and *Rescaled range* statistics are approached in this heading.

## 2.1 Delay Model Applied to Logistic Equation (Medio)

### 2.1.1 Nonlinearities and Lags

Here, the first-order difference equation of R. May is used and its general form is:  $X_{t+T} = G(X_t)$ , where  $T$  represents the length of the *lag* and  $G$  is a *smoothed one-hump function*. Such an equation can be understood as an aggregate system (i.e. a one-loop feedback system) with two components (1) a nonlinear relationship (here, single-hump functions) and (2) a lag (here, fixed delay). For (1): Nonlinearities are widely used in Economics, for example in rational consumption models, overlapping generations models, optimal growth models, we can mention: Sutzer (1980), Day (1981–1982), Benhabib and Day (1981), Grandmont (1985), Pelikan and Deneckere (1986), Baumol and Benhabib (1988), Lorenz (1989), Boldrin and Woodford (1990), Scheinkman (1990). For (2): The notion of lag is neglected in Economics, however it can be a source of important developments in connection with the notions of aggregates, agent behaviors, stochastic processes, aggregate models and chaotic systems.

#### 2.1.1.1 Lag Distribution of Agents

This notion of “lag” has been revisited by A. Medio and can find a significance in certain economic models (e.g. rational consumption or overlapping generations models).<sup>1</sup> Suppose that there is an unspecified great number of agents and that all these agents *respond to a certain stimulation with given discrete lags. The lengths of lags are different for various agents, and are distributed in a random way in the population*. The subject here is the “reaction time” of agents. In a global model, in the whole population the reaction times are aggregate. In the described case, we can model the reaction time by means of a random variable, real, positive or null, that is called  $T$  (in accordance with the equation posed at the beginning of chapter) and will represent the Global length of the lag. A random variable is characterized by its probability distribution (if it is known).

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<sup>1</sup> Ref. to Medio publications, in particular: Medio (1992).

### 2.1.1.2 Convolution of the Lag and Reaction of Agents

Let  $X(t)$  be a variable that is function of another variable  $Z(t)$  through a continuously distributed lag.<sup>2</sup> (Note that  $Z$  may indicate the same variable  $X$  at some time different from  $t$ .)  $m$  can be understood as a “weighting function” (i.e. a kind of “moment”) that formalizes the strength of impact that values of  $Z$  in the more or less distant past have on the value of  $X$  (i.e. a kind of temporal correlation). The equation of the lag can given by:

$$X(t) = \int_0^\infty m(s)Z(t-s)ds \quad (2.1)$$

with  $m$  continuous on  $\mathbb{R}$ , and  $\int_0^\infty m(s)ds = 1$ . In practice,  $s$  is bounded by  $t$  and the previous equation becomes  $X(t) = \int_0^t m(s)Z(t-s)ds$ . This equation can be taken as a (commutative) convolution.<sup>3</sup> Given the polynomial  $v(p) = a_0p^n + \dots + a_n$ , with  $\mathcal{L}$  the Laplace transform that is written  $\mathcal{L}[m(s)] = 1/v(p) = \mathcal{L}[\mathcal{L}^{-1}[1/v(p)]]$ . Then  $m(s)$  is defined as the inverse Laplace transform of  $1/v(p)$ , thus,  $m(s) = \mathcal{L}^{-1}[1/v(p)]$ , then  $X = m \star Z$  is the solution of the differential equations  $v(D)X(t) = Z(t)$  with  $D \equiv d./dt$  and the initial conditions  $X = X' = \dots = X^{n-1} = 0$ . If the lag is an *exponential lag* of order  $n$ , we have  $\frac{1}{v(p)} = \left(\frac{\tau p}{n} + 1\right)^{-n}$ . Thus the differential equation  $v(D)X(t) = Z(t)$  is written  $\left(\frac{\tau D}{n} + 1\right)^n X(t) = Z(t)$  or also

$$X(t) = \left(\frac{\tau D}{n} + 1\right)^{-n} Z(t), \quad (2.2)$$

where  $n \in \mathbb{Z}^+$  and  $\tau$  is the time constant of the lag. For multiple exponential lags,  $m(t)$  can be calculated through the inverse Laplace transform, we have:  $m(t) = \left(\frac{n}{\tau}\right)^n \frac{t^{n-1}}{(n-1)!} e^{-nt/\tau}$ . For  $n \geq 2$ ,  $m(t)$  has a one-hump shape:

- (a) When  $n = 1$ , we deal with the ordinary differential system which can be written in the following way:  $\dot{X}(t) = \gamma(Z(t) - X(t))$  where  $\gamma$  represents the speed of adjustment of the model (note that  $1/\gamma = \tau$ ).
- (b) When  $n$  becomes large, the weighting function tends to a Dirac delta function and the exponential lag tends to a fixed delay of length  $\tau$ . Then we have:  $\lim_{n \rightarrow \infty} \left(\frac{\tau D}{n} + 1\right)^{-n} = e^{-\tau D}$  (Fig. 2.1).

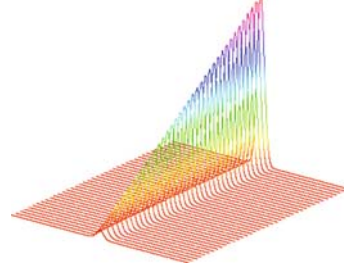
At this stage, we have to combine the delay and the model. If we place the lag in the generic initial model  $X_{t+T} = G(X_t)$ , this can be written:

$$X_n = \left(\frac{D}{n} + 1\right)^{-n} G(X_n). \quad (2.3)$$

<sup>2</sup> The lag can be understood as a shift in the reaction time of agents.

<sup>3</sup> See Appendix.

**Fig. 2.1** Weighting function of different exponential lags



After the application of the factor  $(\frac{D}{n} + 1)^{-n}$  to  $G(X_n)$  we encounter the value of  $X_n$  again. We can turn over the equation, and move the lag term on the other side of the equality and change the sign of the exponent. Then we write the following equivalent relations that describe a differential equation system:

$$((D/n) - 1)X_i = X_{i-1}, \quad \text{with } i = 2 \dots n, \quad (2.4)$$

$$((D/n) - 1)X_1 = G(X_n), \quad (2.5)$$

where  $G(X_n)$  is a one-hump function. The system can also be written:

$$\begin{aligned} ((D/n) + 1)X_2 &= X_1, \\ &\vdots \\ ((D/n) + 1)X_1 &= G(X_n). \end{aligned} \quad (2.6)$$

### 2.1.2 Application to the Logistic Equation

The logistic equation is written  $X_{n+1} = \alpha X_n(1 - X_n)$ . Consequently:  $G(X_n) = \alpha X_n(1 - X_n)$  and the lag takes the form  $(\frac{D}{n} + 1)^{-n}$ ,  $X_n = ((D/n) + 1)^{-n}(\alpha X_n(1 - X_n))$ . We face a *nonlinear functional relation* and a *delay-function*. We will see that the application of the lag to the logistic model shifts the chaotic zone.

#### 2.1.2.1 Solution of the Logistic Model

At the equilibrium we have  $X_1 = X_2 = X_3 = \dots = \bar{X}$ , thus for  $X_{n+1} = \alpha X_n(1 - X_n)$ , there are the equilibrium solutions:  $\bar{X} = 0$  and  $\bar{X} = 1 - (1/\alpha)$ . The stability of these equilibrium points was previously studied in Chap. 1. Finally, we write the differential system with the delay function as follows:

### 2.1.2.2 Differential System Combining Logistic Model and Delay Funct

The system of ordinary differential equations of dimension  $n$  is:

$$\begin{aligned} ((D/n) + 1)X_2 &= X_1, \\ &\vdots \\ ((D/n) + 1)X_1 &= \alpha X_n(1 - X_n). \end{aligned} \quad (2.7)$$

For a system of dimension 10 and of length  $N$ , we write:

$$\begin{aligned} ((D/10) + 1)X_2 &= X_1, \\ &\vdots \\ ((D/10) + 1)X_1 &= \alpha X_{10}(1 - X_{10}). \end{aligned} \quad (2.8)$$

By simple multiplication, the system is written:

$$\begin{aligned} ((\dot{X}_2/10) + X_2) &= X_1, \\ &\vdots \\ ((\dot{X}_{10}/10) + X_{10}) &= X_9, \\ ((\dot{X}_1/10) + X_1) &= \alpha X_{10}(1 - X_{10}). \end{aligned} \quad (2.9)$$

The differential system can be written in matrix form:

$$[\dot{X}] = [A] * [X]. \quad (2.10)$$

### 2.1.2.3 Figures of Various Simulations for $\alpha \in [3, \dots, 5]$

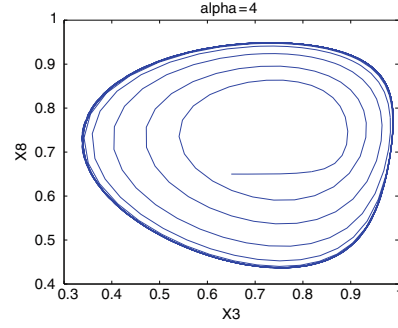
Pictures of differential system solutions for different  $\alpha$  values (Figs. 2.2–2.9).

