Chapter 2
Mechanical Background

2.1 Modeling

The multibody system is an appropriate model for the locomotion systems considered in this book. Therefore, the basic kinematics and dynamics of such systems are briefly introduced in this chapter. The starting point is a rigid body as a basis element of a multibody system.

Definition 2.1. A *multibody system (MBS)* is a finite set of rigid bodies that are physically and/or geometrically interconnected with each other and with a ground not belonging to the MBS. Physical coupling is specified by the applied forces/torques (e.g., spring and damper forces). Constraints describe the geometric interconnections (e.g., prismatic and rotational joints). ◇

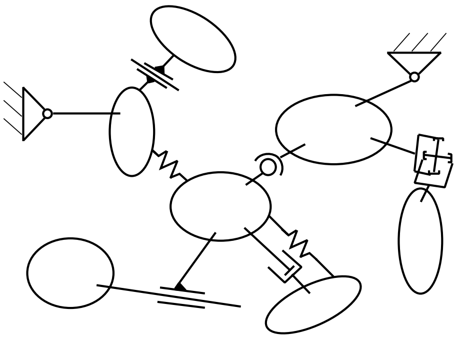

Fig. 2.1 Example of a multibody system

The MBS is the most commonly used mechanical model for the analysis of mechatronic systems. There are different possibilities to classify MBSs, for example, according to the system's degrees of freedom or to the type of constraints. One

K. Zimmermann et al., *Mechanics of Terrestrial Locomotion*,
DOI 10.1007/978-3-540-88841-3_2,

of the most important is the classification based on the topology of the system, i.e., based on the alignment of the bodies.

a) MBS with a kinematic tree structure

The MBS with a kinematic tree structure consists of pairs of bodies, each having only one sequence of linking joints and force coupling elements (without duplication) (also: “open kinematic chain”; example: robot with spherical coordinates).

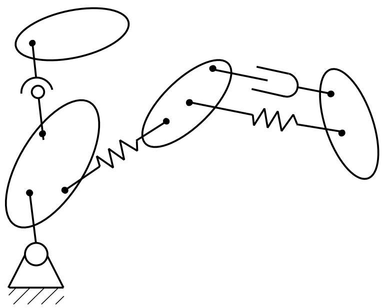

Fig. 2.2 MBS in the form of an open kinematic chain

b) Constrained mechanical system

An MBS with a kinematic tree structure and additional constraints on the bodies is called a “constrained mechanical system” (also: “closed kinematic chain”; example: four-bar linkage).

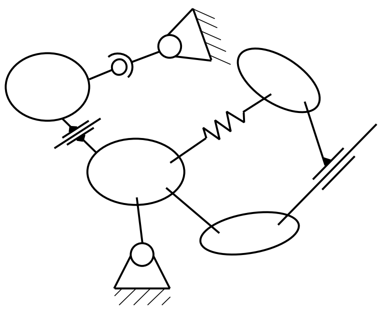

Fig. 2.3 MBS in the form of a closed kinematic chain

2.2 Kinematics of Multibody Systems

2.2.1 Kinematics of Rigid Bodies

For the analysis of a rigid body, a body-fixed coordinate system Σ' is introduced with the origin at an arbitrary but fixed point Ω of the rigid body. The position of the rigid body in an inertial coordinate system Σ is determined by the set of all position vectors

$$\overrightarrow{0P} = \vec{r}(P) = \vec{r}_\Omega + \vec{x} = \vec{r}_\Omega + x_i \vec{E}_i \,. \tag{2.1}$$

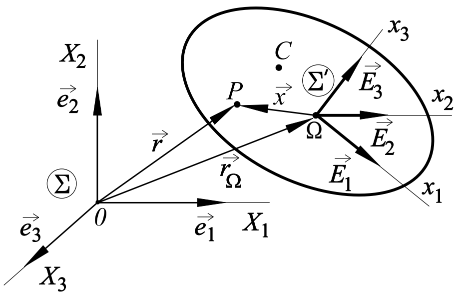

Fig. 2.4 A rigid body with an inertial and a body-fixed coordinate system

Remark 2.1 EINSTEIN's summation convention is used where appropriate, i.e., when an index occurs more than once in the same expression, the expression is implicitly summed over all possible values for that index. ◇

According to the definition of the rigid body it holds:

$$x_i = const \quad \text{for} \;\; i = 1,2,3 \,.$$

The *movement of the rigid body* is given by:

$$\vec{r}_\Omega = \vec{r}_\Omega(t)\,, \quad \vec{E}_i = \vec{E}_i(t) \quad (i = 1,2,3)\,. \tag{2.2}$$

The velocity vector follows from the time derivative of the position vector:

$$\dot{\vec{r}}(t) = \dot{\vec{r}}_\Omega + x_i \dot{\vec{E}}_i \,.$$

The time derivative of the body-fixed unit vectors $\vec{E}_i (i = 1,2,3)$ can be obtained by the orthonormal rotation matrix $\mathbf{E} = (E_{ij}) \;\; (i,j = 1,2,3)$, which describes the orientation of the body-fixed coordinate system Σ' relative to the inertial coordinate system Σ. For orthonormal matrices it is given that $\mathbf{E}^{-1} = \mathbf{E}^T$ and, therefore, it follows that:

$$\dot{\vec{E}}_i = \dot{E}_{ik} \vec{e}_k = \dot{E}_{ik} E_{jk} \vec{E}_j = \omega_{ij} \vec{E}_j \,. \tag{2.3}$$

The *angular velocity vector* follows from the three main elements of the skew symmetric matrix $\omega = (\omega_{ij})$

$$\vec{\omega} = \omega_{23}\,\vec{E}_1 + \omega_{31}\,\vec{E}_2 + \omega_{12}\,\vec{E}_3 \tag{2.4}$$

where

$$\dot{\vec{E}}_i = \vec{\omega} \times \vec{E}_i\,, \quad (i = 1,2,3)\,. \tag{2.5}$$

First, two special cases are considered:

a) Translation: The body-fixed unit vectors $\vec{E}_i$ are not time-dependent for $i = 1,2,3$. The vector $\vec{x}_2 - \vec{x}_1$ is a constant vector. The movement of a point P_2 is obtained by parallel translation of the movement of another point P_1, see Fig. 2.5.

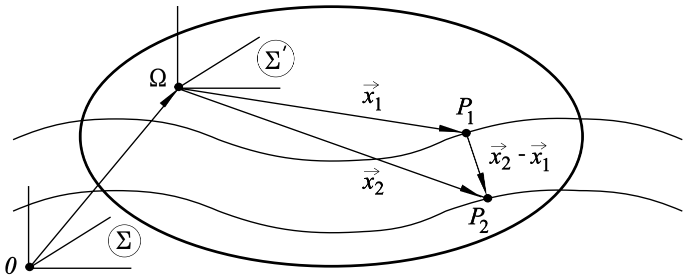

Fig. 2.5 Special case of rigid body motion: translation

b) Rotation: The vector $\vec{r}_\Omega$ is not time-dependent. Then - without loss of generality - the coincidence of the coordinate origins $0 = \Omega$ can be assumed, i.e., $\vec{r}_\Omega = \vec{0}$. It holds that $\vec{r}(t) = x_i\,\vec{E}_i(t)$, where $x_i = const$ characterizes the rigid body. This means the body performs a (rotatory) movement of the body-fixed frame $\vec{E}_i$ about the origin Ω.

Summarizing, the *general movement* of a rigid body consists of a translation and a rotation about an arbitrary fixed point Ω of the body. The *velocity* vector is:

$$\dot{\vec{r}}(t) = \dot{\vec{r}}_\Omega + \dot{\vec{x}} = \dot{\vec{r}}_\Omega + \vec{\omega} \times \vec{x}\,. \tag{2.6}$$

The vector $\dot{\vec{r}}_\Omega$ denotes the translational part, which is the same for all points of the body; $\vec{\omega} \times \vec{x}$ is the rotational part of the velocity, which depends on P. The straight line $\{\Omega, \vec{\omega}\}$ is called the *instantaneous axis of rotation* of the body. The acceleration vector

$$\ddot{\vec{r}}(t) = \ddot{\vec{r}}_\Omega + \dot{\vec{\omega}} \times \vec{x} + \vec{\omega} \times (\vec{\omega} \times \vec{x}) \tag{2.7}$$

can be obtained for any point of the rigid body.

Exercise 2.1.

Using formulas (2.3) and (2.4) find the vector of the angular velocity $\vec{\omega}_k$ for each body $k = 1,2,3,4$ of the robot presented in Fig. 2.10. Show the validity of the

formula $\vec{\omega}_k = \vec{\omega}_{k,0} = \vec{\omega}_{k,k-1} + \vec{\omega}_{k-1,0}$!
Here, $\vec{\omega}_{i,j}$ means the vector of the angular velocity during the motion of the body with number i relative to the body with number j. The body with number 0 is the ground.

