Chapter 2
Classical Yang–Mills Black Hole Hair in Anti-de Sitter Space

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Abstract The properties of hairy black holes in Einstein–Yang–Mills (EYM) theory are reviewed, focusing on spherically symmetric solutions. In particular, in asymptotically anti-de Sitter space (adS) stable black hole hair is known to exist for $\mathfrak{su}(2)$ EYM. We review recent work in which it is shown that stable hair also exists in $\mathfrak{su}(N)$ EYM for arbitrary N, so that there is no upper limit on how much stable hair a black hole in adS can possess.

2.1 Introduction

We begin by very briefly reviewing the "no-hair" conjecture and motivating the study of hairy black holes.

2.1.1 The "no-hair" Conjecture

The black hole "no-hair" conjecture [142] states that (see, for example, [51, 52, 77–79, 118] for detailed reviews and comprehensive lists of references): *All stationary, asymptotically flat, four-dimensional black hole equilibrium solutions of the Einstein equations in vacuum or with an electromagnetic field are characterized by their mass, angular momentum, and (electric or magnetic) charge.*

According to the no-hair conjecture, black holes are therefore extraordinarily simple objects, whose geometry (exterior to the event horizon) is a member of the Kerr–Newman family and completely determined by just three quantities (mass, angular momentum and charge). Furthermore, these quantities are *global charges* which can (at least in principle) be measured at infinity, far from the black hole event horizon. If a black hole is formed by the gravitational collapse of a dying star, the

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initial star will be a highly complex object described by many different parameters. The final, equilibrium, black hole is, by contrast, rather simple and described by a very small number of quantities. During the process of the formation of a black hole, an enormous amount of (classical) information about the star which collapsed has therefore been lost. Similarly, if a complicated object is thrown down a black hole event horizon, once the system settles down, the only changes in the final state will be changes in the total mass, total angular momentum and total charge. Advances in astrometry [174] and future gravitational wave detectors [5] may even be able to probe the validity of the "no-hair" conjecture for astrophysical black holes by verifying that the mass, angular momentum and quadrupole moment Q_2 of the black hole satisfy the relation $Q_2 = J^2/M$ which holds for Kerr black holes.

The "no-hair" conjecture, stated above, has been proved by means of much complicated and beautiful mathematics (as reviewed in, for example, [51, 52, 77–79, 118]), subject to the assumptions of stationarity, asymptotic flatness, four-dimensional spacetime and the electrovac Einstein equations. It is perhaps unsurprising that if one or more of these assumptions is relaxed, then the conjecture does not necessarily hold. For example, if a negative cosmological constant is included, so that the spacetime is no longer asymptotically flat but instead approaches anti-de Sitter (adS) space at infinity, then the event horizon of the black hole is not necessarily spherical, giving rise to "topological" black holes (see, for example, [18, 64, 97, 98, 103, 112, 165]). More recently, the discovery of "black ring" solutions in five spacetime dimensions ([60], see [61] for a recent review) and the even more complicated "black Saturn" [59] solutions indicates that Einstein–Maxwell theory has a rich space of black solutions in higher dimensions, which are not given in terms of the Myers-Perry [121] metric (which is the generalization of the Kerr–Newman geometry to higher dimensions).

2.1.2 Hairy Black Holes

In this article we consider what happens when the other condition in the "no-hair" conjecture, namely that the Einstein equations involve electrovac matter only, is relaxed. The "generalized" version of the no-hair conjecture [79] states that all stationary black hole solutions of the Einstein equations with any type of self-gravitating matter field are determined uniquely by their mass, angular momentum and a set of global charges. Even in asymptotically flat space, this conjecture does not hold, even for the simplest type of self-gravitating matter, a scalar field. The first such counterexample is the famous BBMB black hole [12, 13, 27] which has the same metric as the extremal Reissner–Nordström black hole but possesses a conformally coupled scalar field. However, this solution is controversial due to the divergence of the scalar field on the event horizon [158] and is also highly unstable [48]. Therefore, in some ways the first "hairy" black hole is considered to be the Gibbons solution [71], which describes a Reissner–Nordström black hole with a non-trivial dilaton field. While there are many results which rule out scalar field hair in quite general models, particularly in asymptotically flat spacetimes (see, for example, [14] for a review), in recent years many other examples of black holes with non-trivial scalar field hair have been found. For example, minimally coupled scalar field hair has been found when the cosmological constant is positive [161] or negative [162] and non-minimally coupled scalar field hair has also been considered (see, for example, [176, 177] and references therein).

In this short review, we will focus on another particular matter model, Einstein–Yang–Mills theory (EYM), where the matter is described by a non-Abelian (Yang-Mills) gauge field. It is now well-known that this theory possesses "hairy" black hole solutions, whose metric is not a member of the Kerr–Newman family (see [171] for a detailed review). Furthermore, unlike the Kerr–Newman black holes, the geometry exterior to the event horizon is not determined uniquely by global charges measureable at infinity, although only a small number of parameters are required in order to describe the metric and matter field (see Sect. 2.3 for further details). All the asymptotically flat black hole solutions of pure EYM theory discovered to date are unstable [47] (however, there are examples of asymptotically flat, stable hairy black holes in variants of the EYM action, such as Einstein–Skyrme [22, 58, 80, 81], Einstein-non-Abelian-Proca [73, 110, 159, 160, 163] and Einstein–Yang–Mills–Higgs [1] theories). This means that, while the "letter" of the no-hair theorem is violated in this case (as there exist solutions which are not described by the Kerr–Newman metric), its "spirit" is intact, as stable equilibrium black holes remain simple objects, described by a few parameters if not exactly of the Kerr–Newman form (see [21] for a related discussion along these lines).

The situation is radically different if one considers EYM solutions in asymptotically adS space, rather than asymptotically flat space. For $\mathfrak{su}(2)$ EYM, at least some black hole solutions with hair are stable [25, 26, 175]. These stable black holes require one new parameter (see Sect. 2.4) to completely describe the geometry exterior to their event horizons. Therefore, one might still argue that the true "spirit" of the "no-hair" conjecture remains intact and that stable equilibrium black holes are comparatively simple objects, described by just a few parameters.

One is therefore led to a natural question: are there hairy black hole solutions in adS which require an infinite number of parameters to fully describe the geometry and matter exterior to the event horizon? In other words, is there a limit to how much hair a black hole in adS can be given? This is the question we will be seeking to address in this article.

2.1.3 Scope of this Article

The subject of hairy black holes in EYM theory and its variants is very active, with many new solutions appearing each year. The review [171], written in 1998, is very detailed and thorough and contains a comprehensive list of references to solutions known at that time. We have therefore not sought to be complete in our references prior to that date, and have, instead, chosen to highlight a few solutions (the selection being undoubtedly personal). Even considering just work after 1998, we have been unable to do justice to the huge body of work in this area (for example, the seminal paper [7] has 172 arXiv citations between 1999 and the time of writing) and have instead chosen some examples of solutions. As well as [171], reviews of various aspects of solitons and black holes in EYM can be found in [21, 66, 72, 152, 153, 166].

The outline of this article is as follows. In Sect. 2.2 we will outline $\mathfrak{su}(N)$ EYM theory, including our ansatz for the gauge field and the form of the field equations. We will then, in Sect. 2.3, briefly review some of the properties of the well-known asymptotically flat solutions of this theory. Our main focus in this article are asymptotically adS black holes, and we begin our discussion of these in Sect. 2.4 by reviewing the key features of the $\mathfrak{su}(2)$ EYM black holes in adS, before moving on to describe very recent work on $\mathfrak{su}(N)$, asymptotically adS, EYM black holes in Sect. 2.5. Our conclusions are presented in Sect. 2.6. Throughout this article the metric has signature $(-,+,+,+)$ and we use units in which $4\pi G = c = 1$.

2.2 $\mathfrak{su}(N)$ Einstein–Yang–Mills Theory

In this section we gather together all the formalism and field equations we shall require for our later study of black hole solutions.

2.2.1 Ansatz, Field Equations and Boundary Conditions

In this article we shall be interested in four-dimensional $\mathfrak{su}(N)$ EYM theory with a cosmological constant, described by the following action, given in suitable units:

$$S_{\mathrm{EYM}} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[R - 2\Lambda - \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}\right], \tag{2.1}$$

where R is the Ricci scalar of the geometry and Λ the cosmological constant. Here we have chosen the simplest type of EYM-like theory, many variants have been studied in the literature (see, for example, [171] for a selection of examples).

Varying the action (2.1) gives the field equations

$$\begin{aligned} T_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}; \\ 0 &= D_\mu F_\nu{}^\mu = \nabla_\mu F_\nu{}^\mu + \left[A_\mu, F_\nu{}^\mu\right]; \end{aligned} \tag{2.2}$$

where the YM stress–energy tensor is

$$T_{\mu\nu} = \mathrm{Tr}\, F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}\mathrm{Tr}\, F_{\lambda\sigma}F^{\lambda\sigma}. \tag{2.3}$$

In this article we consider only static, spherically symmetric black hole geometries, with metric given, in standard Schwarzschild-like co-ordinates, as

$$ds^2 = -\mu S^2\, dt^2 + \mu^{-1}\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2, \tag{2.4}$$

where the metric functions μ and S depend on the radial co-ordinate r only. In the presence of a negative cosmological constant $\Lambda < 0$, we write the metric function μ as

$$\mu(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}. \tag{2.5}$$

The most general, spherically symmetric, ansatz for the $\mathfrak{su}(N)$ gauge potential is [99]:

$$A = \mathscr{A}\, dt + \mathscr{B} dr + \frac{1}{2}\left(C - C^H\right) d\theta - \frac{i}{2}\left[\left(C + C^H\right)\sin\theta + D\cos\theta\right] d\phi, \tag{2.6}$$

where $\mathscr{A}$, $\mathscr{B}$, C and D are all $(N \times N)$ matrices and C^H is the Hermitian conjugate of C. The matrices $\mathscr{A}$ and $\mathscr{B}$ are purely imaginary, diagonal, traceless and depend only on the radial co-ordinate r. The matrix C is upper triangular, with non-zero entries only immediately above the diagonal:

$$C_{j,j+1} = \omega_j(r) e^{i\gamma_j(r)}, \tag{2.7}$$

for $j = 1, \cdots, N-1$. In addition, D is a constant matrix:

$$D = \mathrm{Diag}\,(N-1, N-3, \cdots, -N+3, -N+1)\,. \tag{2.8}$$

Here we are primarily interested only in purely magnetic solutions, so we set $\mathscr{A} \equiv 0$. We may also take $\mathscr{B} \equiv 0$ by a choice of gauge [99]. From now on we will assume that all the $\omega_j(r)$ are non-zero (see, for example, [69, 94–96] for the possibilities in asymptotically flat space if this assumption does not hold). In this case one of the Yang–Mills equations becomes [99]

$$\gamma_j = 0 \qquad \forall j = 1, \cdots, N-1. \tag{2.9}$$

Our ansatz for the Yang–Mills potential therefore reduces to

$$A = \frac{1}{2}\left(C - C^H\right) d\theta - \frac{i}{2}\left[\left(C + C^H\right)\sin\theta + D\cos\theta\right] d\phi, \tag{2.10}$$

where the only non-zero entries of the matrix C are

$$C_{j,j+1} = \omega_j(r). \tag{2.11}$$

The gauge field is therefore described by the $N-1$ functions $\omega_j(r)$. We comment that our ansatz (2.10) is by no means the only possible choice in $\mathfrak{su}(N)$ EYM. Techniques for finding *all* spherically symmetric $\mathfrak{su}(N)$ gauge potentials can be found in [6], where all irreducible models are explicitly listed for $N \leq 6$.