For each value of alpha, we have to face a rectangular matrix made with vectors  $X$  of dimension  $10 \times N$ . A weight  $(\frac{D}{10} + 1)$  is applied to each vector of the matrix, and this weight changes the trajectory.

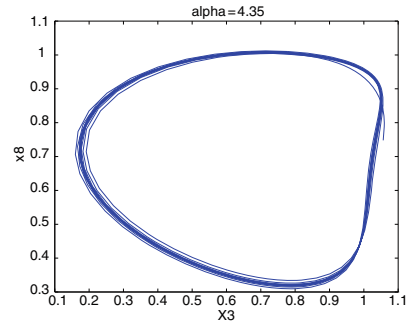
### 2.1.2.4 Shift of Bifurcation Points, and Periodicity

For the logistic model the ultimate bifurcation before chaos occurs for  $\alpha \simeq 3.56$ . In our case, for a system dimension equal to  $10 \times N$ , this occurs for  $\alpha \simeq 5$ . The period-doublings visible in the graphs of the preceding section are more tardy under the effect of delay function. The periodic behaviors and their periods are identifiable with the numbers of distinct orbits in the figures. Starting from the value equal to 5, the trajectory seems to describe different and distinct orbits which do not overlap, contrary to what occurred for  $\alpha = 4$ , for example, where if we eliminate the

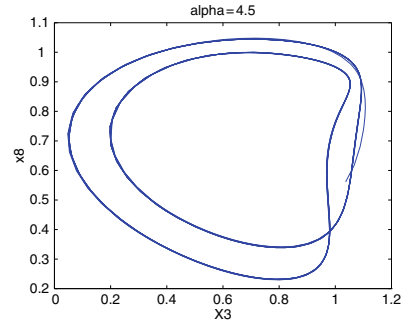
**Fig. 2.2**  $\alpha = 4$ . Transitory behavior was preserved before convergence



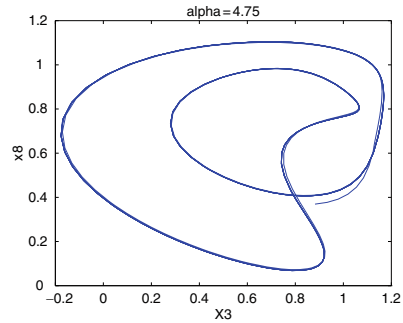
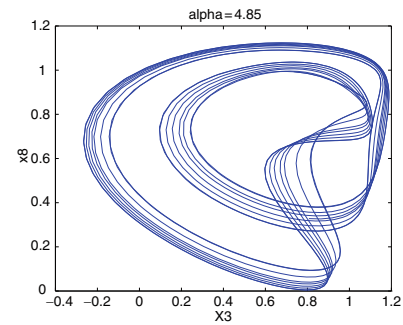
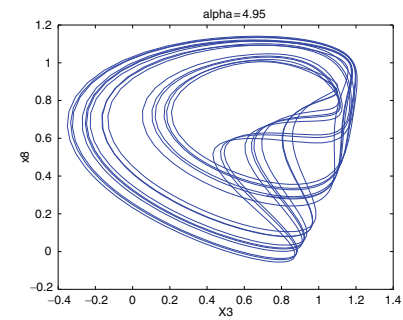
**Fig. 2.3**  $\alpha = 4.35$ . Transitory behavior suppressed to make the orbit visible



**Fig. 2.4** Period-doubling.  $\alpha = 4.5$ . Asymptotic behavior

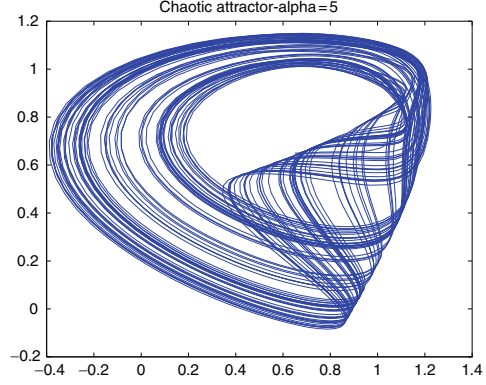


transitory behaviors before convergence, we have a “*dense closed*” orbit without *unhooking* and *period-doubling*. Moreover, in order to underline the periodicities, we can analyse the spectrum of these trajectories and calculate the Lyapunov exponent to highlight the “moment” during which the system rocks towards a pure chaotic behavior, as we could do it in the first chapter for the logistic equation itself. We did not represent the *sensitive dependence on initial conditions of the system* which is the characteristic of chaos nor the value of the *Lyapunov exponent* during the evolution of the system on the route towards chaotic zone. The capacity dimension can also be measured rather easily. The set of these elements converges

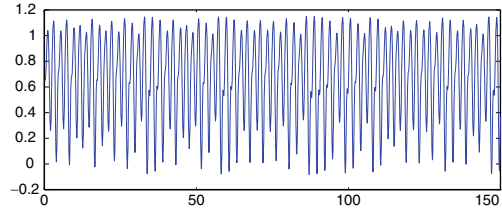
**Fig. 2.5**  $\alpha = 4.75$ **Fig. 2.6**  $\alpha = 4.85$ **Fig. 2.7**  $\alpha = 4.95$ 

towards a value of  $\alpha$  equal to 5 to characterize the critical point. *The value of the capacity dimension for  $\alpha = 5$  is close to 2.15, which is a non-integer value and characterizes the presence of a simple attractor.* The principle of the exponential lag makes it possible to model a large variety of economic situations by controlling the parameters  $T$  and  $n$ . The nonlinearities of one-hump functions coupled to a delay function produce chaos. One of the characteristics of this exponential lag is that the variance which was used to build it is low. Thus we are close to the aspect of a Dirac function. In conclusion, it is possible to say that a high order lag, for example  $n = 10$  is a condition for chaos occurrence. The type of model that we have just seen can find applications in macroeconomics, in particular in the models of

**Fig. 2.8**  $\alpha = 5$ . Orbit and chaotic attractor, with unhooking



**Fig. 2.9** Behavior of  $X(8)$  with  $n = 10$



production–consumption, for example the developments of Grandmont models on the equilibrium cycles (“Overlapping-generations models”).

## 2.2 Singular Spectrum Analysis

### 2.2.1 Singular Spectrum Analysis Principle: “Windowing”, Eigenvector and Projection

The SSA method is built from a technique known in matrix algebra as the “singular value decomposition”. This method consists in projecting a time series on a basis of eigenvectors extracted from this same time-series. Or more exactly, it is the projection of a matrix trajectory, built from the initial time series, on the eigenvectors of an intermediate matrix, itself built from the studied experimental series. The SSA was used in signal theory, and its applications to the dynamical system theory have been introduced by Broomhead and King in 1986 (see Chap. 1). Their version of the Takens theorem states that the space containing the image of the map  $\Phi_{F,v}$  is called the embedding-space and its dimension  $n$  is called embedding dimension.<sup>4</sup>

<sup>4</sup> See Takens theorem. Recall:  $\Phi_{F,v} : A \longrightarrow \mathbb{R}^{2m+1}$ .  
 $\Phi_{F,v}(y) = (v(y), v(\phi_1(y)), \dots, v(\phi_{2m}(y)))^T$ .



This is the reconstruction of phase space of solutions of an arbitrary dynamical system. And the dimension of this reconstructed phase space is also the dimension of the embedding space. It is pointed out that the conclusions of the Takens theorem impose the following constraint  $n \geq 2m + 1$  with:

- $m$ : Dimension of the attractor
- $n$ : Dimension of the embedding space

Consider for example the values taken by a variable  $Y(t)$  with regular time intervals denoted  $\tau$ , then we can write this without dimensioning the number of observations in the following way  $Y(t), Y(t + \tau), Y(t + 2\tau), \dots$ . If the set is dimensioned, we can write a group of observations of the variable  $Y_p = Y(p\tau)$  as follows with  $p = 1, \dots, n$  and where  $\tau$  is a “step” corresponding to a “periodic measurement of the variable”. We will use the concept of  $(n, \tau)$ -window presented in the first chapter (see Takens theorem). Here, it is pointed out that a window is the combination of two criteria: the number of selected measurements that becomes, in fact, the length of the series and the step  $\tau$  which is the periodicity of the measurement. Thus, the periodicity of the measurement with the number of measurements makes visible  $n$  elements of the initial time series, which is therefore sampled on intervals of length  $\tau$ . An optimal approach in a discrete case makes us take  $\tau$  equal to 1, i.e. each measurement of the value of the variable, without descending to the value of the step below the unit. Thus for a  $(n, \tau)$ -window, we make visible  $n$  elements of the initial time series. In the continuous case, things are different since we can construct windows with the length we select and show exactly what we want about the model. This makes much more easier the handling of trajectories. Our limits in this construction are rather those fixed by what we wish to exhibit of the trajectory, i.e. in general the attractor and in particular to operate its reconstruction whose principal constraint is  $n \geq 2m + 1$ . The subject is to manage scale problems that are important in this type of construction. For the choice of a  $(n, \tau)$ -window which obviously is of the size  $n$ , we must generate, by means of the model equation, a discrete time series of a sufficient length so that the window can “exist” and in a significant way.