2.2.2 *Kinematics of Multibody Systems with Open-loop Structures*

First, an MBS with a kinematic tree structure is considered:

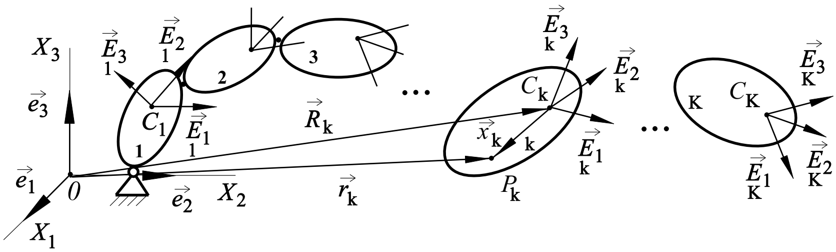

Fig. 2.6 MBS with an open-loop structure and body-fixed cartesian coordinate systems

The rigid bodies of the system are numbered $1,\dots,K$. The inertial coordinate system in the reference body 0 is denoted by $\{0,\vec{e}_i\}$, the center of mass of a body k $(k=1,2,\dots,K)$ is denoted by C_k, and a body-fixed coordinate system is denoted by $\left\{C_k,\underset{k}{\vec{E}_i}\right\}$. This means

$$\overrightarrow{0P}_k = \vec{r}_k = \vec{R}_k + \vec{x}_k = \vec{R}_k + x_i\underset{k}{\vec{E}_i}\,, \quad \overrightarrow{0C}_k = \vec{R}_k = \underset{k}{X_i}\vec{e}_i\,, \quad \underset{k}{\vec{E}_i} = \underset{k}{E_{ij}}\vec{e}_j\,. \tag{2.8}$$

For each free rigid body $\underset{k}{X_i},\underset{k}{E_{ij}}$ depends on the 6 parameters $\underset{k}{x_1},\underset{k}{x_2},\dots,\underset{k}{x_6}$:

$$\underset{k}{X_i} = \underset{k}{X_i}(\underset{k}{x_1},\underset{k}{x_2},\underset{k}{x_3})\,, \quad \underset{k}{E_{ij}} = \underset{k}{E_{ij}}(\underset{k}{x_4},\underset{k}{x_5},\underset{k}{x_6})\,, \tag{2.9}$$

with $\underset{k}{x_1},\underset{k}{x_2},\underset{k}{x_3}$ being for example cylindrical or spherical coordinates and $\underset{k}{x_4},\underset{k}{x_5},\underset{k}{x_6}$ for example EULER or CARDAN angles. The geometric constraints of the bodies are given by the equations

$$Z^m(\underset{k}{x_1},\underset{k}{x_2},\dots,\underset{k}{x_6}) = 0\,,\ \ (m=1,\dots,M<6K\,;\, s=1,\dots,6)\,, \tag{2.10}$$

with rank $\left(\frac{\partial Z^m}{\partial x_s}\right) = M$.

They are satisfied using n parameters q^a by

$$\underset{k}{x_s} = \underset{k}{x_s}(q^a)\,, \;\; s = 1,\ldots,6\,; a = 1,\ldots,n\,; k = 1,\ldots,K\,, \tag{2.11}$$

which means the following identity holds: $Z^m\left(\underset{k}{x_s}(q^a)\right) \equiv 0$.

Definition 2.2. If there is a one-to-one mapping from the set of all possible positions at time t to the set of parameters $(q^1, q^2, \ldots, q^n)$, it follows that

$$\overrightarrow{0P}_k = \vec{r}_k = \vec{r}_k(x_i, q^1, q^2, \ldots, q^n, t)\,. \tag{2.12}$$

The number n is the *degree of freedom (DOF)* of the MBS, and the parameters q^a are called *generalized coordinates*. ◇

The *position of the MBS* is described

a) in the EUCLIDean space $\mathbb{E}^3$ by:

$\vec{R}_k = \underset{k}{X_i}(q^a)\,\vec{e}_i$ – position of the centers of mass of each body,
$\underset{k}{\vec{E}_i} = \underset{k}{E_{ij}}(q^a)\,\vec{e}_j$ – orientation of the body-fixed frames of each body,

b) in the configuration space $\mathbb{R}^n$ by:

$$\left\{ q^a \,\middle|\, \underset{k}{x_s} = \underset{k}{x_s}(q^a)\,,\; s = 1,\ldots,6,\; a = 1,\ldots,n,\; k = 1,\ldots,K \right\}.$$

The MBS is called *scleronomous* if $\vec{r}_k = \vec{r}_k(x_i, q^1, q^2, \ldots, q^n)$; otherwise, it is called *rheonomous*, i.e., $\vec{r}_k = \vec{r}_k(x_i, q^1, q^2, \ldots, q^n, t)$.

2.2.3 Holonomic and Non-Holonomic Constraints

For numerous technical systems it makes sense to use more than n parameters to describe the position of the system, i.e., see (2.12),

$$\overrightarrow{0P}_k = \vec{r}_k = \vec{r}_k(x_i, q^1, q^2, \ldots, q^m, t)\,,\; (m > n)\,. \tag{2.13}$$

Then, these m parameters are not independent, and in mathematical terms this means there are constraints for $q^1, q^2, \ldots, q^m$. The classification of the constraints into two classes is of fundamental relevance for the description of the MBS.

Definition 2.3. Constraints of the form

$$f^b(q^1, q^2, \ldots, q^m) = f^b(\vec{q}) = 0\,,\;\; (b = 1, 2, \ldots, r < m)\,, \tag{2.14}$$

with $\vec{q} = (q^1, q^2, \ldots, q^m)$ and rank $\left(\frac{\partial f^b}{\partial q^a}\right) = r\,,\;\; (a = 1, 2, \ldots, m)$ are called *holonomic*. Other forms of constraints containing velocities that cannot be transformed into this form are called *non-holonomic*, i.e., these constraints cannot be integrated to yield constraints in positions. ◇

A holonomic constraint is also called *geometric* because it limits the system's position. Using equation (2.14) we can find $n = m - r$ independent generalized coordinates $q^a \;\; (a = 1, 2, \ldots, n = \text{DOF})$ that completely describe the position of the MBS.

Non-holonomic constraints limit the velocities and do not impose restrictions on the position (coordinates) of the system.

For locomotion systems that are considered in the following chapters, there are often non-integrable differential (i.e., non-holonomic) constraints in $\dot{q}^a$ of the form

$$f^b{}_a(\vec{q})\,\dot{q}^a = 0\,,\;\; (a = 1, 2, \ldots, m;\;\; b = 1, 2, \ldots, r < m)\,,\;\; \text{rank}(f^b{}_a) = r\,. \tag{2.15}$$

These constraints cannot be reduced to the form (2.14) with an integration procedure.

Example 2.1
We consider a wheel rolling without slip along a straight line, see Fig. 2.7.

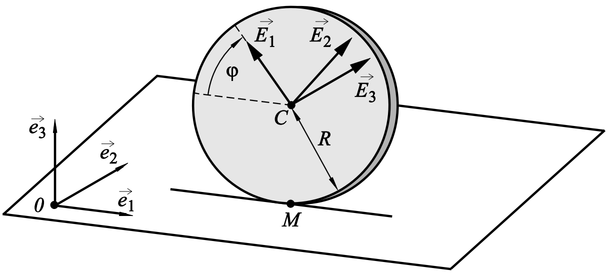

Fig. 2.7 Rolling wheel on a straight line

The position of the wheel is given by the vector of generalized coordinates $\vec{q} = (q^1, q^2, q^3) = (x_c, y_c, \varphi)$. The coordinates (x_c, y_c) identify the center of mass C. From equation (2.6) and given that the point M is the instantaneous center of rotation, i.e., $\dot{\vec{r}}_M = \vec{0}$, it follows that

$$\dot{\vec{r}}_c = \dot{x}_c\,\vec{e}_1 + \dot{y}_c\,\vec{e}_2 = \dot{\vec{r}}_M + \vec{\omega} \times \overrightarrow{MC} = \vec{\omega} \times \overrightarrow{MC} = (\dot{\varphi}\,\vec{E}_3) \times (R\,\vec{e}_3) = R\,\dot{\varphi}\,\vec{e}_1\,,$$

and for the coordinates of the velocity

$$\dot{x}_c = R\,\dot{\varphi}\,, \qquad \dot{y}_c = 0\,. \tag{2.16}$$

Integration of (2.16) yields

$$x_c = R\,\varphi + C_1 \qquad y_c = C_2\,, \tag{2.17}$$

with the initial values $x_c(0) = 0$, $y_c(0) = R$, $\varphi(0) = 0$.
We obtain

$$x_c - R\,\varphi = 0\,, \qquad y_c - R = 0$$

or

$$q^1 - R\,q^3 = 0\,, \qquad q^2 - R = 0\,. \tag{2.18}$$

Hence, the system in Fig. 2.7 is holonomic.

Example 2.2
We again discuss the problem of a wheel rolling without slipping, not following any particular curve, i.e., the trajectory of the point of contact is not prescribed, see Fig. 2.8.

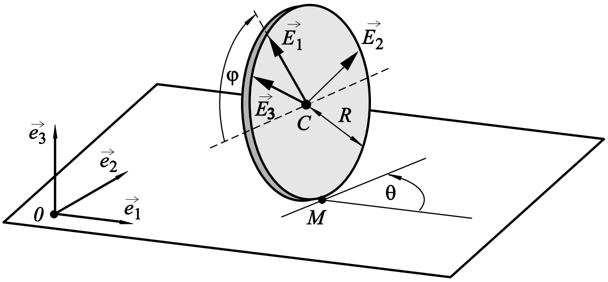

Fig. 2.8 Rolling wheel on a plane

Now, the position and the orientation is determined by the vector of generalized coordinates $\vec{q} = (q^1\,, q^2\,, q^3\,, q^4) = (x_c\,, y_c\,, \varphi\,, \theta)$. The angle θ gives the orientation of the plane of the wheel. Because the point M is again the instantaneous center of rotation, the velocity of the wheel center is

$$\begin{aligned}\dot{\vec{r}}_c &= \dot{x}_c\,\vec{e}_1 + \dot{y}_c\,\vec{e}_2 = \dot{\vec{r}}_M + \vec{\omega} \times \overrightarrow{MC} = (\dot{\theta}\,\vec{e}_3 + \dot{\varphi}\,\vec{E}_3) \times (R\,\vec{e}_3) \\ &= \big(\dot{\theta}\,\vec{e}_3 + \dot{\varphi}\,(-\sin\theta\,\vec{e}_1 + \cos\theta\,\vec{e}_2)\big) \times R\,\vec{e}_3 \\ &= R\,\dot{\varphi}\,\cos\theta\,\vec{e}_1 + R\,\dot{\varphi}\,\sin\theta\,\vec{e}_2\,.\end{aligned}$$

Thus the constraints are

$$\dot{x}_c = R\,\dot{\varphi}\cos\theta\,, \quad \dot{y}_c = R\,\dot{\varphi}\sin\theta \tag{2.19}$$

or

$$\dot{q}^1 - R\cos q^4\,\dot{q}^3 = 0\,, \quad \dot{q}^2 - R\sin q^4\,\dot{q}^3 = 0\,. \tag{2.20}$$

The relationship between the velocities (2.20) is non-integrable, and thus the system must be classified as non-holonomic.