With the ansatz (2.10), there are $N-1$ non-trivial Yang–Mills equations for the $N-1$ functions ω_j:

$$r^2\mu\omega_j'' + \left(2m - 2r^3 p_\theta - \frac{2\Lambda r^3}{3}\right)\omega_j' + W_j\omega_j = 0 \tag{2.12}$$

for $j = 1, \ldots, N-1$, where a prime $'$ denotes $\mathrm{d}/\mathrm{d}r$,

$$p_\theta = \frac{1}{4r^4}\sum_{j=1}^{N}\left[\left(\omega_j^2 - \omega_{j-1}^2 - N - 1 + 2j\right)^2\right], \tag{2.13}$$

$$W_j = 1 - \omega_j^2 + \frac{1}{2}\left(\omega_{j-1}^2 + \omega_{j+1}^2\right), \tag{2.14}$$

and $\omega_0 = \omega_N = 0$. The Einstein equations take the form

$$m' = \mu G + r^2 p_\theta, \qquad \frac{S'}{S} = \frac{2G}{r}, \tag{2.15}$$

where

$$G = \sum_{j=1}^{N-1}\omega_j'^2. \tag{2.16}$$

Altogether, then, we have $N+1$ ordinary differential equations for the $N+1$ unknown functions $m(r)$, $S(r)$ and $\omega_j(r)$. The field equations (2.12) and (2.15) are invariant under the transformation

$$\omega_j(r) \rightarrow -\omega_j(r) \tag{2.17}$$

for each j independently, and also under the substitution:

$$j \rightarrow N - j. \tag{2.18}$$

We are interested in black hole solutions of the field equations (2.12) and (2.15). We assume there is a regular, non-extremal, black hole event horizon at $r = r_h$, where $\mu(r)$ has a single zero. This fixes the value of $m(r_h)$ to be:

$$2m(r_h) = r_h - \frac{\Lambda r_h^3}{3}. \tag{2.19}$$

However, the field equations (2.12) and (2.15) are singular at the black hole event horizon $r = r_h$ and at infinity $r \rightarrow \infty$. We therefore need to impose boundary conditions on the field variables $m(r)$, $S(r)$ and $\omega_j(r)$ at these singular points. When the cosmological constant Λ is zero, local existence of solutions of the field equations in neighbourhoods of these singular points has been rigorously proved [100, 125]. This proof can be extended to the case when the cosmological constant is negative [8, 11].

We assume that the field variables $\omega_j(r)$, $m(r)$ and $S(r)$ have regular Taylor series expansions about $r = r_h$:

$$
\begin{aligned}
m(r) &= m(r_h) + m'(r_h)\,(r - r_h) + O\,(r - r_h)^2\,; \\
\omega_j(r) &= \omega_j(r_h) + \omega_j'(r_h)\,(r - r_h) + O\,(r - r_h)^2\,; \\
S(r) &= S(r_h) + S'(r_h)\,(r - r_h) + O\,(r - r_h)\,.
\end{aligned} \tag{2.20}
$$

Setting $\mu(r_h) = 0$ in the Yang–Mills equations (2.12) fixes the derivatives of the gauge field functions at the horizon:

$$
\omega_j'(r_h) = -\frac{W_j(r_h)\omega_j(r_h)}{2m(r_h) - 2r_h^3 p_\theta(r_h) - \frac{2\Lambda r_h^3}{3}}. \tag{2.21}
$$

Therefore the expansions (2.20) are determined by the $N+1$ quantities $\omega_j(r_h)$, r_h, $S(r_h)$ for fixed cosmological constant Λ. For the event horizon to be non-extremal, it must be the case that

$$
2m'(r_h) = 2r_h^2 p_\theta(r_h) < 1 - \Lambda r_h^2, \tag{2.22}
$$

which weakly constrains the possible values of the gauge field functions $\omega_j(r_h)$ at the event horizon. Since the field equations (2.12) and (2.15) are invariant under the transformation (2.17), we may consider $\omega_j(r_h) > 0$ without loss of generality.

At infinity, we require that the field variables $\omega_j(r)$, $m(r)$ and $S(r)$ converge to constant values as $r \to \infty$ and have regular Taylor series expansions in r^{-1} near infinity:

$$
m(r) = M + O\left(r^{-1}\right); \qquad S(r) = 1 + O\left(r^{-1}\right); \qquad \omega_j(r) = \omega_{j,\infty} + O\left(r^{-1}\right). \tag{2.23}
$$

If the spacetime is asymptotically flat, with $\Lambda = 0$, then the values of $\omega_{j,\infty}$ are constrained to be

$$
\omega_{j,\infty} = \pm\sqrt{j(N-j)}. \tag{2.24}
$$

This condition means that the asymptotically flat black holes have no magnetic charge at infinity, or, in other words, these solutions have no global magnetic charge. Therefore, at infinity, they are indistinguishable from Schwarzschild black holes. However, if the cosmological constant is non-zero, so that the geometry approaches (a)dS at infinity, then there are no *a priori* constraints on the values of $\omega_{j,\infty}$. In general, therefore, the (a)dS black holes will be magnetically charged. It should be noted that the boundary conditions in the case when the cosmological constant Λ is positive are more complex, as there is a cosmological horizon between the event horizon and infinity.

2.2.2 Some "trivial" Solutions

Although the field equations (2.12) and (2.15) are highly non-linear and rather complicated, they do have some trivial solutions which can easily be written down:

Schwarzschild(-(a)dS) Setting

$$\omega_j(r) \equiv \pm\sqrt{j(N-j)} \tag{2.25}$$

for all j gives the Schwarzschild(-(a)dS) black hole with

$$m(r) = M = \text{constant} \tag{2.26}$$

We note that, by setting $M = 0$, pure Minkowski ($\Lambda = 0$) or (a)dS ($\Lambda \neq 0$) space is also a solution.

Reissner–Nordström(-(a)dS) Setting

$$\omega_j(r) \equiv 0 \tag{2.27}$$

for all j gives the Reissner–Nordström(-(a)dS) black hole with metric function

$$\mu(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}, \tag{2.28}$$

where the magnetic charge Q is fixed by

$$Q^2 = \frac{1}{6}N(N+1)(N-1). \tag{2.29}$$

Embedded $\mathfrak{su}(2)$ *solutions* For our later numerical and analytic work, an additional special class of solutions turns out to be extremely useful. We begin by setting

$$\omega_j(r) = \pm\sqrt{j(N-j)}\,\omega(r) \qquad \forall j = 1,\ldots,N-1, \tag{2.30}$$

then follow [100] and define

$$\lambda_N = \sqrt{\frac{1}{6}N(N-1)(N+1)}, \tag{2.31}$$

and then rescale the field variables as follows:

$$\begin{aligned} R = \lambda_N^{-1} r; \qquad \tilde{\Lambda} = \lambda_N^2 \Lambda; \qquad \tilde{m}(R) = \lambda_N^{-1} m(r); \\ \tilde{S}(R) = S(r); \qquad \tilde{\omega}(R) = \omega(r). \end{aligned} \tag{2.32}$$

Note that we rescale the cosmological constant Λ (this is not necessary in [100] as there $\Lambda = 0$). The field equations satisfied by $\tilde{m}(R)$, $\tilde{S}(R)$ and $\tilde{\omega}(R)$ are then

$$\begin{aligned} \frac{d\tilde{m}}{dR} &= \mu\tilde{G} + R^2\tilde{p}_\theta; \\ \frac{1}{\tilde{S}}\frac{d\tilde{S}}{dR} &= -\frac{2\tilde{G}}{R}; \\ 0 &= R^2\mu\frac{d^2\tilde{\omega}}{dR^2} + \left[2\tilde{m} - 2R^3\tilde{p}_\theta - \frac{2\tilde{\Lambda}R^3}{3}\right]\frac{d\tilde{\omega}}{dR} + \left[1-\tilde{\omega}^2\right]\tilde{\omega}; \end{aligned} \tag{2.33}$$

where we now have

$$\mu = 1 - \frac{2\tilde{m}}{R} - \frac{\tilde{\Lambda} R^2}{3}, \tag{2.34}$$

and

$$\tilde{G} = \left(\frac{d\tilde{\omega}}{dR} \right)^2, \qquad \tilde{p}_\theta = \frac{1}{2R^4} \left(1 - \tilde{\omega}^2 \right)^2. \tag{2.35}$$

The (2.33) are precisely the $\mathfrak{su}(2)$ EYM field equations. Furthermore, the boundary conditions (2.20) and (2.23) also reduce to those for the $\mathfrak{su}(2)$ case.

2.2.3 Dyonic Field Equations

As will be discussed in Sect. 2.4.3, if either $N > 2$ or we have a negative cosmological constant Λ, then we do not need to restrict ourselves to considering only purely magnetic equilibrium gauge potentials. If the electric part of the gauge potential (2.6), $\mathscr{A}$, is non-zero, there is still sufficient gauge freedom to set $\mathscr{B} = 0$ in (2.6) [99]. Then, provided none of the ω_j vanish identically, one of the Yang–Mills equations again tells us that all the γ_j are identically zero. Following [99] it is convenient to define new real variables $\alpha_j(r)$ by

$$\mathscr{A}_{jj} = i \left[-\frac{1}{N} \sum_{k=1}^{j-1} k \alpha_k + \sum_{k=j}^{N-1} \left(1 - \frac{k}{N} \right) \alpha_k \right] \tag{2.36}$$

so that the matrix $\mathscr{A}$ is automatically purely imaginary, diagonal and traceless. In this case the Yang–Mills equations (2.12) now take the form [99]

$$r^2 \mu \omega_j'' + \left(2m - 2r^3 p_\theta - \frac{2\Lambda r^3}{3} \right) \omega_j' + W_j \omega_j + \frac{\mu}{r^2} \alpha_j^2 \omega_j = 0, \tag{2.37}$$

and there are additional Yang–Mills equations for the α_j, namely [99]

$$\left[r^2 S^{-1} \left(\mu S \alpha_j \right)' \right]' = 2\alpha_j \omega_j^2 - \alpha_{j-1} \omega_{j-1}^2 - \alpha_{j+1} \omega_{j+1}^2. \tag{2.38}$$

The Einstein equations retain the form (2.15) but the quantities p_θ (2.13) and G (2.16) now read [99]

$$\begin{aligned} p_\theta &= \frac{1}{4r^4} \sum_{j=1}^{N} \left[\left(\omega_j^2 - \omega_{j-1}^2 - N - 1 + 2j \right)^2 + \left(\frac{r^2}{S} \left(\mu S \mathscr{A}_{jj} \right)' \right)^2 \right] \\ G &= \sum_{j=1}^{N-1} \left[\omega_j'^2 + \alpha_j^2 \omega_j^2 \right]. \end{aligned} \tag{2.39}$$

2.2.4 Perturbation Equations

We are also interested in the stability of the static, equilibrium solutions. For simplicity, we consider only linear, spherically symmetric perturbations of the purely magnetic solutions. We return to the general gauge potential of the form (2.6), and the metric (2.4), where now all functions depend on time t as well as r. There is still sufficient gauge freedom to enable us to set $\mathscr{A} \equiv 0$. This choice of gauge is particularly useful as then we shall shortly see that the perturbation equations decouple into two sectors, the "gravitational" and "sphaleronic" sectors [102]. We consider perturbations about the equilibrium solutions of the form

$$\omega_j(t,r) = \omega_j(r) + \delta\omega_j(t,r), \tag{2.40}$$

where $\omega_j(r)$ are the equilibrium functions and $\delta\omega_j(t,r)$ are the linear perturbations. There are similar perturbations for the other equilibrium quantities m and S, and in addition we have the perturbations $\delta\gamma_j(t,r)$ and $\delta\beta_j(t,r)$, the latter being the entries along the diagonal of the matrix $\mathscr{B}$ (2.6):

$$\mathscr{B} = \mathrm{Diag}\left(i\delta\beta_1, \cdots, i\delta\beta_N\right). \tag{2.41}$$

Note that the $\delta\beta_j$ are not independent because the matrix $\mathscr{B}$ is traceless, so

$$\delta\beta_1 + \cdots + \delta\beta_N = 0, \tag{2.42}$$

but it simplifies the derivation of the perturbation equations to retain all the $\delta\beta_j$ for the moment. We ignore all terms involving squares or higher powers of the perturbations. The full derivation of the perturbation equations is highly involved and the details will be presented elsewhere [11]. Instead here we summarize the key features of the perturbation equations. As usual, we will employ the "tortoise" co-ordinate r_*, defined by

$$\frac{dr_*}{dr} = \frac{1}{\mu S}, \tag{2.43}$$

where μ and S are the equilibrium metric functions.