(a) *Classical method of reconstruction and trajectory matrix.* It is possible to construct a set that is sequence of vectors in the embedding space of dimension  $n$ , which is written:  $\{x_i \in \mathbb{R}^n \mid i = 1, \dots, N\}$ . Each vector  $x_i$  of this set is a point in  $\mathbb{R}^n$ . The set contains thus  $N$  points, each one of the dimension in  $\mathbb{R}^n$ . Consequently, we represent the set called “ $n$ -history” by a rectangular matrix of dimension  $N \times n$ . We can write the constraint on the set:

$$N = N_o - n + 1, \quad (2.11)$$

with  $N_o$ : number of observations (i.e. the length of the series),  $N$ : dimension of the reconstructed series, and  $n$ : dimension of the embedding space. These sequences are

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Knowing that  $n \geq 2m + 1$ .  $\phi_t$  is a flow of  $F$ .  $\phi$  is a map representing the dynamics such that:  $y_{j+1} = \phi(y_j)$ . And the function  $v(y)$  can be taken as a measurement made on the system at the point  $y \in A$ , i.e.  $v(\phi_t(y))$  would be equal to an observation of  $Y$  at time  $i$ .

used to build a matrix  $X$ , which is the trajectory matrix of dimension  $N \times n$ . If we pose the trajectory matrix in the following way:

$$X = N^{-\frac{1}{2}} \begin{pmatrix} X_1^T \\ \vdots \\ X_N^T \end{pmatrix}, \quad \text{or } X = N^{-\frac{1}{2}} \begin{pmatrix} (x_1^1, \dots, x_1^n) \\ \vdots \\ (x_N^1, \dots, x_N^n) \end{pmatrix}, \quad (2.12)$$

with  $\dim(X) = (N, n)$ , and the factor  $N^{-\frac{1}{2}}$  being introduced by convenience. “By plotting the columns of  $X$  against the principal directions of the embedding space while respecting  $n \geq 2m + 1$ , we obtain the reconstructed attractor” according to Takens method.

(b) *The SSA method.* The previous method has evolved. A better reconstruction technique has replaced the Takens method. We will describe it hereafter. In spite of a rather long presentation, the resulting system is of a quite simple handling. Consider a set of vectors

$$\{s_i \in \mathbb{R}^N / i = 1, \dots, n\} \quad (2.13)$$

such as by their action on  $X$ , they generate a new set of vectors “linearly independent and orthogonal”<sup>5</sup> which is

$$\{c_i \in \mathbb{R}^n / i = 1, \dots, n\}. \quad (2.14)$$

It is possible to assume that the vectors  $\{c_i\}$  are also “normal” and provide a basis in  $\mathbb{R}^n$

$$s_i^T X = \sigma_i c_i^T, \quad (2.15)$$

where  $\{\sigma_i\}$  is a set of real constants used to normalize the vectors. From the algebra of matrices we know that:

$$s_i^T X X^T s_j = \sigma_i \sigma_j \delta_{ij}, \quad (2.16)$$

where  $\delta$  is the Kronecker symbol.<sup>6</sup> In addition, the matrix  $H = X X^T$  is real and symmetrical and, moreover, its eigenvalues form an orthogonal basis for  $\mathbb{R}^N$ . More precisely the eigenvectors of  $H$  satisfy the preceding equation  $s_i^T X = \sigma_i c_i^T$ . The extraction of eigenvectors and eigenvalues is made from the square matrix  $H$  whose

<sup>5</sup> We will refer to the following different notions: basis of vectors, linearly independent vectors, singular values, and “rank” of a group of vectors. The rank of a family of  $p$  vectors  $(V_1, V_2, \dots, V_p)$  is the greatest number of linearly independent vectors among them. It is also said: row of the  $p$  vectors. It is also said: Rank of the  $p$  vectors  $(V_1, V_2, \dots, V_p)$ . The rank is lower or equal to  $p$ . If  $A$  is a matrix of vectors, we note by  $k = \text{rank}(A)$  the number of singular values of  $A$  (which are larger than the  $[\max(\text{size}(A)) \cdot \text{norm}(A)]$ ).

<sup>6</sup>  $\delta$  is the Kronecker symbol: A family of vectors  $(v_1, v_2, \dots, v_s)$  of  $\mathbb{R}^n$  is said orthogonal system if for all  $(i, j), i \neq j, v_i \cdot v_j = \delta_{ij}$  (where  $\delta$  is the symbol of Kronecker:  $\delta_{ij} = 0$  if  $i \neq j$ ,  $\delta_{ij} = 1$ , if  $i = j$ ). Any orthogonal system of non-null vectors is free.

dimensions are very large, since they are those of the length of  $X$ . Then, we will see that it is easier to work with a matrix that is a variant of  $H$  and of which the dimension is much lower. This new square matrix will note  $V$  and its dimension will be equal to the width of matrix  $X$ . We can write:<sup>7</sup>

$$Hs_i = \sigma_i^2 s_i. \quad (2.17)$$

Then we have the set  $\{s_i\}$  of the corresponding eigenvectors of  $H$ , and the set  $\{\sigma_i^2\}$  of the corresponding eigenvalues of  $H$  all real and non-negative ( $H$  being positive-semidefinite).  $H$  can be written as follows:

$$H = N^{-1} \begin{pmatrix} X_1^T X_1 & \cdots & X_1^T X_N \\ \vdots & \ddots & \vdots \\ X_N^T X_1 & \cdots & X_N^T X_N \end{pmatrix} \quad (2.18)$$

with  $\dim(H) : (N, n) * (n, N) = (N, N)$ . The writing in the simplified form is:

$$H = X^T X. \quad (2.19)$$

$H$  can be understood as a correlation matrix between pairs of vectors (generated by the window of size  $n$ ). With this technique, we have to face an important difficulty: indeed,  $H$  has a dimension  $N \times N$  usually very large and consequently its diagonalization is often impossible to practice.<sup>8</sup> To remove this serious constraint in particular, a new technique was developed.

(c) *SSA 2nd method.* A more effective method to obtain the desired result is the following: it is to take the “transpose” of the equation  $s_i^T X = \sigma_i c_i^T$ , thus:

$$X^T s_i = \sigma_i c_i. \quad (2.20)$$

If we pre-multiply by  $X$ :  $XX^T s_i = \sigma_i X c_i$ . By using the equation  $Hs_i = \sigma_i^2 s_i$  and by simplifying:

$$X c_i = \sigma_i s_i. \quad (2.21)$$

And if we pre-multiply the preceding equation by  $X^T$  and we introduce it into the equation  $s_i^T X = \sigma_i c_i^T$  the following equation is obtained:

$$V c_i = \sigma_i^2 c_i, \quad (2.22)$$

<sup>7</sup> Let us note that:

$$f(V) - \lambda V = 0; AV - \lambda V = 0. Hs_i - \sigma_i^2 s_i = 0; (H - \sigma_i^2) s_i = 0$$

Collinearity:

$$AV - \lambda V$$

↓

$$Hs_i - \sigma_i^2 s_i.$$

<sup>8</sup> See appendix about the diagonalization of matrices.

where  $V \equiv X^T X \in \mathbb{R}^{n \times n}$ , the extraction of eigenvectors and eigenvalues is carried out on the square matrix  $V$

$$V = \begin{pmatrix} X_1^T X_1 & \cdots & X_n^T X_n \\ \vdots & \ddots & \vdots \\ X_n^T X_1 & \cdots & X_n^T X_n \end{pmatrix}. \quad (2.23)$$

The equation  $Vc_i = \sigma_i^2 c_i$  allows to deduce  $\{c_i\}$  as the set of eigenvectors of  $V$  and  $\{\sigma_i^2\}$  as the set of the corresponding eigenvalues of  $V$ . As said for the matrix  $H$ , note that  $V$  can be taken as a covariance matrix of observations. The number of “steps” ( $\tau$ ) will be equal to the dimension of the “embedding”  $n$  being appreciably smaller than  $N$ , the equation  $Vc_i = \sigma_i^2 c_i$  is much easier to treat than the equation  $HS_i = \sigma_i^2 s_i$ . The numerical calculation of eigenvectors and eigenvalues of  $H$  was often very long, sometimes even impossible, due to the high dimensions of the matrix, directly connected with the length of observed vectors. For example, for an experimental series constructed from 1,000 measurements, the matrix  $H$  becomes a square matrix of dimensions  $1,000 \times 1,000$  (i.e. millions of values) which has to be diagonalized to obtain the eigenvectors and eigenvalues. Often the calculations by  $H$  fail. On the other hand, the use of the matrix  $V$  (which is of a considerably reduced size) facilitates the calculation by its simplification. Let  $C$  be the matrix whose columns are composed by the  $c_i$  and  $\Sigma^2 = \text{diag}(\sigma_1^2, \dots, \sigma_n^2)$ , where the  $\sigma_i^2$  are ordered from the largest to the smallest, i.e.  $\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_n^2 \geq 0$ . Consequently, the equation  $Vc_i = \sigma_i^2 c_i$  can be written:

$$VC = C\Sigma^2 \quad (2.24)$$

$X^T X c_i \equiv c_i \text{diag}(\sigma_1^2, \dots, \sigma_n^2)$ . Using the definition of  $V$ , consider  $V \equiv X^T X \in \mathbb{R}^{n \times n}$ , we have:

$$(XC)^T (XC) = \Sigma^2. \quad (2.25)$$

*The matrix  $XC$  represents the trajectory matrix projected on the basis  $\{c_i\}$ . The subject is the choice of  $\{c_i\}$  as a basis for the projection, i.e. to project the trajectory matrix onto the space spanned by the eigenvectors of the covariance matrix of the time series. This projection is optimal because the columns of the trajectory matrix are independent  $(XC)^T (XC) = \Sigma^2$  and minimize the mean square error of the projection. (“Thus the plots are not squeezed any more onto the diagonal and the projections on the planes  $(I, J)$  and  $(i + p, j + p)$  are not equal any more, like with the Takens method” Medio 1992.) We will find an application of the method above in the section that follows. There are other developments of this method which approached the time-series with a background noise that disturbs the analysis. The statistical approaches that aim at sorting the eigenvalues of the matrix  $V$ , make it possible to “denoise” the reconstruction, but they are not depicted in the present work.*