2.3 Dynamics of Multibody Systems

The *synthetic* and the *analytical* methods are two methods to classify almost all formalisms for describing the dynamics of MBSs. The synthetic method applies the principle of linear momentum (NEWTON's second law) and the principle of angular momentum to all free-cut partial bodies in order to derive the differential equations describing the motion.

On the other hand, analytical methods use one closed concept of the complete system (for example, the kinetic energy). Some selected analytical methods that are well-suited for efficiently deriving the system equations for locomotion systems are introduced in the following sections.

2.3.1 Synthetic Method

2.3.1.1 Principle of Linear Momentum

The mass elements dm are under the action of the forces $d\vec{F}$, see Fig. 2.9. The position of the center of mass is given by

$$\vec{r}_c = \frac{\int\limits_{(V)} \vec{r}\,dm}{\int\limits_{(V)} dm}\,, \tag{2.21}$$

the sum of the acting forces by $\vec{F} = \int_{(V)} d\vec{F}$.

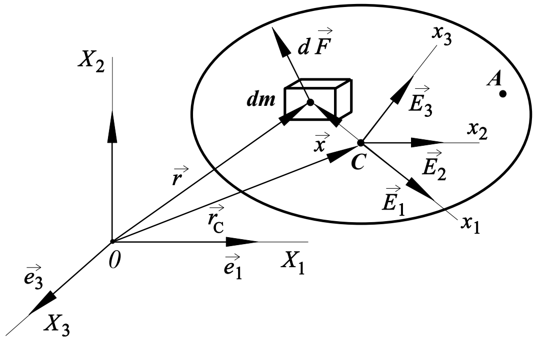

Fig. 2.9 Rigid body with distributed forces in an inertial coordinate system

The total momentum of the rigid body is defined as

$$\vec{p} = \int\limits_{(V)} \dot{\vec{r}}\, dm = m\dot{\vec{r}}_c\,. \tag{2.22}$$

Principle of linear momentum:
The center of mass of a rigid body moves as a particle, the mass of which coincides with the mass of the body and which is acted upon by a force equal to the resultant external force applied to the body:

$$\dot{\vec{p}} = m\ddot{\vec{r}}_c = \vec{F}\,. \tag{2.23}$$

2.3.1.2 Principle of Angular Momentum

The principle of angular momentum describes the rotation of a rigid body analogous to the principle of linear momentum used for translational motion. According to EULER this second principle is an independent one (in general not derivable from the principle of linear momentum). According to Fig. 2.9 the angular momentum and the moment of forces with respect to the origin *0* of the inertial coordinate system are introduced as

$$\vec{D}_0 = \int\limits_{(V)} \vec{r} \times \dot{\vec{r}}\, dm\,, \quad \vec{M}_0 = \int\limits_{(V)} \vec{r} \times d\vec{F}\,. \tag{2.24}$$

With respect to the center of mass C, the corresponding equations are

$$\vec{D}_c = \int\limits_{(V)} \vec{x} \times \dot{\vec{x}}\, dm\,, \quad \vec{M}_c = \int\limits_{(V)} \vec{x} \times d\vec{F}\,. \tag{2.25}$$

Principle of angular momentum:
The total time derivative of the angular momentum vector $\vec{D}_0$ is equal to the moment $\vec{M}_0$ of the resulting force acting on the body with respect to the origin *0* of the inertial coordinate system

$$\dot{\vec{D}}_0 = \vec{M}_0 \,. \tag{2.26}$$

The total time derivative of the angular momentum vector $\vec{D}_c$ is equal to the moment $\vec{M}_c$ of the resulting force $\vec{F}$ with respect to the body's center of mass C

$$\dot{\vec{D}}_c = \vec{M}_c \,. \tag{2.27}$$

The principle of angular momentum of the form "derivative of angular momentum is equal to moment" holds true for the center of mass of the body. However, $\dot{\vec{D}}_A = \vec{M}_A$ does *not* hold in general for an arbitrary point A of the rigid body. With respect to the angular momentum for an arbitrary point A, the principle takes the form:

$$\dot{\vec{D}}_A = \vec{M}_A + m\ddot{\vec{r}}_A \times \vec{a} \,, \tag{2.28}$$

with $\vec{r}_A = \overrightarrow{0A}$ and $\vec{a} = \overrightarrow{AC}$, see Fig. 2.9.

For practical use the principle of angular momentum is needed in form of vector coordinates. For the angular momentum vector $\vec{D}_c$ using (2.6) we obtain

$$\vec{D}_c = \int\limits_{(V)} \vec{x} \times \dot{\vec{x}}\, dm = \int\limits_{(V)} \vec{x} \times (\vec{\omega} \times \vec{x})\, dm = J_{ik}\, \omega_i\, \vec{E}_k \,. \tag{2.29}$$

Here, $\mathbf{J} = (J_{ik})$ is the inertial tensor consisting of the mass moments of inertia

$$\begin{aligned} J_{11} &= \int\limits_{(V)} \left((x_2)^2 + (x_3)^2\right) dm \,, \\ J_{22} &= \int\limits_{(V)} \left((x_1)^2 + (x_3)^2\right) dm \,, \\ J_{33} &= \int\limits_{(V)} \left((x_1)^2 + (x_2)^2\right) dm \end{aligned} \tag{2.30}$$

and the products of inertia

$$\begin{aligned} J_{12} = J_{21} &= -\int\limits_{(V)} x_1 x_2\, dm \,, \\ J_{13} = J_{31} &= -\int\limits_{(V)} x_1 x_3\, dm \,, \\ J_{23} = J_{32} &= -\int\limits_{(V)} x_2 x_3\, dm \,. \end{aligned} \tag{2.31}$$

Then, the inertia matrix is

$$\mathbf{J} = (J_{ik}) = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix} . \tag{2.32}$$

The products of inertia J_{ik} $(i \neq k;\, i,k = 1,2,3)$ vanish if the body-fixed coordinate system $\Sigma : \{C, \vec{E}_i\}$ is a principle axis system. From $\dot{\vec{D}}_c = (J_{ik}\,\omega_i\,\vec{E}_k)^{\cdot} = \vec{M}_c$ follow the three equations

$$\begin{aligned} J_{1i}\,\dot{\omega}_i - J_{2j}\,\omega_j\,\omega_3 + J_{3k}\,\omega_k\,\omega_2 &= M_{c_1}\,, \\ J_{2i}\,\dot{\omega}_i - J_{1j}\,\omega_j\,\omega_3 - J_{3k}\,\omega_k\,\omega_1 &= M_{c_2}\,, \\ J_{3i}\,\dot{\omega}_i - J_{1j}\,\omega_j\,\omega_2 + J_{2k}\,\omega_k\,\omega_l &= M_{c_3}\,. \end{aligned} \tag{2.33}$$

Exercise 2.2.
For wheeled locomotion systems expressions are needed for the mass moments of inertia for axes parallel and perpendicular to the wheel plane. Assuming the wheel is a thin disk with mass m and radius R, calculate the mass moments of inertia J_{33} and $J_{11} = J_{22}$ for the wheel shown in Fig. 2.7!

For the important special case of rotation of a rigid body about a fixed principle axis (without loss of generality assumed to be $\vec{E}_3 = \vec{e}_3$) and a constant mass moment of inertia J_{33}, the principle of angular momentum takes the form

$$J_{33}\,\dot{\omega} = M_{c_3}\,. \tag{2.34}$$

The following transformation equations of the inertia tensor describe the transition to a new coordinate system $\{C, \vec{E}_i\}$ obtained by parallel translational displacement (known as STEINER's theorem or the parallel axes theorem) or rotation of the original coordinate system $\{C, \vec{E}_i\}$.

- Parallel translational displacement (STEINER's theorem) $\{C, \vec{E}_i\} \to \{C', \vec{E}_i\}$ with the vector $\vec{a} = a_i\,\vec{E}_i$:

$$J_{i'k'} = J_{ik} + m\,(a_j\,a_j\,\delta_{ik} - a_i\,a_k)\,. \tag{2.35}$$

- Rotation $\{C, \vec{E}_i\} \to \{C, \vec{E}_{i'}\}$ with the rotation matrix $\mathbf{E} = (E_{i'j})$, $(i' = 1', 2', 3';\, j = 1,2,3)$:

$$J_{i'k'} = E_{i'i}\,E_{k'k}\,J_{ik}\,. \tag{2.36}$$

Exercise 2.3.
The locomotion system presented in Section 4.4 consists of four bars in a rhombus configuration, see Fig. 4.20. Compute the mass moments of inertia for a slender bar with mass m and length l with respect to axes perpendicular to the axis

of the bar and passing through (a) the center of mass and (b) the end of the bar (STEINER's theorem).

2.3.2 Analytical Method

2.3.2.1 D'ALEMBERT's Principle

The concept of virtual displacement is an essential element in the formulation of the principles of mechanics.