2.2.4.1 Sphaleronic Sector

The sphaleronic sector consists of the $2N-1$ perturbations $\delta\beta_j$, $j = 1, \ldots, N$ and $\delta\gamma_j$, $j = 1, \ldots, N-1$. We define new variables $\delta\Phi_j$ by

$$\delta\Phi_j = \omega_j \delta\gamma_j. \tag{2.44}$$

The perturbation equations for the sphaleronic sector arise solely from the Yang–Mills equations, and comprise

$$
\begin{aligned}
\delta\ddot{\beta}_j = {} & \frac{S}{r^2}\left[\omega_{j-1}\partial_{r_*}\left(\delta\Phi_{j-1}\right) - \omega_j\partial_{r_*}\left(\delta\Phi_j\right)\right] \\
& + \frac{S}{r^2}\left[\left(\partial_{r_*}\omega_j\right)\delta\Phi_j - \left(\partial_{r_*}\omega_{j-1}\right)\delta\Phi_{j-1}\right] \\
& + \frac{\mu S^2}{r^2}\left[\omega_j^2\left(\delta\beta_{j+1} - \delta\beta_j\right) - \omega_{j-1}^2\left(\delta\beta_j - \delta\beta_{j-1}\right)\right];
\end{aligned}
\tag{2.45}
$$

$$
\begin{aligned}
\delta\ddot{\Phi}_j = {} & \partial_{r_*}^2\left(\delta\Phi_j\right) - \frac{1}{\omega_j}\left(\partial_{r_*}^2\omega_j\right)\delta\Phi_j + \mu S\omega_j\partial_{r_*}\left(\delta\beta_j - \delta\beta_{j+1}\right) \\
& + \left[\mu\left(\partial_{r_*}S\right)\omega_j + \left(\partial_{r_*}\mu\right)S\omega_j + 2\mu S\left(\partial_{r_*}\omega_j\right)\right]\left(\delta\beta_j - \delta\beta_{j+1}\right);
\end{aligned}
\tag{2.46}
$$

together with the *Gauss constraint*

$$
0 = \partial_{r_*}\left(\delta\dot{\beta}_j\right) + \left[\frac{2\mu S}{r} - \frac{\partial_{r_*}S}{S}\right]\delta\dot{\beta}_j + \frac{S}{r^2}\left[\omega_j\delta\dot{\Phi}_j + \omega_{j-1}\delta\dot{\Phi}_{j-1}\right],
\tag{2.47}
$$

where a dot denotes $\partial/\partial t$. It is important to note that the cosmological constant Λ only appears in these equations through the metric function μ (2.5), and therefore the perturbation equations (2.45) and (2.46) and the Gauss constraint (2.47) have exactly the same form as derived in [47] for arbitrary gauge groups in asymptotically flat space.

2.2.4.2 Gravitational Sector

The gravitational sector consists of the perturbations of the metric functions $\delta\mu$ and δS as well as the perturbations of the remaining gauge field functions $\delta\omega_j$. Both the Einstein equations and the remaining Yang–Mills equations are involved in this sector. For an arbitrary gauge group and asymptotically flat space, the perturbation equations in this sector have been considered in [47]. In asymptotically adS, we also find that the metric perturbations can be eliminated to give a set of equations governing the perturbations $\delta\omega_j$, which can be written in matrix form

$$
\underline{\delta\ddot{\omega}} = \partial_{r_*}^2\left(\underline{\delta\omega}\right) + \mathscr{M}_G\underline{\delta\omega},
\tag{2.48}
$$

where $\underline{\delta\omega} = \left(\delta\omega_1, \ldots, \delta\omega_{N-1}\right)^T$ and the $(N-1)\times(N-1)$ matrix $\mathscr{M}_G$ has entries

$$
\begin{aligned}
\mathscr{M}_{G,j,j} = {} & \frac{\mu S^2}{r^2}\left[W_j - 2\omega_j^2\right] + \frac{4}{\mu S r}\Upsilon\left(\partial_{r_*}\omega_j\right)^2 + \frac{8S}{r^3}W_j\omega_j\left(\partial_{r_*}\omega_j\right); \\
\mathscr{M}_{G,j,j+1} = {} & \frac{\mu S^2}{r^2}\omega_j\omega_{j+1} + \frac{4}{\mu S r}\Upsilon\left(\partial_{r_*}\omega_j\right)\left(\partial_{r_*}\omega_{j+1}\right) \\
& + \frac{8S}{r^3}\left[W_j\omega_j\left(\partial_{r_*}\omega_{j+1}\right) + W_{j+1}\omega_{j+1}\left(\partial_{r_*}\omega_j\right)\right]; \\
\mathscr{M}_{G,j,k} = {} & \frac{4}{\mu S r}\Upsilon\left(\partial_{r_*}\omega_j\right)\left(\partial_{r_*}\omega_k\right) + \frac{8S}{r^3}\left[W_j\omega_j\left(\partial_{r_*}\omega_k\right) + W_k\omega_k\left(\partial_{r_*}\omega_j\right)\right];
\end{aligned}
\tag{2.49}
$$

where $k \neq j, j+1$, and Υ is given in terms of the equilibrium metric functions μ and S as follows:

$$\Upsilon = \frac{1}{\mu}\partial_{r_*}\mu + \frac{1}{S}\partial_{r_*}S + \frac{\mu S}{r}. \tag{2.50}$$

2.3 Asymptotically Flat/de Sitter Solutions for $\mathfrak{su}(N)$ EYM

We now turn to black hole solutions of the EYM field equations, beginning by briefly reviewing some of the key features of solutions in asymptotically flat or asymptotically de Sitter space.

2.3.1 Asymptotically Flat, Spherically Symmetric $\mathfrak{su}(2)$ Solutions

Apart from the trivial solutions given above (2.25) and (2.27), the first black hole solutions of the EYM field equations were found by Yasskin [182], and correspond to embedding the Reissner–Nordström electromagnetic gauge field into a higher–dimensional gauge group. The metric of these solutions is still Reissner–Nordström. Yasskin conjectured that his solutions were the only ones possible. This conjecture was only shown to be false 25 years later [19, 101, 168, 169]. That the discovery of hairy black holes in $\mathfrak{su}(2)$ EYM took so long may be attributed to the conjecture that there were no soliton solutions in this model. This conjecture is based on the fact that there are no solitons in pure gravity (see, for example, [78, 104]); no solitons in Einstein–Maxwell theory [77], no pure YM solitons in flat spacetime [53, 56] and no EYM solitons in three spacetime dimensions [57]. However, once Bartnik and McKinnon [7] had discovered non-trivial EYM solitons in four-dimensional spacetime, Yasskin's no-hair conjecture for EYM theory was quickly shown to be false [19].

For $\mathfrak{su}(2)$ EYM, it has been shown [23, 62, 67] that non-trivial solutions (i.e., solutions in which the gauge field is not essentially Abelian) must have a purely magnetic gauge potential, which is described by a single gauge field function $\omega(r)$ (2.10). Note that the ansatz (2.10) for $\mathfrak{su}(2)$ is not the same as the Witten ansatz [179] which was used in the original papers [7, 19], but it gives equivalent field equations. In this case the $\mathfrak{su}(2)$ EYM equations have the form

$$\begin{aligned}
\frac{dm}{dr} &= \left(1-\frac{2m}{r}\right)\left(\frac{d\omega}{dr}\right)^2 + \frac{1}{2r^2}\left(1-\omega^2\right)^2; \\
\frac{1}{S}\frac{dS}{dr} &= -\frac{2}{r}\left(\frac{d\omega}{dr}\right)^2; \\
0 &= r^2\left(1-\frac{2m}{r}\right)\frac{d^2\omega}{dr^2} + \left[2m - \frac{\left(1-\omega^2\right)^2}{r}\right]\frac{d\omega}{dr} + \left[1-\omega^2\right]\omega.
\end{aligned} \tag{2.51}$$

It is the highly non-linear nature of these equations which allows for non-trivial soliton and hairy black hole solutions, which may be thought of heuristically as arising from a balancing of the gravitational and gauge field interactions (see [82] for a recent discussion). The non-linear nature of the equations also means, however, that (apart from the solutions for the Yang–Mills field on a fixed Schwarzschild metric [28, 34]) solutions can only be found numerically.

The numerical work in [7, 19, 101, 168, 169] found discrete families of solutions [156], indexed by the event horizon radius r_h (with $r_h = 0$ for solitons) and n, the number of zeros of the single gauge field function ω, each pair (r_h, n) identifying a solution of the field equations. A key feature of the solutions is that $n > 0$, so that the gauge field function must have at least one zero (or "node"). Later analytic work [29, 149–151] rigorously proved these numerical features. The black holes are "hairy" in the sense that they have no magnetic charge [23, 62, 67] and are therefore indistinguishable at infinity from a standard Schwarzschild black hole. However, the "hair", that is, the non-trivial structure in the matter fields, extends some way out from the event horizon, leading to the "no-short-hair" conjecture [122].

Although initially controversial [20, 24, 152, 173], rapidly it was accepted that both the soliton [154] and the black hole solutions [155] are unstable. This instability is not unexpected if we consider the solutions as arising from a balancing of the gauge field and gravitational interactions. Studies of the non-linear stability of the solutions [183, 184] reveal that the gauge field "hair" either radiates away to infinity or falls down the black hole event horizon, leaving, as the end-point, a bald Schwarzschild black hole. Due to this instability, the black holes, while they violate the "letter" of the no-hair conjecture, may be thought of as not contradicting its "spirit", and one might be led to conjecture that all *stable* black holes are fixed by their mass, angular momentum and conserved charges.