### 2.2.2 SSA Applied to the Logistic Equation with Delay Function

#### 2.2.2.1 Projections and Reconstructions of the Initial Series

Consider the delay model applied to the logistic equation, whose system is:

$$X_{t+1} = \alpha X_t(1 - X_t), \quad X_n = ((D/n) + 1)^{-n} G(X_n) \quad \text{with } n = 10, \alpha = 5 \quad (2.26)$$

then we have  $((D/n) + 1)^n + 1^n X = \alpha X(1 - X)$ . Consider the trajectory of  $X_1$  for  $\alpha = 5$  to be positioned at the beginning of the chaotic regime, and then we will compute outputs for  $\alpha = 3$ ,  $\alpha = 4$ ,  $\alpha = 20$ . (Recall: Takens constraint for the attractor reconstruction is  $n \geq 2m + 1$ .)

(a) Observe the logistic system attractor for a function delay  $\alpha = 5$  (Fig. 2.10).

$X$  (above) is the solution of the delay model applied to the logistic equation for a parameter equal to 5.  $X$  has the shape of a rectangular matrix. Starting from the first vector  $x_1$  of  $X$ , we will apply the SSA. Let us visualize  $x_1$  (Fig. 2.11).

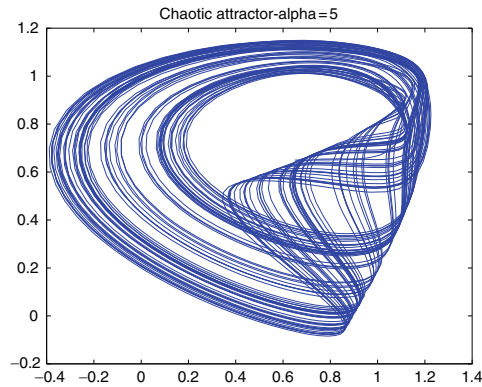
(b) Then, let us calculate the trajectory matrix by the first method with  $N = N_o - n + 1$ ,  $N = 1000$ ,  $n = 10$ ,  $N_o = 1,009$ ,  $\dim(X) = (N, n) = (1000, 10)$ :

$$X = 1000^{-\frac{1}{2}} \begin{pmatrix} X_1^T \\ \vdots \\ X_{1000}^T \end{pmatrix} = 1000^{-\frac{1}{2}} \begin{pmatrix} x_1^T(1) & \cdots & x_1^T(10) \\ \vdots & \ddots & \vdots \\ x_{1000}^T(1) & \cdots & x_{1000}^T(10) \end{pmatrix}, \quad (2.27)$$

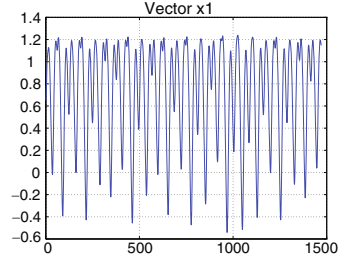
$$H = X^T X, \quad (2.28)$$

$$\dim(H) = (N, n) * (n, N) = (N, N). \dim(H) = (1000, 10) * (10, 1000) = (1000, 1000)$$

$$H = 1000^{-1} \begin{pmatrix} X_1^T X_1 & \cdots & X_1^T X_{1000} \\ \vdots & \ddots & \vdots \\ X_{1000}^T X_1 & \cdots & X_{1000}^T X_{1000} \end{pmatrix}. \quad (2.29)$$



**Fig. 2.10** Logistic system attractor for a function delay with  $\alpha = 5$

**Fig. 2.11** Behavior of  $x_1$ 

(c) Let us compute the new trajectory matrix  $V$ . According to what precedes, note that as the matrix  $H$  is heavy to handle, we use the following more efficient way: Given  $\{s_i \in \mathbb{R}^{1000}/i = 1, \dots, 10\}$  and  $\{c_i \in \mathbb{R}^{10}/i = 1, \dots, 10\}$  with  $\dim(V) = (n, N) * (N, n) = (n, n)$ ,  $\dim(V) = (1000, 10) * (10, 1000) = (10, 10)$

$$V = X^T X = \begin{pmatrix} X_1^T X_1 & \cdots & X_1^T X_{10} \\ \vdots & \ddots & \vdots \\ X_{10}^T X_1 & \cdots & X_{10}^T X_{10} \end{pmatrix}. \quad (2.30)$$

(d) *Results.* The vectors and matrices contributing to the decomposition are:

$X$  which is the initial matrix of vectors

$X^T$ : the transpose of  $X$

$V = X \cdot X^T$ : the product of both preceding vectors

$c$ : eigenvectors of  $V$

$\sigma^2$ : eigenvalues of  $V$

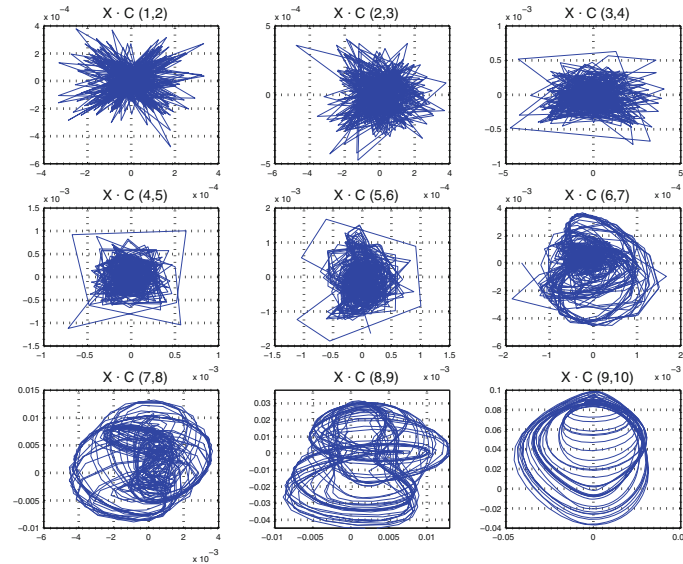
$V \cdot c$ : product of the matrix  $V$  with its eigenvectors

$\sigma^2 \cdot c$ : product of the eigenvalues with eigenvectors

$X \cdot c$ : product of the initial matrix of vectors with vectors of matrix  $V$

$(X \cdot c)^T \cdot (X \cdot c)$ : product of transpose of previous projection with itself

If we extract the eigenvalues from  $V$ , by taking the diagonal of the matrix of eigenvalues  $\sigma^2$  composed of  $\sigma_1^2, \dots, \sigma_n^2$ , the values of the diagonal are: [ 0 0 0 0 0 0 0.0004 0.0148 0.3185 6.9033]. These eigenvalues are naturally ordered in an ascending order. And the construction of  $\Sigma^2$  which is carried out in a descending order of eigenvalues does not give more information. But such a case is obviously not frequent in practice, especially if we increase the size of  $n$ . When the matrix  $X \cdot c$  is constructed, several types of Poincaré sections are shown from the different components of  $X \cdot c$  and with various steps. These projections of the attractor reconstructed by the SSA method are depicted in Fig. 2.12 ( $\alpha = 5$ ). The results of this SSA construction could be compared with the projections resulting from the Takens method – not shown here –: in the Takens method, the figures would show orbit projections more or less *squeezed* on the diagonal according to selected “steps”. Figure 2.12 of pairwise components of the matrix  $X \cdot c$ , we observe very different results. Indeed, in such a case, we do not observe any more squeezing on the diagonal. Note that each plane constructed with two different vectors from the matrix  $X \cdot c$  provides different plots, whereas the Takens method always showed similar orbits.



**Fig. 2.12** Pairwise components of  $X \cdot c$  by disregarding time for  $\alpha = 5$

Thus, the projection of  $X$  on the eigenvectors  $c_i$  of  $C$  allows to isolate different structures of the signal, which correspond to the many aspects of the signal. This is due to the eigenvectors of  $V$  that are individually different. Periodicities and stationarities can be observed in the vectors  $X \cdot c$ . Note that the more the sequence number of the vector increases, the more its amplitude above and below zero increases. Then, we construct a sample of Poincaré sections similar to the previous construction and we observe eigenvectors of  $V$  which constitute the basis on which  $X$  is projected.

(f) Case  $\alpha = 3.5$ . The behavior of the system for this value of alpha is not chaotic but periodic. The orbit exhibits a spiral form for  $X \cdot c(9, 10)$ . Figure 2.13 shows different components of  $XC$  by disregarding time.