According to Sections 2.2.2 and 2.2.3 the position vector

$$\overrightarrow{0P} = \vec{r}(x_i, q^1, q^2, \ldots, q^m, t)$$

can be written as a function of the body-fixed coordinates x_i of the point P in a rigid body of the MBS, the generalized coordinates q^a $(a = 1, 2, \ldots, m)$ and the time t. For the *effective* or *actual displacement* of the position vector $\vec{r}$, it holds that

$$d\vec{r} = \frac{\partial \vec{r}}{\partial q^a} dq^a + \frac{\partial \vec{r}}{\partial t} dt\,. \tag{2.37}$$

Definition 2.4. The set of differential changes of the position vector $\vec{r}$ to a variation δq^a for fixed time t is called *virtual displacement* $\delta\vec{r}$

$$\delta\vec{r} = \frac{\partial \vec{r}}{\partial q^a} \delta q^a\,. \tag{2.38}$$

◇

In contrast to the actual displacement $d\vec{r}$, the virtual displacement $\delta\vec{r}$ is a possible (i.e., compatible with the constraints) displacement at a fixed time t.

The virtual work δA of a force $\vec{F}$ can be obtained by the scalar multiplication of $\vec{F}$ with the virtual displacement $\delta\vec{r}$. The virtual work of the ideal constraint forces $\vec{F}^{(C)}$ is zero, i.e., $\vec{F}^{(C)} \cdot \delta\vec{r} = 0$. This statement forms the content of

D'ALEMBERT's principle for rigid bodies:

$$\int\limits_{(V)} (d\vec{F} - \ddot{\vec{r}}\,dm)\,\delta\vec{r} = 0\,,\ \ \forall\,\delta\vec{r}\,, \tag{2.39}$$

where $d\vec{F}$ is an applied active force at the mass element dm of the rigid body. D'ALEMBERT's principle is valid for systems with holonomic and non-holonomic constraints. It is not dependent on the choice of the coordinate system since it contains a scalar product in its formulation.

2.3.2.2 LAGRANGE's Equations of the 2nd Kind

We consider a MBS with holonomic constraints which can be described by n independent generalized coordinates q^a $(a = 1,2,\ldots,n = \mathrm{DOF})$. LAGRANGE's equations of the 2nd kind can be derived from D'ALEMBERT's principle.

Using the HELMHOLTZ-Identities $\frac{\partial \dot{\vec{r}}}{\partial \dot{q}^a} = \frac{\partial \vec{r}}{\partial q^a}$ and $\frac{d}{dt}\left(\frac{\partial \vec{r}}{\partial q^a}\right) = \frac{\partial \dot{\vec{r}}}{\partial q^a}$ we obtain for the expression $\ddot{\vec{r}}\,\frac{\partial \vec{r}}{\partial q^a}$, which is contained in (2.39)

$$\ddot{\vec{r}}\,\frac{\partial \vec{r}}{\partial q^a} = \frac{1}{2}\,\frac{d}{dt}\left(\frac{\partial \dot{\vec{r}}^2}{\partial \dot{q}^a}\right) - \frac{1}{2}\left(\frac{\partial \dot{\vec{r}}^2}{\partial q^a}\right). \tag{2.40}$$

Introducing the generalized forces

$$Q_a = \int\limits_{(V)} d\vec{F}\,\frac{\partial \vec{r}}{\partial q^a} \tag{2.41}$$

and the kinetic energy of the rigid body

$$T = \frac{1}{2}\int\limits_{(V)} \dot{\vec{r}}^2\,dm\,, \tag{2.42}$$

from (2.38)

$$\int\limits_{(V)} (d\vec{F} - \ddot{\vec{r}}\,dm)\cdot\frac{\partial \vec{r}}{\partial q^a}\,\delta q^a = 0\,,\ \ \forall\,\delta q^a$$

concerning (2.40) - (2.42) it follows that

$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^a}\right) - \left(\frac{\partial T}{\partial q^a}\right) - Q_a\right)\delta q^a = 0\,. \tag{2.43}$$

Because the virtual displacements δq^a are independent for all a, we obtain LAGRANGE's equations of the 2nd kind in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^a}\right) - \left(\frac{\partial T}{\partial q^a}\right) = Q_a\,,\ \ (a = 1,2,\ldots,n)\,. \tag{2.44}$$

The generalized forces Q_a can be divided into 4 classes

$$Q_a = {Q_a}^1 + {Q_a}^2 + {Q_a}^3 + {Q_a}^4\,.$$

- 1st class ${Q_a}^1$:
  All generalized forces ${Q_a}^1 = {Q_a}^1(q^a,t)$ belong to this class if a potential function $U(q^a,t)$ (so-called potential energy) exists such that

$$Q_a{}^1 = -\frac{\partial U}{\partial q^a}. \tag{2.45}$$

$Q_a{}^1$-forces are potential forces.
Example: Spring force

$$U(q^a) = \frac{1}{2} c\,(q^a)^2 \;\rightarrow\; Q_a{}^1 = -c\,q^a\,.$$

- 2nd class $Q_a{}^2$:
  This class consists of forces $Q_a{}^2 = Q_a{}^2(\dot{q}^a, q^a, t)$ that can be calculated by the specification

$$Q_a{}^2 = -\frac{\partial D}{\partial \dot{q}^a}, \tag{2.46}$$

  where $D(\dot{q}^a, q^a, t)$ is called the dissipation function and $Q_a{}^2$-forces are dissipative forces.
  Example: STOKES friction force

$$D(\dot{q}^a) = \frac{1}{2} d\,(\dot{q}^a)^2 \rightarrow Q_a{}^2 = -d\dot{q}^a\,. \tag{2.47}$$

- 3rd class $Q_a{}^3$:
  $Q_a{}^3 = Q_a{}^3(\dot{q}^a, q^a, t)$ can be calculated by the formula

$$Q_a{}^3 = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}^a}\right) - \frac{\partial V}{\partial q^a}, \tag{2.48}$$

  using the generalized potential $V(\dot{q}^a, q^a, t)$.
  Example: LORENTZ force acting on an electron with the elementary charge e and the velocity $\dot{\vec{r}} = \dot{x}\vec{e}_x + \dot{y}\vec{e}_y + \dot{z}\vec{e}_z$ in a magnetic field $\vec{B} = B_z\,\vec{e}_z$

$$V(\dot{q}^a, q^a, t) = h_{ab}\, q^b\, \dot{q}^a = \frac{1}{2}\, e B_z\,(y\dot{x} - x\dot{y})\,, \tag{2.49}$$

  where h_{ab} are the coefficients of a bilinear form in q^a and $\dot{q}^a$.
- 4th class $Q_a{}^4$:
  All remaining forces that do not belong to classes $Q_a{}^1$ - $Q_a{}^3$ are classified as 4th class forces. They can only be obtained from the definition (2.41), i.e., in the special case of N single forces,

$$Q_a{}^4 = \sum_{i=1}^{N} \vec{F}_i\, \frac{\partial \vec{r}}{\partial q^a}, \tag{2.50}$$

  where $\vec{r}_i$ is the radius vector to the force application point.
  Example: COULOMB friction force, because of the discontinuity at the point $v = 0$, see Section 2.4.3.3.

Exercise 2.4.
Using (2.44) and (2.49) define the trajectory that describes an electron with the elementary charge e in a magnetic field $\vec{B} = B_z \vec{e}_z$. The initial conditions are $\vec{r}(0) = \vec{0}\,, \dot{\vec{r}}(0) = v_0\,\vec{e}_x$.

The generalized forces can also be obtained from the virtual work using the formula $\delta A = Q_a\,\delta q^a$. This way is efficient in particular for ${Q_a}^4$-forces. Using the LAGRANGE function $L = T - U - V$ the LAGRANGE's equations of the 2nd kind can be obtained

- for non-conservative systems

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \left(\frac{\partial L}{\partial q^a}\right) = -\frac{\partial D}{\partial \dot{q}^a} + {Q_a}^4\,, \quad (a = 1\,,2\,,\ldots,n)\,, \tag{2.51}$$

- for conservative system $(D \equiv 0\,,\ \ {Q_a}^4 \equiv 0)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \left(\frac{\partial L}{\partial q^a}\right) = 0\,, \quad (a = 1\,,2\,,\ldots,n)\,. \tag{2.52}$$

Kinetic energy is an important component for describing the dynamics of an MBS as well as being an element of the LAGRANGE's equations. Using definition (2.42) of the kinetic energy for rigid bodies, a more workable form should be given. Using (2.6) implies

$$\begin{aligned} T &= \frac{1}{2}\int\limits_{(V)} \dot{\vec{r}}^{\,2}\,dm = \frac{1}{2}\int\limits_{(V)} (\dot{\vec{r}}_\Omega + \vec{\omega}\times\vec{x})^2\,dm \\ &= \frac{1}{2}\int\limits_{(V)} {\dot{\vec{r}}_\Omega}^{\,2}\,dm + \int\limits_{(V)} \dot{\vec{r}}_\Omega\,(\vec{\omega}\times\vec{x})\,dm + \frac{1}{2}\int\limits_{(V)} (\vec{\omega}\times\vec{x})^2\,dm \end{aligned}$$

and finally

$$T = T_{\text{Trans}} + T^* + T_{\text{Rot}} = \frac{1}{2}m{\dot{\vec{r}}_\Omega}^{\,2} + m\dot{\vec{r}}_\Omega\,(\vec{\omega}\times\overrightarrow{\Omega C}) + \frac{1}{2}J_{ik}^{(\Omega)}\,\omega_i\,\omega_k\,, \tag{2.53}$$

where $J_{ik}^{(\Omega)}$ are the elements of the inertia tensor with respect to the system with the origin at the point Ω and $\vec{\omega} = \omega_i\,\vec{E}_i$, see Fig. 2.4. In the special case of $\Omega = C$, i.e., the center of mass C is taken for the arbitrary chosen point Ω, the general form (2.53) of the kinetic energy takes the following form:

$$T = T_{\text{Trans}} + T_{\text{Rot}} = \frac{1}{2}\,m{\dot{\vec{r}}_c}^{\,2} + \frac{1}{2}\,J_{ik}\,\omega_i\,\omega_k\,, \tag{2.54}$$

which is used for practical applications.

