# Chapter 15
# Elastic Beams

## 15.1 Phenomenological Approach

**Beams.** Consider a space curve, $\mathring{\Gamma}$, in three-dimensional space and a region, $\mathring{V}$, which is formed by motion of a flat figure, $S$, along $\mathring{\Gamma}$; at every point, $S$ is orthogonal to $\mathring{\Gamma}$ (Fig. 15.1). Denote the diameter of $S$ (the maximum distance between two points of $S$) by $h$, the length of $\mathring{\Gamma}$ by $L_0$ and the minimum curvature-torsion radius of $\mathring{\Gamma}$ by $R$. If

$$\frac{h}{R} \ll 1, \quad \frac{h}{L_0} \ll 1,$$

then an elastic body occupying in its undeformed state the region, $\mathring{V}$, is called an elastic beam.

Let the beam be deformed by some external forces; either the surface forces or the displacements are given at the ends of the beam. It seems plausible that the three-dimensional elastic problem can be approximated by a one-dimensional problem,

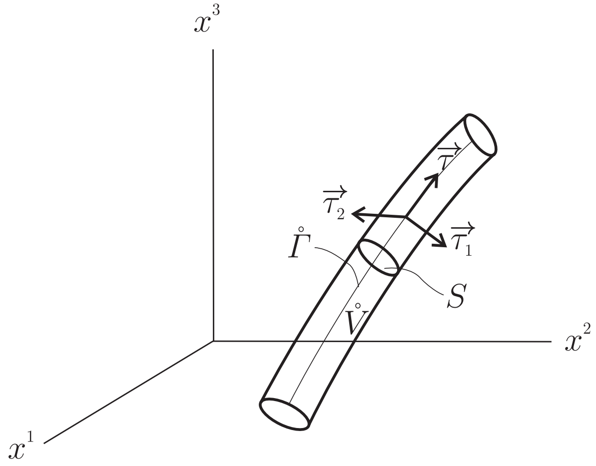


**Fig. 15.1** Notation in beam theory

V.L. Berdichevsky, *Variational Principles of Continuum Mechanics*,
Interaction of Mechanics and Mathematics, DOI 10.1007/978-3-540-88469-9_2,

which contains only the functions of the longitudinal coordinate, $\xi$, along $\mathring{\Gamma}$ (we discuss here only the statics of beams).

We will first consider a heuristic beam theory, which was founded by Kirchhoff and Clebsch, and then discuss the variational problem that must be solved to find energy of the beam in one-dimensional theory. This will be followed by the systematic derivation of the one-dimensional beam theory from the three-dimensional elasticity theory using the variational-asymptotic method.

**Kinematics of beams.** The beam will be modeled by a space curve, $\Gamma$, at every point of which an orthogonal vector triad is attached; one of the triad vectors is tangent to the curve. For a given position of the curve, the triad is defined up to rotation around the tangent vector. The corresponding degree of freedom serves to describe the relative rotation of the transverse cross-sections of the beam. So, the beam has four functional degrees of freedom.

Let

$$x^i = r^i(\xi)$$

be the parametric equations of the curve $\Gamma$ in some Cartesian coordinate system $x^i$, $\tau^i(\xi)$ the unit tangent vector to $\Gamma$, and $\tau^i_\alpha(\xi)$, $\alpha = 1, 2$, the two other vectors of the triad; as in the previous chapter, Greek indices[1] run through values 1, 2. The position of the curve $\Gamma$ in the undeformed state is denoted by $\mathring{\Gamma}$. Its parametric equations are

$$x^i = \mathring{r}^i(\xi),$$

$\mathring{\tau}^i(\xi)$ and $\mathring{\tau}^i_\alpha(\xi)$ being the vectors of the orthonormal triad at $\mathring{\Gamma}$. The parameter $\xi$ is identified with the arc length in the undeformed state; thus

$$\mathring{\tau}^i = \frac{d\mathring{r}^i}{d\xi}.$$

The parameter $\xi$ changes on the segment $0 \le \xi \le L_0$.

To introduce the strain measures for the beam, we first need to define the beam curvatures. The vectors, $\mathring{\tau}^i$ and $\mathring{\tau}^i_\alpha$, as the vectors of an orthogonal triad, satisfy the conditions

$$\mathring{\tau}_i \mathring{\tau}^i = 1, \quad \mathring{\tau}^i_\alpha \mathring{\tau}_{i\beta} = \delta_{\alpha\beta}, \quad \mathring{\tau}_i \mathring{\tau}^i_\alpha = 0. \tag{15.1}$$

Let us project the vectors $d\mathring{\tau}^i/d\xi$ and $d\mathring{\tau}^i_\alpha/d\xi$ on to the vectors $\mathring{\tau}^i$, $\mathring{\tau}^i_\alpha$. Taking the derivative of the first equation (15.1) with respect to $\xi$,

[1] Small Greek indices number the two vectors of the triad that are orthogonal to the tangent vector. They are also used for the projections on these vectors. Greek indices can be put in either upper or lower position in correspondence with the general rule of summation over repeating upper and lower indices; quantities with upper and lower indices coincide; in particular $\tau^i_\alpha = \tau^{i\alpha}$, $\tau^i_\alpha = \tau_{i\alpha}$.

$$\mathring{\tau}_i \frac{d\mathring{\tau}^i}{d\xi} = 0,$$

we see that the vector $d\mathring{\tau}^i/d\xi$ is orthogonal to the vector $\mathring{\tau}^i$ and, consequently, has nonzero projections only on the vectors $\mathring{\tau}^i_\alpha$. Denote these projections by $-\mathring{\omega}^\alpha$:

$$\frac{d\mathring{\tau}^i}{d\xi} = -\mathring{\omega}^\alpha \mathring{\tau}^i_\alpha. \tag{15.2}$$

Analogously, the vector $d\mathring{\tau}^i_\alpha/d\xi$ is orthogonal to the vector $\mathring{\tau}^i_\alpha$ and, therefore has nonzero projections on the vectors,[2] $\mathring{\tau}^i$ and $e_{\alpha\cdot}^{\cdot\beta}\mathring{\tau}^i_\beta$. The projections of $d\mathring{\tau}^i_\alpha/d\xi$ on $\mathring{\tau}^i$ are equal to $\mathring{\omega}_\alpha$, as follows from the third equation (15.1) differentiated with respect to $\xi$:

$$\frac{d\mathring{\tau}^i}{d\xi}\mathring{\tau}^i_\alpha + \mathring{\tau}^i \frac{d\mathring{\tau}^i_\alpha}{d\xi} = 0 \qquad \text{or} \qquad \frac{d\mathring{\tau}^i}{d\xi}\mathring{\tau}^i_\alpha = \mathring{\omega}_\alpha.$$

Therefore,

$$\frac{d\mathring{\tau}^i_\alpha}{d\xi} = \mathring{\omega}_\alpha \mathring{\tau}^i + \mathring{\omega} e_{\alpha\cdot}^{\cdot\beta}\mathring{\tau}^i_\beta. \tag{15.3}$$

The quantities, $\mathring{\omega}_1$ and $\mathring{\omega}_2$, will be called the curvatures, and $\mathring{\omega}$ the torsion of the beam. In order to avoid confusion, we note that often $\mathring{\omega}_1$ and $\mathring{\omega}_2$ denote the projections of $d\mathring{\tau}^i/d\xi$ onto $-\mathring{\tau}^i_2$ and $\mathring{\tau}^i_1$, correspondingly. The notation used in the text allows us to simplify the tensor form of the basic relations.

Curvatures and torsion, $\mathring{\omega}_\alpha$ and $\mathring{\omega}$, determine the triad uniquely by the system of ordinary differential equations (15.2), (15.3), if $\mathring{\tau}^i$ and $\mathring{\tau}^i_\alpha$ are given at one point of $\mathring{\Gamma}$.

For a given $\mathring{\Gamma}$, there is an arbitrariness in the choice of the triad: the vectors, $\mathring{\tau}^i_\alpha$, can be rotated around the tangent vector. Accordingly, the curvatures change. If the vectors $\mathring{\tau}^i_\alpha$ are chosen in such a way that $\mathring{\omega}_2$ is equal to zero, then the vector $\mathring{\tau}^i_1$ is called the principal normal vector, $\mathring{\tau}^i_2$ the binormal vector, $-\mathring{\omega}_1$ the curvature of $\mathring{\Gamma}$, and $\mathring{\omega}$ the torsion of $\mathring{\Gamma}$. In beam theory, it is convenient to link the vectors $\mathring{\tau}^i_\alpha$ to the geometric or the physical properties of the cross-section of the beam, for example by directing $\mathring{\tau}^i_\alpha$ along the symmetry axes of the cross-section (in case when such axes exist). If, after the vectors $\mathring{\tau}^i_\alpha$ are linked to the properties of the cross-section, it turns out that $\mathring{\omega} \neq 0$, the beam is called naturally twisted or pre-twisted.

For a deformed state, the formulae analogous to (15.2) and (15.3) are

$$\frac{d\tau^i}{ds} = -\omega^\alpha \tau^i_\alpha, \qquad \frac{d\tau^i_\alpha}{ds} = \omega_\alpha \tau^i + \omega e_{\alpha\cdot}^{\cdot\beta}\tau^i_\beta, \tag{15.4}$$

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[2] As before, $e_{\alpha\beta} = e_{\alpha\cdot}^{\cdot\beta} = e^{\alpha\beta}$ are the two-dimensional Levi-Civita symbols: $e_{11} = e_{22} = 0$, $e_{12} = -e_{21} = 1$.

where $s$ is the arc length along $\Gamma$.

The elongation of the axis of the beam is described by the function $s = s(\xi)$. The function

$$\gamma = \frac{1}{2}\left(\left(\frac{ds}{d\xi}\right)^2 - 1\right) = \frac{1}{2}\left(\frac{dr^i}{d\xi}\frac{dr_i}{d\xi} - 1\right)$$

will be used as the elongation measure. Since

$$\frac{ds}{d\xi} = \sqrt{1 + 2\gamma},$$

the formulae (15.4) can be conveniently written in terms of the derivatives of $\tau^i$ and $\tau^i_\alpha$ with respect to $\xi$ as

$$\frac{d\tau^i}{d\xi} = -\sqrt{1+2\gamma}\,\omega^\alpha \tau^i_\alpha, \quad \frac{d\tau^i_\alpha}{d\xi} = \sqrt{1+2\gamma}\,\omega_\alpha \tau^i + \sqrt{1+2\gamma}\,\omega e_{\alpha.}^{\cdot\beta}\tau^i_\beta. \quad (15.5)$$

Comparison of (15.2), (15.3) and (15.5) shows that the curvature and torsion measures may be introduced in the following way:

$$\Omega_\alpha = \sqrt{1+2\gamma}\,\omega_\alpha(s(\xi)) - \mathring{\omega}_\alpha(\xi), \quad \Omega = \sqrt{1+2\gamma}\,\omega\,(s(\xi)) - \mathring{\omega}(\xi). \quad (15.6)$$

If $\gamma(\xi)$, $\Omega_\alpha(\xi)$, $\Omega(\xi)$, the curvature and torsion, $\mathring{\omega}_\alpha(\xi)$ and $\mathring{\omega}(\xi)$, and the vectors $\tau^i$, and $\tau^i_\alpha$ at one of the points of the curve $\Gamma$ are known, then $\tau^i(\xi)$ and $\tau^i_\alpha(\xi)$ can be uniquely computed from the system of ordinary differential equations (15.5). The curve $\Gamma$ is found then from the differential equations,

$$\frac{dr^i}{d\xi} = \sqrt{1+2\gamma(\xi)}\tau^i(\xi).$$

The deformed and undeformed positions of the beam (and the corresponding vector triad) coincide if, and only if, the strain measures $\gamma$, $\Omega_\alpha$ and $\Omega$ are identically equal to zero.

The curvature and torsion measures may be expressed in terms of the derivatives of the triad vectors with respect to $\xi$ from (15.5) and (15.6):

$$\Omega_\alpha = \tau_i \frac{d\tau^i_\alpha}{d\xi} - \mathring{\omega}_\alpha = -\tau^i_\alpha \frac{d\tau_i}{d\xi} - \mathring{\omega}_\alpha, \quad \Omega = \frac{1}{2}e^{\alpha\beta}\frac{d\tau^i_\alpha}{d\xi}\tau_{i\beta} - \mathring{\omega}. \quad (15.7)$$

In (15.7) the vector $\tau^i$ is defined by the functions $r^i(\xi)$ according to the equations

$$\tau^i = \frac{1}{\sqrt{1+2\gamma}}\frac{dr^i}{d\xi}, \quad \gamma = \frac{1}{2}\left(\frac{dr^i}{d\xi}\frac{dr_i}{d\xi} - 1\right). \quad (15.8)$$

**Variational principle.** Since the strain measures, $\gamma$, $\Omega_\alpha$ and $\Omega$, completely determine the deformed state of the beam, it is natural to assume that the elastic energy density per unit length of the initial state $\Phi$ is a function of $\gamma$, $\Omega_\alpha$ and $\Omega$:

$$\Phi = \Phi(\gamma, \Omega_\alpha, \Omega).$$

The external forces do work on the variations of $r^i$ and $\tau^i_\alpha$. Denote the corresponding "generalized forces" by $Q_i$ and $Q^\alpha_i$, and, for simplicity, assume that the ends of the beam are clamped: $r^i(\xi)$ and $\tau^i_\alpha(\xi)$ take the fixed values $r^i_0$, $r^i_1$, $\tau^i_{\alpha 0}$, $\tau^i_{\alpha 1}$, respectively. The forces are supposed to be "dead forces", i.e. they do not depend on the deformed state of the beam but may depend on $\xi$. The deformed state of the beam is a stationary point of the functional

$$\int_0^{L_0} \Phi(\gamma, \Omega_\alpha, \Omega)\, d\xi - \int_0^{L_0} \left(Q_i r^i + Q^\alpha_i \tau^i_\alpha\right) d\xi \tag{15.9}$$

on the set of all admissible positions of the curve $\Gamma$ and all orientations of the vectors of the triad, which are subject to the constraints

$$\tau_i \tau^i_\alpha = 0, \quad \tau^i_\alpha \tau_{i\beta} = \delta_{\alpha\beta} \tag{15.10}$$

in which $\tau^i$ is the vector (15.8).

**Variation of the strain measures.** Six functions $\tau^i_\alpha$ satisfy the five constraints (15.10). Therefore, there is only one independent variation among six variations, $\delta\tau^i_\alpha$. It corresponds to the infinitesimally small rotations of the vectors $\vec{\tau}_\alpha$ around the vector $\vec{\tau}$. The infinitesimally small angle of rotation will be denoted by $\delta\varphi$ :

$$\delta\varphi = \frac{1}{2} e^{\alpha\beta} \tau^i_\beta \delta\tau_{i\alpha}.$$

So, there are four independent variations, $\delta r^i$ and $\delta\varphi$, and, correspondingly, four equilibrium equations.