(g) Case  $\alpha = 4$ . Same remarks as previously, the orbit shows periodicities at the same time on the  $V \cdot c$  vectors and also on the eigenvectors. The matrix  $XC$  exhibits all the periodicities in course of time (not shown here). Figure 2.14 show the various components of  $XC$  by disregarding time.

### 2.2.3 SSA Applied to a Financial Series (Cac40)

The method is based on the decomposition of a time series by using a basis generated by the initial time series itself. In Economics and Finance, it is usual to observe time series whose terms are autocorrelated. To study the series, it is often necessary to suppress the trend.

In order to stationarize the time series, a regression on time can be used or the  $n$ th-differences. Here we stationarize by the first-differences. As previously, we compute

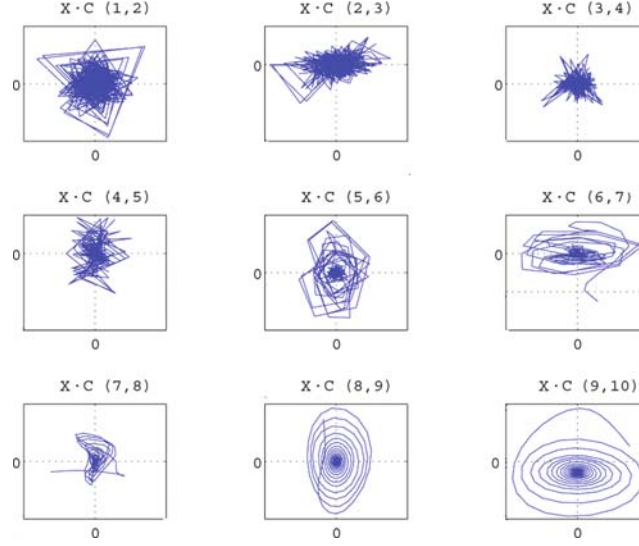


Fig. 2.13 Pairwise components of XC for  $\alpha = 3.5$

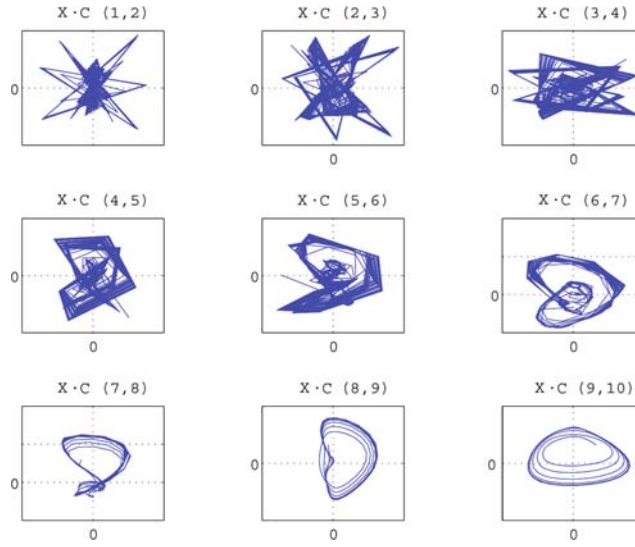
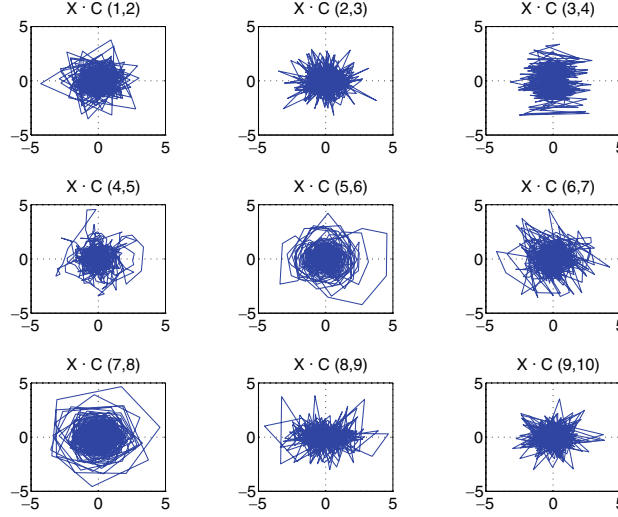


Fig. 2.14 Pairwise components of XC for  $\alpha = 4$

the following vectors:  $X, X^T, (V = X * X^T), c, \sigma_i^2, (V * c), (\sigma_i^2 * c), (X * c)$ . If the diagonal is extracted ( $\sigma_1^2, \dots, \sigma_n^2$ ) from matrix of eigenvalues of  $V$ , we obtain: [770.5 761.4 826.2 850.2 965.3 952.8 952.8 1,005.4 1018.5 693.5 647.1]. Note that the eigenvalues are not naturally ordered. In Fig.2.15 we project matrix  $X$  on the matrix of eigenvectors of  $V$ .





**Fig. 2.15** Nine main components of  $X \cdot C$  for SSA applied to a stock index (Cac40)

### 2.2.3.1 Role of Eigenvalues, and Filtering of Background Noises

One of the elements contributing to the SSA is  $\Sigma^2 = \text{diag}(\sigma_1^2, \dots, \sigma_n^2)$  (i.e. the vector containing the eigenvalues of the matrix  $V$ ), whose eigenvalues  $\sigma_i^2$  are classified in descending order from the largest to the smallest, i.e.  $\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_n^2 \geq 0$ . After their classification, the elements  $\sigma_i^2$  of  $\Sigma^2$  are: [1,018.5 1,005.4 965.3 952.8 850.2 826.2 770.5 761.4 693.5 647.1]. From these ordered eigenvalues, we obtain a diagonal matrix:  $\Sigma^2$ . Here, the size of the window or the dimension of the embedding space is 10. One of the properties of the SSA method is the filtering of *background noise* of a time series. According to the words of A. Medio (1992, p. 186), this method allows “the identification of directions along of which the deterministic component of motion takes place, which we shall henceforth call significant (or deterministic) directions”, whereas the infinite rest will be denoted by “stochastic directions”. These assertions were applied to unidimensional systems in which deterministic chaos appear (e.g. logistic equation). Here, we will not compare the eigenvalues and the dimension of phase-space orbits of a dynamics, which has been largely presented by A. Medio in 1992 in “Chaotic dynamics”. We will just say that we can split the embedding space  $n$  into a space in which the attractor is immersed, i.e. a subspace called  $d$  where the orbits exist without background noise and a stochastic subspace  $(n - d)$  in which the only involved motion is the noise. We also mention<sup>9</sup> that the rank<sup>10</sup> of the matrix  $V$  gives the higher limit of the dimension

<sup>9</sup> By considering the Broomhead and King assumptions.

<sup>10</sup> The rank of a matrix  $V$  is the number of linearly independent columns in the matrix  $V$ . For a square matrix, this number is always equal to the number of linearly independent lines. If the matrix is rectangular  $m \times n$ , then the rank is lower or equal to the  $\min(m, n)$ .

of the subspace explored by the deterministic component of the trajectory. Thus, the dimension  $d$  of the deterministic subspace is obtained by computing the *rank of the matrix*  $V$ . In the case of a series without background noise, a hypothesis has been enunciated saying that  $d$  (with  $d \leq n$ ) is equal to the number of strictly positive eigenvalues of  $V$ , and the rest of eigenvalues is equal to the number of  $(n - d)$  whose value is equal to zero. In this particular case (i.e. differentiated Cac40), we observe that there is no eigenvalue equal to zero. It was different for the vector  $x_1$  resulting from the delay model applied to the logistic equation (for  $\alpha = 5$ ), for which six values among ten were equal to zero. Observe below the eigenvalues (after classification in descending order), (1) for the delay model, (2) then for the stock index:

Logistic. Eq/Delay model:

[6.9033 0.3185 0.0148 0.0004 0 0 0 0 0 0]

First-difference of Cac40:

[1,018.5 1,005.4 965.3 952.8 850.2 826.2 770.5 761.4 693.5 647.1]

However, the previous assumptions about the eigenvalues proved to be incomplete and non-exhaustive. In addition, the noise level of the background noise is not dependent on the dimension  $n$  of the embedding space. Thus, when we increase the size of the embedding space (by increasing the size of the window- $(n, \tau)$  or by reducing the size of the interval  $\tau$  which is used to sample), then the level of background noise is lowered and new eigenvalues appear. From these observations, we can work on the eigenvalues  $\sigma_i^2$  and treat them to distinguish (in a noisy signal) the *deterministic* part corresponding to the cleaned signal, and the part corresponding to the background noise.  $\sigma_i^2 = (\sigma_i^D)^2 + (\sigma^B)^2$  ( $D$ : determinist,  $B$ : noise). The SSA is an important source of research for complex dynamics in Economics and Finance.