2.3.2.3 Multibody Systems with Additional Constraints

In the previous section we only considered systems with holonomic constraints, described by n independent coordinates, i.e., the virtual displacements δq^a are independent for all a. Now, we assume that for the MBS r additional non-holonomic constraints exist of the form

$$f^b{}_a(\vec{q})\,\dot{q^a} = 0 \ , \ \ (a = 1,2,...,n \ ; \ b = 1,2,...,r < n) \ , \ \text{rank}\,(f^b{}_a) = r \qquad (2.55)$$

with $\vec{q} = (q^1, q^2, ..., q^n)$. These constraints are called homogeneous. In practice MBSs also exist with additional constraints in an inhomogeneous form, i.e.,

$$f^b{}_a(\vec{q})\dot{q^a} = g^b(\vec{q})\,.$$

An example for such an MBS is shown in Fig. 2.10. The robot has 5 primary degrees of freedom and possesses a kitchen knife in its gripper for cutting meat.

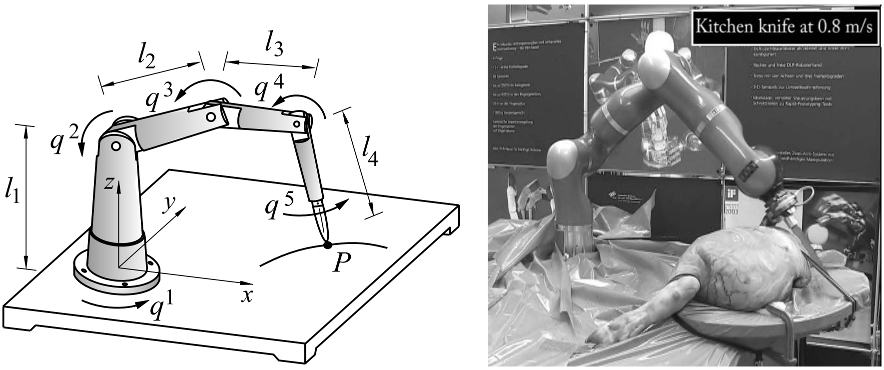

Fig. 2.10 Five-axis serial manipulator with additional constraints (left) and a practical example (right) [59]

Exercise 2.5.
Formulate the equations for the additional constraints of the robot shown in Fig. 2.10. Assume a knife-edge condition (no-side-slip condition) for the point P moving in the x-y-plane.

In the case of additional constraints (2.55) the virtual displacements δq^a are not independent for all a, and the conclusion leading from equation (2.43) to the LAGRANGE equations (2.44) is not possible. However, the idea for drawing a conclusion about the dynamics of the MBS with constraints should be the same as before.

Thus, we divide the sum in (2.55) into two parts:

$$f^b{}_{a_1}\,\dot{q}^{a_1} + f^b{}_{a_2}\,\dot{q}^{a_2} = 0\,,\ \ (a_1 = 1,2,\ldots,n-r;\ \ a_2 = n-r+1,\ldots,n) \tag{2.56}$$

or

$$f^b{}_{a_1}\,dq^{a_1} + f^b{}_{a_2}\,dq^{a_2} = 0\,.$$

Considering the definition of the virtual displacements, it follows that

$$f^b{}_{a_1}\,\delta q^{a_1} + f^b{}_{a_2}\,\delta q^{a_2} = 0\,. \tag{2.57}$$

Multiplying this relation by the arbitrary coefficients λ_b, we obtain

$$\lambda_b\,f^b{}_{a_1}\,\delta q^{a_1} + \lambda_b\,f^b{}_{a_2}\,\delta q^{a_2} = 0\,. \tag{2.58}$$

Subtracting (2.58) from the sum (2.43)

$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{a_1}}\right) - \left(\frac{\partial T}{\partial q^{a_1}}\right) - Q_{a_1}\right)\delta q^{a_1} + \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{a_2}}\right) - \left(\frac{\partial T}{\partial q^{a_2}}\right) - Q_{a_2}\right)\delta q^{a_2} = 0$$

we get

$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{a_1}}\right) - \left(\frac{\partial T}{\partial q^{a_1}}\right) - Q_{a_1} - \lambda_b\,f^b{}_{a_1}\right)\delta q^{a_1} + \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{a_2}}\right) - \left(\frac{\partial T}{\partial q^{a_2}}\right) - Q_{a_2} - \lambda_b\,f^b{}_{a_2}\right)\delta q^{a_2} = 0\,.$$

The $n-r$ virtual displacements δq^{a_1} are independent, and the r ones δq^{a_2} are dependent (rank$(f^b{}_{a_2}) = r$). Next, we choose r arbitrary coefficients λ_b such that $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{a_2}}\right) - \left(\frac{\partial T}{\partial q^{a_2}}\right) - Q_{a_2} - \lambda_b\,f^b{}_{a_2} = 0$. Then, because of the independency of δq^{a_1}, LAGRANGE's equations have the following form:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{a_1}}\right) - \left(\frac{\partial T}{\partial q^{a_1}}\right) &= Q_{a_1} + \lambda_b\,f^b{}_{a_1}\,,\ \ (a_1 = 1,2,\ldots,n-r)\,,\\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{a_2}}\right) - \left(\frac{\partial T}{\partial q^{a_2}}\right) &= Q_{a_2} + \lambda_b\,f^b{}_{a_2}\,,\ \ (a_2 = n-r+1,\ldots,n)\end{aligned} \tag{2.59}$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{a}}\right) - \left(\frac{\partial T}{\partial q^{a}}\right) = Q_a + \lambda_b\,f^b{}_a\,,\ \ (a = 1,2,\ldots,n)\,. \tag{2.60}$$

The terms $R_a = \lambda_b\,f^b{}_a$ represent the reaction forces due to the r constraints (2.55). The unknown functions in (2.60) are n generalized coordinates q^a and r so-called *Lagrangian multipliers* λ_b. These unknowns can be determined by n equations (2.60) and r constraints (2.55).

2.3.2.4 VORONETS' Equations

For special applications when the reaction forces are not of interest, it makes sense to eliminate the r unknown multipliers λ_b in (2.60). This tedious procedure results in the VORONETS' equations, which are applicable to locomotion systems as well. From equations (2.56) we find for the last r generalized velocities (assuming a regular matrix ${f^b}_{a_2}$)

$$\dot{q}^{a_2} = {\alpha^{a_2}}_{a_1}\, \dot{q}^{a_1}\,, \tag{2.61}$$

with ${\alpha^{a_2}}_{a_1} = {\alpha^{a_2}}_{a_1}(q^1, q^2, \ldots, q^n)$. Then, equations (2.57) takes the form

$${\alpha^{a_2}}_{a_1}\, \delta q^{a_1} - \delta q^{a_2} = 0$$

and multiplying this relation (analogous to Section 2.3.2.3) by an arbitrary coefficient λ_{a_2}, we find

$$\lambda_{a_2}\, {\alpha^{a_2}}_{a_1}\, \delta q^{a_1} - \lambda_{a_2}\, \delta q^{a_2} = 0\,.$$

In this case LAGRANGE's equations (2.59) are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{a_1}}\right) - \left(\frac{\partial T}{\partial q^{a_1}}\right) &= Q_{a_1} + \lambda_{a_2}\, {\alpha^{a_2}}_{a_1} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{a_2}}\right) - \left(\frac{\partial T}{\partial q^{a_2}}\right) &= Q_{a_2} - \lambda_{a_2}\,. \end{aligned} \tag{2.62}$$

Using (2.61) we consider the kinetic energy of the MBS as a function of n generalized coordinates q^a $(a = 1, 2, \ldots, n)$ and $n-r$ generalized velocities $\dot{q}^{a_1}$ $(a_1 = 1, 2, \ldots, n-r)$:

$$\begin{aligned} T &= T\left(t, q^1, q^2, \ldots, q^n, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^n\right) \\ &= T\left(t, q^1, q^2, \ldots, q^n, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^{n-r}, \dot{q}^{n-r+1}, \ldots, \dot{q}^n\right) \\ &= T'\left(t, q^1, q^2, \ldots, q^n, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^{n-r}\right). \end{aligned} \tag{2.63}$$

We define the derivatives of the "new" kinetic energy T'

$$\frac{\partial T'}{\partial \dot{q}^{a_1}} = \frac{\partial T}{\partial \dot{q}^{a_1}} + \frac{\partial T}{\partial \dot{q}^{a_2}}\frac{\partial \dot{q}^{a_2}}{\partial \dot{q}^{a_1}} = \frac{\partial T}{\partial \dot{q}^{a_1}} + \frac{\partial T}{\partial \dot{q}^{a_2}}\, {\alpha^{a_2}}_{a_1}\,,$$

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{q}^{a_1}}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{a_1}}\right) + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{a_2}}\right) {\alpha^{a_2}}_{a_1} + \frac{\partial T}{\partial \dot{q}^{a_2}}\frac{d{\alpha^{a_2}}_{a_1}}{dt}\,.$$