From (15.8), the variation of the tangent vector is

$$\delta\tau^i = \frac{d\delta r^i}{ds} - \tau^i \tau_k \frac{d\delta r^k}{ds}. \tag{15.11}$$

It follows from (15.11) that the variation of the tangent vector is orthogonal to the tangent vector: $\tau_i \delta\tau^i = 0$; the same conclusion, of course, is obtained by varying the equation $\tau_i \tau^i = 1$.

Denote the projections of the vector $\delta\vec{\tau}$ onto the vectors $\vec{\tau}_\alpha$ by $\delta\theta^\alpha$: $\tau^\alpha_i \delta\tau^i \equiv \delta\theta^\alpha$. According to (15.11),

$$\delta\theta_\alpha = \tau_{i\alpha}\frac{d\delta r^i}{ds}.$$

Varying (15.10) and (15.8) we obtain the following expressions for the variations of the vectors $\tau^i$ and $\tau^i_\alpha$:

$$\delta\tau^i = \tau^i_\alpha\delta\theta^\alpha, \qquad \delta\tau^i_\alpha = -\delta\theta_\alpha\tau^i + \tau^{i\beta}e_{\alpha\beta}\delta\varphi. \tag{15.12}$$

From (15.7) and (15.12), we find the variations of the strain measures:

$$\delta\Omega_\alpha = \sqrt{1+2\gamma}\left[-\frac{d\delta\theta_\alpha}{ds} + e_{\alpha\beta}\omega^\beta\delta\varphi + \omega e_{\alpha\beta}\delta\theta^\beta\right],$$
$$\delta\Omega = \sqrt{1+2\gamma}\left[\frac{d\delta\varphi}{ds} + e^{\alpha\beta}\omega_\beta\delta\theta_\alpha\right], \quad \delta\gamma = (1+2\gamma)\,\tau_i\frac{d\delta r^i}{ds}. \tag{15.13}$$

For the clamped beam ends,

$$\delta r^i = \delta\tau^i = \delta\tau^i_\alpha = 0 \quad \text{at} \quad \xi = 0, L_0.$$

According to (15.12), at the ends of the beam the variations $\delta\tau^i$ and $\delta\varphi$ vanish:

$$\delta\tau^i = \delta\theta_\alpha = \delta\varphi = 0 \quad \text{at} \quad \xi = 0, L_0. \tag{15.14}$$

**Governing equations of beam theory.** Denote the derivatives of energy density by $T$, $M^\alpha$, and $M$ :

$$T = \sqrt{1+2\gamma}\frac{\partial\Phi}{\partial\gamma}, \quad M^\alpha = \frac{\partial\Phi}{\partial\Omega_\alpha}, \quad M = \frac{\partial\Phi}{\partial\Omega}. \tag{15.15}$$

They have the meaning of tension, bending moments and torque, respectively. $M^1$ is the bending moment in the plane $\vec{\tau}, \vec{\tau}_1$, and $M^2$ is the bending moment in the plane $\vec{\tau}, \vec{\tau}_2$.

For the variation of the energy of the beam, we find using (15.13):

$$\begin{aligned}\delta\int_0^{L_0}\Phi d\xi = \int_0^{L_0}\Bigg[&\frac{T}{\sqrt{1+2\gamma}}(1+2\gamma)\,\tau_i\frac{d\delta r^i}{ds} + \\ &+M^\alpha\sqrt{1+2\gamma}\left(-\frac{d\delta\theta_\alpha}{ds} + e_{\alpha\beta}\omega^\beta\delta\varphi + \omega e_{\alpha\beta}\delta\theta^\beta\right) + \\ &+M\sqrt{1+2\gamma}\left(\frac{d\delta\varphi}{ds} + e^{\alpha\beta}\omega_\beta\delta\theta_\alpha\right)\Bigg]d\xi.\end{aligned}$$

In order to integrate by parts, it is convenient to change the integration variable, $\xi$, to $s$. Since

$$ds = \sqrt{1 + 2\gamma} d\xi,$$

we have

$$\delta \int_0^{L_0} \Phi d\xi = \int_0^L \left[ T\tau_i \frac{d\delta r^i}{ds} + M^\alpha \left( -\frac{d\delta\theta_\alpha}{ds} + e_{\alpha\beta}\omega^\beta \delta\varphi + \omega e_{\alpha\beta}\delta\theta^\beta \right) + \right.$$
$$\left. + M \left( \frac{d\delta\varphi}{ds} + e^{\alpha\beta}\omega_\beta \delta\theta_\alpha \right) \right] ds.$$

Here $L$ is the beam length in the deformed state. After integration by parts of the second and the third terms and use of the end conditions (15.14) we obtain

$$\delta \int_0^{L_0} \Phi d\xi = \int_0^L \left[ T\tau_i \frac{d\delta r^i}{ds} + \delta\theta^\alpha \left( \frac{dM_\alpha}{ds} + e_{\alpha\beta} \left( M\omega^\beta - M^\beta \omega \right) \right) + \delta\varphi \left( -\frac{dM}{ds} + e_{\alpha\beta} M^\alpha \omega^\beta \right) \right] ds.$$

Equating variation of energy to the work of the external forces and using the formula for variation $\delta\theta_\alpha$ (15.12), after additional integration by parts we arrive at the governing system of equations of the beam theory:

$$\frac{d}{ds} \left[ T\tau^i + \left( \frac{dM_\alpha}{ds} + e_{\alpha\beta} \left( M\omega^\beta - M^\beta\omega \right) + R_\alpha \right) \tau^{i\alpha} \right] + \frac{Q^i}{\sqrt{1+2\gamma}} = 0, \quad (15.16)$$

$$\frac{dM}{ds} - e_{\alpha\beta} M^\alpha \omega^\beta + Q = 0. \quad (15.17)$$

The bending and the twisting moments of the external forces $R^\alpha$ and $R$ are linked to the “generalized forces” $Q_i^\alpha$ from (15.9) by the equalities

$$R^\alpha = \frac{Q_i^\alpha \tau^i}{\sqrt{1+2\gamma}}, \quad Q = \frac{e_{\alpha.}^{\cdot\beta} Q_i^\alpha \tau_\beta^i}{\sqrt{1+2\gamma}}.$$

To form a closed system of equations, (15.16) and (15.17) must be supplemented by the constitutive equations (15.15) and the kinematical formulae (15.7).

Equation (15.16) admits a reduction of order. Indeed, let $q^i$ be such that

$$\frac{dq^i}{d\xi} = Q^i.$$

Then

$$T\tau^i + \left( \frac{dM_\alpha}{ds} + e_{\alpha\beta} \left( M\omega^\beta - M^\beta\omega \right) + R_\alpha \right) \tau^{i\alpha} + q^i = 0. \quad (15.18)$$

If $Q^i \equiv 0$, then $q^i$ are some constants that should be found from the boundary conditions.

Projecting (15.18) on the triad, we obtain the usual form of the equilibrium equations (15.16):

$$T + q_i \tau^i = 0, \quad \frac{dM_\alpha}{ds} + e_{\alpha\beta}\left(M\omega^\beta - M^\beta \omega\right) + R_\alpha + q_i \tau^i_\alpha = 0. \tag{15.19}$$

**Physically linear theory.** The beam model is determined by prescribing the energy density, $\Phi(\gamma, \Omega_\alpha, \Omega)$. If the amplitude of the deformations is small, then for many materials $\Phi$ may be taken as a quadratic form with respect to $\gamma$, $\Omega_\alpha$ and $\Omega$:

$$2\Phi = \bar{E}\gamma^2 + A^{\alpha\beta}\Omega_\alpha\Omega_\beta + C\Omega^2 + 2\gamma\left(A^\alpha\Omega_\alpha + A\Omega\right) + 2B^\alpha\Omega_\alpha\Omega. \tag{15.20}$$

The deformation measures, $\gamma$, $\Omega_\alpha$ and $\Omega$, are kinematically independent; therefore, all the terms in the expression (15.20) are significant.

The constitutive equations (15.15) become

$$\begin{aligned} T &= \bar{E}\gamma + A^\alpha\Omega_\alpha + A\Omega, \\ M^\alpha &= A^{\alpha\beta}\Omega_\beta + A^\alpha\gamma + B^\alpha\Omega, \\ M &= C\Omega + A\gamma + B^\alpha\Omega_\alpha. \end{aligned} \tag{15.21}$$

Here we used the fact that in physically linear theory $\gamma \ll 1$, and $\sqrt{1+2\gamma}$ may be replaced by unity. The coefficient $\bar{E}$ has the meaning of the effective Young modulus, $A^{\alpha\beta}$ the bending rigidity, $C$ the torsion rigidity, and the coefficients $A^\alpha$, $A$ and $B^\alpha$ characterize the interaction effects between bending, torsion and extension.

**Physically and geometrically linear theory.** Denote by $u_i$ the displacements of the beam axis, $u^i = r^i - \mathring{r}^i$. Suppose that the derivatives of the displacements $du^i/d\xi$ are small in comparison to unity. Let also the increments of the triad vectors $\tau^i_\alpha - \mathring{\tau}^i_\alpha$ be small. Then the rotation angle of the cross-section $\frac{1}{2}e^{\alpha\beta}\left(\tau^i_\alpha - \mathring{\tau}^i_\alpha\right)\mathring{\tau}_{i\beta}$ is also small. If the derivatives of the displacements and $\varphi$ have a certain order of smallness $\Delta$, then $\gamma, \Omega_\alpha, \Omega \sim \Delta$. In linear theory one neglects all terms on the order of $\Delta$ in comparison with unity. $T$, $M_\alpha$ and $M$ are on the order of $\Delta$. Therefore in the equilibrium equations one can replace $\omega_\alpha$ by $\mathring{\omega}_\alpha$ and $\omega$ by $\mathring{\omega}$, and differentiation over $s$ by differentiation over $\xi$ :

$$\begin{aligned} &T = -q_i \mathring{\tau}^i, \\ &\frac{dM_\alpha}{d\xi} + e_{\alpha\beta}\left(M\mathring{\omega}^\beta - M^\beta\mathring{\omega}\right) = -R_\alpha - q_i \mathring{\tau}^i_\alpha, \\ &\frac{dM}{d\xi} - e_{\alpha\beta}M^\alpha\mathring{\omega}^\beta = -Q. \end{aligned} \tag{15.22}$$

The most simplifications occur in the kinematic relations linking $\gamma$, $\Omega_\alpha$, and $\Omega$ with displacements and $\varphi$. Setting $\delta r^i = u^i$, $\delta\theta = \varphi$ in the formulae for variations (15.13), we obtain the linearized expressions for $\Omega_\alpha$, $\Omega$ and $\gamma$ :

$$\Omega_\alpha = -\frac{d}{d\xi}\left(\mathring{\tau}_{i\alpha}\frac{du^i}{d\xi}\right) + e_{\alpha\beta}\mathring{\omega}^\beta\varphi + \mathring{\omega}e_{\alpha\cdot}^{\cdot\beta}\mathring{\tau}_\beta^i\frac{du_i}{d\xi},$$

$$\Omega = \frac{d\varphi}{d\xi} + e^{\alpha\beta}\mathring{\omega}_\beta\mathring{\tau}_\alpha^i\frac{du_i}{d\xi}, \quad \gamma = \mathring{\tau}_i\frac{du^i}{d\xi}. \tag{15.23}$$

The system of equations of the linear beam theory comprises the constitutive equations (15.21), the equilibrium equations (15.22), and the kinematical relations (15.23).

**Physically nonlinear effects.** As will be seen, usually the coefficients $A^\alpha$, $A$ and $B^\alpha$ are equal to zero, and the leading interaction terms are, in fact, cubic. Although these terms are small, they describe effects which are missing in the classical theory (15.20). For example, consider the cubic term, $B\gamma\Omega^2$, in energy

$$2\Phi = \bar{E}\gamma^2 + A^{\alpha\beta}\Omega_\alpha\Omega_\beta + C\Omega^2 + 2B\gamma\Omega^2.$$

It generates an additional term $B\Omega^2$ in constitutive equation for the tension $T$, and additional term, $2B\gamma\Omega$, in torque:

$$T = \bar{E}\gamma + B\Omega^2, \qquad M = C\Omega + 2B\gamma\Omega.$$

The additional terms describe the well-known experimental fact that any twist is accompanied by some tension, while the simultaneous twist and elongation of the beam cause an additional torque, $2B\gamma\Omega$. The latter effect becomes pronounced only for finite strains. The asymmetry of the term $B\gamma\Omega^2$ with respect to $\gamma$ and $\Omega$ results in the asymmetry of the interaction effect it describes: if an elongation, $\gamma$, is inflicted upon an initially non-twisted beam ($\Omega = 0$), then the torque remains equal to zero, while there is always nonzero tension, caused by the twist, $B\Omega^2$.

**The Kirchhoff-Clebsch theory.** Let energy be quadratic with respect to the deformation measures. The coefficients of the quadratic form (15.20) depend on the elastic moduli and the geometry of the cross-section of the beam. If $E$ is the characteristic value of the elastic moduli, then from the dimension reasoning, $\bar{E} \sim Eh^2$, $A^{\alpha\beta} \sim Eh^4$, $C \sim Eh^4$, $A^\alpha \sim Eh^3$, $B^\alpha \sim Eh^4$. The energy terms have different orders of the cross-section size, $h$, and one can try to simplify the theory using the smallness of the parameter $h$. This can be done by means of the variational-asymptotic method. To describe the idea, we set the external forces, $Q_i$ and $Q_i^\alpha$, to be equal to zero and assume that the beam is deformed by prescribing its displacements at the beam ends. The first step is to minimize the formally leading term of the energy functional

$$\int_0^{L_0} \frac{1}{2} \bar{E}\gamma^2 d\xi. \tag{15.24}$$

Assume that among the admissible deformed curves, $\Gamma$, there are curves of the length equal to the length of the undeformed curve, $\mathring{\Gamma}$. This is obviously true for curvilinear beams if the displacements of its ends are sufficiently small, and is not true for straight beams subject to elongation. With the assumption made, the minimum of the functional (15.24) is equal to zero and reached at the functions, $\bar{r}^i(\xi)$, for which

$$2\gamma = \frac{d\bar{r}^i}{d\xi}\frac{d\bar{r}_i}{d\xi} - 1 = 0. \tag{15.25}$$

Therefore, the set $\mathcal{M}_0$ of the general scheme of the variational-asymptotic method comprises the functions, $\bar{r}^i(\xi)$, taking on the assigned values at the ends of the beam and satisfying the incompressibility condition (15.25).

Fixing $\bar{r}^i$, we seek for the next term in the expansion

$$r^i = \bar{r}^i + r'^i$$

where $r'^i$ is much less than $\bar{r}^i$.

One can show (we do not pause to do this) that, in many problems, $r'^i$ are uniquely defined by $\bar{r}^i$ (i.e., in terms of the variational-asymptotic scheme, $\mathcal{M}_0 = \mathcal{N}$) and make small contributions to energy. This leads to the Kirchhoff-Clebsch beam theory: the deformed state of the beam is the stationary point of the energy functional,

$$\int_0^L \frac{1}{2}(A^{\alpha\beta}\Omega_\alpha\Omega_\beta + C\Omega^2 + 2B\gamma\Omega^2)ds, \tag{15.26}$$

on the set of functions $\bar{r}^i(\xi)$ and $\tau^i_\alpha$, satisfying the kinematic conditions at the ends of the beam and the incompressibility-inextensibility condition (15.25).