Originally these hairy black holes were shown to be unstable using numerical techniques [155] but the instability can also be shown analytically [68, 170]. In the $\mathfrak{su}(2)$ case, the perturbation equations (2.45), (2.46) and (2.48) simplify considerably. The sphaleronic sector reduces to a single equation (see Sect. 2.4.2 below for further details)

$$-\ddot{\zeta} = -\partial_{r_*}^2 \zeta + \left[\frac{\mu S^2}{r^2} \left(1 + \omega^2\right) + \frac{2}{\omega^2} \left(\frac{d\omega}{dr_*} \right)^2 \right] \zeta, \tag{2.52}$$

while, on eliminating the metric perturbations, the gravitational sector also has just one equation:

$$\begin{aligned} -\delta\ddot{\omega} = & -\partial_{r_*}^2 (\delta\omega) \\ & + \frac{\mu S^2}{r^2} \left[3\omega^2 - 1 - 4r\omega'^2 \left(\frac{1}{r} - \frac{\left(1-\omega^2\right)^2}{r^3} \right) + \frac{8}{r}\omega\omega' \left(\omega^2 - 1\right) \right] \delta\omega. \end{aligned} \tag{2.53}$$

The instability has been compared to that of the flat-space Yang–Mills sphaleron [170], which has a single unstable mode. The situation is slightly more complicated here, due to the two sectors of perturbations. The sphaleronic sector certainly, as its name suggests, mimics the perturbations of the flat-space sphaleron. It can be shown [167] that the number of instabilities in the sphaleronic sector equals n, the number of zeros of the gauge field function ω. The same is true in the gravitational sector, as conjectured in [102] and can be shown using catastrophe theory, by considering the more general EYM–Higgs solutions [115]. The above concerns only spherically symmetric perturbations. It is known that the flat-space sphaleron has instabilities only in the spherically symmetric sector [4]. Extending this to the $\mathfrak{su}(2)$ EYM black holes requires complicated analysis [143], using a curvature-based formalism developed in [43, 144, 145].

Using the isolated horizons formalism, these "hairy" black holes can be interpreted as bound states of ordinary black holes with the Bartnik–MacKinnon solitons [3, 54, 55]. In particular, the soliton masses are given in terms of the masses of the corresponding black holes [55], and the instability of the colored black holes arises naturally from the instability of the corresponding solitons [3, 54].

Since these initial discoveries a plethora of new, asymptotically flat, hairy black hole solutions have been found in Einstein–Yang–Mills theory and its variants (see [171] for a review of those solutions discovered prior to 1999). Most of these are, indeed, unstable. However, there are notable exceptions, including (a) the Skyrme black hole [22, 58, 80, 81] where the existence of an integer-valued topological winding number renders the solutions stable, (b) Einstein–Yang–Mills–Higgs black holes in the limit of infinitely strong coupling of the Higgs field [1] and (c) a particular branch of Einstein-non-Abelian-Proca black holes [73, 110, 159, 160, 163]. We will not consider additional matter fields further in this article.

2.3.2 Non-spherically Symmetric, Asymptotically Flat $\mathfrak{su}(2)$ Solutions

One of the surprising aspects of the failure of black hole uniqueness in EYM is that almost every step in the uniqueness theorem in Einstein–Maxwell theory has a counterexample in EYM (see [79] for detailed discussions on this topic, and [45, 128, 153, 156, 157] for examples of some results from Einstein–Maxwell theory which do generalize). An important example of this is Israel's theorem [86, 87], which states that the geometry outside the event horizon of a static black hole must be spherically symmetric. This is not true in EYM: there are static black hole solutions which are not spherically symmetric but only axisymmetric [90] (in more general matter models, static black holes do not necessarily possess any symmetries at all [138, 139]). These solutions are found numerically by writing the metric in isotropic co-ordinates

$$ds^2 = -f(r,\theta)\,dt^2 + \frac{m(r,\theta)}{f(r,\theta)}dr^2 + \frac{m(r,\theta)r^2}{f(r,\theta)}d\theta^2 + \frac{L(r,\theta)r^2\sin^2\theta}{f(r,\theta)}d\phi^2, \quad (2.54)$$

and using the following ansatz for the $\mathfrak{su}(2)$ gauge field [137]

$$
\begin{aligned}
A = \frac{1}{2r}\Big\{ & \tau_\phi^p \left[H_1(r,\theta)\,dr + (1 - H_2(r,\theta))\,r\,d\theta\right] \\
& - p\left[\tau_r^p H_3(r,\theta) + \tau_\theta^p\,(1 - H_4(r,\theta))\right] r\sin\theta\,d\phi \Big\},
\end{aligned} \tag{2.55}
$$

where

$$
\begin{aligned}
\tau_r^p &= \underline{\tau}.\left(\sin\theta\cos p\phi, \sin\theta\sin p\phi, \cos\theta\right), \\
\tau_\theta^p &= \underline{\tau}.\left(\cos\theta\cos p\phi, \cos\theta\sin p\phi, -\sin\theta\right), \\
\tau_\phi^p &= \underline{\tau}.\left(-\sin p\phi, \cos p\phi, 0\right),
\end{aligned} \tag{2.56}
$$

with

$$
\underline{\tau} = (\tau_x, \tau_y, \tau_z), \tag{2.57}
$$

where τ_x, τ_y, τ_z are the usual generators of $\mathfrak{su}(2)$. Here, p is a winding number, with $p = 1$ corresponding to spherically symmetric solutions (with the gauge potential written in a different form to that we have used in (2.10)). Substituting the ansatz into the field equations gives a complicated set of partial differential equations, solutions of which are exhibited in [90]. Static, axisymmetric soliton solutions also exist [65, 85, 91].

It is less surprising that rotating black holes also exist in this model [92, 93], generalizing the Kerr–Newman metric (as predicted in [156]). These solutions are indexed by the winding number p (2.56) and a node number n. They carry no magnetic charge, but all have non-zero electric charge [156, 157]. The question of whether there are rotating solitons in pure $\mathfrak{su}(2)$ EYM has yet to be conclusively settled, however. Rotating soliton solutions have been found in EYM–Higgs theory [127], but not in pure EYM theory. Although rotating solitons are predicted perturbatively [44], the consensus in the literature is now that it seems unlikely that rotating soliton solutions do exist [17].

2.3.3 Asymptotically Flat $\mathfrak{su}(N)$ *Solutions*

We shall next consider generalizations of the $\mathfrak{su}(2)$ YM gauge group. The simplest such generalization is to consider $\mathfrak{su}(N)$ EYM. The results of [62, 67] do not extend to this larger gauge group, and it is possible to have solutions with electric charge [69], which correspond to a superposition of electrically charged Reissner–Nordström and the $\mathfrak{su}(2)$ EYM black holes. Numerical solutions of the field equations have been found in the following papers: [69, 94–96]. As N increases, the possible structures of the gauge field potential (2.6) become ever more complicated. A method for computing all spherically symmetric $\mathfrak{su}(N)$ gauge field potentials is given in [6], where all the irreducible possibilities are enumerated for $N \leq 6$. As in the $\mathfrak{su}(2)$ case, black hole solutions are found at discrete points in the parameter space $\{\omega_j(r_h), j = 1 \ldots N-1\}$.

There is comparatively little analytic work for more general gauge groups. Local existence of solutions of the field equations (2.12) and (2.15) near the black hole event horizon and at infinity has been proven for gauge group $\mathfrak{su}(N)$ [100], and subsequently extended to arbitrary compact gauge group [124, 125]. The existence of non-trivial black hole solutions to the field equations has been proven rigorously only in the $\mathfrak{su}(3)$ case [140, 141], although there are arguments that hairy black hole solutions exist for all N [116]. In the $\mathfrak{su}(3)$ case, Ruan [140, 141] has proved that there are infinitely many hairy black hole solutions, indexed by the numbers of zeros (n_1, n_2), respectively, of the two gauge field functions (ω_1, ω_2). Furthermore, provided that the radius of the event horizon is sufficiently large, there is a black hole solution for any combination of (n_1, n_2). The global properties of the solutions for arbitrary compact gauge group are studied in [126]. However, it will come as no surprise to learn that all these solutions, in asymptotically flat space, and for any compact gauge group, are unstable [46, 47]. To show instability it is sufficient to find a single unstable mode, and therefore the work in [46, 47] studies the simpler, sphaleronic sector of perturbations (see Sect. 2.2.4).

2.3.4 Asymptotically de Sitter $\mathfrak{su}(2)$ *EYM Solutions*

Another natural generalization of asymptotically flat $\mathfrak{su}(2)$ EYM is the inclusion of a non-zero cosmological constant Λ. When the cosmological constant is positive, soliton [172] and black hole [164] $\mathfrak{su}(2)$ EYM solutions have been found (other numerical solutions are presented in [41, 119]). These solutions possess a cosmological horizon and approach de Sitter space at infinity (for a complete classification of the possible spacetime structures, see [30]). The phase space of solutions is again discrete, and the single gauge field function ω must have at least one zero. Unsurprisingly, these solutions again turn out to be unstable [42, 63, 164]. Given this instability, the asymptotically de Sitter solutions have received rather less attention in the literature, but some analytic work can be found in [105–107].

2.4 Asymptotically anti-de Sitter Solutions for $\mathfrak{su}(2)$ EYM

We now turn to the main focus of this article: asymptotically anti-de Sitter solutions. We begin by reviewing some of the properties of black holes in $\mathfrak{su}(2)$ EYM.

2.4.1 Spherically Symmetric, Asymptotically adS, $\mathfrak{su}(2)$ *EYM Solutions*

Black hole solutions of $\mathfrak{su}(2)$ EYM with a negative cosmological constant were first studied in [175], and subsequently in [25, 26]. The field equations now take the form

$$\frac{dm}{dr} = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)\left(\frac{d\omega}{dr}\right)^2 + \frac{1}{2r^2}\left(1-\omega^2\right)^2;$$
$$\frac{1}{S}\frac{dS}{dr} = -\frac{2}{r}\left(\frac{d\omega}{dr}\right)^2;$$
$$0 = r^2\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)\frac{d^2\omega}{dr^2} + \left[2m - \frac{2\Lambda r^3}{3} - \frac{\left(1-\omega^2\right)^2}{r}\right]\frac{d\omega}{dr} + \left[1-\omega^2\right]\omega. \tag{2.58}$$

The inclusion of a negative cosmological constant means that boundary conditions at infinity (2.23) are considerably less stringent than in the asymptotically flat case; it is therefore unsurprising that it is easier to find solutions in asymptotically adS.

The space of solutions in adS is very different to that in asymptotically flat space. Instead of finding solutions at discrete values of $\omega(r_h)$, solutions exist in continuous, open intervals. Furthermore, for sufficiently large $|\Lambda|$, we now find solutions in which the single gauge field function $\omega(r)$ has no zeros. A typical example of such a solution is shown in Fig. 2.1, further examples can be found in [175]. These properties of the space of solutions of the (2.58) are proved in [175].