With respect to (2.62), it follows that

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot q^{a_1}}\right)$$
$$= \frac{\partial T}{\partial q^{a_1}} + Q_{a_1} + \lambda_{a_2}\,\alpha^{a_2}{}_{a_1} + \frac{\partial T}{\partial q^{a_2}}\,\alpha^{a_2}{}_{a_1} + Q_{a_2}\alpha^{a_2}{}_{a_1} - \lambda_{a_2}\,\alpha^{a_2}{}_{a_1} + \frac{\partial T}{\partial \dot q^{a_2}}\,\frac{d\alpha^{a_2}{}_{a_1}}{dt}$$
$$= \frac{\partial T}{\partial q^{a_1}} + Q_{a_1} + \frac{\partial T}{\partial q^{a_2}}\,\alpha^{a_2}{}_{a_1} + Q_{a_2}\,\alpha^{a_2}{}_{a_1} + \frac{\partial T}{\partial \dot q^{a_2}}\,\frac{d\alpha^{a_2}{}_{a_1}}{dt}\,.$$

It remains to eliminate the term $\frac{\partial T}{\partial q^a}$ with $(a = a_1\,, a_2)$:

$$\frac{\partial T'}{\partial q^a} = \frac{\partial T}{\partial q^a} + \frac{\partial T}{\partial \dot q^c}\,\frac{\partial \dot q^c}{\partial q^a} = \frac{\partial T}{\partial q^a} + \frac{\partial T}{\partial \dot q^c}\,\frac{\partial \alpha^c{}_d}{\partial q^a}\dot q^d\,,$$

$$\frac{\partial T}{\partial q^a} = \frac{\partial T'}{\partial q^a} - \frac{\partial T}{\partial \dot q^c}\,\frac{\partial \alpha^c{}_d}{\partial q^a}\dot q^d\,,\ (c = n-r+1\,,\ldots,n;\ \ d = 1\,,\ldots,n-r)\,.$$

Now, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot q^{a_1}}\right) =& \frac{\partial T'}{\partial q^{a_1}} + Q_{a_1} + \frac{\partial T'}{\partial q^{a_2}}\,\alpha^{a_2}{}_{a_1} + Q_{a_2}\,\alpha^{a_2}{}_{a_1}\\ &+ \frac{\partial T}{\partial \dot q^{a_2}}\,\frac{d\alpha^{a_2}{}_{a_1}}{dt} - \frac{\partial T}{\partial \dot q^{a_2}}\,\frac{\partial \alpha^{a_2}{}_d}{\partial q^{a_1}}\,\dot q^d - \left(\frac{\partial T}{\partial \dot q^{a_2}}\,\frac{\partial \alpha^{a_2}{}_d}{\partial q^c}\,\dot q^d\right)\alpha^c{}_{a_1}\\ =& \frac{\partial T'}{\partial q^{a_1}} + Q_{a_1} + \frac{\partial T'}{\partial q^{a_2}}\,\alpha^{a_2}{}_{a_1} + Q_{a_2}\,\alpha^{a_2}{}_{a_1}\\ &+ \frac{\partial T}{\partial \dot q^{a_2}}\left[\frac{d\alpha^{a_2}{}_{a_1}}{dt} - \left(\frac{\partial \alpha^{a_2}{}_d}{\partial q^{a_1}} + \frac{\partial \alpha^{a_2}{}_d}{\partial q^c}\,\alpha^c{}_{a_1}\right)\dot q^d\right]\end{aligned} \tag{2.64}$$

with

$$\frac{d\alpha^{a_2}{}_{a_1}}{dt} = \frac{\partial \alpha^{a_2}{}_{a_1}}{\partial t} + \frac{\partial \alpha^{a_2}{}_{a_1}}{\partial q^l}\,\frac{\partial q^l}{\partial t} = 0 + \frac{\partial \alpha^{a_2}{}_{a_1}}{\partial q^{l_1}}\,\dot q^{l_1} + \frac{\partial \alpha^{a_2}{}_{a_1}}{\partial q^{l_2}}\,\dot q^{l_2}\,,$$
$$(l_1 = 1\,,\ldots,n-r\ \ ;\ \ l_2 = n-r+1\,,\ldots,n)\,.$$

Taking into account (2.61) $\dot q^{l_2} = \alpha^{l_2}{}_{l_1}\,\dot q^{l_1}$ with new indices l_1 and l_2, which take the same set of values as a_1 and a_2, we obtain for $\frac{d\alpha^{a_2}{}_{a_1}}{dt}$ the formula

$$\frac{d\alpha^{a_2}{}_{a_1}}{dt} = \frac{\partial \alpha^{a_2}{}_{a_1}}{\partial q^{l_1}}\,\dot q^{l_1} + \frac{\partial \alpha^{a_2}{}_{a_1}}{\partial q^{l_2}}\,\alpha^{l_2}{}_{l_1}\,\dot q^{l_1} = \left(\frac{\partial \alpha^{a_2}{}_{a_1}}{\partial q^d} + \frac{\partial \alpha^{a_2}{}_{a_1}}{\partial q^c}\,\alpha^c{}_d\right)\dot q^d\,,$$

where the indices l_1 and l_2 were changed to d and c, respectively. We put this formula into equation (2.64). The expression in the square brackets in (2.64) takes the form

$$\left[\frac{d\alpha^{a_2}{}_{a_1}}{dt} - \left(\frac{\partial \alpha^{a_2}{}_d}{\partial q^{a_1}} + \frac{\partial \alpha^{a_2}{}_d}{\partial q^c}\, \alpha^c{}_{a_1}\right) \dot{q}^d\right]$$
$$= \left(\frac{\partial \alpha^{a_2}{}_{a_1}}{\partial q^d} + \frac{\partial \alpha^{a_2}{}_{a_1}}{\partial q^c}\, \alpha^c{}_d - \frac{\partial \alpha^{a_2}{}_d}{\partial q^{a_1}} - \frac{\partial \alpha^{a_2}{}_d}{\partial q^c}\, \alpha^c{}_{a_1}\right) \dot{q}^d \,.$$

Let $\beta^{a_2}{}_{a_1 d}$ denote the coefficient of $\dot{q}^d$ on the right-hand side, i.e.,

$$\beta^{a_2}{}_{a_1 d} = \frac{\partial \alpha^{a_2}{}_{a_1}}{\partial q^d} + \frac{\partial \alpha^{a_2}{}_{a_1}}{\partial q^c}\, \alpha^c{}_d - \frac{\partial \alpha^{a_2}{}_d}{\partial q^{a_1}} - \frac{\partial \alpha^{a_2}{}_d}{\partial q^c}\, \alpha^c{}_{a_1} \,, \qquad (2.65)$$
$$(a_1\,, d = 1\,, \ldots, n-r \quad \text{and} \quad a_2\,, c = n-r+1\,, \ldots, n)\,.$$

We note that the coefficients $\beta^{a_2}{}_{a_1 d}$ satisfy the relation: $\beta^{a_2}{}_{lk} = -\beta^{a_2}{}_{kl}$.

Finally, we obtain the VORONETS' equations of the form

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{q}^{a_1}}\right) - \frac{\partial T'}{\partial q^{a_1}} - \frac{\partial T'}{\partial q^{a_2}}\, \alpha^{a_2}{}_{a_1} = Q_{a_1} + Q_{a_2}\, \alpha^{a_2}{}_{a_1} + \frac{\partial T}{\partial \dot{q}^{a_2}}\, \beta^{a_2}{}_{a_1 d}\, \dot{q}^d \,. \qquad (2.66)$$

In the special case when the kinetic energy T' and the coefficients in the constraints $\alpha^{a_2}{}_{a_1}$ do not depend on the generalized coordinates q^{a_2}, using (2.66) we arrive at CHAPLYGIN's equations:

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{q}^{a_1}}\right) - \frac{\partial T'}{\partial q^{a_1}} = Q_{a_1} + Q_{a_2}\, \alpha^{a_2}{}_{a_1} + \frac{\partial T}{\partial \dot{q}^{a_2}} \left(\frac{\partial \alpha^{a_2}{}_{a_1}}{\partial q^d} - \frac{\partial \alpha^{a_2}{}_d}{\partial q^{a_1}}\right) \dot{q}^d \,. \qquad (2.67)$$

2.3.2.5 APPELL's Equations

To obtain APPELL's equations we use D'ALEMBERT's principle (2.39) in the form

$$\int\limits_{(V)} \ddot{\vec{r}}\, dm\, \delta\vec{r} = \int\limits_{(V)} d\vec{F}\, \delta\vec{r} \;, \quad \forall \delta\vec{r}. \qquad (2.68)$$

Writing (2.61) in the form of variations

$$\delta q^{a_2} = \alpha^{a_2}{}_{a_1}\, \delta q^{a_1} \qquad (2.69)$$

and taking into account equation (2.38), we find

$$\delta\vec{r} = \frac{\partial \vec{r}}{\partial q^a}\, \delta q^a = \frac{\partial \vec{r}}{\partial q^{a_1}}\, \delta q^{a_1} + \frac{\partial \vec{r}}{\partial q^{a_2}}\, \delta q^{a_2} = \left(\frac{\partial \vec{r}}{\partial q^{a_1}} + \frac{\partial \vec{r}}{\partial q^{a_2}}\, \alpha^{a_2}{}_{a_1}\right) \delta q^a_1 \,.$$