**Initially straight incompressible beams.** There is a case when Kirchhoff-Clebsch theory admits a considerable simplification. Let the energy density $\Phi$ be the sum of the bending energy and the twist energy,

$$2\Phi = A^{\alpha\beta}\Omega_\alpha\Omega_\beta + C\Omega^2,$$

the bending rigidity tensor be spherical, $A^{\alpha\beta} = A\delta^{\alpha\beta}$, and the beam is straight in the undeformed state ($\mathring{\omega}_\alpha = 0$). We are going to show that the energy density $\Phi$ can be written just as a quadratic form with respect to second derivatives of $r^i$ :

$$2\Phi = A\frac{d^2r^i}{d\xi^2}\frac{d^2r_i}{d\xi^2} + C\Omega^2.$$

Therefore, the nonlinearity of the theory remains only in the incompressibility constraint (15.25) and the expression for $\omega$ in terms of $\tau_\alpha^i$ (15.7).

Indeed, using the decomposition of the Kroneker delta,

$$\delta_i^j = \tau_i \tau^j + \tau_i^\alpha \tau_\alpha^j,$$

the first formula (15.7), and the equations

$$\tau_i = \frac{dr_i}{d\xi}, \quad \tau_i \frac{d\tau^i}{d\xi} = 0,$$

we can write

$$A^{\alpha\beta}\Omega_\alpha \Omega_\beta = A\tau_\alpha^i \frac{d\tau_i}{d\xi} \tau^{j\alpha} \frac{d\tau_j}{d\xi} = A\left(\delta^{ij} - \tau^i \tau^j\right) \frac{d\tau_i}{d\xi} \frac{d\tau_j}{d\xi} = A \frac{d^2 r^i}{d\xi^2} \frac{d^2 r_i}{d\xi^2}$$

as claimed.

## 15.2 Variational Problem for Energy Density

Construction of energy density, $\Phi$, in terms of the elastic characteristics and the geometry of the beam cross-section is not as elementary as for shells where the computation of energy density is reduced to solution of some algebraic problem and integration. It turns out that for beams the energy density is the minimum value in some variational problem for a functional defined on functions of the cross-sectional coordinates. In this section we discuss the formulation of this variational problem and some of its consequences. The derivation of the variational problem from three-dimensional elasticity by the variational-asymptotic method is given in the next section. We consider only the first approximation, where the small contributions on the order of $h/R$, $h/l$ and $\varepsilon$ ($l$ being the characteristic length of the stress state along the beam, and $\varepsilon$ the strain amplitude) are ignored.

Locally, a curved beam with a large curvature-torsion radius $R$ ($R \gg h$) can be viewed as a cylinder with constant cross-section $S$. Denote the coordinates in the cross-section by $\xi^\alpha$; the third coordinate, $\xi^3$, is directed along the cylinder axis. The free energy density of the material, $F$, is a function of strains:

$$F = F\left(\varepsilon_{\alpha\beta}, \varepsilon_{\alpha 3}, \varepsilon_{33}\right).$$

We assume that the material is physically linear, and $F$ is a quadratic form of $\varepsilon_{\alpha\beta}$, $\varepsilon_{\alpha 3}$, and $\varepsilon_{33}$. We define three functions: the longitudinal energy:

$$F_{||}(\varepsilon_{33}) = \min_{\varepsilon_{\alpha\beta}, \varepsilon_{\alpha 3}} F\left(\varepsilon_{\alpha\beta}, \varepsilon_{\alpha 3}, \varepsilon_{33}\right), \tag{15.27}$$

the shear energy:

$$F_{\angle}\left(\varepsilon_{\alpha 3}, \varepsilon_{33}\right)=\min_{\varepsilon_{\alpha\beta}}\left(F\left(\varepsilon_{\alpha\beta}, \varepsilon_{\alpha 3}, \varepsilon_{33}\right)-F_{||}\left(\varepsilon_{33}\right)\right), \tag{15.28}$$

and the transverse energy:

$$F_{\perp}=F-F_{||}-F_{\angle}. \tag{15.29}$$

Accordingly, the energy density is the sum

$$F=F_{||}+F_{\angle}+F_{\perp}. \tag{15.30}$$

The direct computation given in the next section yields the expressions:

$$\begin{aligned}
F_{||} &= \frac{1}{2}E\varepsilon_{33}^{2},\\
F_{\angle} &= \frac{1}{2}G^{\alpha\beta}\left(2\varepsilon_{\alpha 3}-C_{\alpha}\varepsilon_{33}\right)(\alpha\rightarrow\beta),\\
F_{\perp} &= \frac{1}{2}C^{\alpha\beta\gamma\delta}\left(\varepsilon_{\alpha\beta}+C_{\alpha\beta}\varepsilon_{33}+C_{\alpha\beta}^{\sigma}2\varepsilon_{\sigma 3}\right)(\alpha,\beta\rightarrow\gamma,\delta).
\end{aligned} \tag{15.31}$$

The coefficients appearing in (15.31) can be taken as primary elastic characteristics of the material. For inhomogeneous beams, they can be functions of the cross-sectional coordinates, $\xi^{\alpha}$. Their explicit relations to the conventional elastic moduli are given in the next section.

To comprehend better the meaning of such energy splitting, we note that the longitudinal and shear energies do not depend on the cross-sectional in-plane strains, $\varepsilon_{\alpha\beta}$. Therefore, in-plane stresses are

$$\sigma^{\alpha\beta}=\frac{\partial F}{\partial\varepsilon_{\alpha\beta}}=\frac{\partial F_{\perp}}{\partial\varepsilon_{\alpha\beta}}=C^{\alpha\beta\gamma\delta}\left(\varepsilon_{\gamma\delta}+C_{\gamma\delta}\varepsilon_{33}+C_{\gamma\delta}^{\sigma}2\varepsilon_{\sigma 3}\right).$$

Similarly[3],

$$\sigma^{\alpha 3}=\frac{\partial F}{\partial(2\varepsilon_{\alpha 3})}=\frac{\partial F_{\perp}}{\partial(2\varepsilon_{\alpha 3})}+\frac{\partial F_{\angle}}{\partial(2\varepsilon_{\alpha 3})}=\sigma^{\gamma\delta}C_{\gamma\delta}^{\alpha}+G^{\alpha\beta}\left(2\varepsilon_{\beta 3}-C_{\beta}\varepsilon_{33}\right).$$

These relations show that the transversal energy, $F_{\perp}$, depends only on the stress components, $\sigma^{\alpha\beta}$, and $F_{\perp}$ is equal to zero if and only if $\sigma^{\alpha\beta}=0$. If $\sigma^{\alpha\beta}=0$, then the shear energy, $F_{\angle}$, depends only on $\sigma^{\alpha 3}$, and $F_{\angle}=0$ if and only if $\sigma^{\alpha 3}=0$. The

[3] See the note on the differentiation of a scalar function over components of symmetric tensors in Sect. 3.3.

longitudinal energy, $F_{||}$, is the energy which the body has in case when $\sigma^{\alpha\beta} = 0$ and $\sigma^{\alpha 3} = 0$. Therefore, the coefficient $E$ has the meaning of the Young modulus of anisotropic elastic body.

Let $u_\alpha\left(\xi^\beta\right)$ and $u\left(\xi^\beta\right)$ be some functions of cross-sectional coordinates. Consider the functional

$$\Theta\left(u_\alpha, u\right) = \Theta_\angle\left(u\right) + \Theta_\perp\left(u_\alpha, u\right),$$

$$\Theta_\angle\left(u\right) = \frac{1}{2}\int_S G^{\alpha\beta}\left(u_{,\alpha} + \Omega e_{\sigma\alpha}\xi^\sigma + C_\alpha\left(\gamma + \Omega_\sigma\xi^\sigma\right)\right)\left(\alpha \to \beta\right) d\xi^1 d\xi^2$$

$$\Theta_\perp\left(u_\alpha, u\right) = \frac{1}{2}\int_S C^{\alpha\beta\gamma\delta}\left(u_{(\alpha,\beta)} + C_{\alpha\beta}\left(\gamma + \Omega_\sigma\xi^\sigma\right)\right.$$
$$\left. + C^\lambda_{\alpha\beta}\left(u_{,\lambda} + \Omega e_{\sigma\lambda}\xi^\sigma\right)\left(\alpha, \beta \to \gamma, \delta\right)\right) d\xi^1 d\xi^2.$$

Comma in indices denotes differentiation over cross-sectional coordinates.

The functional $\Theta\left(u^\alpha, u\right)$ depends on the parameters $\gamma$, $\Omega_\alpha$ and $\Omega$. Denote its minimum value by $\Psi\left(\gamma, \Omega_\alpha, \Omega\right)$ :

$$\Psi\left(\gamma, \Omega_\alpha, \Omega\right) = \min_{u_\alpha, u} \Theta. \tag{15.32}$$

Now we can formulate the key formula of the one-dimensional beam theory: the energy density of the beam is

$$\Phi\left(\gamma, \Omega_\alpha, \Omega\right) = \frac{1}{2}\int_S E\left(\gamma + \Omega_\sigma\xi^\sigma\right)^2 d\xi^1 d\xi^2 + \Psi\left(\gamma, \Omega_\alpha, \Omega\right). \tag{15.33}$$

The derivation of this formula is given further while here we consider some of its consequences.

Functional $\Theta\left(u_\alpha, u\right)$ is invariant with respect to transformations

$$u \to u + c, \quad u_\alpha \to u_\alpha + c_\alpha + \varkappa e_{\alpha\beta}\xi^\beta,$$

$c_\alpha$, $c$, $\varkappa$ being constants. In order to select the unique minimizer (we will see in the next section its physical meaning) we set the constraints

$$\langle u\rangle = 0, \quad \langle u_\alpha\rangle = 0, \quad \left\langle u_{\alpha,\beta}\right\rangle e^{\alpha\beta} = 0. \tag{15.34}$$

**Dual variational problem.** In general, finding the minimum of the functional $\Theta$ is equivalent to solving the Neuman-type boundary value problem for a system of three elliptic equations of the second order with the variable coefficients. The transition to the dual variational problem allows one to replace the system of three equation of the second order by a system of two equations, one of the second order with respect to $u$ and one of the fourth order with respect to the stress function $\chi$.

Let the function $u$ be fixed. Then, determining of $u_\alpha$ is reduced to minimization of $\Theta_\perp$ with respect to $u_\alpha$. Let us write $\Theta_\perp$ as

$$\Theta_\perp = \max_{\sigma^{\alpha\beta}} \int_S \left\{ \sigma^{\alpha\beta} \left[ u_{(\alpha,\beta)} + C_{\alpha\beta}(\gamma + \Omega_\sigma \xi^\sigma) + C^\lambda_{\alpha\beta}(u_{,\lambda} + \Omega e_{\sigma\lambda}\xi^\sigma) \right] \right. \\ \left. - \frac{1}{2} C^{(-1)}_{\alpha\beta\gamma\delta} \sigma^{\alpha\beta}\sigma^{\gamma\delta} \right\} d\xi^1 d\xi^2, \tag{15.35}$$

where $C^{(-1)}_{\alpha\beta\gamma\delta}$ is the inverse tensor of the tensor $C^{\alpha\beta\gamma\delta}$, and maximum is sought over all symmetric tensor fields $\sigma^{\alpha\beta} = \sigma^{\beta\alpha}$. Then

$$\min_{u_a} \Theta_\perp = \min_{u_\alpha} \max_{\sigma^{\alpha\beta}} \{\cdot\}, \tag{15.36}$$

where $\{\cdot\}$ is the integral in the right-hand side of (15.35). Changing the order of maximization and minimization in (15.36) (which is possible when the assumptions of the general scheme of Sect. 5.8 for integral functionals are satisfied) and calculating $\min_{u_\alpha} \{\cdot\}$, we find that it is equal to

$$\Theta^*_\perp(\sigma^{\alpha\beta}, u) = \int_S \left\{ \sigma^{\alpha\beta} \left[ C_{\alpha\beta}(\gamma + \Omega_\sigma \xi^\sigma) + C^\lambda_{\alpha\beta}(u_{,\lambda} + \Omega e_{\sigma\lambda}\xi^\sigma) \right] \right. \\ \left. - \frac{1}{2} C^{(-1)}_{\alpha\beta\gamma\delta} \sigma^{\alpha\beta}\sigma^{\gamma\delta} \right\} d\xi^1 d\xi^2, \tag{15.37}$$

if

$$\sigma^{\alpha\beta}{}_{,\beta} = 0 \qquad \text{in } S, \ \sigma^{\alpha\beta}\nu_\beta = 0 \qquad \text{on } \partial S, \tag{15.38}$$

and $-\infty$ if (15.38) does not hold. Therefore,

$$\min_{u^\alpha} \Theta_\perp = \max_{\sigma^{\alpha\beta}} \Theta^*_\perp(\sigma^{\alpha\beta}, u), \tag{15.39}$$

where maximum is sought over all $\sigma^{\alpha\beta}$ satisfying the constraints (15.38).

Each solution of (15.38) may be written as (see Sect. 6.6)

$$\sigma^{\alpha\beta} = e^{\alpha\mu} e^{\beta} \chi_{,\mu\nu},$$

where $\chi$ is some function in $S$ satisfying the boundary constraints

$$\chi_{,\alpha} = const \text{ on } \partial S. \tag{15.40}$$

For a given stress state, function $\chi$ is determined up to an arbitrary linear function. One can choose this function in such a way that for a simply connected region,

$$\chi_{,\alpha} = 0 \text{ on } \partial S,$$

while for multiple-connected region, $\chi_{,\alpha} = 0$ at one of the components of the boundary of $S$. So

$$\min_{u_\alpha, u} \Theta = \min_{u_\alpha, u}(\Theta_\angle + \Theta_\perp) = \min_{u} \max_{\chi} \left[\Theta_\angle(u) + \Theta^*_\perp(\chi, u)\right], \tag{15.41}$$

where the functional $\Theta^*_\perp(\chi, u)$ is obtained by substituting the expressions for $\sigma^{\alpha\beta}$ in terms of $\chi$ into (15.37):

$$\Theta^*_\perp(\chi, u) = \int_S \Big\{ e^{\alpha\mu} e^{\beta\nu} \chi_{,\mu\nu} \left[ C_{\alpha\beta}(\gamma + \Omega_\sigma \xi^\sigma) + C_{\alpha\beta}{}^{\lambda}(u_{,\lambda} + \Omega e_{\sigma\lambda} \xi^\sigma)\right]$$
$$- \frac{1}{2} C^{(-1)}_{\alpha\beta\gamma\delta} e^{\alpha\mu} e^{\beta\nu} e^{\gamma\lambda} e^{\delta\kappa} \chi_{,\mu\nu} \chi_{,\lambda\kappa} \Big\} d\xi^1 d\xi^2.$$

The Euler equations of the minimax problem (15.41) are the two equations of the second and the fourth orders for $u$ and $\chi$.