## 2.3 Fractional Brownian Motions

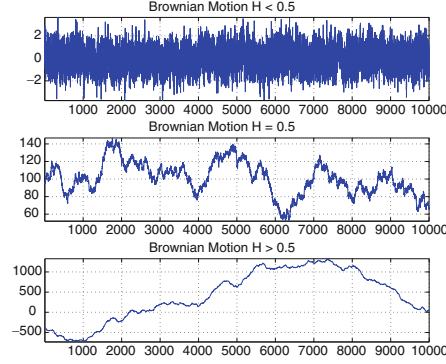
### 2.3.1 Brownian Motion and Random Walk

A Brownian motion can be defined as a *random series*  $x(t)$  with Gaussian increases and whose variance

$$\text{var}[x(t_2) - x(t_1)] \text{ is proportional to } |t_2 - t_1|^{2H}$$

with  $2H = 1/2$ . Although the internal structure of Brownian motion is different according to the value of  $H$  ( $0 < H < 1$ ), in a generic way, we speak of a fractional Brownian motion, whatsoever  $H$ . Figure 2.16 shows examples of three Brownian motions respectively for  $H = 0.1, 0.5$  and  $0.72$ .

Without presenting its genesis, it is possible to say that  $H$  is a statistical indicator known under the name of the Hurst exponent. In particular, for an experimental series, the goal of this indicator was to dissociate random walk from non-random walk. The Hurst subject was that the experimental dynamics in Nature do generally



**Fig. 2.16** Examples of Brownian motions respectively for  $H = 0.1$ ,  $0.5$  and  $0.72$

not follow random walks. The Hurst-Test was used by analogy on financial markets in order to test the concept of random walk, which in econometrics has been used for a long time to characterize financial series. We will see later on how the Hurst statistic also plays a role in the following concepts: Persistence, Long memory of the series (part II), ARFIMA models (part II), Self-similarities (part III, and fractal series).

### 2.3.1.1 Rescaled Range Statistic and Fractional Brownian Motion

The long-term temporal dependence was approached by the statistic called “Rescaled Range” that corresponds to a ratio: the ratio of the *extent* of a series to a standard deviation. This is the extent of partial sums of variations to the average (of a time series) divided by its standard deviation. Given a series  $(x_1, x_2, \dots, x_n)$  and the average of sample  $\bar{x}_n = \frac{1}{n} \sum_{j=1}^n x_j$ , we write  $R$  and  $S$ :

$$R = \max_{1 \leq k \leq n} \sum_{j=1}^k (x_j - \bar{x}_n) - \min_{1 \leq k \leq n} \sum_{j=1}^k (x_j - \bar{x}_n), \quad (2.31)$$

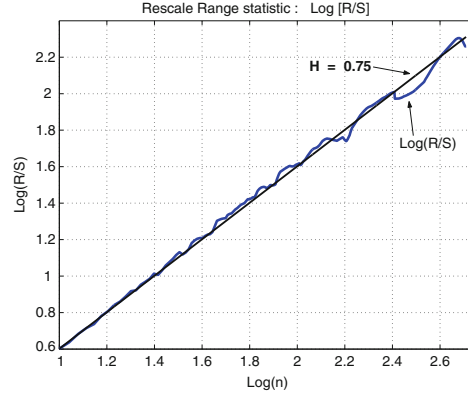
$$S = \left( \frac{1}{n} \sum_j (x_j - \bar{x}_n)^2 \right)^{1/2}. \quad (2.32)$$

( $k$  = shifts). Thus the R/S statistic is written:

$$R/S = \frac{\max_{1 \leq k \leq n} \sum_{j=1}^k (x_j - \bar{x}_n) - \min_{1 \leq k \leq n} \sum_{j=1}^k (x_j - \bar{x}_n)}{\left[ \frac{1}{n} \sum_j (x_j - \bar{x}_n)^2 \right]^{1/2}}.$$

The relation between these statistics and the Hurst exponent can be written:

$$R/S = a \cdot n^H, \quad (2.33)$$



**Fig. 2.17** Log (R/S) of a Brownian motion  $H = 0.75$

$a$  is a constant. If the series is a random walk, then  $H = 0.5$ . On the other hand, when  $H > 0.5$ , it is not a random walk. This statistical criterion, which implicitly rests on the autocorrelation concept, would allow to identify the stochastic aspect of a series. This test is also expressed in a logarithmic form:

$$\log(R/S) = H \cdot \log(n) + b. \quad (2.34)$$

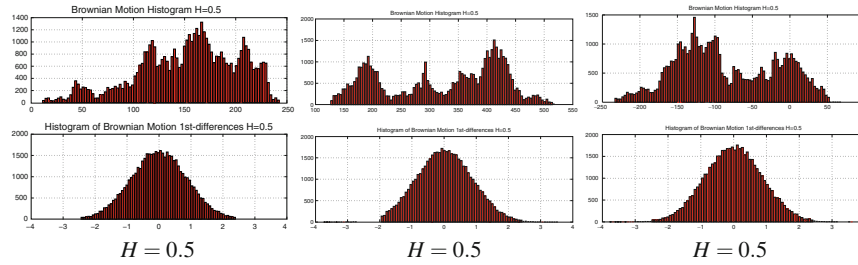
By using a log-log scale, we show the graph of the statistic (in relation to the number of observations of the time series). From the preceding equation and for  $b = 0$ , we deduce  $H = \log(R/S)/\log(n)$ . Thus, we can read the estimate of  $H$  compared with the value chosen to construct the series. Plane  $[\log_{10}(n), \log_{10}(R/S)]$  (Fig. 2.17).

We mentioned above that when  $H = 0.5$ , we have to face a random walk, and not for  $H > 0.5$ . This assertion means that for  $H = 0.5$ , the variable of the series is not autocorrelated, as we could see it in a statistical analysis. On the other hand for  $H > 0.5$ , there are autocorrelation or dependence of terms. It is said that: “Each observation carries a memory of events which precedes it, this is a long term memory: the most recent events have an impact larger than those which are prior to them. What happens today has an influence on the future; the present is a consequence of the past. The time plays an important role” (Abraham-Frois and Berrebi 1995).<sup>11</sup>

$$C = 2^{2H-1} - 1. \quad (2.35)$$

$C$  is an autocorrelation of long period or a correlation of future values with the past values. If  $H = 0.5$  then  $C = 0$ , there is not a “temporal correlation” between the terms of the series, we are thus faced with a random process of a random walk. And it is noted that the characterization of the process is done without using the probability law of the series.

<sup>11</sup> *Remark:* These remarks will be able to echo in econometrics concerning concepts of *process of the DS type* (Difference Stationary) where the method used to make the time series stationary is done by differentiation, and of *TS type* (Trend Stationary) where the method used to make the time series stationary is done by a regression over time.



**Fig. 2.18** Empirical distributions (histograms) of three arbitrary Brownian motions for  $H = 0.5$ . Below histograms of their 1st-differences

If  $H$  is not equal to 0.5 and in particular when it is higher than 0.5, the correlation is an increasing function of  $H$ . It is said that there is *persistence*, it is also said that the Brownian motion is fractional and for this reason, *there are global and even local tendencies* that emerge from the series. The series has frequent “exits” of the tunnel described around the average of values which precede. These exits are kinds of fractures (“unhookings”) in relation to the prior walk of the process which does not have a periodicity. This random component of a process, which was stated as non-stochastic, requires to use the probability laws to describe it, and the statistic  $H$  is understood as the probability so that two consecutive events occur.<sup>12</sup> On the other hand, if  $H > 0.5$ , there is an occurrence of non-periodic cycles, and the more  $H$  is high, the more the aperiodic oscillations frequently deviate from the average of values that the time series took prior to each fracture. In Fig.2.18, we show for  $H = 0.5$  the histograms (empirical distributions) of three Brownian motions (50,000 steps) and below, the histograms of their first-differences (increases). Note that signal distributions have particular forms and the distributions of their first-differences seemingly tend towards a structure of the Gaussian type (to verify with normality test).

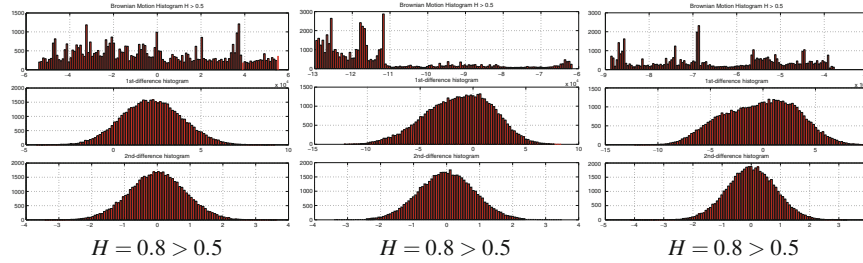
In Fig. 2.19, (1) we show for  $H > 0.5$  the histograms of three Brownian motions, (2) the histograms of their first-differences and (3) the histograms of their second-differences.

In this case, on the other hand, for  $H = 0.8 > 0.5$ , only the distributions of second-differences appear to tend towards a structure of the Gaussian type.

### 2.3.2 Capacity Dimension of a Fractional Brownian Motion

Although the characterization of a Brownian motion is complex, even if we have given a definition above, we can depict a Brownian motion in the following way. Let us imagine an object represented by a point that moves and that at every moment

<sup>12</sup> Ref: Abraham-Frois and Peters: “For  $H = 0.6$ , there is a probability of 60% so that if the last change were positive the following movement will be also” (cf. Abraham-Frois and Berrebi 1995, p. 330).