We now examine the structure of the space of solutions, more details of which can be found in [8, 9, 175]. There are three parameters describing the solutions, r_h, Λ and $\omega(r_h)$. In order to plot two-dimensional figures, we fix either r_h or Λ and vary the other two quantities. For $\mathfrak{su}(2)$ black holes, the constraint (2.22) on the value of the gauge field function at the event horizon reads

$$\left(\omega(r_h)^2 - 1\right)^2 < r_h^2\left(1 - \Lambda r_h^2\right). \tag{2.59}$$

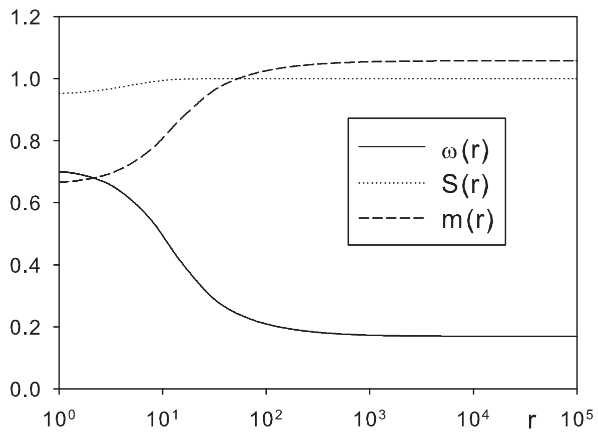

Fig. 2.1 An example of an $\mathfrak{su}(2)$ EYM black hole in adS in which the gauge field function $\omega(r)$ has no zeros. Here, $\Lambda = -1$, $r_h = 1$ and $\omega(r_h) = 0.7$

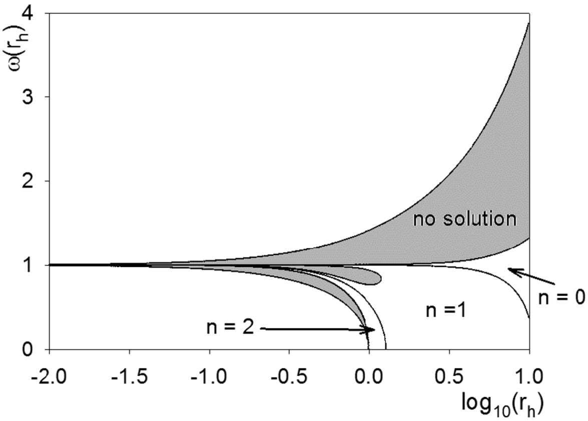

Fig. 2.2 The space of $\mathfrak{su}(2)$ black hole solutions when $\Lambda = -0.01$, for varying r_h. The *shaded region* indicates values of the gauge field function $\omega(r_h)$ at the event horizon for which the constraint (2.59) is satisfied, but for which we find no well-behaved black hole solution. The number of zeros n of the gauge field function ω are indicated in those regions of the phase space where we find black hole solutions. Elsewhere on the diagram, the constraint (2.59) is not satisfied. Between the region where $n = 2$ and the *shaded region* we find black hole solutions with $n = 3$, 4 and 5, but these regions are too small to indicate on the graph. Taken from [9]

Whether we are varying r_h or Λ, we perform a scan over all values of ω_h which satisfy (2.59). First, we show in Fig. 2.2 the space of black hole solutions for fixed $\Lambda = -0.01$ and varying event horizon radius r_h. The outermost curves in Fig. 2.2 are where the inequality (2.59) is saturated. Immediately inside these curves we have a shaded region, which represents values of $(r_h, \omega(r_h))$ for which the constraint (2.59) is satisfied, but for which we are unable to find black hole solutions which remain regular all the way out to infinity. Where we do find solutions, we indicate in Fig. 2.2 the number of zeros of the gauge field function $\omega(r)$. The solution for which $\omega(r_h) = 1$ is simply the Schwarzschild-adS black hole, while that for $\omega(r_h) = 0$ is the magnetically charged Reissner–Nordström-adS black hole (see Sect. 2.2.2). As $r_h \to 0$, the constraint (2.59) implies that $\omega(r_h) \to 1$, as can be seen in Fig. 2.2. The black hole solutions become solitons in this limit. However, for this value of Λ, there are different soliton solutions, with ω having different numbers of zeros [31], a feature which is not readily apparent from Fig. 2.2. We find similar behavior on varying r_h for different values of Λ.

If we now fix the event horizon radius to be $r_h = 1$ and vary Λ, the solution space is shown in Fig. 2.3, with a close-up for smaller values of $|\Lambda|$ in Fig. 2.4.

Again, in Figs. 2.3 and 2.4 we have shaded those regions where the constraint (2.59) is satisfied, but no regular black hole solutions could be found. Where we do find solutions, the number of zeros of the gauge field function $\omega(r)$ is indicated in the figures. As $\Lambda \to 0$, the phase space breaks up into discrete points, which correspond to the asymptotically flat "colored" $\mathfrak{su}(2)$ black holes described in Sect. 2.3.1 [19]. For sufficiently large $|\Lambda|$, we find solutions in which the gauge field function has no zeros.

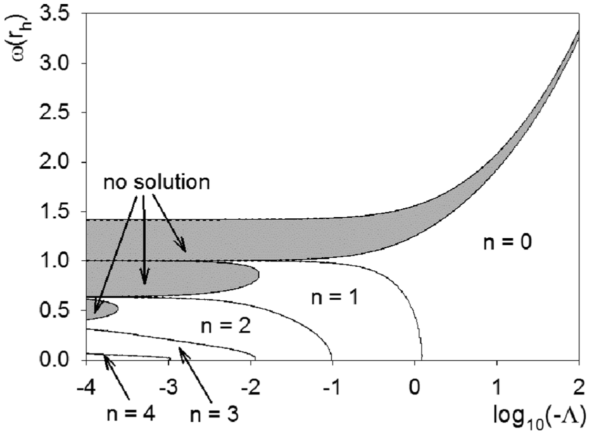

Fig. 2.3 Phase space of $\mathfrak{su}(2)$ black holes with $r_h = 1$ and varying Λ. The *shaded region* indicates values of the gauge field function $\omega(r_h)$ at the event horizon for which the constraint (2.59) is satisfied, but for which we find no well-behaved black hole solution. The number of zeros n of the gauge field function ω are indicated in those regions of the phase space where we find black hole solutions. Elsewhere on the diagram, the constraint (2.59) is not satisfied. As well as the regions where $n = 0, \ldots, 4$ as marked on the diagram, we find a *small region* in the *bottom left* of the plot where $n = 5$. This region is too small to indicate on the current figure, but can be seen in Fig. 2.4. Taken from [9]

The spectrum of black hole solutions (that is, the relationship between the mass M and magnetic charge Q of the black holes) was first studied in [26]. We plot in Fig. 2.5 the black hole mass versus magnetic charge for black holes with $r_h = 1$ and varying values of Λ (cf. Fig. 8 in [26]). For large values of $|\Lambda|$, there are only nodeless solutions and the spectrum is simple, with the black holes being uniquely

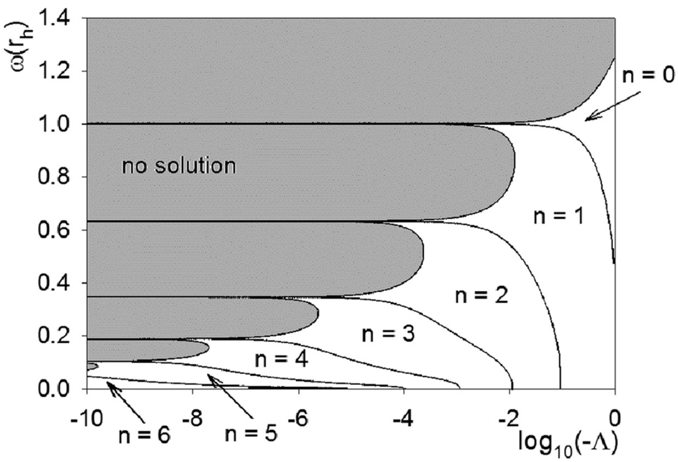

Fig. 2.4 Close-up of the phase space of $\mathfrak{su}(2)$ black holes with $r_h = 1$ and smaller values of Λ. In the *bottom left* of the plot there is a *small region* of solutions for which $n = 7$, but the region is too small to be visible. Taken from [9]

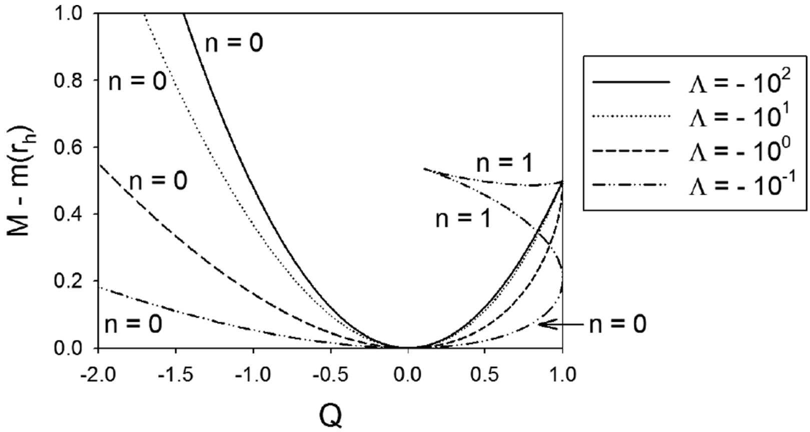

Fig. 2.5 Black hole mass M and magnetic charge Q for $\mathfrak{su}(2)$ EYM black holes with $r_h = 1$ and varying Λ (cf. Fig. 8 in [26])

specified by Λ, r_h and Q_M. As $|\Lambda|$ decreases, the spectrum becomes more complicated. For example, looking at the $\Lambda = -0.1$ curve in Fig. 2.5 we see that a branch structure emerges. The lower M curve for $\Lambda = -0.1$ consists of $n = 0$ (nodeless) solutions, and extends from negative Q up to $Q = 1$. When $Q = 1$, a branch of $n = 1$ solutions appears, which have larger mass. As Q decreases along this branch of solutions, the mass M increases, until a bifurcation point is reached and a second branch of $n = 1$ solutions appears, with even larger mass, and with the charge increasing as M increases. For smaller values of $|\Lambda|$, we find ever more complicated spectra, which appear to become "fractal" as $|\Lambda| \to 0$ [26, 114]. In view of the catastrophe theory analysis of other hairy black hole solutions [159, 160, 163], one might anticipate that the stability of the solutions changes at the points in the spectrum where two branches of solutions meet, but this has yet to be fully investigated in the literature (see [31] for an in-depth stability analysis of the soliton solutions). We therefore next consider the stability of these black holes.

2.4.2 Stability of the Spherically Symmetric Solutions

As discussed in Sect. 2.3.1, for the asymptotically flat $\mathfrak{su}(2)$ EYM black holes, it has been shown that the number of instabilities is twice the number of zeros of the gauge field function $\omega(r)$. Therefore, one might anticipate that at least some solutions when $\omega(r)$ has no zeros could be stable. For the $\mathfrak{su}(2)$ EYM case, the perturbation equations (2.45), (2.46) and (2.48) simplify considerably. In the sphaleronic sector, there is a single $\delta\Phi$ (2.44) and two further perturbations $\delta\beta_1$, $\delta\beta_2$, although these are not independent (2.42), so we may consider just $\delta\nu = \delta\beta_2 - \delta\beta_1$. The sphaleronic sector perturbations equations (2.45) and (2.46) then reduce to

$$\delta\ddot{v} = \frac{2S}{r^2}\left[\omega\partial_{r_*}(\delta\Phi) - (\partial_{r_*}\omega)\,\delta\Phi\right] - \frac{2\mu S^2}{r^2}\omega^2\delta v; \tag{2.60}$$

$$\begin{aligned}\delta\ddot{\Phi} = {} & \partial_{r_*}^2(\delta\Phi) - \frac{1}{\omega}\left(\partial_{r_*}^2\omega\right)\delta\Phi - \mu S\omega\partial_{r_*}(\delta v) \\ & + \left[\mu\left(\partial_{r_*}S\right)\omega + \left(\partial_{r_*}\mu\right)S\omega - 2\mu S\left(\partial_{r_*}\omega\right)\right]\delta v;\end{aligned} \tag{2.61}$$

and the Gauss constraint (2.47) is now

$$0 = \partial_{r_*}(\delta\dot{v}) + \left[\frac{2\mu S}{r} - \frac{\partial_{r_*}S}{S}\right]\delta\dot{v} + \frac{S}{r^2}\omega\delta\dot{\Phi}. \tag{2.62}$$