As a rule, the variational problem under consideration may be solved only by numerical methods. However, there is a number of important particular cases, where the problem admits considerable simplifications or the exact solutions. The most important simplifications are related to the homogeneity properties, the existence of the plane of elastic symmetry perpendicular to the beam central line and of the central symmetry of elastic characteristics and the geometry of the cross-section. We begin from consideration of homogeneous beams.

**Homogeneous beams.** A beam is called homogeneous if $C^{\alpha\beta\gamma\delta}$, $G^{\alpha\beta}$, $C_{\alpha\beta}$, $C_{\alpha\beta}{}^{\lambda}$ and $C_\alpha$ do not depend on $\xi^\alpha$.

Let us show that for general anisotropy and arbitrary geometry of the cross-section, the minimum value of the functional $\Theta$ can be found analytically up to a multiplicative constant; to compute this constant, one has to solve some variational problem. To obtain this result, we change the required function, $u \to v$:

$$u = -C_\alpha \xi^\alpha \gamma - \frac{1}{2} C_\alpha \Omega_\beta \left( \xi^\alpha \xi^\beta - \frac{\langle \xi^\alpha \xi^\beta \rangle}{|S|} \right) + v. \tag{15.42}$$

By $\langle\cdot\rangle$ we denote further the integral over cross-section

$$\langle\cdot\rangle = \int_S \cdot d\xi^1 d\xi^2,$$

by $|S|$ the area of the cross-section.

The functional $\Theta_\angle$ becomes

$$\Theta_{\angle} = \frac{1}{2}\langle G^{\alpha\beta}\left(v_{,\alpha} + \tilde{\Omega}e_{\sigma\alpha}\xi^{\sigma}\right)(\alpha \to \beta)\rangle,$$

where $\tilde{\Omega}$ denotes the parameter:

$$\tilde{\Omega} = \Omega - \frac{1}{2}e^{\mu\nu}C_{\mu}\Omega_{\nu}.$$

Substitution of (15.42) into the functional $\Theta_{\perp}$ results in the relation

$$\Theta_{\perp} = \frac{1}{2}\int\limits_{S} C^{\alpha\beta\gamma\delta}\left[u_{(\alpha,\beta)} + D_{\alpha\beta} + D_{\alpha\beta\lambda}\xi^{\lambda} + C_{\alpha\beta}{}^{\lambda}v_{,\lambda}\right][\alpha,\beta \to \gamma,\delta]d\xi^1 d\xi^2 \ . \tag{15.43}$$

Here

$$D_{\alpha\beta} = \left(C_{\alpha\beta} - C_{\alpha\beta}{}^{\lambda}C_{\lambda}\right)\gamma, \qquad D_{\alpha\beta\lambda} = C_{\alpha\beta}\Omega_{\lambda} - C^{\sigma}_{\alpha\beta}C_{(\sigma}\Omega_{\lambda)} + C^{\sigma}_{\alpha\beta}\Omega e_{\lambda\sigma}.$$

Formula (15.43) suggests the substitution $u_\alpha \to v_\alpha$:

$$u_{\alpha} = -D_{\alpha\beta}\xi^{\beta} - \frac{1}{2}a_{\alpha\beta\gamma}\left(\xi^{\beta}\xi^{\gamma} - \frac{\langle\xi^{\beta}\xi^{\gamma}\rangle}{|S|}\right) + v_{\alpha}. \tag{15.44}$$

where $a_{\alpha\beta\gamma}$ are some constants yet to be defined. The constants $a_{\alpha\beta\gamma}$ are symmetric with respect to the indices $\beta, \gamma$. We set $a_{\alpha\beta\gamma}$ to be a solution of the linear system of equations

$$a_{(\alpha\beta)\gamma} \equiv \frac{1}{2}(a_{\alpha\beta\gamma} + a_{\beta\alpha\gamma}) = D_{\alpha\beta\gamma}. \tag{15.45}$$

It can be checked by direct inspection that for any tensor $a_{\alpha\beta\gamma}$ of the third order, which is symmetric with respect to the last two indices, the identity holds:

$$a_{\alpha\beta\gamma} = a_{(\alpha\beta)\gamma} + a_{(\alpha\gamma)\beta} - a_{(\beta\gamma)\alpha}. \tag{15.46}$$

Therefore, the solution of the system of equations (15.45) is

$$a_{\alpha\beta\gamma} = D_{(\alpha\beta)\gamma} + D_{(\alpha\gamma)\beta} - D_{(\beta\gamma)\alpha}. \tag{15.47}$$

After the changes of variables, the functional $\Theta$, due to (15.45), becomes

$$\begin{aligned}\Theta = \frac{1}{2}\int\limits_{S} &\left[G^{\alpha\beta}\left(v_{,\alpha} + \tilde{\Omega}e_{\sigma\alpha}\xi^{\sigma}\right)(\alpha \to \beta)\right.\\ &\left.+C^{\alpha\beta\gamma\delta}\left(v_{(\alpha,\beta)} + C_{\alpha\beta}{}^{\lambda}v_{,\lambda}\right)(\alpha,\beta \to \gamma,\delta)\right]d\xi^1 d\xi^2\end{aligned} \tag{15.48}$$

The minimizing element of the functional $\Theta$ is proportional to $\tilde{\Omega}$; therefore

$$\Psi = \frac{1}{2} C \tilde{\Omega}^2,$$

where $C$ is the minimum value of the functional

$$C = \min_{\bar{v}, \bar{v}_\alpha} \int_S \left( G^{\alpha\beta} \left( \bar{v}_{,\alpha} + e_{\sigma\alpha} \xi^\sigma \right) (\alpha \to \beta) + \right.$$
$$\left. + C^{\alpha\beta\gamma\delta} \left( \bar{v}_{(\alpha,\beta)} + C_{\alpha\beta}{}^{\lambda} \bar{v}_{,\lambda} \right) (\alpha, \beta \to \gamma, \delta) \right) d\xi^1 d\xi^2. \tag{15.49}$$

Here, $\bar{v} = v/\tilde{\Omega}$, $\bar{v}_\alpha = v_\alpha/\tilde{\Omega}$. The constant $C$ is called the torsional rigidity of the beam.

So, for a homogeneous beam, $\Psi$ does not depend on $\gamma$, while $\Omega_\alpha$ and $\Omega$ enter into $\Psi$ through the combination $\tilde{\Omega}$:

$$\Psi = \frac{1}{2} C \left( \Omega - \frac{1}{2} e^{\mu\nu} C_\mu \Omega_\nu \right)^2. \tag{15.50}$$

**Elliptic cross-section.** Let us find the torsional rigidity for a beam with an elliptic cross-section, $b_{\alpha\beta} \xi^\alpha \xi^\beta \le 1$, where $b_{\alpha\beta}$ is a positive symmetric tensor.[4] Let us find the functions, $\bar{v}_\alpha$ and $\bar{v}$, from the system of equations

$$G^{\alpha\beta} \left( \bar{v}_{,\beta} + e_{\sigma\beta} \xi^\sigma \right) = a e^{\alpha\lambda} b_{\lambda\mu} \xi^\mu, \qquad \bar{v}_{(\alpha,\beta)} + C^{\lambda}_{\alpha\beta} \bar{v}_{,\lambda} = 0, \tag{15.51}$$

where $a$ is a constant yet to be defined. From the first equation (15.51), we have

$$\bar{v}_{,\beta} = \left( a G^{(-1)}_{\alpha\beta} e^{\alpha\lambda} b_{\lambda\mu} - e_{\mu\beta} \right) \xi^\mu. \tag{15.52}$$

These are two equations for one function, $\bar{v}$. We choose $a$ to make these equations compatible: the derivative $\partial^2 \bar{v} / \partial \xi^1 \partial \xi^2$ found from (15.52) for $\beta = 1$ and the derivative $\partial^2 \bar{v} / \partial \xi^2 \partial \xi^1$found from (15.52) for $\beta = 2$ must be equal, or

$$e^{\mu\beta} \left( \bar{v}_{,\beta} \right)_{,\mu} = \left( a G^{(-1)}_{\alpha\beta} e^{\alpha\lambda} b_{\lambda\mu} - e_{\mu\beta} \right) e^{\mu\beta} = 0.$$

Hence,

$$a = \frac{2}{G^{(-1)}_{\alpha\beta} e^{\alpha\lambda} e^{\mu\beta} b_{\lambda\mu}}. \tag{15.53}$$

From (15.52), we find that

[4] If the coordinates $\xi^1$, $\xi^2$ are directed along the axes of the ellipse, it will not result in simplifications, because there is an arbitary anisotropy of the elastic properties. Due to this, all relations are written in tensor form.

$$\bar{v} = \frac{1}{2}\left(aG_{\beta\alpha}^{(-1)}e^{\alpha\lambda}b_{\lambda\mu} - e_{\mu\beta}\right)\left(\xi^{\mu}\xi^{\beta} - \left\langle\xi^{\mu}\xi^{\beta}\right\rangle \Big/ |S|\right). \tag{15.54}$$

Substituting (15.54) into the second equation (15.51) results in

$$\bar{v}_{(\alpha,\beta)} + C_{\sigma\lambda}^{\lambda}\left(aG_{\alpha\beta}{}^{(-1)}e^{\sigma\nu}b_{\nu\mu} - e_{\mu\lambda}\right)\xi^{\mu} = 0. \tag{15.55}$$

The solution of (15.55) is

$$\bar{v}_{\alpha} = \frac{1}{2}\bar{a}_{\alpha\beta\gamma}\xi^{\beta}\xi^{\gamma}, \tag{15.56}$$

where the constant tensor, $\bar{a}_{\alpha\beta\gamma}$, is symmetric with respect to the indices $\beta, \gamma$ and satisfies the system of linear equations

$$\bar{a}_{(\alpha\beta)\gamma} = -C_{\alpha\beta}^{\lambda}\left(aG_{\sigma\lambda}^{(-1)}e^{\sigma\nu}b_{\nu\gamma} - e_{\gamma\lambda}\right).$$

This system is analogous to the system of equations (15.45), and its solution is given by the formula (15.47).

It is easy to check that the functions, $\bar{v}$ and $\bar{v}_{\alpha}$, satisfying equations (15.51), also satisfy the Euler equations for the functional (15.49) and the corresponding natural boundary conditions. Substituting them into (15.49) we get

$$C = a^2 G_{\alpha\beta}^{(-1)}e^{\alpha\lambda}e^{\beta\sigma}b_{\lambda\mu}b_{\sigma\nu}I^{\mu\nu},$$

where $I^{\mu\nu}$ are the moments of the cross-section,

$$I^{\mu\nu} = \int \xi^{\mu}\xi^{\gamma}d\xi^1 d\xi^2.$$

In the principal coordinates of the ellipse with the semi-axes, $b_1, b_2$, $b_{11} = \frac{1}{b_1^2}$, $b_{22} = \frac{1}{b_2^2}$, $b_{12} = 0$, $I_{11} = \frac{1}{4}\pi b_1^3 b_2$, $I_{22} = \frac{1}{4}\pi b_2^3 b_1$, and

$$C = \frac{\pi b_1^3 b_2^3}{G_{11}^{(-1)}b_1^2 + G_{22}^{(-1)}b_2^2} = \frac{4}{G_{11}^{(-1)}(I_{22})^{-1} + G_{22}^{(-1)}(I_{11})^{-1}}. \tag{15.57}$$

Note that we did not assume that the tensors $G^{\alpha\beta}$ and $I^{\alpha\beta}$ are coaxial. If they are, then $G_{11}^{(-1)} = (G_{11})^{-1}$, $G_{22}^{(-1)} = (G_{22})^{-1}$ and

$$C = \frac{\pi G_{11}G_{22}b_1^3 b_2^3}{G_{22}b_1^2 + G_{11}b_2^2}.$$

In the case of a transversally-isotropic material, $G^{\alpha\beta} = G\delta^{\alpha\beta}$, and

$$C = \frac{\pi G b_1^3 b_2^3}{b_1^2 + b_2^2} = \frac{4G}{(I_{11})^{-1} + (I_{22})^{-1}} = \frac{4G}{I_\alpha^{(-1)\alpha}},$$

where $I_{\alpha\beta}^{(-1)}$ is the inverse tensor of $I^{\alpha\beta}$.

**Estimate of the torsion rigidity of homogeneous anisotropic beam with an arbitrary cross-section.** Let us show that for arbitrary geometry of the beam cross-section the following estimate for the torsional rigidity is true:

$$C \le \frac{4}{G_{\alpha\beta}{}^{(-1)} e^{\alpha\mu} e^{\beta\nu} I_{\mu\nu}^{(-1)}}. \tag{15.58}$$

The inequality (15.58) is a generalization to the anisotropic case of the Nikolai's inequality,

$$C \le \frac{4 G I_{11} I_{22}}{I_{11} + I_{22}},$$

or, in tensor notation,

$$C \le \frac{4G}{I_\alpha^{(-1)\alpha}}.$$

Let us use the functions (15.54) and (15.56) as the trial functions in the variational principle for torsional rigidity, with $b_{\alpha\beta}$ being some constants. This gives the following upper estimate of the torsional rigidity:

$$C \le \frac{4 \bar{G}^{\mu\nu} b_{\mu\lambda} b_{\nu\sigma} I^{\lambda\sigma}}{\left(\bar{G}^{\mu\nu} b_{\mu\nu}\right)^2}. \tag{15.59}$$

Here $\bar{G}^{\mu\nu} = G_{\alpha\beta}^{(-1)} e^{\alpha\mu} e^{\beta\nu}$.

Let us minimize the right-hand side of (15.59) with respect to $b_{\alpha\beta}$. Since the right-hand side of (15.59) is not affected by the transformation $b_{\alpha\beta} \to \lambda b_{\alpha\beta}$, this problem is equivalent to minimization of the quadratic form $4\bar{G}^{\mu\nu} b_{\mu\lambda} b_{\nu\sigma} I^{\lambda\sigma}$, with the constraint $\bar{G}^{\mu\nu} b_{\mu\nu} = 1$. It is easy to see that the minimum value of the quadratic form is reached at the tensor $b_{\mu\nu} = I_{\mu\nu}^{(-1)} (\bar{G}^{\lambda\sigma} I_{\lambda\sigma}^{(-1)})^{-1}$. This proves the assertion made.

The inequality (15.58) allows one to establish some extremal features of the torsional rigidity. Consider the torsional rigidities of the beams with different cross-sections, and, possibly, with different elastic moduli, but such that the contraction $G_{\alpha\beta}^{(-1)} e^{\alpha\mu} e^{\beta\nu} I_{\mu\nu}^{(-1)}$ is the same. From the comparison of the relations (15.57) and (15.58), it is seen that on this set of beams, the beams with an elliptic cross-section have the maximum torsional rigidity. If, in addition, the tensor $G^{\alpha\beta}$ is spherical (and $G_{\alpha\beta}^{(-1)} = G^{(-1)} \delta_{\alpha\beta}$), then $G_{\alpha\beta}^{(-1)} e^{\alpha\mu} e^{\beta\nu} I_{\mu\nu}^{(-1)} = G^{-1} I_\alpha^{(-1)\alpha}$, and, strengthening the estimate (15.58) by means of the inequality

$$\frac{1}{I_{\mu}^{(-1)\mu}} \leq \frac{1}{4} I_{\mu}^{\mu},$$

we get

$$C \leq G I_{\mu}^{\mu}.$$

This inequality becomes an equality for circular beams; consequently, of all the cross-sections with the same polar moment $I_{\mu}^{\mu}$, the circle has the maximum torsional rigidity.