**Fig. 2.19** Histograms of three arbitrary Brownian motions for  $H > 0.5$ . Below histograms of their 1st-differences. Below histograms of their 2nd-differences

makes a jump (or a step) in an unspecified direction (knowing that here, it is a two-dimensional or three-dimensional Brownian motion). More exactly, we imagine that this displacement is that of a drunk walker as it is usual to symbolize it. Viewed from a big distance, one does not know if it is a continuous curve where each step is represented by a point, or if it is discontinuous, i.e. if each point corresponds to 100 or 1,000 steps for example. If we compare the trajectories taken step by step, all the 100 steps or all the 1,000 steps, we will observe that the trajectories resemble each other. It is usual to say that the trajectory is a fractal curve and *its capacity dimension (or fractal dimension) is equal to 2* ( $D = 2$ ). This type of trajectory has been highlighted by Jean Perrin at the beginning of the twentieth century, he observed the movements of atoms under the microscope. One of the consequences of *this capacity dimension equal to 2 is that the trajectory tends to “blacken” the plane* while advancing in the course of time. And if despite everything the length of the time series is not sufficient, we will be able to observe portions of *the plane “filled” uniformly by the trajectory*. A “singular geometry” can be associated with this type of trajectory which is (certainly too quickly) defined as a random walk. The selected example of a walker, lets precise that the step taken by the walker excludes all the “long distance jumps”. We can choose another way of symbolizing this type of movement borrowed from the physical science by observing the trajectory of an atom or a molecule belonging to a gas from which the temperature is different from the absolute zero. The random movements of atoms consecutive to the presence of a non-zero ambient temperature are called *phenomena of diffusion* which are related to Physics but also the biology for example. Generally, any matter undergoes this diffusion phenomenon through an increase in the ambient temperature. Above, we evoked the notion of *the long distance jump*, but the length of the walker’s steps is always close for two arbitrary steps, which may be slightly different if we measured them with precision. But this way of representing the movement excludes *the long distance jump*. This type of long distance jump has been approached by Paul Levy who in the 1930s studied fundamental variants of the Brownian motion where these long jumps are allowed, in opposition to what is stated above. The probability of these jumps is weak, however they have the very important capacity to modify the general structure of a movement. This type of event is called *Levy jump* or *Levy flight*.

### 2.3.2.1 Capacity Dimension of Trajectories, Gauss and Levy Laws

This concept of Levy jump can find transpositions in Nature. The most usual case is the displacement of an albatross above the ocean, seeking the fish shoals to nourish itself. The displacement of the bird is done in three dimensions. However if we observe the trajectory in a plane (i.e. two-dimensions), we cannot observe the movements (elevation) in the vertical dimension any more. The characteristic is that the bird that found a fish shoal will describe displacements of weak length and width during a time, the majority of its displacements at this place being vertical (not represented). Then, having finished at this place, it will move much further in the plane until finding another fish shoal. This aspect of the set in the plane corresponds to concentrations of movements in the form of very localized loops which will be separated by long curves without loops. The distance between two fish shoals is much larger than the one inside the same fish shoal. In short, there will be many short flights and from time to time long distance flights. This observation was done by two teams, one from the British Antarctic Survey from Cambridge and the other from the Boston University. Unlike the Brownian motions described by the displacement of a walker, where the steps have close lengths, the Levy jumps produce immense steps (in the trajectory) even if they have a weak probability to occur. This type of Levy jump offers trajectories whose *capacity dimension or fractal dimension is lower than 2* ( $D < 2$ ). *A Levy jump will thus be defined as a movement where the probability  $P(s > S)$  of a jump  $s$  of distance higher than  $S$  varies like a negative power of  $S$ :  $P(s > S) = S^{-D}$ , with  $D < 2$ .*<sup>13</sup>

This principle of the Levy distribution is used in astronomy in the study of the distribution of “celestial objects”. The trajectories of the stock exchange indexes do obviously not follow the Gauss law. The strong fluctuations of the stock exchange, the shocks and the explosions over long periods could be integrated into the probability law. Sapoval (2001) gives a recent example of the Frankfurt Stock Exchange index, for which we notice *brutal explosions* and *self-similarities* on different scales. He notices that many stock market indexes have representations of this type and have the statistical property of the *self-affinity*. (The stake is to consider that the stock-exchange trajectories result from a subjacent deterministic dynamics, like the trajectories with Levy jumps.)

### 2.3.2.2 Capacity Dimension and Fractional Integration

The fractional integration parameter  $d$  is defined as follows, in connection with the Hurst exponent  $H$  evoked in the preceding sections:  $H = d + 1/2$ . In connection with the Levy law, we can indicate the functional relation between the fractal dimension and the Hurst exponent:  $D = 2 - H$  or, the relation between the fractal dimension and fractional integration:  $D = 3/2 - d$ . There exists a relation between the Levy law and fractional integration:  $D = 3/2 - d < 2$ ,  $-d < -3/2 + 2$ ,

---

<sup>13</sup> The central limit theorem is impugned when the expectation value:  $E(s^2) \rightarrow \infty$ .

$d < 3/2 - 2$ ,  $d < -1/2$ . This case corresponds to the lower bound of the parameter of fractional integration  $d$  of a process ARFIMA(0,d,0) for which we have to face “anti-persistence” phenomena. On the contrary, the relation between the Levy law and the Hurst exponent is immediate:  $D < 2$ ,  $2 - H < 2$ ,  $H > 0$ .  $H$  will have to be strictly higher than 0, what appears to be a too large sample to be significant. The relation between the fractal dimension and the exponent is also written in certain works (Mandelbrot 1975, p. 114) and under some conditions no longer described as a difference but as a ratio:  $D = 1/H$ .

### 2.3.2.3 Definition of a Brownian Motion by Mandelbrot

**Definition 2.1 (Mandelbrot, Brownian motion).** If  $x$  is a point of the plane,  $x(t)$  is called a Brownian motion it is a succession of small displacements which are mutually independent and “isotropic”.<sup>14</sup> The last characteristic means that all the directions for the displacement of the point in the plane are possible. For any couple of moments  $t$  and  $t' > t$ , one dissociates two points of vector  $x$  by  $x(t)$  to  $x(t')$  and considers that (Mandelbrot 1975, p. 45):

- (1) The direction as well as the length of the trajectory are independent of the initial position and of the position of each previous point.
- (2) The vector must be isotropic.
- (3) The length of the vector is such that its projection on an unspecified axis obeys the Gaussian distribution of density:  $\frac{1}{\sqrt{2\pi|t'-t|}} \exp \frac{-x^2}{2|t'-t|}$ .

### 2.3.3 Introduction to Persistence and Loops Concepts

We implicitly evoked in different previous sections the fact that the Brownian motion is spread in a natural space, i.e. in a plane or in a volume. This is why in this section we chose to represent some Brownian motions in dimension higher than 1. Brownian motions have been generated by the same simulator (simulator seeds influence the results).

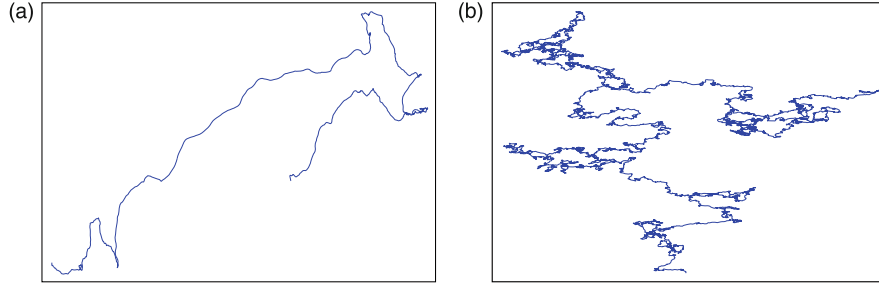
#### 2.3.3.1 Brownian Motions: Highly Persistent, Fairly Persistent and Weakly Persistent

*Highly persistent.* The Hurst exponent chosen for the simulation hereafter is close to 0.85. We are beyond the value 0.5 of  $H$  for which we define a random walk

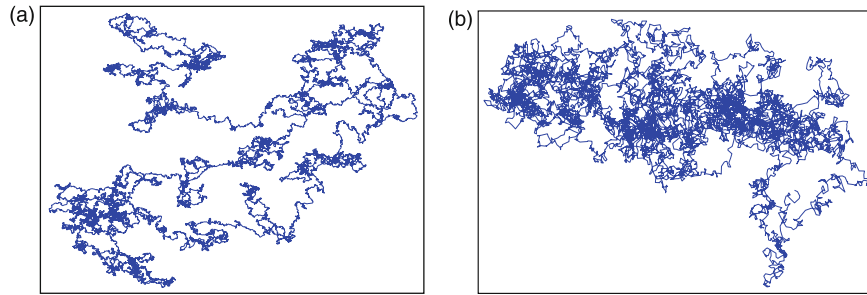
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<sup>14</sup> An *isotropic line*: an isotropic line passes through the circular points at infinity. Isotropic lines are perpendicular to themselves.