By introducing a new variable ζ (note our notation above is different from that used in [175])

$$\zeta = \frac{r^2}{S}\delta v, \tag{2.63}$$

the sphaleronic sector then reduces to a single equation [175]

$$-\ddot{\zeta} = -\partial_{r_*}^2\zeta + \left[\frac{\mu S^2}{r^2}\left(1+\omega^2\right) + \frac{2}{\omega^2}\left(\frac{d\omega}{dr_*}\right)^2\right]\zeta, \tag{2.64}$$

while the gravitational sector (2.48) also has just one equation:

$$\begin{aligned}-\delta\ddot{\omega} = {} & -\partial_{r_*}^2(\delta\omega) \\ & + \frac{\mu S^2}{r^2}\left[3\omega^2 - 1 - 4r\omega'^2\left(\frac{1}{r} - \Lambda r - \frac{\left(1-\omega^2\right)^2}{r^3}\right) + \frac{8}{r}\omega\omega'\left(\omega^2-1\right)\right]\delta\omega.\end{aligned} \tag{2.65}$$

The sphaleronic sector equation (2.64) is exactly the same as that in the asymptotically flat $\mathfrak{su}(2)$ EYM case (2.52), but the gravitational sector equation (2.53) unsurprisingly is modified by the presence of non-zero Λ. Both (2.64) and (2.65) have the standard Schrödinger form

$$-\ddot{\Psi} = -\partial_{r_*}^2\Psi + \mathscr{U}\Psi, \tag{2.66}$$

with potential $\mathscr{U}$. For the sphaleronic sector, when the gauge field function $\omega(r)$ has no zeros, it is immediately clear that the potential $\mathscr{U}$ is positive, so there are no instabilities in this sector (this result does not hold in the asymptotically flat case because the zeros of $\omega(r)$ in that case mean that $\mathscr{U}$ is not regular). The gravitational sector potential is more complex to analyze, but, for sufficiently large $|\Lambda|$ and $\omega(r_h) > 1/\sqrt{3}$, it can be shown that the potential is positive and there are no instabilities in this sector either. Therefore there are at least some hairy black holes which are stable under linear, spherically symmetric, perturbations. It can further be proved that at least some of these solutions remain stable when non-spherically symmetric perturbations are considered [146, 178] but the analysis is highly involved and so we do not attempt to summarize it here.

It should be remarked that it is unlikely that *all* nodeless black hole solutions are stable, although this has not been investigated in the literature. An in-depth study of the corresponding solitonic solutions [31] has revealed that some soliton solutions for which $\omega(r)$ has no zeros, although they do not have any instabilities in the sphaleronic sector, do possess unstable modes in the gravitational sector. A scaling behavior analysis of the solitonic solutions [83] has shown that the stable soliton solutions can be approximated well by the stable solitons which exist on pure adS space. On the other hand, the unstable solitons are interpreted as the unstable Bartnik–MacKinnon solitons [7] dressed with solitons on pure adS.

2.4.3 Other Asymptotically Anti-de Sitter $\mathfrak{su}(2)$ *EYM Solutions*

2.4.3.1 Dyonic Solutions

In asymptotically adS, it is no longer the case that the only genuinely non-Abelian solutions must have vanishing electric part in the gauge potential (2.6), so the results of [62, 67] do not extend to non-asymptotically flat solutions. As well as the magnetically charged solutions described above, dyonic black holes were discussed in [25, 26], which we shall not consider further here. The stability of the dyonic solutions remains an open question as the perturbation equations do not decouple into two sectors in this case, making analysis difficult.

2.4.3.2 Topological Black Holes

As in Einstein–Maxwell theory, topological black hole solutions exist for $\mathfrak{su}(2)$ EYM in adS [16]. The metric in this case reads

$$ds^2 = -\mu S^2\,dt^2 + \mu^{-1}\,dr^2 + r^2\,d\theta^2 + r^2 f^2(\theta)\,d\phi^2, \tag{2.67}$$

where

$$f(\theta) = \begin{cases} \sin\theta & \text{for} \quad k=1, \\ \theta & \text{for} \quad k=0, \\ \sinh\theta & \text{for} \quad k=-1, \end{cases} \tag{2.68}$$

and

$$\mu = k - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}. \tag{2.69}$$

The ansatz for the purely magnetic gauge field potential is now [16]

$$A = \tau_x \omega(r)\,d\theta + \left[\tau_y \omega(r) + \tau_z \frac{d\ln f}{d\theta}\right] f(\theta)\,d\phi. \tag{2.70}$$

When $\Lambda = 0$, only spherically symmetric solutions with $k = 1$ are possible, but for $\Lambda < 0$, solutions with both $k = 0$ and $k = -1$ have been found [16]. All the solutions are nodeless, which can be easily proved from the field equations [16]. It is found in [16] that all the $k = 0$ solutions are stable under spherically symmetric perturbations in both the sphaleronic and the gravitational sectors. The same is true for the $k = -1$ solutions for which $\omega > 1$ as $r \to \infty$ [16].

2.4.3.3 Non-spherically Symmetric Solutions

As in the asymptotically flat case, there are both soliton [129] and black hole [136] solutions which are static but not spherically symmetric, so that the metric and gauge potential take the form (2.54) and (2.55). Rotating black holes have also been found [113], and there are also rotating dyonic soliton solutions [131].

2.5 Asymptotically Anti-de Sitter Solutions for $\mathfrak{su}(N)$ EYM

In the previous section we found that stable hairy black holes exist in $\mathfrak{su}(2)$ EYM with a sufficiently large and negative cosmological constant. A natural question is therefore whether there are stable hairy black hole solutions of $\mathfrak{su}(N)$ EYM in adS, and we examine this question in this section.

2.5.1 Spherically Symmetric Numerical Solutions

For any fixed N, the field equations (2.12) and (2.15) can be solved numerically using standard techniques. We will outline briefly some of the key features of the black hole solutions for $\mathfrak{su}(3)$ EYM. Details of the corresponding soliton solutions and the solution space for $\mathfrak{su}(4)$ EYM can be found in [9].

For $\mathfrak{su}(3)$ EYM, there are two gauge field functions $\omega_1(r)$ and $\omega_2(r)$, and therefore four parameters describing black hole solutions: r_h, Λ, $\omega_1(r_h)$ and $\omega_2(r_h)$. Using the symmetry of the field equations (2.17), we set $\omega_1(r_h), \omega_2(r_h) > 0$ without loss of generality. The constraint (2.22) on the values of the gauge field functions at the horizon becomes, in this case

$$\left[\omega_1(r_h)^2 - 2\right]^2 + \left[\omega_1(r_h)^2 - \omega_2(r_h)^2\right]^2 + \left[2 - \omega_2(r_h)^2\right]^2 < 2r_h^2\left(1 - \Lambda r_h^2\right). \quad (2.71)$$

Two typical black hole solutions are shown in Figs. 2.6 and 2.7. The metric functions behave in a very similar way to the $\mathfrak{su}(2)$ solutions, smoothly interpolating between their values at the horizon and at infinity. We note that $S(r)$ in particular converges very rapidly to 1 as $r \to \infty$. In Fig. 2.6, we show an example of a black hole solution in which both gauge field functions have no zeros. We note that both

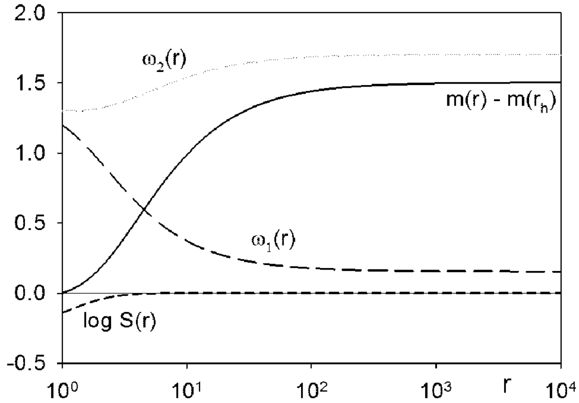

Fig. 2.6 Typical $\mathfrak{su}(3)$ black hole solution, with $r_h = 1$, $\Lambda = -1$, $\omega_1(r_h) = 1.2$ and $\omega_2(r_h) = 1.3$. In this example, both gauge field functions have no zeros. Taken from [9]

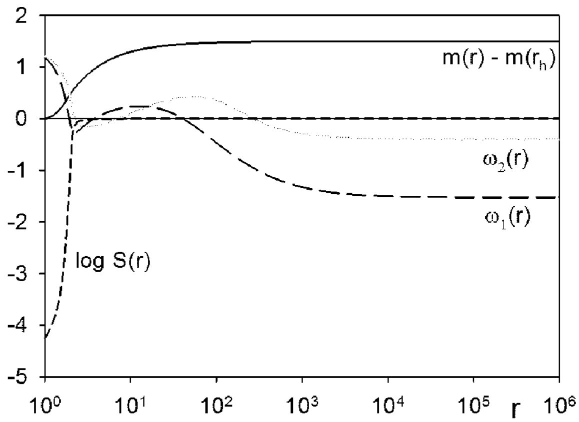

Fig. 2.7 Example of an $\mathfrak{su}(3)$ black hole solution, with $r_h = 1$, $\Lambda = -0.0001$, $\omega_1(r_h) = 1.184$ and $\omega_2(r_h) = 1.216$. In this case, both gauge field functions have three zeros. Taken from [9]

gauge field functions are monotonic, however, one is monotonically increasing and the other monotonically decreasing. In our second example (Fig. 2.7) both gauge field functions have three zeros. Although, in both our examples the two gauge field functions have the same number of zeros, we also find solutions where the two gauge field functions have different numbers of zeros (see Figs. 2.8 and 2.9).

We now examine the space of black hole solutions. Since we have four parameters, in order to produce two-dimensional figures, we need to fix two parameters in each case. We find that varying the event horizon radius produces similar behavior to the $\mathfrak{su}(2)$ case, so for the remainder of this section we fix $r_h = 1$ and consider the phase space for different, fixed values of Λ, scanning all values of $\omega_1(r_h)$, $\omega_2(r_h)$ such that the constraint (2.71) is satisfied. From the discussion in Sect. 2.2, we have embedded $\mathfrak{su}(2)$ black hole solutions when, from (2.30)

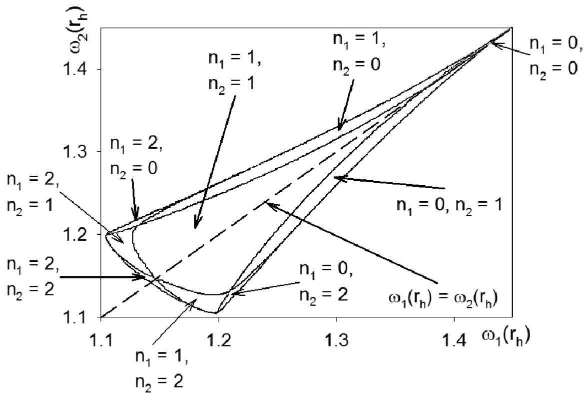

Fig. 2.8 Solution space for $\mathfrak{su}(3)$ black holes with $r_h = 1$ and $\Lambda = -0.1$. The numbers of zeros of the gauge field functions for the various regions of the solution space are shown. For other values of $\omega_1(r_h)$, $\omega_2(r_h)$ we find no solutions. There is a very small region containing solutions in which both gauge field functions have no zeros, in the *top-right-hand corner* of the plot. Taken from [9]

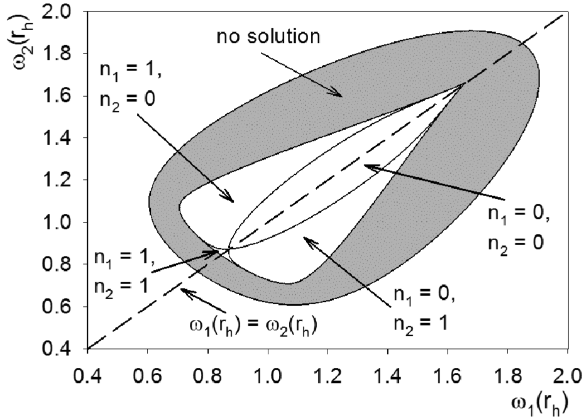

Fig. 2.9 Solution space for $\mathfrak{su}(3)$ black holes with $r_h = 1$ and $\Lambda = -1$. The *shaded region* indicates where the constraint (2.71) is satisfied but we do not find black hole solutions. Outside the *shaded region* the constraint (2.71) does not hold. Where there are solutions, we have indicated the number of zeros of the gauge field functions within the different regions. For this value of Λ there is a large region in which both gauge field functions have no zeros. Taken from [9]

$$\omega_1(r) = \sqrt{2}\omega(r) = \omega_2(r) \tag{2.72}$$

which occurs when $\omega_1(r_h) = \omega_2(r_h)$.