**Heterogeneous beams.** We limit the consideration of heterogeneous beams with the case when the beam has a plane of elastic symmetry perpendicular to the central line. The "two-dimensional" elastic moduli tensors with an odd number of indices are equal to zero ($C_{\alpha\beta}^{\gamma} = 0, C_{\alpha} = 0$) and the minimization problem for the functional $\Theta$ splits into two independent problems: the minimization problem for the functional

$$\Theta_{\angle}(u) = \frac{1}{2} \int_{S} \left( G^{\alpha\beta} \left( u_{,\alpha} + \Omega e_{\sigma\alpha} \xi^{\sigma} \right) (\alpha \to \beta) \right) d\xi^1 d\xi^2,$$

and the minimization problem for the functional

$$\Theta_{\perp}(u_{\alpha}) = \frac{1}{2} \int_{S} \left( C^{\alpha\beta\gamma\delta} \left( u_{(\alpha,\beta)} + C_{\alpha\beta}(\gamma + \Omega_{\sigma} \xi^{\sigma}) \right) (\alpha, \beta \to \gamma, \delta) \right) d\xi^1 d\xi^2. \tag{15.60}$$

The first problem is, in essence, the well-known Saint-Venant torsion problem (see [223]). The minimizing function of the functional $\Theta_{\angle}(u)$ is proportional to $\Omega$; therefore,

$$\min_{u} \Theta_{\angle}(u) = \frac{1}{2} C \Omega^2,$$

where the torsion rigidity, $C$, is the minimum value of the functional

$$C = \min_{u} \int_{S} G^{\alpha\beta} (u_{,\alpha} + e_{\sigma\alpha} \xi^{\sigma})(\alpha \to \beta) \; d\xi^1 d\xi^2.$$

The second problem corresponds to a problem of two-dimensional elasticity. Since the minimizing functions depend on $\gamma$ and $\Omega_{\alpha}$ linearly, the minimum value of $\Theta_{\perp}$ is a quadratic form with respect to $\gamma$ and $\Omega_{\alpha}$:

$$\Psi_{\perp} = \min_{u_{\alpha}} \Theta_{\perp} = \frac{1}{2} \left( E_{\perp} \gamma^2 + 2 E_{\perp}^{\alpha} \gamma \Omega_{\alpha} + E_{\perp}{}^{\alpha\beta} \Omega_{\alpha} \Omega_{\beta} \right).$$

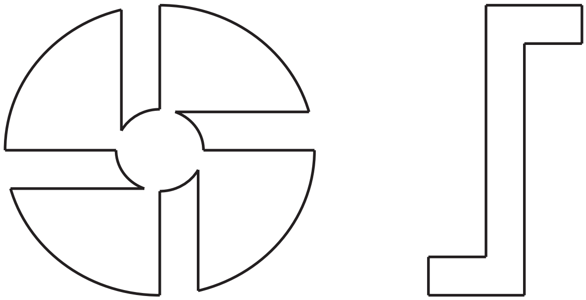

**Fig. 15.2** Examples of central-symmetric cross-sections which do not possess two axes of symmetry

So, for the beams with an elastic symmetry plane perpendicular to the central line, the function $\Psi$ is a sum of the torsion energy, $C\Omega^2/2$, and a contribution to the extension and bending energies $\Psi_\perp$; function $\Psi$ does not contain the interaction terms between torsion and extension, $\gamma \cdot \Omega$, and between torsion and bending, $\Omega \cdot \Omega_\alpha$.

More substantial conclusions can be made if the cross-section and the elastic properties of the beam also have the central symmetry. The cross-section is called central-symmetric if, for every point with the coordinates $\xi^\alpha$, it contains a point with the coordinates $-\xi^\alpha$. Note that the cross-section may be central-symmetric and not have two symmetry axes (two such cross-sections are shown in Fig. 15.2).

For functions defined in a central-symmetric regions, the notion of evenness may be introduced: a function is even if it has the same values at the points $\xi^\alpha$ and -$\xi^\alpha$. By definition, the elastic characteristics of the beam are central-symmetric if $C$, $G^{\alpha\beta}$ and $C^{\alpha\beta}$ are even functions of $\xi^\alpha$.

Let us write $u_\alpha$ as a sum of odd and even functions (they are denoted by one and two primes, respectively), $u_\alpha = u'_\alpha + u''_\alpha$. Then the functional $\Theta_\perp$ becomes a sum of two functionals, one of which depends only on $u'_\alpha$ and the other only on $u''_\alpha$:

$$\Theta_\perp = \Theta'_\perp + \Theta''_\perp,$$
$$\Theta'_\perp = \frac{1}{2}\int_S C^{\alpha\beta\gamma\delta}\left(u'_{(\alpha,\beta)} + C_{\alpha\beta}\gamma\right)(\alpha,\beta \to \gamma,\delta)\, d\xi^1 d\xi^2,$$
$$\Theta''_\perp = \frac{1}{2}\int_S C^{\alpha\beta\gamma\delta}\left(u''_{(\alpha,\beta)} + C_{\alpha\beta}\Omega_\sigma\xi^\sigma\right)(\alpha,\beta \to \gamma,\delta)\, d\xi^1 d\xi^2.$$

Functionals $\Theta'_\perp$ and $\Theta''_\perp$ can be minimized independently. Minimum of $\Theta'_\perp$ is proportional to $\gamma^2$, minimum of $\Theta''_\perp$ is a quadratic form of $\Omega_\alpha$. So

$$2\Psi_\perp = E_\perp\gamma^2 + E_\perp^{\alpha\beta}\Omega_\alpha\Omega_\beta,$$

and energy does not contain the interaction terms between $\gamma$, $\Omega_\alpha$ and $\Omega$. As is easy to see, the similar conclusion can be made if $C^{\alpha\beta\gamma\delta}$ are even functions of $\xi^\alpha$, $G^{\alpha\beta}$ are arbitrary while $C^{\alpha\beta}$ are either odd or even functions of $\xi^\alpha$.

**Heterogeneous beams with constant Poisson coefficients.** It turns out that, for the beams with constant Poisson coefficients, the following remarkable feature holds: $\Psi$ contains only the torsion energy

$$\Psi = \frac{1}{2} C \Omega^2 \quad \text{and} \quad \Psi_\perp = 0. \tag{15.61}$$

To prove that, we use the fact that any value of the functional $\Theta$ gives an upper bound of $\Psi$. We take

$$u_\alpha = -C_{\alpha\beta} \gamma \xi^\beta - \frac{1}{2} a_{\alpha\beta\gamma} \xi^\beta \xi^\gamma. \tag{15.62}$$

The constants $a_{\alpha\beta\gamma}$ are chosen as the solution of the linear system of equations

$$a_{(\alpha\beta)\gamma} = C_{\alpha\beta} \Omega_\gamma. \tag{15.63}$$

For these functions, $\Theta_\perp = 0$. Consequently, $\Psi_\perp = \min\limits_{u_\alpha} \Theta_\perp = 0$.

The minimizing functions $u_\alpha$ have a universal form for an arbitrary cross-section and an arbitrary dependence of the elastic moduli $C^{\alpha\beta\gamma\delta}$ on the coordinates; this form is found from (15.62), (15.63) and (15.46):

$$u_\alpha = -C_{\alpha\beta} \gamma \xi^\beta - \left( C_{\alpha(\beta} \Omega_{\gamma)} - \frac{1}{2} C_{\beta\gamma} \Omega_\alpha \right) \xi^\beta \xi^\gamma. \tag{15.64}$$

**Criterion of vanishing of $\Psi_\perp$.** The quadratic form, $\Psi_\perp$, is identically equal to zero not only for beams with constant Poisson coefficients, but also for some beams with variable Poisson coefficients. Let us prove the following assertion.

Let the region $S$ be divided into two parts, $S_1$ and $S_2$, by a smooth line $L$. In both sub-regions, the Poisson coefficients, $C_{\alpha\beta}$, are continuous, but they may be discontinuous on the line $L$. Then, in order for $\Psi_\perp$ to be equal to zero, it is necessary and sufficient that there is a function $c(\xi^\alpha)$ such that in the regions of continuity of the Poisson coefficients,

$$C_{\alpha\beta} = c_{,\alpha\beta} \tag{15.65}$$

while on the discontinuity line,[5]

$$[c_{,\alpha}] = const. \tag{15.66}$$

[5] Recall that the symbol [$A$] denotes the difference between the values of $A$ on the two sides of the discontinuity line.

*Sufficiency.* Let us set

$$u_\alpha = -c_{,\alpha}\gamma - c_{,\alpha}\Omega_\sigma\xi^\sigma + c\Omega_\alpha + \omega e_{\alpha\sigma}\xi^\sigma + a_\alpha, \tag{15.67}$$

where $\omega$, $a_\alpha$ are constants which may have different values in $S_1$ and $S_2$. Then, inside $S_1$ and $S_2$,

$$u_{(\alpha,\beta)} + C_{\alpha\beta}\gamma + C_{\alpha\beta}\Omega_\sigma\xi^\sigma = 0. \tag{15.68}$$

Let us choose the constants $\omega$ and $a_\alpha$ in such a way that $u_\alpha$ be continuous on $L$,

$$-r_\alpha\gamma - r_\alpha\Omega_\sigma\xi^\sigma + [c]\Omega_\alpha + [\omega]e_{\alpha\sigma}\xi^\sigma + [a_\alpha] = 0, \tag{15.69}$$

where $r_\alpha$ denote the constants equal to $[c_{,\alpha}]$.

From the kinematic compatibility conditions

$$\frac{d[c]}{d\sigma} = r_\alpha\frac{d\xi^\alpha}{d\sigma},$$

where $\sigma$ is a parameter on $L$, it follows that

$$[c] = r_\alpha\xi^\alpha + r, \quad r = const. \tag{15.70}$$

The constant terms in (15.69) cancel out if we set

$$[a_\alpha] = r_\alpha\gamma + r\Omega_\alpha. \tag{15.71}$$

The terms linear with respect to $\xi^\sigma$ have the form $(r_\sigma\Omega_\alpha - r_\alpha\Omega_\sigma + [\omega]e_{\alpha\sigma})\xi^\sigma$. Therefore, setting

$$[\omega] = e^{\sigma\alpha}r_\sigma\Omega_\alpha, \tag{15.72}$$

we completely satisfy (15.69). The formulae (15.71) and (15.72) give the differences between the constants $\omega$ and $a_\alpha$ in the regions $S_1$ and $S_2$. If one sets the constraints on $u_\alpha$ (15.34), then these constants are completely defined.

The functions $u_\alpha$ are admissible and the functional $\Theta_\perp$ is equal to zero at these functions. Consequently, $\Psi_\perp = 0$.

*Necessity.* Let $\Psi_\perp = 0$. Then since the quadratic form $F_\perp$ is non-degenerate, the equalities (15.68) hold. The parameters $\gamma$ and $\Omega_\alpha$ are independent; therefore, as follows from (15.68), the compatibility conditions for the tensors $C_{\alpha\beta}$ and $C_{\alpha\beta}\xi^\sigma$ should be satisfied in $S_1$ and $S_2$ to guarantee that $C_{\alpha\beta}$ and $C_{\alpha\beta}\xi^\sigma$ may be presented as a symmetric part of the gradient of a vector field. For an arbitrary tensor $\varepsilon_{\alpha\beta}$, which admits such representation, $\varepsilon_{\alpha\beta} = u_{(\alpha,\beta)}$, the necessary and sufficient compatibility conditions are

$$\Delta\varepsilon_{\alpha\beta} + \varepsilon^{\lambda}_{\lambda,\alpha\beta} - \varepsilon^{\mu}_{\alpha,\beta\mu} - \varepsilon^{\mu}_{\beta,\alpha\mu} = 0, \tag{15.73}$$

where $\Delta$ is the two-dimensional Laplace operator. Substitution of $C_{\alpha\beta}$ and $C_{\alpha\beta}\xi^{\sigma}$ into (15.73) results in the system of equations

$$\Delta C_{\alpha\beta} + C^{\lambda}_{\lambda,\alpha\beta} - C^{\mu}_{\alpha,\beta\mu} - C^{\mu}_{\beta,\alpha\mu} = 0, \tag{15.74}$$

$$2C_{\alpha\beta,\lambda} + \left(C^{\nu}_{\nu,\alpha} - C^{\nu}_{\alpha,\nu}\right)\delta_{\lambda\beta} + \left(C^{\nu}_{\nu,\beta} - C^{\nu}_{\beta,\nu}\right)\delta_{\lambda\alpha} - C_{\alpha\lambda,\beta} - C_{\beta\lambda,\alpha} = 0. \tag{15.75}$$

Contracting the equalities (15.75) with respect to $\alpha$, $\beta$, we get

$$C^{\nu}_{\nu,\lambda} = C^{\nu}_{\lambda,\nu}. \tag{15.76}$$

Equation (15.75) can be simplified using (15.76)[6]

$$2C_{\alpha\beta,\lambda} = C_{\alpha\lambda,\beta} + C_{\beta\lambda,\alpha}. \tag{15.77}$$

Let us write these equations substituting indices, $\beta \to \lambda$, $\lambda \to \beta$:

$$2C_{\alpha\lambda,\beta} = C_{\alpha\beta,\lambda} + C_{\lambda\beta,\alpha}, \tag{15.78}$$

and plug $C_{\alpha\lambda,\beta}$ in (15.77) from (15.78). Due to symmetry $C_{\alpha\beta}$, we obtain

$$C_{\beta\alpha,\lambda} = C_{\beta\lambda,\alpha}.$$

Hence, there exists a vector $C_{\alpha}$ such that

$$C_{\alpha\beta} = C_{\alpha,\beta}.$$

The symmetry of $C_{\alpha\beta}$ yields that vector $C_{\alpha}$ is potential,

$$C_{\alpha} = c_{,\alpha},$$

and, therefore, (15.65) hold. Equations (15.74) and (15.75) are satisfied for $C_{\alpha\beta} = c_{,\alpha\beta}$.