**Fig. 2.20** (a) “Highly” persistent,  $H = 0.85$ ; (b) “Fairly” persistent,  $H = 0.6$



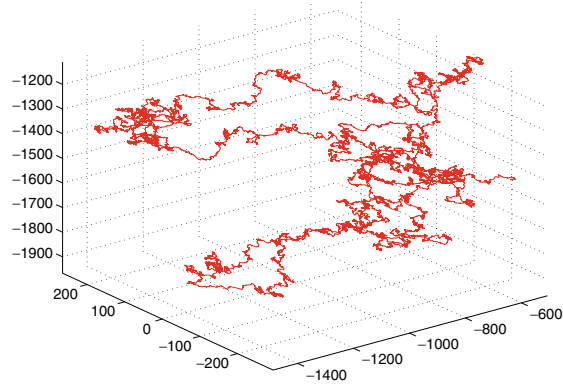
**Fig. 2.21** (a) Brownian motion,  $H = 0.6$ ; (b) Brownian motion for  $H = 0.51$

in the sense of Hurst. The value  $H = 0.85$  provides a fractal dimension near  $D = 2 - 0.85 = 1.15$ . The second approach also provides a dimension lower than 2,  $D = 1/0.85 = 1.17$ . The construction of such a time series, according to the terms of Mandelbrot “discourages very strongly, without forbidding them”, the formation of loops, because we “forced this trajectory to be very persistent”. We can observe above in Fig. 2.20a that the loops are rare. The low frequency “drift” is very high.

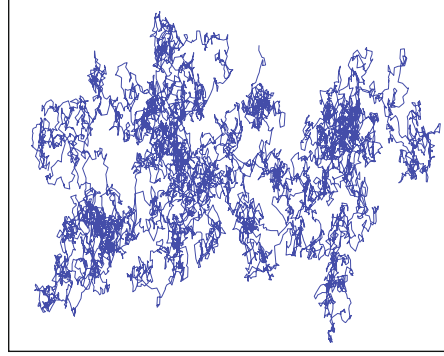
*Fairly persistent.* The Hurst exponent chosen in this case is around 0.6. We are beyond the value 0.5 of  $H$  for which we define a random walk in the sense of Hurst. The value  $H = 0.6$  provides a fractal dimension near  $D = 2 - 0.6 = 1.4$ . The second approach also provides a dimension lower than 2,  $D = 1/0.6 = 1.66$ . We observe the proliferation of many small loops that the trajectory describes on itself. *The more the Hurst exponent approaches 0.5, or the more the fractal dimension approaches 2, the more the curve forms convolutions.* The time series becomes *dense* (see Fig. 2.20b built by means 1,000 steps). Figures 2.21a and 2.22 represent fairly persistent Brownian motions built by means of 10,000 steps, the first one is a two-dimensional motion (see Fig. 2.21a) and the second one is a three-dimensional Brownian motion (see Fig. 2.22):

*Weakly persistent.* The Hurst exponent chosen in this case is around 0.51. We are very close to the value 0.5 of  $H$  for which we define a random walk in the sense

**Fig. 2.22** Three-dimensional Brownian motion,  $H = 0.6$



**Fig. 2.23** Brownian motion for  $H = 0.5$



of Hurst. The value  $H = 0.51$  provides a fractal dimension about  $D = 2 - 0.51 = 1.49$ . the second approach also provides a dimension lower than 2 but very close,  $D = 1/0.51 = 1.96$ . *The number of loops that the trajectory describes on itself is growing. The more the fractal dimension approaches 2, the more the curve forms convolutions, the time series becomes more and more dense.* The low frequency “drift” becomes very variable and is almost invisible in our case (see Fig. 2.21b).

*Brownian motion for  $H = 0.5$ .* The different convolutions are similar. *Such a situation is similar to the statistical analysis where the spectrum is white, we have to face a white noise, by the average or after the differentiation of the time series.* Low frequency drift is invisible. We will be able to visualize the density of the trajectory that blackens and uniformly fills the plane (see Fig. 2.23).

### 2.3.4 Comment on DS/TS Process and Brownian Motions

In this section, we open a parenthesis that has the form of a question. The “memory” notion of observations mentioned above (which appears when  $H > 0.5$ , i.e. when

“the present is a consequence of the past”) interrogates about the “tendency” notion in a time series. *The analysis of the trend in econometrics*, or more exactly *the analysis of the stationarity* by means of the Dickey and Fuller (Unit root) statistic, leads to choose for an arbitrary series between the two tendency types. The series are identified by these tests as, either (1) or (2):

- (1) DS type (Difference stationary), i.e. stationary by difference. The “stationarization” is done by differentiation.<sup>15</sup> We speak of the *stochastic stationarity* that concerns the *non stationary random processes*.
- (2) TS type (Trend stationary), i.e. with “stationary trend”. The “stationarization” (or detrend) is done by a regression on time. We say that there is a *deterministic non-stationarity*.

The stock markets and economic time series are processes that are rarely analysed as being stationary (or even Gaussian). The non-stationarity (non-stationariness) can result from moments of the first order (i.e. Expectation) or of the second order (i.e. Variance). Before the end of the 1970s, there was no analytical method to study non-stationariness. The Box–Jenkins method of graphic analysis, that makes it possible to visualize the tendencies or the cycles and the seasonalities, is interesting but not sufficient. In 1984, works of Nelson and Plosser analysed the non-stationarity (non-stationariness) by means of two processes: TS and DS processes.

#### 2.3.4.1 TS Processes: Non-Stationarity of Deterministic Type

Such a process is written  $x_t = f_t + \varepsilon_t$ , where  $f_t$  is a polynomial function of time, linear or nonlinear and  $\varepsilon_t$  is a stationary process. An example of an elementary process TS of order 1 is written:  $x_t = a_0 + a_1t + \varepsilon_t$ . If  $\varepsilon_t$  is a white noise (Gaussian or not)  $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ , the process is determined by:

$$E(x_t) = a_0 + a_1t + E(\varepsilon_t) = a_0 + a_1t, \quad (2.36)$$

$$V(x_t) = E(a_0 + a_1t + \varepsilon_t - (a_0 + a_1t))^2 = \sigma_\varepsilon^2, \quad (2.37)$$

$$\text{cov}(x_t, x_{t'}) = 0 \text{ for } t \neq t'. \quad (2.38)$$

Such a process is non-stationary because the expectation depends on time (in a linear way), as one can see above,  $t$  is provided with a coefficient. The expectation is calculated at every moment and obviously depends on  $t$ . We speak of (non-stationariness) non-stationary of the deterministic type. The parameters of the tendency has  $a_0$  and  $a_1$ , we can estimate them by the method of least squares. The estimators obtained are the “Best Linear Unbiased Estimators (BLUE)” which make forecasts possible. The detrend is done by removing from the value of  $x_t$  the estimation  $\hat{a}_0 + \hat{a}_1t$  at each moment  $t$ . In this type of process, it is said that after a *random shock*, known as *transitory*, the series *re-takes* its walk around its tendency curve.

<sup>15</sup> *Stationarization* (mathematical barbarism or neologism): To make a time series stationary.

### 2.3.4.2 DS Processes: Random Walk

Such a process is called “random walk” (and often used concerning stock markets). Here, the process becomes stationary by a filter using the differences:  $(1 - B)^d = \beta + \varepsilon_t$ , usually the filter is of order 1 ( $d = 1$ ).  $\varepsilon_t$  is a stationary process or a white noise and  $\beta$  is a constant symbolizing the possible drift of the process. The process is written:  $x_t = x_{t-1} + \beta + \varepsilon_t$ ,  $\varepsilon_t$  is a white noise (Gaussian or not). If the process is without drift ( $\beta = 0$ ), then:  $x_t = x_0 + \sum_{i=1}^t \varepsilon_i$ , thus  $E(x_t) = x_0$ ,  $V(x_t) = t\sigma_\varepsilon^2$ ,  $cov = \sigma_\varepsilon^2 \times \text{Min}(t, t')$  for  $t \neq t'$ . If the process has a drift ( $\beta \neq 0$ ), then we write:  $x_t = x_0 + \beta t + \sum_{i=1}^t \varepsilon_i$ , thus  $E(x_t) = x_0 + \beta t$ ,  $V(x_t) = t\sigma_\varepsilon^2$ ,  $cov = \sigma_\varepsilon^2 \times \text{Min}(t, t')$  for  $t \neq t'$ .

### 2.3.4.3 Fractional Brownian Motion and DS Process

The fractional Brownian motions are mainly non-stationary. For a Hurst exponent ( $H > 0.5$ ), the process is not analysed as a random walk but as a more or less persisting process, and the correlation  $C$  is an “increasing function of  $H$  for  $0.5 < H < 1$ ”. It is known that in this case, the movement has frequent exits of the tunnel described by its average and its variance. These exits are fractures in relation to its prior walk. It was also said that these “exits” have a *random periodicity*. (It is the random component of a process which for  $H > 0.5$  is considered as *non-stochastic* by Hurst.) The more  $H$  approaches 1, the more the non-periodic cycles deviate frequently from the average of values that the time series took before. For  $H = 0.5$ , we have pure randomness as in this case the correlations  $C = 0$ . The present does not influence the future and is not influenced by the past. This is known as a random walk.

Traditional Statistics analyse DS processes as a random walk. We can study the stationarity by means of the Dickey and Fuller (Unit root) Test of some samples of fractional Brownian motions for  $H = 0.5$  and  $H > 0.5$ . In the cases where the series exhibits a non-stationarity, usually the conclusion is that we are faced with a DS process, i.e. a random walk that becomes stationary by means of a filter using the differences (test not presented here). Later we will present the traditional statistical tests of stationarity, in Part II and they will be applied to a stock index.



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