Introducing the vectors

$$\vec{g}_{a_1} = \vec{g}_{a_1}(x_i\,, q^1\,, \ldots, q^n\,, t) = \frac{\partial \vec{r}}{\partial q^{a_1}} + \frac{\partial \vec{r}}{\partial q^{a_2}}\, \alpha^{a_2}{}_{a_1} \;, \;\; (a_1 = 1\,, 2\,, \ldots, n-r) \qquad (2.70)$$

it holds that

$$\delta\vec{r} = \vec{g}_{a_1}\,\delta q^{a_1}\,. \tag{2.71}$$

The velocity of a mass element dm of the MBS takes the form

$$\dot{\vec{r}} = \frac{\partial\vec{r}}{\partial t} + \frac{\partial\vec{r}}{\partial q^a}\dot{q}^a = \frac{\partial\vec{r}}{\partial t} + \frac{\partial\vec{r}}{\partial q^{a_1}}\dot{q}^{a_1} + \frac{\partial\vec{r}}{\partial q^{a_2}}\dot{q}^{a_2} = \frac{\partial\vec{r}}{\partial t} + \left(\frac{\partial\vec{r}}{\partial q^{a_1}} + \frac{\partial\vec{r}}{\partial q^{a_2}}\,\alpha^{a_2}{}_{a_1}\right)\dot{q}^{a_1},$$

and with (2.70) it follows that

$$\dot{\vec{r}} = \frac{\partial\vec{r}}{\partial t} + \vec{g}_{a_1}\,\dot{q}^{a_1}\;,\;\;(a_1 = 1\,,2\,,\ldots,n-r)\,. \tag{2.72}$$

After differentiating $\dot{\vec{r}}$ with respect to time, the acceleration of the mass element dm is expressed by

$$\begin{aligned}\ddot{\vec{r}} &= \frac{d}{dt}\left(\frac{\partial\vec{r}}{\partial t}\right) + \frac{d\vec{g}_{a_1}}{dt}\,\dot{q}^{a_1} + \vec{g}_{a_1}\,\ddot{q}^{a_1}\\ &= \frac{\partial^2\vec{r}}{\partial t^2} + \frac{\partial^2\vec{r}}{\partial t\partial q^a}\,\dot{q}^a + \frac{\partial\vec{g}_{a_1}}{\partial t}\,\dot{q}^{a_1} + \frac{\partial\vec{g}_{a_1}}{\partial q^c}\,\dot{q}^c\,\dot{q}^{a_1} + \vec{g}_{a_1}\,\ddot{q}^{a_1}\,. \end{aligned} \tag{2.73}$$

From equation (2.73) we obtain

$$\frac{\partial\ddot{\vec{r}}}{\partial\ddot{q}^{a_1}} = \vec{g}_{a_1}\,. \tag{2.74}$$

With respect to (2.71) and (2.74), the left-hand-side of equation (2.68) takes the form

$$\int\limits_{(V)} \ddot{\vec{r}}\,dm\,\delta\vec{r} = \int\limits_{(V)} \ddot{\vec{r}}\,dm\,\vec{g}_{a_1}\,\delta q^{a_1} = \int\limits_{(V)} \ddot{\vec{r}}\,\frac{\partial\ddot{\vec{r}}}{\partial q^{a_1}}\,dm\,\delta q^{a_1}\,. \tag{2.75}$$

Using (2.71) the right-hand-side of equation (2.68) can be expressed by

$$\int\limits_{(V)} d\vec{F}\,\delta\vec{r} = \int\limits_{(V)} d\vec{F}\,\vec{g}_{a_1}\,\delta q^{a_1}\,. \tag{2.76}$$

Introducing the acceleration energy

$$S = \frac{1}{2}\int\limits_{(V)} \ddot{\vec{r}}^{\,2} dm\,,\quad S = S\left(q^1,\ldots,q^n,\dot{q}^1,\ldots,\dot{q}^{\,n-r},\ddot{q}^1,\ldots,\ddot{q}^{\,n-r}\right) \tag{2.77}$$

and the generalized forces

$$\Pi_{a_1} = \int\limits_{(V)} d\vec{F}\,\vec{g}_{a_1} \tag{2.78}$$

from D'ALEMBERT's principle (2.68) taking into account (2.75) to (2.78) we obtain the equation

$$\left(\Pi_{a_1} - \frac{\partial S}{\partial \ddot{q}^{a_1}} \right) \delta q^{a_1} = 0 . \tag{2.79}$$

Since the virtual displacements δq^{a_1} are independent, relation (2.79) implies APPELL's equations

$$\frac{\partial S}{\partial \ddot{q}^{a_1}} = \Pi_{a_1} \ , \ (a_1 = 1, 2, \ldots, n-r) . \tag{2.80}$$

2.4 Forces Related to Locomotion

2.4.1 Force of Gravity

The force of gravity (Fig. 2.11) is the force that causes a massively large object to attract another object towards itself. All objects on Earth are under the influence of a gravitational force directed towards the center of the earth. Near the surface ($h \ll R$), where the locomotion systems considered in the following chapters operate, the force of gravity is

$$\vec{F}_G = m\vec{g} , \tag{2.81}$$

where m is the mass of the object and g is the gravitational acceleration ($|\vec{g}| = 9.80665 \frac{m}{s^2}$).

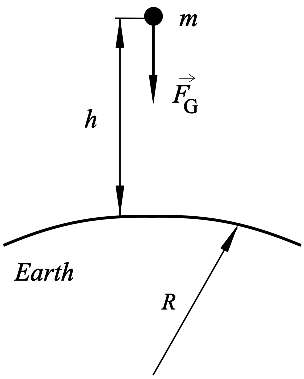

Fig. 2.11 Force of gravity

Of course, the force of gravity is only one of many forces acting on a locomotion system on Earth. Nevertheless, it is of interest in this context that some walking machines have been developed for which gravity is the only motion-generating force, so-called *passive walkers*. McGEER [93] has shown that a simple planar mechanism with two legs can walk down a slight slope with no other energy input or control. This system consists of two coupled pendula. The pivot leg acts as an inverted pendulum, and the swinging leg acts as a free pendulum attached to the pivot leg at the hip. Given sufficient mass at the hip, the system will have a stable limit cycle; that is, a nominal trajectory that repeats itself and will return to this trajectory even if perturbed slightly. After McGEER's pioneering work on passive dynamic walking, other researchers have developed new prototypes, including systems with knees, see for example passive walkers from the Delft University of Technology [45], [159], see Fig. 2.12.

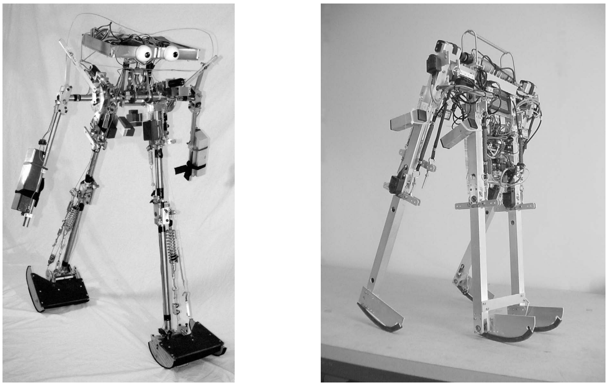

Fig. 2.12 A passive-dynamics-based walker (left) and the passive walker "MIKE" with pneumatic McKIBBEN muscles (right), Delft University of Technology [Photos courtesy of S. COLLINS and M. WISSE]

2.4.2 Spring Force

The spring force is the force exerted by a compressed or stretched spring upon any object attached to it. For most springs under consideration, the magnitude of the force is directly proportional to the amount of stretch or compression; therefore, spring forces take the form

$$\vec{F}_c = -c\,(\lambda - \lambda_0)\,\vec{e}_x = -c\,x\,\vec{e}_x\,, \tag{2.82}$$

with the constant stiffness c, the actual spring length λ and the original spring length λ_0, see Fig. 2.13.

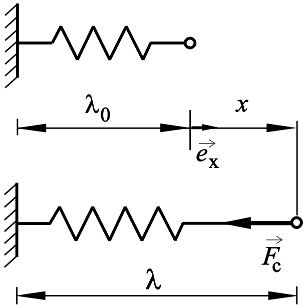

Fig. 2.13 Spring force

Technicians have developed various forms of springs for different applications, see Fig. 2.14. For motion generation most pedal and non-pedal systems follow the principle of undulatory locomotion, see Section 6.1. For this type of locomotion, a periodic deformation of the shape of the system is needed. Thus, the spring representing the elasticity in the locomotion system is an important element in vibration-driven systems.

Fig. 2.14 Various forms of springs [Photo courtesy of WAFIOS AG Reutlingen]

In biomechanics the spring is a typical element for modeling the muscle-ligament complex of humans and animals. BLICKHAN and SEYFARTH used spring-mass models to describe the long jump, see Fig. 2.15. Knowing the velocity, the leg stiffness, and the angle of attack, a very precise prediction of jump length is possible.

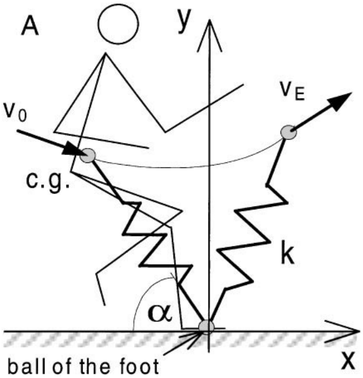

Fig. 2.15 Spring-mass model describing jumping [127]

WITTE et al. [162] explain the mechanism responsible for energy storage during the locomotion of horses with simplified models, including various springs. In these models, the extremities move like pendula and are compressed like springs during contact with the ground, see also Section 5.2.2.

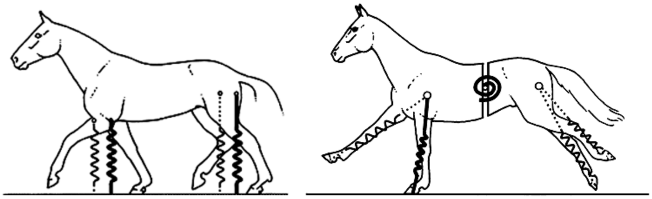

Fig. 2.16 Vibration models with various springs describing the gaits of a horse [162]

The muscle is of great interest to engineers since it is effectively a linear actuator. Muscles have many features and an outstanding strength-to-weight ratio, which is useful in practical actuators. Because of its specific characteristics and parameters, the performance of muscle is significantly different from the performance of current artificial actuators. Different mechanical models have been developed in order to understand the working principles of muscles in biomechanics. The dynamic behavior of a muscle was first described using mass-damper-spring models by HILL [65] and then by HUXLEY [69], ZAJAC et al. [166]. SCHMALZ [125] considered various models for the muscle-tendon complex, which consists of elastic elements (springs) both in series (SEC) and in parallel (PEC) with a contractile element (CC), see Fig. 2.17.