The equalities (15.67) follow from the relations (15.65) and (15.68): the displacement, $u_{\alpha}$, is the sum of the general solution of the homogeneous equations, rigid motions (the last two terms of (15.67)) and a solution of the inhomogeneous

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[6] If there were no coefficient 2 on the left-hand side of (15.77), these equations can be interpreted as the conditions that the Kristoffel's symbols of the two-dimensional metric, $C_{\alpha\beta}$, vanish; that is possible only for $C_{\alpha\beta} = const$.

equations (15.68) (the first three terms in (15.67)). It remains to show that the condition of the continuity of $u_\alpha$ results in the constraints (15.66). Indeed, since the parameters $\gamma$ and $\Omega_\alpha$ are arbitrary, $[\omega] = \theta\gamma + \theta^\alpha\Omega_\alpha$, $[a_\alpha] = r_\alpha\gamma + r_\alpha^\lambda\Omega_\lambda$ where $\theta, \theta^\alpha, r_\alpha$ and $r_\alpha^\beta$ are constants. The equations $[u_\alpha] = 0$, are reduced to the system of equations

$$\left[c_{,\alpha}\right] = r_\alpha + \theta e_{\alpha\sigma}\xi^\sigma, \quad [c]\delta_\alpha^\beta = (r_\alpha + \theta e_{\alpha\sigma}\xi^\sigma)\xi^\beta - \theta^\beta e_{\alpha\sigma}\xi^\sigma - r_\alpha^\beta. \qquad (15.79)$$

Let us set $\alpha = 1, \beta = 2$ and $\alpha = 2, \beta = 1$ in the second equations (15.79). Then, on the line of discontinuity,

$$r_1\xi^2 + \theta(\xi^2)^2 - \theta^2\xi^2 - r_1^2 = 0, \quad r_2\xi^1 - \theta(\xi^1)^2 + \theta^1\xi^1 - r_2^1 = 0.$$

At least one of these relations has to be an identity with respect to the coordinates $\xi^1, \xi^2$; otherwise, the first one defines the line $\xi^2 = const$, while the second one the line $\xi^1 = const$. Let the first relation be identically satisfied. Then $\theta = 0$, and it follows from the first equation (15.79) that $[c_{,\alpha}] = const$. The same result is obtained if the second relation is identically satisfied. For $[c_{,\alpha}] = const$, the continuity of $u_\alpha$ may be achieved by the choice of the constants $\omega$ and $a_\alpha$ in the regions $S_1$ and $S_2$, as has already been checked.

It is obvious that analogous conclusions also hold for several lines of discontinuity. Note several consequences of the criterion obtained.

1. In order for $\Psi_\perp \equiv 0$ for a heterogeneous beam with discontinuous Poisson coefficients, it is necessary that the conditions

$$\left[C_{\alpha\beta}\right]\tau^\beta = 0 \qquad (15.80)$$

   be satisfied on the discontinuity line, where $\tau^\beta$ is the tangent vector to the discontinuity line.
   The relations (15.80) are obtained by differentiation of (15.66) along the discontinuity line.
2. For a transversely isotropic body, $\Psi_\perp \equiv 0$ if and only if the Poisson coefficient $\nu$ is constant.
   Indeed, if $C_{\alpha\beta} = \nu\delta_{\alpha\beta}$, then (15.65) becomes

$$c_{,12} = 0, \;\; c_{,11} = c_{,22}$$

   and has a unique solution, $\nu = c_{,11} = c_{,22} = const$; there is no curve for which the conditions (15.80) can be satisfied for $[\nu] \neq 0$.
3. In order for $\Psi_\perp \equiv 0$ for a heterogeneous beam with piece-wise constant Poisson coefficients, it is necessary that

$$\det \left\|[C_{\alpha\beta}]\right\| = 0$$

and the discontinuity lines are straight and perpendicular and the principal axis of the tensor $[C_{\alpha\beta}]$ for which the corresponding eigenvalue does not equal to zero.

This assertion is an elementary consequence of (15.80).

**Summary.** Now we outline the most important results obtained.

According to (15.115) and (15.87), for homogeneous anisotropic beams,

$$\Phi = \frac{1}{2}\left(E|S|\gamma^2 + EI^{\alpha\beta}\Omega_\alpha\Omega_\beta + C\left(\Omega - \frac{1}{2}e^{\mu\nu}C_\mu\Omega_\nu\right)^2\right), \tag{15.81}$$

where $E$ is the longitudinal Young modulus, $I^{\alpha\beta}$ are the moments of the cross-section, torsional rigidity, $C$, is the minimum value of the functional (15.49), and $C_\mu$ the "Poisson coefficient vector".

If the beam has a plane of elastic symmetry perpendicular to the central line, then $C_\alpha = 0$ and the expression (15.81) becomes the classical one:

$$\Phi = \frac{1}{2}\left(E|S|\gamma^2 + EI^{\alpha\beta}\Omega_\alpha\Omega_\beta + C\Omega^2\right). \tag{15.82}$$

Otherwise, there is the interaction term between torsion and bending, $-Ce^{\mu\nu}C_\mu\Omega_\nu\Omega$, in the expression for energy, while the effective bending rigidities are

$$EI^{\alpha\beta} + \frac{1}{4}Ce^{\mu\alpha}e^{\nu\beta}C_\mu C_\nu.$$

Note that the interaction of torsion and bending and the increase in the effective bending rigidity occur only for bending in the plane perpendicular to the vector $C_\mu$.

For heterogeneous beams with an elastic symmetry plane perpendicular to the central line, the function $\Phi$, along with the torsion energy, contains contributions to the extension and bending energies, and the formula for $\Phi$ becomes

$$\Phi = \frac{1}{2}\left[\left(\int_S Ed\xi^1 d\xi^2 + E_\perp\right)\gamma^2 + 2\left(\int_S E\xi^\alpha d\xi^1 d\xi^2 + E_\perp^a\right)\Omega_\alpha\gamma + \right.$$
$$\left. + \left(\int_S E\xi^\alpha\xi^\beta d\xi^1 d\xi^2 + E_\perp{}^{\alpha\beta}\right)\Omega_\alpha\Omega_\beta + C\Omega^2\right].$$

Additional rigidities $E_\perp$, $E_\perp^a$, $E_\perp^{\alpha\beta}$ are found from the solution of the minimization problem for the functional $\Theta_\perp$.

Emphasize that all these results pertain to geometrically non-linear theory. Non-linearity enters through the dependence of deformation measures, $\gamma$, $\Omega_\alpha$ and $\Omega$, on $r^i(\xi)$ and $\tau_\alpha^i(\xi)$.

## 15.3 Asymptotic Analysis of the Energy Functional of Three-Dimensional Elasticity

**Geometry of undeformed state.** We will use the curvilinear system of coordinates, $\xi^\alpha$, $\xi$, in the region $\mathring{V}$, introduced by the equations

$$x^i = \mathring{x}^i\,(\xi^\alpha, \xi) = \mathring{r}^i(\xi) + \mathring{\tau}^i_\alpha \xi^\alpha. \tag{15.83}$$

The points $\xi^\alpha$ are the points of the beam cross-section $S$ which could be, in general, a multi-connected region. It is assumed that the point with the coordinates $\xi^\alpha = 0$ coincides with the center of gravity of $S$, i.e. $\langle \xi^\alpha \rangle = 0$.

The projections on the axis $\xi$ are marked by index 3, which will sometimes be omitted (in particular, $\xi^3 \equiv \xi$). The coordinates $\xi^\alpha$, $\xi$ are assumed to be Lagrangian.

To find the components of the metric tensor in the Lagrangian coordinate system, we first find the derivatives of functions (15.83). Using (15.3), we have

$$\mathring{x}^i_{,\alpha} = \mathring{\tau}^i_\alpha, \quad \mathring{x}^i_{,\xi} = \left(1 + \mathring{\omega}_\beta \xi^\beta\right) \mathring{\tau}^i + \mathring{\omega} e^{\cdot\alpha}_{\beta\cdot} \xi^\beta \mathring{\tau}^i_\alpha.$$

As usual, the comma before the Greek indices denotes derivative with respect to $\xi^\alpha$, and the comma before $\xi$ denotes the derivative with respect to $\xi$. Thus,

$$\mathring{g}_{\alpha\beta} = \delta_{\alpha\beta}, \quad \mathring{g}_{\alpha 3} = \mathring{\omega} e_{\beta\alpha} \xi^\beta, \quad \mathring{g}_{33} = \left(1 + \mathring{\omega}_\beta \xi^\beta\right)^2 + \mathring{\omega} \xi_\alpha \xi^\alpha, \tag{15.84}$$
$$\mathring{g} \equiv \det \left\| \mathring{g}_{\alpha\beta} \right\| = \left(1 + \mathring{\omega}_\beta \xi^\beta\right)^2.$$

To determine the contravariant components of the metric tensor we have to find the solutions of the linear system of equations

$$\mathring{g}^{\alpha\beta} \mathring{g}_{\beta\gamma} + \mathring{g}^{\alpha 3} \mathring{g}_{3\gamma} = \delta^\alpha_\gamma, \quad \mathring{g}^{3\beta} \mathring{g}_{\beta\gamma} + \mathring{g}^{33} \mathring{g}_{3\gamma} = 0, \quad \mathring{g}^{3\beta} \mathring{g}_{\beta 3} + \mathring{g}^{33} \mathring{g}_{33} = 1. \tag{15.85}$$

From the second equation (15.85) and (15.84), we have $\mathring{g}^{3\beta} = -\mathring{g}^{33} \mathring{g}_{3\beta}$. Substituting this into the third equation (15.85) and using (15.84) for $\mathring{g}_{3\beta}$, we obtain

$$\mathring{g}^{33} = \frac{1}{\left(1 + \mathring{\omega}_\beta \xi^\beta\right)^2}, \quad \mathring{g}^{3\gamma} = \frac{\mathring{\omega} e^{\gamma\cdot}_{\cdot\sigma} \xi^\sigma}{\left(1 + \mathring{\omega}_\beta \xi^\beta\right)^2}. \tag{15.86}$$

From the first equation (15.85), and from the equations (15.84) and (15.86), we get the expression for the other components of the metric tensor:

$$\mathring{g}^{\alpha\beta} = \delta^{\alpha\beta} + \frac{\mathring{\omega} e^{\alpha\cdot}_{\cdot\gamma} e^{\beta\cdot}_{\cdot\delta} \xi^\gamma \xi^\delta}{(1 + \mathring{\omega}_\sigma \xi^\sigma)^2}. \tag{15.87}$$

We will assume that $\mathring{\omega}_\alpha$ and $\mathring{\omega}$ are smooth functions of $\xi$. The best constant, $R$, in the inequalities

$$|\mathring{\omega}_\alpha| \leq \frac{1}{R}, \quad \left|\frac{d\mathring{\omega}_\alpha}{d\xi}\right| \leq \frac{1}{R^2}, \quad |\mathring{\omega}| \leq \frac{1}{R}, \quad \left|\frac{d\mathring{\omega}}{d\xi}\right| \leq \frac{1}{R^2}, \tag{15.88}$$

is called the characteristic radius of curvature-torsion.

**Three-dimensional energy functional.** We are going to consider the problem of equilibrium of an elastic beam under the actions of dead surface and body forces assuming that the beam ends are clamped:

$$x^i(\xi^\alpha, 0) = r_0^i + \tau_{\alpha 0}^i \xi^\alpha, \quad x^i(\xi^\alpha, L_0) = r_1^i + \tau_{\alpha 1}^i \xi^\alpha, \tag{15.89}$$

where $\tau_{\alpha 0}^i \tau_{i\beta 0} = \delta_{\alpha\beta}$, $\tau_{\alpha 1}^i \tau_{i\beta 1} = \delta_{\alpha\beta}$.

The equilibrium positions of the beam are the stationary points of the functional

$$I = \int\limits_{\mathring{V}} F d\mathring{V} - \int\limits_{\mathring{V}} g_i x^i (\xi^\alpha, \xi) d\mathring{V} - \int\limits_{\partial\mathring{V}} f_i x^i (\xi^\alpha, \xi) dA, \tag{15.90}$$

on the set of all continuously differentiable functions, $x^i(\xi^\alpha, \xi)$, satisfying the end conditions (15.89).

The energy density per unit volume, $F$, is assumed to be a positive quadratic form of the strains:

$$\begin{aligned} 2F = E^{abcd}\varepsilon_{ab}\varepsilon_{cd} = E^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + E^{3333}\varepsilon_{33}\varepsilon_{33} + 4E^{\alpha 3\beta 3}\varepsilon_{\alpha 3}\varepsilon_{\beta 3} + \\ + 2E^{\alpha\beta 33}\varepsilon_{\alpha\beta}\varepsilon_{33} + 4E^{\alpha\beta\gamma 3}\varepsilon_{\alpha\beta}\varepsilon_{\gamma 3} + 4E^{\alpha 333}\varepsilon_{\alpha 3}\varepsilon_{33}. \end{aligned} \tag{15.91}$$

We expect that the functional (15.90) tends to a "one-dimensional" functional as $h \to 0$. The dependence of the external body and surface forces, $g_i$ and $f_i$, and the elastic moduli tensor on $h$ is to be specified; we will discuss these dependences later.

Let us make the change of variable $\xi^\alpha \to \zeta^\alpha$: $\xi^\alpha = h\zeta^\alpha$. After this change, the region of values of $\zeta^\alpha$ does not depend on $h$, and $h$ enters the functional explicitly. The region of values of $\zeta^\alpha$ is also denoted by $S$.

**Change of required functions.** We may begin in the same way as we have done in construction of the shell theory by looking for the set $\mathcal{N}$ of the general scheme of the variational-asymptotic method. Then in the first step we would find that the functions $x^i(\xi^\alpha, \xi)$ do not depend on $\xi^\alpha$: $x^i = r^i(\xi)$; in the second step that $x'^i(\xi^\alpha, \xi)$ are linear functions of $\xi^\alpha$: $x'^i = \tau_\alpha^i(\xi)\xi^\alpha$, where the vectors $\tau_1^i$ and $\tau_2^i$ are orthogonal to each other and orthogonal to the vector $dr^i/d\xi$, and, hence, have an additional degree of freedom, rotation around the tangent vector of $\Gamma$. In the next step, the functions $x''^i$ are completely determined by $r^i$ and $\tau_\alpha^i$ and, therefore, the set $\mathcal{N}$ consists of the functions $r^i(\xi)$ and $\tau_\alpha^i(\xi)$. We omit these considerations, having "guessed" the set $\mathcal{N}$ and move on to the appropriate change of the required functions.

Let us introduce the functions

$$r^i(\xi) = \frac{\langle x^i(\xi^\alpha, \xi)\rangle}{|S|}. \tag{15.92}$$

The functions $r^i$ have the meaning of the components of the average position vector of the cross-section in the deformed state. The curve $\Gamma$ with the equation $x^i = r^i(\xi)$ is called the deformed center line of the beam. The parameter $\xi$ is not the arc length of the deformed center line. The elongation of the center line is characterized by the axial strain $\gamma$ introduced above.