In Figs. 2.8, 2.9 and 2.10 we plot the phase space of solutions for fixed event horizon radius $r_h = 1$ and varying cosmological constant $\Lambda = -0.1$, -1 and -5, respectively. In each of Figs. 2.8, 2.9 and 2.10 we plot the dashed line $\omega_1(r_h) = \omega_2(r_h)$, along which lie the embedded $\mathfrak{su}(2)$ black holes. It is seen in all these figures that the solution space is symmetric about this line, as would be expected from the

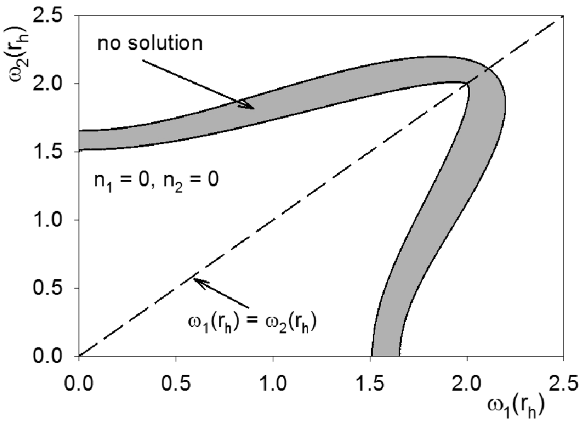

Fig. 2.10 Solution space for $\mathfrak{su}(3)$ black holes with $r_h = 1$ and $\Lambda = -5$. It can be seen that for the vast majority of the phase space for which the constraint (2.71) is satisfied, we have black hole solutions in which both gauge field functions have no zeros. Taken from [9]

symmetry (2.18) of the field equations. The solution space is found to be symmetric about the line $\omega_1(r_h) = \omega_2(r_h)$ not only in terms of where we find solutions but also in terms of the numbers of zeros of the gauge field functions. To state this precisely, suppose that at the point $\omega_1(r_h) = a_1$, $\omega_2(r_h) = a_2$ we find a black hole solution in which $\omega_1(r)$ has n_1 zeros and $\omega_2(r)$ has n_2 zeros. Then, at the point $\omega_1(r) = a_2$, $\omega_2(r) = a_1$, we find a black hole solution in which $\omega_1(r)$ has n_2 zeros and $\omega_1(r)$ has n_1 zeros. This is clearly seen in Figs. 2.8 and 2.9 and follows from the symmetry (2.18) of the field equations. As we increase $|\Lambda|$, we find (see Figs. 2.8, 2.9, and 2.10) that the solution space expands as a proportion of the space of values of $\omega_1(r_h)$, $\omega_2(r_h)$ satisfying the constraint (2.71). It can also be seen from Figs. 2.8, 2.9 and 2.10 that the number of nodes of the gauge field functions decreases as $|\Lambda|$ increases and that the space of solutions becomes simpler. For $\Lambda = -0.1$, there is a very small region of the solution space where both gauge field functions have no zeros. This region expands as we increase $|\Lambda|$, until for $\Lambda = -5$, both gauge field functions have no zeros for all the solutions we find.

The solution space becomes progressively more complicated as N increases, due to the increased number of parameters required to describe the solutions. However, the key feature described above is found; namely that for sufficiently large $|\Lambda|$, all the solutions we find are such that all the gauge field functions ω_j have no zeros. These solutions are of particular interest since one might hope that at least some of them might be stable.

As with the $\mathfrak{su}(2)$ black holes we may consider the spectra of black hole solutions by plotting the relationship between the mass M and the magnetic charge Q of the solutions (see Fig. 2.5 for the $\mathfrak{su}(2)$ case). As may be expected, for higher N the spectra are even more complicated than for $\mathfrak{su}(2)$. In Fig. 2.11 we plot some of the possible values of M and Q for $\mathfrak{su}(3)$ EYM black holes with $\Lambda = -0.1$ and $r_h = 1$. In Fig. 2.11 we have color coded the various possible numbers of zeros of

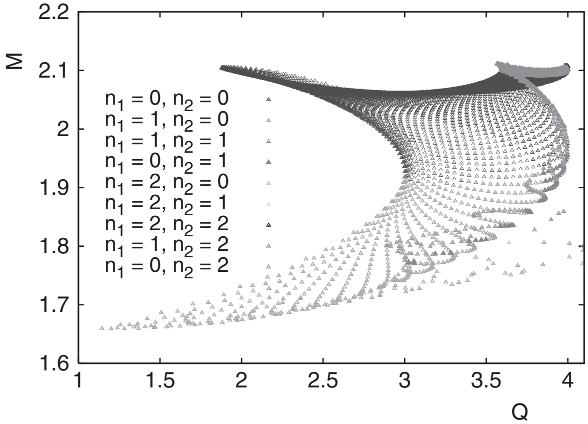

Fig. 2.11 Black hole mass M versus magnetic charge Q for $\mathfrak{su}(3)$ EYM black holes with $r_h = 1$ and $\Lambda = -0.1$. There are many different combinations of number of zeros of the gauge field functions (see Fig. 2.8), which are indicated by different colors. Here we have performed a scan over a grid of possible values of the gauge field functions at the event horizon, $\omega_1(r_h)$, $\omega_2(r_h)$, leading to discrete points in the spectrum. This is to enable the complicated structure of the spectrum to be seen

the gauge field functions (cf. Fig. 2.8). We have used a discrete grid of initial values of the gauge field functions at the event horizon $(\omega_1(r_h), \omega_2(r_h))$ and plotted discrete points so that at least some of the structure can be seen. In this case, because we have a four-parameter $(\Lambda, r_h, \omega_1(r_h), \omega_2(r_h))$ space of solutions of the field equations, even when Λ and r_h are fixed, we obtain two-dimensional regions in the (M, Q) plane, rather than curves as in the $\mathfrak{su}(2)$ case. It can be seen from Fig. 2.11 that the spectrum is very complicated, with the regions corresponding to different numbers of zeros of the gauge field functions overlapping. It is certainly the case that the black holes cannot be uniquely characterized by the four parameters (Λ, r_h, M, Q).

2.5.2 Analytic Work

For any fixed value of N, it is possible to examine the space of solutions numerically. However, we would like to know whether there are solutions for *all* N, and, in particular, whether for all N there are some solutions for which all the gauge field functions have no zeros, which we expect to be the case for sufficiently large $|\Lambda|$. Answering this question for general N requires analytic rather than numerical work.

In [175], the existence of black hole solutions for which the gauge function $\omega(r)$ had no zeros was proven analytically in the $\mathfrak{su}(2)$ case. Since $\mathfrak{su}(2)$ solutions can be embedded as $\mathfrak{su}(N)$ solutions via (2.30), we have automatically an analytic proof of the existence of nodeless $\mathfrak{su}(N)$ EYM black holes in adS. However, these embedded solutions are “trivial” in the sense that they are described by just three parameters: r_h, Λ and $\omega(r_h)$. The question is therefore whether the existence of “non-trivial” (that is, genuinely $\mathfrak{su}(N)$) solutions in which all the gauge field functions $\omega_j(r)$ have no zeros can be proven analytically. The answer to this question is affirmative and involves a generalization to $\mathfrak{su}(N)$ of the continuity-type argument used in [175]. The details are lengthy and will be presented elsewhere [11]. Here we simply outline the key steps in the proof.

The main idea of the proof is sketched in Fig. 2.12. We wish to find black hole solutions which are regular on the event horizon, regular everywhere outside the event horizon and regular at infinity. The proof proceeds via the following steps:

1. We first prove (generalizing the analysis of [100] to include Λ) that the field equations (2.12) and (2.15) and initial conditions at the event horizon (2.20) possess, locally in a neighborhood of the horizon, solutions which are analytic in r, r_h, Λ and the parameters $\omega_j(r_h)$. As might be expected, the analysis of [100] requires only minor modifications to include a negative cosmological constant.
2. This enables us to prove that, in a sufficiently small neighborhood of any embedded $\mathfrak{su}(2)$ solution in which $\omega(r)$ has no nodes, there exists (at least in a neighborhood of the event horizon) an $\mathfrak{su}(N)$ solution in which all the $\omega_j(r)$ have no nodes.
3. Using the analyticity properties of the solutions of the field equations, we then show that these $\mathfrak{su}(N)$ solutions can be extended out to large $r_L \gg r_h$, provided the initial parameters $\omega_j(r_h)$ are sufficiently close to those of an embedded $\mathfrak{su}(2)$ solution in which $\omega(r)$ has no zeros. Furthermore, by analyticity, none of the $\omega_j(r)$ will have any zeros between the event horizon r_h and r_L.
4. The key part of the proof lies in then showing that these $\mathfrak{su}(N)$ solutions can be further extended out to $r \to \infty$ and that they satisfy the boundary conditions (2.23) at infinity. This part of the analysis uses the properties of the Yang–Mills field equations (2.12) in the asymptotically adS regime. As in the $\mathfrak{su}(2)$ case [175], these have very different properties from the asymptotically flat case, and this makes it much easier to prove the existence of solutions. Furthermore, it can be shown that the gauge field functions $\omega_j(r)$ will have no zeros for $r \geq r_L$.

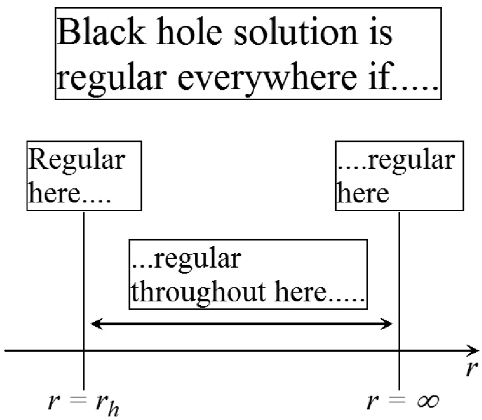

Fig. 2.12 Sketch of the main steps in the proof of the existence of non-trivial $\mathfrak{su}(N)$ EYM black holes in adS for which all the gauge field functions have no zeros. We wish to find black hole solutions which are regular on the event horizon regular everywhere outside the event horizon and regular at infinity. We thank J. E. Baxter for providing this sketch

In summary, this process gives genuinely $\mathfrak{su}(N)$ black hole solutions in which all the gauge field functions have no zeros and which are characterized by the $N+1$ parameters r_h, Λ and $\omega_j(r_h)$.