At the curve $\Gamma$ we introduce two unit vectors $\tau^i_\alpha$ which are orthogonal to each other and are also orthogonal to the vector

$$\tau^i = \frac{1}{\sqrt{1+2\gamma}} \frac{dr^i}{d\xi},$$

and then make the following change of the required functions, $x^i(\xi^\alpha, \xi) \to y^i(\xi^\alpha, \xi)$:

$$x^i(\xi^\alpha, \xi) = r^i(\xi) + \tau^i_\alpha(\xi)\xi^\alpha + hy^i(\xi^\alpha, \xi). \tag{15.93}$$

Due to the definition of $r^i$ (15.92), the functions $y^i(\xi^\alpha, \xi)$ satisfy the constraints

$$\langle y^i \rangle = 0. \tag{15.94}$$

The extra degree of freedom, which appears in prescribing $\tau^i_\alpha$, allows us to put an additional constraint on $y^i$. For definiteness, as such we take the equality

$$\langle y_{\alpha|\beta}\rangle e^{\alpha\beta} = 0, \quad (y_a \equiv \tau^i_\alpha y_i). \tag{15.95}$$

The vertical line before the Greek indices denotes a derivative with respect to $\zeta^\alpha$. The equality (15.95) means that the "average" rotation of the cross-section is described by rotation of the vectors $\tau^i_\alpha$.

Equation (15.93) establishes a one-to-one correspondence between all functions $x^i(\xi^\alpha, \xi)$ and all sets of triplets $\left\{r^i(\xi), \tau^i_\alpha(\xi), y^i(\xi^\alpha, \xi)\right\}$, satisfying the constraints (15.10), (15.8), (15.94) and (15.95). Indeed, for given $r^i$, $\tau^i_\alpha$ and $y^i(\xi^\alpha, \xi)$ one can find $x^i(\xi^\alpha, \xi)$. Let us show the opposite: for given $x^i(\xi^\alpha, \xi)$, it is possible to find uniquely $r^i$, $\tau^i_\alpha$ and $y^i$ which satisfy the constraints (15.10), (15.8), (15.94) and (15.95). First, we determine $r^i$ by (15.92). Then we take some vectors $\tau'^i_\mu$ which satisfy the equalities (15.10). These vectors may be rotated, $\tau'^i_\mu \to \tau^i_\alpha$, $\tau^i_\alpha = \tau'^i_\mu s^\mu_\alpha$, with $s^\mu_\alpha$ being the components of some two-dimensional orthogonal matrix performing the rotation for angle $\sigma$. We define the angle $\sigma$ by the equation

$$\langle x^i_{,\alpha}\rangle \tau_{i\beta} e^{\alpha\beta} = \langle x^i_{,\alpha}\rangle \tau'_{i\mu} s^\mu_\beta e^{\alpha\beta} = 0. \tag{15.96}$$

Equation (15.96) is of the form $A\cos\sigma + B\sin\sigma = 0$, where $A$ and $B$ are expressed in terms of $\langle x^i_{,\alpha}\rangle \tau'_{i\mu}$, and, consequently, is always solvable. The solution defines the vectors $\tau^i_\alpha$ uniquely. Finally, we introduce $y^i$ by the equation $hy^i = x^i(\xi^\alpha, \xi) - r^i(\xi) - \tau^i_\alpha(\xi)\xi^\alpha$. Then, the relation (15.91) holds, and the functions $y^i$ satisfy the constraints (15.94) and (15.95).

At the beam ends, the functions $r^i$ and $\tau^i_\alpha$ should be chosen according to the boundary conditions (15.89):

$$\tau^i(0) = r^i_0, \quad r^i(L_0) = r^i_1, \quad \tau^i_\alpha(0) = \tau^i_{\alpha 0}, \quad \tau^i_\alpha(L_0) = \tau^i_{\alpha 1}. \tag{15.97}$$

Also,

$$y^i(\xi^\alpha, 0) = y^i(\xi^\alpha, L_0) = 0. \tag{15.98}$$

Let us define the bending-twist amplitude, $\varepsilon_\Omega$, and the central line elongation amplitude $\varepsilon_\gamma$ as

$$\varepsilon_\Omega = h \max_\xi \sqrt{\Omega_\alpha \Omega^\alpha + \Omega^2}, \quad \varepsilon_\gamma = \max_\xi |\gamma|.$$

The number $\varepsilon = \varepsilon_\Omega + \varepsilon_\gamma$ characterizes the magnitude of three-dimensional strains.

The characteristic length of the deformed state $l$ is introduced as the best constant in the system of inequalities

$$\left|\frac{d\omega_\alpha}{d\xi}\right| \le \frac{\varepsilon_\Omega}{l}, \quad \left|\frac{d\omega}{d\xi}\right| \le \frac{\varepsilon_\Omega}{l}, \quad \left|\frac{d\gamma}{d\xi}\right| \le \frac{\varepsilon_\gamma}{l}, \quad |y^i_{,\xi}| \le \frac{1}{l} \max_{\mathring{V}} \sqrt{y^j_{|\alpha} y_j^{|\alpha}}. \tag{15.99}$$

The characteristic length $l$ is a function of $\xi$. Suppose that the strain state of the beam is such that $l$ can be on the order of $h$ only at the end regions of the beam, while away from the ends of the beam

$$\frac{h}{l} \ll 1.$$

The size of the end regions is, by our assumption, on the order of $h$. Regarding the external forces, we accept the conditions

$$f_i = O\left(\underline{\mu}\frac{h}{l}\varepsilon\right), \quad g_i = O\left(\underline{\mu}\frac{\varepsilon}{l}\right), \tag{15.100}$$

and, for simplicity, limit our consideration to the body forces which are constant over the cross-section. In these estimates, $\underline{\mu}$ is the characteristic value of the components of the elastic moduli tensor.

Let us show that longitudinal, transverse and shear energies defined by (15.27), (15.28) and (15.29) are

$$F_{\|} = \frac{1}{2} E_{\|} \varepsilon_{33}^2, \quad F_{\angle} = \frac{1}{2} G_{\angle}^{\alpha\beta} \left(2\varepsilon_{\alpha 3} + E_\alpha \varepsilon_{33}\right)(\alpha \to \beta),$$

$$F_{\perp} = \frac{1}{2} E^{\alpha\beta\gamma\delta} \left(\varepsilon_{\alpha\beta} + E_{\alpha\beta}\varepsilon_{33} + E^{\sigma}_{\alpha\beta} 2\varepsilon_{\sigma 3}\right)(\alpha, \beta \to \gamma, \delta), \tag{15.101}$$

and find the relationship between the coefficients $E_{\|}$, $G_{\angle}^{\alpha\beta}$, $E_\alpha$, $E_{\alpha\beta}$, $E^{\sigma}_{\alpha\beta}$ and the components of the elastic modulus tensor.

The first, fourth and fifth terms in (15.91) contain $\varepsilon_{\alpha\beta}$. Extracting a complete square, let us write these terms as

$$E^{\alpha\beta\gamma\delta} \left(\varepsilon_{\alpha\beta} + E_{\alpha\beta}\varepsilon_{33} + E^{\sigma}_{\alpha\beta} 2\varepsilon_{\sigma 3}\right)(\alpha, \beta \to \gamma, \delta) - E^{\alpha\beta\gamma\delta} E_{\alpha\beta} E_{\gamma\delta} \varepsilon_{33}\varepsilon_{33} -$$
$$-4E^{\mu\nu\lambda\sigma} E^{\alpha}_{\mu\nu} E^{\beta}_{\lambda\sigma} \varepsilon_{\alpha 3}\varepsilon_{\beta 3} - 4E^{\mu\nu\lambda\sigma} E_{\mu\nu} E^{\alpha}_{\lambda\sigma} \varepsilon_{\alpha 3}\varepsilon_{33} \tag{15.102}$$

(coefficient 2 in front of $\varepsilon_{\sigma 3}$ is introduced in order to simplify further relations).

The coefficients $E_{\alpha\beta}$ and $E^{\sigma}_{\alpha\beta}$ are the solutions of the system of linear equations

$$E^{\alpha\beta\gamma\delta} E_{\gamma\delta} = E^{\alpha\beta 33}, \quad E^{\alpha\beta\gamma\delta} E^{\sigma}_{\gamma\delta} = E^{\alpha\beta\sigma 3}.$$

Denoting the inverse tensor to $E^{\alpha\beta\gamma\delta}$ by $E^{(-1)}_{\alpha\beta\gamma\delta}$, (i.e., $E^{(-1)}_{\alpha\beta\gamma\delta} E^{\gamma\delta\mu\nu} = \delta^{(\mu}_{\alpha} \delta^{\nu)}_{\beta}$), the solution of this system can be written as

$$E_{\gamma\delta} = E^{(-1)}_{\alpha\beta\gamma\delta} E^{\alpha\beta 33}, \quad E^{\sigma}_{\gamma\delta} = E^{(-1)}_{\alpha\beta\gamma\delta} E^{\alpha\beta\sigma 3}.$$

From (15.91) and (15.102), for $F$ we have

$$2F = E^{\alpha\beta\gamma\delta} \left(\varepsilon_{\alpha\beta} + E_{\alpha\beta}\varepsilon_{33} + E^{\sigma}_{\alpha\beta} 2\varepsilon_{\sigma 3}\right)(\alpha, \beta \to \gamma, \delta) + 4G_{\angle}^{\alpha\beta} \varepsilon_{\alpha 3}\varepsilon_{\beta 3}$$
$$+4\left(E^{\alpha 333} - E^{\mu\nu\lambda\sigma} E_{\mu\nu} E^{a}_{\lambda\sigma}\right) \varepsilon_{\alpha 3}\varepsilon_{33} + \left(E^{3333} - E^{\alpha\beta\gamma\delta} E_{\alpha\beta} E_{\gamma\delta}\right) \varepsilon_{33}\varepsilon_{33}. \tag{15.103}$$

Here, the notation is introduced:

$$G_{\angle}^{\alpha\beta} = E^{\alpha 3\beta 3} - E^{\mu\nu\lambda\sigma} E^{\alpha}_{\mu\nu} E^{\beta}_{\lambda\sigma}.$$

The second and third terms of (15.103) contain the strain components $\varepsilon_{\alpha 3}$. Extracting a complete square in these terms, we write them as follows:

$$G_{\angle}^{\alpha\beta} \left(2\varepsilon_{\alpha 3} + E_\alpha \varepsilon_{33}\right)(\alpha \to \beta) - G_{\angle}^{\alpha\beta} E_\alpha E_\beta \varepsilon_{33}\varepsilon_{33},$$

where $E_\alpha$ is the solution of the system of linear equations

$$G_{\angle}^{\alpha\beta} E_\beta = E^{\alpha 333} - E^{\mu\nu\lambda\sigma} E_{\mu\nu} E^{\alpha}_{\lambda\sigma}.$$

This solution is

$$E_\beta = \Theta_{\alpha\beta}\left(E^{\alpha 333} - E^{\mu\nu\lambda\sigma}E_{\mu\nu}E^{\alpha}_{\lambda\sigma}\right),$$

where $\Theta_{\alpha\beta}$ is the inverse tensor of $G^{\alpha\beta}_{\angle}$.

Substituting the expression for the second and the third terms into (15.103), for $F$ we finally get

$$\begin{aligned}2F = \; & E^{\alpha\beta\gamma\delta}\left(\varepsilon_{\alpha\beta}+E_{\alpha\beta}\varepsilon_{33} + E^{\sigma}_{\alpha\beta}2\varepsilon_{\sigma 3}\right)(\alpha,\beta \to \gamma,\delta) \\ & +4G^{\alpha\beta}_{\angle}\left(2\varepsilon_{\alpha 3}+E_\alpha\varepsilon_{33}\right)(\alpha \to \beta) + E_{\|}\varepsilon_{33}\varepsilon_{33},\end{aligned} \tag{15.104}$$

where

$$E_{\|} = E^{3333} - E^{\alpha\beta\gamma\delta}E_{\alpha\beta}E_{\gamma\delta} - G^{\alpha\beta}_{\angle}E_\alpha E_\beta.$$

Let us make a non-degenerated change of variables $\varepsilon_{\alpha\beta}$, $\varepsilon_{\alpha 3}$, $\varepsilon_{33} \to \gamma_{\alpha\beta}$, $\gamma_\alpha$, $\varepsilon_{33}$ in the quadratic form (15.104):

$$\gamma_{\alpha\beta} = \varepsilon_{\alpha\beta} + E_{\alpha\beta}\varepsilon_{33} + E^{\sigma}_{\alpha\beta}2\varepsilon_{\sigma 3}, \quad \gamma_\alpha = 2\varepsilon_{\alpha 3} + E_a\varepsilon_{33}.$$

Since $\gamma_{\alpha\beta}$, $\gamma_\alpha$ and $\varepsilon_{33}$ are independent, and there are no interaction terms between $\gamma_{\alpha\beta}$ and $\gamma_\alpha$, $\gamma_{\alpha\beta}$ and $\varepsilon_{33}$, $\gamma_\alpha$ and $\varepsilon_{33}$ in (15.104), the quadratic form (15.104) is positive, if the inequalities

$$E^{\alpha\beta\gamma\delta}\gamma_{\alpha\beta}\gamma_{\sigma\delta} \geq \underline{E}\gamma_{\alpha\beta}\gamma^{\alpha\beta}, \quad G^{\alpha\beta}_{\angle}\gamma_a\gamma_\beta \geq \underline{G}\gamma_\alpha\gamma^\alpha, \quad E_{\|} > 0 \tag{15.105}$$

hold where $\underline{E}$ and $\underline{G}$ are some positive constants.

For fixed $\varepsilon_{33}$ the minimum of (15.104) with respect to $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\alpha 3}$ is reached for $\gamma_{\alpha\beta} = 0$, $\gamma_\alpha = 0$ and is equal to $\frac{1}{2}E_{\|}(\varepsilon_{33})^2$. Therefore, the first relation (15.101) holds. The two other relations (15.101) are checked similarly. The intermediate equalities which came up in derivation of the formula (15.104) relate the coefficients of the quadratic forms (15.91) and (15.104).

**Two-dimensional elastic moduli tensors.** The two-dimensional tensors $E^{\alpha\beta\gamma\delta}$, $G^{\alpha\beta}_{\angle}$, $E^{\gamma}_{\alpha\beta}$ and $E_{\alpha\beta}$ are subject to the symmetry conditions

$$\begin{gathered}E^{\alpha\beta\gamma\delta} = E^{\beta\alpha\gamma\delta} = E^{\alpha\beta\delta\gamma} = E^{\gamma\delta\alpha\beta}, \quad G^{\alpha\beta}_{\angle} = G^{\beta\alpha}_{\angle}, \\ E^{\gamma}_{\alpha\beta} = E^{\gamma}_{\beta\alpha}, \quad E_{\alpha\beta} = E_{\beta\alpha}.\end{gathered}$$

It is natural to consider these tensors, along with the vector $E_\alpha$ and the scalar $E_{\|}$, as the independent components of the elastic modulus tensor. We call them two-dimensional elastic modulus tensors.

The components of the tensors $E^{\alpha\beta\gamma\delta}$, $G^{\alpha\beta}_{\angle}$ and $E_{\|}$ have the dimensionality of the Young modulus, while the components of the tensors $E^{\gamma}_{\alpha\beta}$, $E_{\alpha\beta}$ and $E_\alpha$ are dimensionless.

The positiveness of elastic energy puts constraints (15.105) on the tensors $E^{\alpha\beta\gamma\delta}$, $G_{\angle}^{\alpha\beta}$ and $E_{\|}$. The dimensionless tensors $E_{\alpha\beta}^{\gamma}$, $E_{\alpha\beta}$ and $E_{\alpha}$ may take on any values.

The elastic moduli are functions of $\xi^{\alpha}$ and $\xi$, and have the form $E = E(\zeta^{\alpha}, \xi, h/R)$. The dependence on the parameter $h/R$ is caused by curvilinearity of the Lagrangian coordinate system. For the limit values of the elastic moduli as $h/R \to 0$, we introduce the following notation: in the limit, $h/R \to 0$,

$$E_{\|} = E(\zeta^{\sigma}, \xi), \qquad G_{\angle}^{\alpha\beta} = G^{\alpha\beta}(\zeta^{\sigma}, \xi), \qquad E^{\alpha\beta\gamma\delta} = C^{\alpha\beta\gamma\delta}(\zeta^{\sigma}, \xi),$$
$$E_{\alpha\beta} = C_{\alpha\beta}(\zeta^{\sigma}, \xi), \qquad E_{\alpha} = C_{\alpha}(\zeta^{\sigma}, \xi), \qquad E_{\alpha\beta}^{\gamma} = C_{\alpha\beta}^{\gamma}(\zeta^{\sigma}, \xi).$$

As has already been mentioned, if the elastic characteristics are symmetric with respect to the plane perpendicular to the central line of the beam, the two-dimensional tensors with an odd number of indices are equal to zero:

$$C_{\alpha} = 0, \quad C_{\alpha\beta}^{\gamma} = 0.$$

If, moreover, the elastic characteristics are invariant with respect to rotation in the cross-sectional plane (i.e., if the body is transversely isotropic), then the tensors $C^{\alpha\beta\gamma\delta}$, $C^{\alpha\beta}$ and $G^{\alpha\beta}$ have the special form

$$C^{\alpha\beta\gamma\delta} = \lambda\delta^{\alpha\beta}\delta^{\gamma\delta} + \mu\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\gamma\beta}\right), G^{\alpha\beta} = G\delta^{\alpha\beta}, \quad C^{\alpha\beta} = \nu\delta^{\alpha\beta}.$$

The elastic properties of a transversely isotropic body are characterized by five parameters, $E$, $G$, $\lambda$, $\mu$ and $\nu$; the parameters, $E$, $G$, $\mu$ and $\lambda + \mu$ are positive, while $\nu$ is arbitrary.

For an isotropic body, $E = 2\mu(1+\nu)$ is the Young modulus, $G = \mu$ the shear modulus, and $\nu = \lambda/2(\lambda+\mu)$ the Poisson coefficient.

In the anisotropic case, $G^{\alpha\beta}$ and $C^{\alpha\beta}$ have the meaning of the shear modulus tensor and the transverse Poisson coefficient tensor, respectively. The dimensionless tensors $C_{\alpha}$ and $C_{\alpha\beta}^{\lambda}$ can also be interpreted as some "Poisson coefficients".

**Asymptotic analysis of the energy functional.** Let us write the strain tensor components in terms of $r^i$, $\tau_{\alpha}^{i}$ and $y^i$. Taking the derivative of (15.93), we have

$$x_{,\alpha}^{i} = \tau_{\alpha}^{i} + y_{|\alpha}^{i}, \quad x_{,\xi}^{i} = \sqrt{1+2\gamma}\left[(1+\omega_{\sigma}\xi^{\sigma})\tau^{i} + \omega e_{\sigma\cdot}^{\cdot\beta}\xi^{\sigma}\tau_{\beta}^{i} + hy_{,s}^{i}\right],$$

where the comma before the index $s$ denotes a derivative with respect to $s$.

Projecting $y^i$ to the vector triad, $y^i = \tilde{y}\tau^i + y^{\lambda}\tau_{\lambda}^{i}$, and using the relation

$$y_{,s}^{i} = \left(\tilde{y}_{,s} + y^{\lambda}\omega_{\lambda}\right)\tau^{i} + \left(y_{,s}^{\lambda} - \omega^{\lambda}\tilde{y} + \omega y^{\sigma}e_{\sigma\cdot}^{\cdot\lambda}\right)\tau_{\lambda}^{i},$$

we obtain

$$\varepsilon_{\alpha\beta} = y_{(\alpha,\beta)} + \frac{1}{2}\tilde{y}_{,\alpha}\tilde{y}_{,\beta} + \frac{1}{2}y_{\alpha}^{i}y_{\lambda,\beta}, \tag{15.106}$$

$$
\begin{aligned}
2\varepsilon_{\alpha 3} = {} & (1+\omega_\sigma \xi^\sigma) y_{|\alpha} + \omega e_{\sigma\alpha}\xi^\sigma + \sqrt{1+2\gamma}\left[h y_{\alpha,s} - h\omega_\alpha \tilde{y} + h\omega e_{\sigma\alpha} y^\sigma + \right.\\
& \left. + \omega e_{\sigma\cdot}^{\cdot\beta}\xi^\sigma y_{\beta|\alpha} + h\left(\tilde{y}_{,s} + \omega^\lambda y_\lambda\right) y_{|\alpha} + h\left(y^\lambda_{,s} - \omega^\lambda \tilde{y} + y^\sigma \omega e_{\sigma\cdot}^{\cdot\lambda}\right) y_{\lambda|\alpha}\right],\\
& \varepsilon_{33} = \gamma + \sqrt{1+2\gamma}\,\Omega_\sigma \xi^\sigma + \left(\sqrt{1+2\gamma} - 1\right)\mathring{\omega}_\sigma \xi^\sigma + \mathring{\omega}_\alpha \Omega_\beta \xi^\alpha \xi^\beta + \\
& + \mathring{\omega}\Omega\xi^\alpha \xi_\alpha + \frac{1}{2}\Omega_\alpha \Omega_\beta \xi^\alpha \xi^\beta + \frac{1}{2}\Omega^2 \xi_\alpha \xi^\alpha + (1+2\gamma)\left[h(1+\omega_\sigma \xi^\sigma)(\tilde{y}_{,s} + y^\lambda \omega_\lambda) + \right.\\
& + h\omega \xi^\sigma e_{\sigma\cdot}^{\cdot\lambda}(y_{\lambda,s} - \omega_\lambda \tilde{y} + y^\sigma e_{\sigma\lambda}\omega) + \frac{1}{2}h^2(\tilde{y}_{,s} + y^\lambda \omega_\lambda)^2 + \\
& \left. + \frac{1}{2}h^2\left(y^\lambda_{,s} - \omega^\lambda \tilde{y} + y^\sigma e_{\sigma\cdot}^{\cdot\lambda}\omega\right)\left(y_{\lambda,s} - \omega_\lambda \tilde{y} + y^\mu e_{\mu\lambda}\omega\right)\right], \quad y \equiv \sqrt{1+2\gamma}\,\tilde{y}.
\end{aligned}
$$

Holding $r^i(\xi)$ and $\tau^i_\alpha(\xi)$, let us seek the functions $y^i(\zeta^\alpha, \xi)$ in the leading approximation. First, we take the approximate expressions for strains:

$$
\varepsilon_{\alpha\beta} = y_{(\alpha|\beta)}, \quad 2\varepsilon_{\alpha 3} = y_{|\alpha} + h\Omega e_{\beta\alpha}\zeta^\beta, \quad \varepsilon_{33} = \gamma + h\Omega_\beta \zeta^\beta. \tag{15.107}
$$

Then, the derivatives of $y_\alpha$ and $y$ with respect to $\xi$ do not enter into the functional, the functional does not keep the end conditions (15.98) for $y_\alpha$, $y$, and the problem of finding $y_\alpha$ and $y$ is reduced to minimizing for each $\xi$ the functional

$$
\Theta - h^2 \int\limits_{\partial S} (f^\alpha y_\alpha + f y)\, d\varphi,
$$

$$
\begin{aligned}
\Theta(y_\alpha, y) = {} & \frac{1}{2}\Big\langle G^{\alpha\beta}\left[y_{|\alpha} + h\Omega e_{\sigma\alpha}\zeta^\sigma + C_\alpha\left(\gamma + h\Omega_\sigma \zeta^\sigma\right)\right][\alpha \to \beta] + \\
& + C^{\alpha\beta\gamma\delta}\Big[y_{(\alpha|\beta)} + C_{\alpha\beta}\left(\gamma + h\Omega_\sigma \zeta^\sigma\right) \\
& + C^\lambda_{\alpha\beta}\left(y_{|\lambda} + h\Omega e_{\sigma\lambda}\zeta^\sigma\right)\Big][\alpha, \beta \to \gamma, \delta]\Big\rangle,
\end{aligned} \tag{15.108}
$$

with the constraints

$$
\langle y_\alpha \rangle = 0, \quad \langle y \rangle = 0, \quad \left\langle y_{\alpha|\beta}\right\rangle e^{\alpha\beta} = 0. \tag{15.109}
$$

Here, $d\varphi$ is the arc length element on $\partial S$ divided by $h$, $f_\alpha \equiv f_i \tau^i_\alpha$, and $f = f_i \tau^i$. The work of external body forces is dropped, since due to the first two constraints (15.109) and the assumption (15.100), it is on the order of $\mu h^2 \varepsilon^2 / lR$ and is small compared to the interaction terms between $y_\alpha$, $y$ and $h\Omega_\sigma$, $h\Omega$ and $\gamma$, which are taken into account in (15.108). The functional $\Theta$ is the integral over the cross-section of the sum of the transverse and shear energies.

Consider first the minimization problem for the functional $\Theta$, i.e. the minimization problem for the functional (15.108) with zero external forces on $\partial S$. The functional $\Theta$ is strictly convex, bounded from below, quadratic functional. Its minimizing element $y_\alpha$, $y$ linearly depends on $h\Omega_\alpha$, $h\Omega$ and $\gamma$; therefore,

$$\check{y}_\alpha \sim \varepsilon, \quad \check{y} \sim \varepsilon. \tag{15.110}$$

The minimum value, $\Psi$, of the functional $\Theta$ is a quadratic form with respect to $h\Omega_\alpha$, $h\Omega$ and $\gamma$.

For nonzero external forces, $y_\alpha$ and $y$ can be written as

$$y_\alpha = \check{y}_\alpha + z_\alpha, \quad y = \check{y} + z. \tag{15.111}$$

The functions, $z_\alpha$, $z$, satisfy the constraints

$$\langle z_\alpha \rangle = 0, \quad \langle z \rangle = 0, \quad \langle z_{\alpha|\beta} \rangle e^{\alpha\beta} = 0. \tag{15.112}$$

Substituting (15.111) into the functional (15.108) results in the expression

$$\begin{aligned} &\Psi - R^\alpha \Omega_\alpha - R\Omega - N\gamma + \langle G^{\alpha\beta} z_{|\alpha} z_{|\beta} + \\ C^{\alpha\beta\gamma\delta} &\left( z_{(\alpha|\beta)} + C^\lambda_{\alpha\beta} z_{|\lambda} \right) (\alpha, \beta \to \gamma, \delta) \rangle - h^2 \int\limits_{\partial S} (f_\alpha z^\alpha + f z) d\varphi, \end{aligned} \tag{15.113}$$

where $R^\alpha$, $R$ and $N$ are coefficients at $\Omega_\alpha$, $\Omega$ and $\gamma$, respectively, in the work of external forces on the displacements $\check{y}_\alpha$, $\check{y}$:

$$R^\alpha \Omega_\alpha + R\Omega + N\gamma = h^2 \int\limits_{\partial S} (f_\alpha \check{y}^\alpha + f \check{y}) d\varphi. \tag{15.114}$$

They can be determined as soon as the dependence of $\check{y}_\alpha$ and $\check{y}$ on $\Omega_\alpha$, $\Omega$ and $\gamma$ is found. In derivation of (15.113), it is used that, due to the Euler equations for $\check{y}_\alpha$, $\check{y}$, there are no interaction terms between $\check{y}_\alpha$, $\check{y}$ and $z_\alpha$, $z$.

The part of the functional (15.113) which is quadratic with respect to $z_\alpha$ and $z$, is equal to zero on the fields $z_\alpha = c_\alpha + e_{\alpha\beta}\zeta^\beta \kappa$, $z = c$, $c_\alpha$, $c$, $\kappa = const$. These fields are excluded by the conditions (15.113), and it can be proved that the functional (15.113) is bounded from below. The minimizing element of the functional (15.113), $\check{z}_\alpha$, $\check{z}$, linearly depends on the external forces and, according to the condition $f^i = O\left(\underline{\mu}\varepsilon h/l\right)$, the values of the last two terms in (15.113) are ignorably small.

Due to the same condition, $R^\alpha\Omega_\alpha + R\Omega + N\gamma$ is on the order of $h\Psi/l$ and can, therefore, be dropped in the first approximation. Note, however, that, as the additional analysis shows, for a cross-section with central symmetry, all other corrections are smaller, and the main refinement of the classical beam theory is in keeping this linear form.

The terms omitted in the expression for strains (15.101) are on the order of $h/R$, $h/l$, $\varepsilon_\Omega$ and $\varepsilon_\gamma$, as can be seen from (15.106) and the estimates (15.110). Therefore, the solution of the minimization problem for the functional $\Theta$ does indeed give the leading approximation of $y_\alpha$ and $y$.

Dropping the small terms in the work of external forces, we get generalized forces $Q_i$ and $Q_i^\alpha$ introduced in the heuristic beam theory:

$$Q_i = h \int\limits_{\partial S} f_i d\varphi + |S| F_i, \quad Q_i^\alpha = h^2 \int\limits_{\partial S} f_i \zeta^\alpha d\varphi.$$

The energy density per unit length of the beam

$$\Phi = \langle F_{\|} + F_{\perp} + F_{\angle} \rangle,$$

where only the terms on the order of $\underline{\mu} h^2 \varepsilon^2$ are retained, is

$$\Phi = \frac{1}{2} \langle E\left(\gamma + h\Omega_\sigma \zeta^\sigma\right)^2 \rangle + \Psi\left(\gamma, \Omega_\alpha, \Omega\right). \tag{15.115}$$

The first term in (15.115) (the average value of the longitudinal energy) characterizes the extension and bending energies related to the elongation of the longitudinal beam fibers; the second term (the average value of transverse energy and shear energy) includes the twist energy and the additional contribution of the extension and bending energies caused by the deformation of the beam in the transverse directions.

The minimization problem for the functional $\Theta$ is a variational problem for a quadratic functional to be minimized with respect to three functions, $y_\alpha$ and $y$, of the cross-sectional coordinates. To construct the "classical" beam theory, one has to find only the minimum value, $\Psi(\gamma, \Omega_\alpha, \Omega)$, of this functional as a function of four parameters, $\gamma$, $\Omega_\alpha$ and $\Omega$. To describe the stresses inside the beam, one has to find the minimizer of $\Theta$ as well.

http://www.springer.com/978-3-540-88468-2

Variational Principles of Continuum Mechanics
II. Applications
Berdichevsky, V.
2009, X, 430 p., Softcover
ISBN: 978-3-540-88468-2