2.5.3 Stability Analysis of the Spherically Symmetric Solutions

The remaining outstanding question is whether these new black holes, with potentially unbounded amounts of gauge field hair, are stable. We consider linear, spherically symmetric perturbations only for simplicity. The analysis of [146, 178] in the $\mathfrak{su}(2)$ case revealed that, for sufficiently large $|\Lambda|$, stability under spherically symmetric perturbations continued to hold also for non-spherically symmetric perturbations, and one might hope that a similar result will hold in the more complex $\mathfrak{su}(N)$ case. However, we leave this for future work. Even for spherically symmetric perturbations, the analysis is highly involved in the $\mathfrak{su}(N)$ case and the details will be presented elsewhere [8, 11]. Here we briefly outline just the key features. The perturbation equations themselves can be found in Sect. 2.2.4.

2.5.3.1 Sphaleronic Sector

The sphaleronic sector consists of the perturbation equations (2.45) and (2.46) together with the Gauss constraint (2.47). The analysis of this sector essentially follows that of [47] in the asymptotically flat case. We begin by defining yet more new variables, $\delta\epsilon_j$, for $j=1,\ldots,N$ by

$$\delta\epsilon_j = r\sqrt{\mu}\,\delta\beta_j, \tag{2.73}$$

then, after much algebra, the sphaleronic sector perturbation equations can be cast in the form

$$-\underline{\ddot{\Psi}} = \mathscr{M}_S\underline{\Psi}, \tag{2.74}$$

where the $(2N-1)$-dimensional vector $\underline{\Psi}$ is defined by

$$\underline{\Psi} = (\delta\epsilon_1,\ldots,\delta\epsilon_N,\delta\Phi_1,\ldots,\delta\Phi_{N-1})\,. \tag{2.75}$$

and $\mathscr{M}_S$ is a self-adjoint, second order, differential operator (involving derivatives with respect to r but not t), depending on the equilibrium functions $\omega_j(r)$, $m(r)$ and $S(r)$. The operator $\mathscr{M}_S$ can be written as the sum of three parts. The first is of the form $\chi^\dagger\chi$ for a particular first-order differential operator χ (whose precise form can be found in [8, 11]) and is therefore manifestly positive and is regular if the gauge field functions ω_j have no zeros. The second part vanishes when applied to a physical perturbation due to the Gauss constraint (2.47). The third part is a matrix $\mathscr{V}$ which does not contain any differential operators. It can be shown that the matrix $\mathscr{V}$ is regular and positive definite provided the unperturbed gauge functions $\omega_j(r)$ have no zeros and satisfy the $N-1$ inequalities

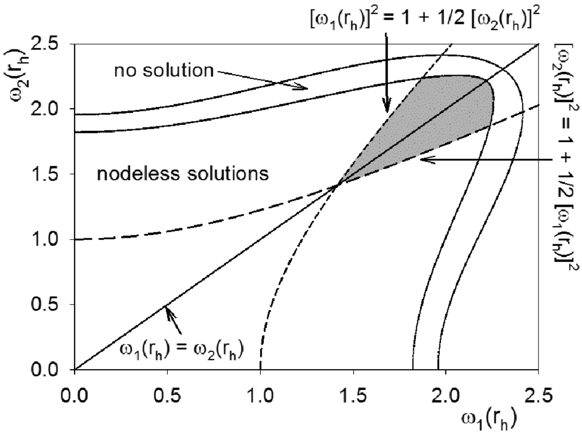

Fig. 2.13 Phase space of black hole solutions in $\mathfrak{su}(3)$ EYM with $\Lambda = -10$ and $r_h = 1$. The *shaded region* shows where solutions exist which satisfy the inequalities (2.76) at the event horizon. Taken from [10]

$$\omega_j^2 > 1 + \frac{1}{2}\left(\omega_{j+1}^2 + \omega_{j-1}^2\right) \tag{2.76}$$

for all $j = 1,\ldots N-1$ and all $r \geq r_h$. The inequalities (2.76) define a non-empty subset of the parameter space. For example, we show in Fig. 2.13 where the inequalities (2.76) are satisfied for the gauge field functions at the event horizon, for the particular case of $\Lambda = -10$ and $r_h = 1$. From Fig. 2.13 we can see that there are some nodeless solutions which satisfy the inequalities (2.76) at the event horizon. For any N, it can also be proved analytically that, for sufficiently large $|\Lambda|$, there are non-trivial $\mathfrak{su}(N)$ solutions, in a neighborhood of some embedded $\mathfrak{su}(2)$ solutions, such that the inequalities (2.76) are satisfied at the event horizon.

However, the requirements of (2.76) are considerably stronger, as the inequalities have to be satisfied for *all* $r \geq r_h$. Our analytic work shows that, in fact, for any N and sufficiently large $|\Lambda|$, there do exist solutions to the field equations for which the inequalities (2.76) are indeed satisfied for all r. This involves proving that for at least some solutions for which the gauge field function values at the event horizon lie within the region where the inequalities (2.76) are satisfied, the gauge field functions remain within this open region. In Fig. 2.14 we show an example of such a solution for $\mathfrak{su}(3)$ EYM.

2.5.3.2 Gravitational Sector

As might be expected, the gravitational sector perturbation equations (2.48) are more difficult to analyze than the sphaleronic sector perturbation equations. For stable solutions, we require the matrix $\mathscr{M}_G$ (2.49) to be negative definite. For sufficiently large $|\Lambda|$, it can be shown that $\mathscr{M}_G$ is indeed negative definite for embedded

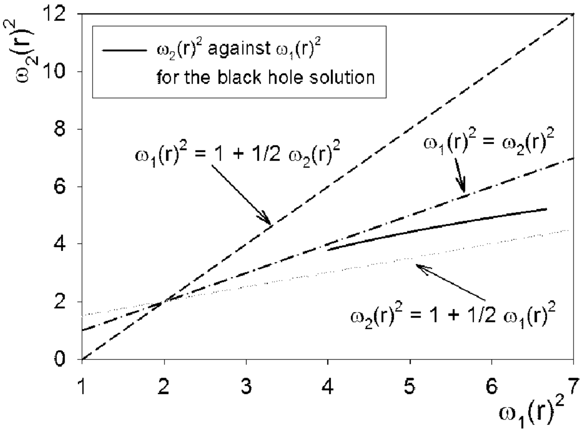

Fig. 2.14 An example of an $\mathfrak{su}(3)$ solution for which the inequalities (2.76) are satisfied for all $r \geq r_h$. In this example, $\Lambda = -10$, $r_h = 1$ and the values of the gauge field functions at the event horizon are $\omega_1(r_h) = 2$, $\omega_2(r_h) = 1.95$. Taken from [10]

$\mathfrak{su}(2)$ solutions, provided that $\omega^2(r) > 1$ for all $r \geq r_h$ (the existence of such $\mathfrak{su}(2)$ solutions is proved, for sufficiently large $|\Lambda|$, in [175]). As described in Sect. 2.5.2 above, our analytic work ensures the existence of genuinely $\mathfrak{su}(N)$ solutions in a sufficiently small neighborhood of these embedded $\mathfrak{su}(2)$ solutions. These $\mathfrak{su}(N)$ solutions are such that the inequalities (2.76) are satisfied for all $r \geq r_h$ (and therefore the solutions are stable under sphaleronic perturbations). The negativity of $\mathscr{M}_G$ can then be extended to these genuinely $\mathfrak{su}(N)$ solutions using an analyticity argument, based on the nodal theorem of [2] (see also [178] for a similar argument for the non-spherically symmetric perturbations of the $\mathfrak{su}(2)$ EYM black holes). The technical details of this argument will be presented elsewhere [11].

The conclusion of the work in this section is that there are at least some genuinely $\mathfrak{su}(N)$ EYM black holes in adS, for sufficiently large $|\Lambda|$, for which all the gauge field functions ω_j have no zeros, and which are stable under spherically symmetric perturbations in both the sphaleronic and the gravitational sectors.

2.6 Summary and Outlook

In this review we have studied classical, hairy black hole solutions of $\mathfrak{su}(N)$ EYM theory, particularly spherically symmetric spacetimes and black holes in adS. We very briefly discussed some of the key aspects of the solutions in asymptotically flat space, which have been extensively reviewed in [171]. Hairy black hole solutions exist for all N, with $N-1$ gauge field degrees of freedom [116], however, all these solutions are unstable [47]. Therefore, while these hairy black holes violate the "letter" of the no-hair conjecture (that is, their geometry is not completely fixed by global charges measurable at infinity), its "spirit" is maintained. In particular, stable equilibrium black holes are comparatively simple objects, described completely by just a few parameters.

The main conclusion of this article is that this is not true in adS. The existence of stable hairy black holes in $\mathfrak{su}(2)$ EYM [175] did not really contradict the "spirit" of the no-hair conjecture, as only a single additional parameter was required to fix the geometry outside the event horizon. However, the recent work [10] which shows that there are stable hairy black holes in $\mathfrak{su}(N)$ EYM in adS for arbitrarily large N changes the picture completely. For sufficiently large $|\Lambda|$, an infinite number of parameters are required in order to describe stable black holes. We might flippantly describe these as "furry" black holes, since they possess copious amounts of hair.

What are the consequences for black hole physics in adS of these "furry" black holes? These need to be explored. Given the huge amount of interest in the adS/CFT correspondence in string theory [111, 180, 181], a natural question is how black hole hair in the bulk asymptotically adS spacetime relates to the dual CFT. In particular, it has been suggested [76] that there should be observables in the dual (deformed) CFT which are sensitive to the presence of black hole hair. Another example of this approach can be found in [70], where an adS/CFT interpretation is given of some stable seven-dimensional black holes with $\mathfrak{so}(5)$ gauge fields. We would expect that, in analogy with the $\mathfrak{su}(2)$ case [49, 50, 74, 84, 113, 130, 132], there are solutions in some super-gravity theories with a gauge group containing an $\mathfrak{su}(N)$ factor, which will need to be studied in the context of adS/CFT. There is evidence [117] that there are non-trivial black hole solutions of $\mathfrak{su}(\infty)$ EYM in adS, giving black holes not just with unbounded amounts of hair, but infinite amounts of hair, at least in the limit $|\Lambda| \to \infty$. It remains to be seen whether exact solutions of the $\mathfrak{su}(\infty)$ field equations can be found for finite $\Lambda < 0$ and whether any of these black holes are stable. If so, then their role in adS/CFT would be puzzling indeed.

Due to space restrictions, there are many aspects of black holes in EYM which we have not been able to discuss. In particular, we have not mentioned the vast number of solutions which involve modifications of the EYM action (2.1), including higher curvature terms (see, for example, [88, 89]) or the inclusion of dilaton (see, for example, [134]), Higgs (see, for example, [15, 108, 109]) or other modifications of the EYM action (see, for example, [120, 147, 148]). Here we have also only studied four-dimensional spacetimes, while recent work has considered EYM in higher-dimensional spacetimes (see, for example, [32, 33, 35–40, 75, 123, 133, 135] and [166] for a review).

The black hole solutions of EYM and its variants certainly exhibit an abundantly rich structure, and no doubt will have more surprises in store for us in the future.

Acknowledgments We thank the organizers of the 4th Aegean Summer School for a most enjoyable and informative conference. The work in Sect. 2.5 was done in collaboration with J. E. Baxter and Marc Helbling. We thank B. Bear, R. F. W. Jackson and the Chester Chronicle for suggesting the terminology "furry" black holes. We also thank Eugen Radu for numerous enlightening discussions. This work was supported by STFC (UK), grant reference numbers PPA/G/S/2003/00082 and PP/D000351/1.

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