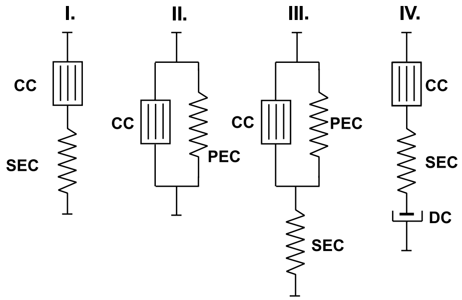

Fig. 2.17 Muscle-tendon models with serial and parallel elastic elements [125]

Biological and technical locomotion systems also use springs as an energy storage. A biologically inspired artificial exoskeleton that supports a walking human is shown in Fig. 2.18.

Fig. 2.18 Exoskeleton supporting walking motion [Photo courtesy of B. SMALE]

2.4.3 Friction Forces

2.4.3.1 General Notes

Friction simultaneously plays a paramount yet antagonistic role in connection with motion. It ultimately causes the standstill of all types of motion both in the nature and in technology. Therefore, engineers generally take great pains to overcome friction for in order to maintain motion. Also, evolution has created remarkable solutions to reduce friction as much as possible. A well-known biological paradigm is the scale structure of the shark skin, which is now being used in a technical setting to decrease the flow resistance of aircrafts.

On the other hand, locomotion becomes impossible without the existence of friction forces. According to NEWTON's second law an external force is essential for the motion of the center of mass, which in the case of walking, crawling, rolling, etc. is the static friction force. Friction of rest, predominantly considered as a hindering effect, can also support movements. Numerous prototypes of artificial worms, presented in Chapter 6, are based on the targeted creation of different static and sliding friction forces between the worm segments and the environment.

When active forces are capable of creating relative motion of a body having contact with a surface or an environment, other forces that resist the motion of the body and have the opposite direction of the velocity vector will appear. These forces are called *friction forces*. Friction forces can also appear when the body is at rest. The question about the direction of such forces will be considered below.

The derivation of the friction forces is a very confusing question; therefore, we will discuss the properties of these forces and their mathematical description methods (without considering their physical nature). Many studies have been done regarding the mathematical description of the different laws of friction, see for example OLSSON, ÅSTRÖM et al. [109] or AWREJCEWICZ & OLEJNIK [13]. Here, we shall briefly consider the laws of friction where the friction force (during motion) depends only on the size and direction of the relative velocity of the body. The force law can be different for the different directions despite the fact that the direction of the friction force is always opposite to the velocity vector (anisotropy of the size of the friction force). In that case the friction force can be determined using the following expression for $\vec{v} \neq \vec{0}$:

$$\vec{F}_{fr} = -F(\vec{v})\,\frac{\vec{v}}{|\vec{v}|}\,. \tag{2.83}$$

Here, $F(\vec{v})$ is a scalar function of the vector argument and continuous for all $\vec{v} \neq \vec{0}$. In the case of *isotropy* (directional independence), the value of the friction force is only a function of the magnitude of velocity and $F(\vec{v}) = F(|\vec{v}|)$.

Friction, which depends on the velocity, can be conditionally divided into *viscous* and *dry*. Physically, this classification is caused by the size of the force that is necessary to bring the body out of rest. Mathematically, this is evidenced by the different behavior of the function used to assign the friction force for $\vec{v} = \vec{0}$.

2.4.3.2 Viscous Friction Force

With viscous friction the motion begins at any size of the applied (active) force, no matter how small. Mathematically, this means that the function $F(\vec{v})$ in (2.83) is continuous for all $\vec{v}$ (also for $\vec{v} = \vec{0}$) and zero for $\vec{v} = \vec{0}$, i.e., $F(\vec{0}) = 0$. Thus, equation

(2.83) can be written in the following form:

$$\vec{F}_{fr} = \begin{cases} -F(\vec{v})\,\frac{\vec{v}}{|\vec{v}|}\,, & \vec{v} \neq \vec{0} \\ \vec{0}, & \vec{v} = \vec{0}. \end{cases} \tag{2.84}$$

In the case of an object's motion along the x-axis with the velocity $\dot{x} = v$, the expression for the isotropic friction force (identical during the motion "back and forth") takes the form:

$$\vec{F}_{fr} = -F\left(|v|\right)\operatorname{sign}(v)\,\vec{e}_x\,,\;\; F(0) = 0\,. \tag{2.85}$$

The dependence $F(|v|)$ is usually described by the power of dependency with the coefficient of viscous friction d and the exponent α, i.e., $F(|v|) = d\,|v|^{\alpha}$. In this case the expression (2.85) becomes

$$\vec{F}_{fr} = -d\,|v|^{\alpha}\operatorname{sign}(v)\,\vec{e}_x\,,\;\; \alpha > 0\,. \tag{2.86}$$

This law describes well the deceleration of bodies in fluids. With not very high motion velocities the friction forces can be described by the linear function of velocity $(\alpha = 1)$. With sufficiently high velocities the friction force is proportional to the square of speed ($\alpha = 2$, see Fig. 2.19).

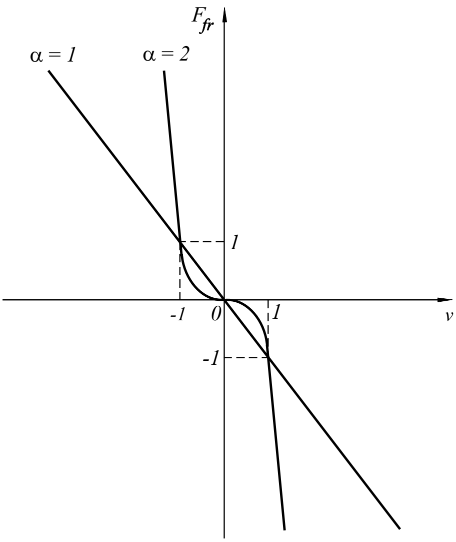

Fig. 2.19 Viscous friction force vs. velocity

If the value of the viscous friction force depends on the direction of motion, then the friction force becomes anisotropic (asymmetric). Figure 2.20 shows the asymmetric dependence between the force of a viscous friction and the velocity $(d_- > d_+)$. This dependence is piecewise linear ($\alpha = 1$), which gives a satisfactory approximation of the real physical phenomenon. Analytically, this dependence can be represented in the following form:

$$F_{fr} = \begin{cases} -d_- v\,, & v < 0 \\ -d_+ v\,, & v > 0\,. \end{cases}$$

If $v = 0$, then the coefficient is arbitrary since $F_{fr}(0) = 0$. Let us note that this dependence is piecewise linear, i.e., generally speaking, non-linear.

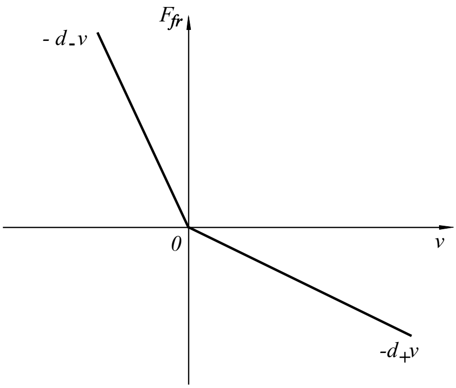

Fig. 2.20 Anisotropic viscous friction force vs. velocity

2.4.3.3 Dry Friction Force

The main difference between dry and viscous friction is the fact that the motion with dry friction cannot start with any value of the applied (acting) force. Therefore, an acting force must exceed a finite value to initiate the motion. The mathematical meaning is that the friction force, according to (2.83), immediately varies its direction but not necessarily its modulus while crossing the zero-velocity axis.

The most established classical dry friction model is from COULOMB. He studied the friction force by the slow mutual displacement of contacted bodies and proposed a simple empirical law: *The friction force during motion does not depend on the velocity, it only depends on the direction of motion (and is always directed against the motion).* According to COULOMB's law the friction force is proportional to the normal load F_N during motion ($v \neq 0$). It is described by the relationship, see Fig. 2.21:

$$F_{fr} = -\mu\, F_N \,\mathrm{sign}(v)\,. \tag{2.87}$$

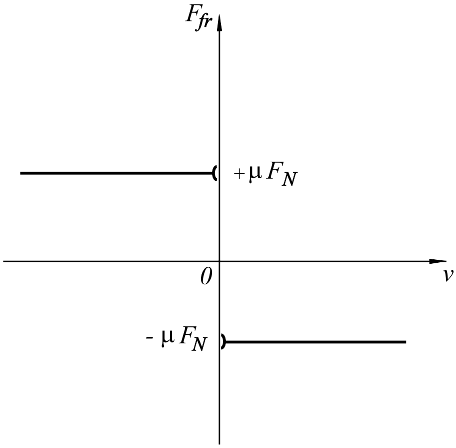

Fig. 2.21 COULOMB's friction model

The coefficient of proportionality μ can be determined experimentally and is called the coefficient of kinetic friction.

The friction model in (2.87) does not specify the friction force in the case of zero velocities. Obviously, this is a drawback, but because of its simplicity it is often used in the first steps of friction modeling. A detailed discussion of this fact, see also the uncertainty of the expression (2.83) with $\vec{v} = \vec{0}$, leads to the consideration of such concepts as the *static friction force* (also called the *friction force at rest* or *stiction*). This idea was introduced by MORIN [99]. It follows from the force equilibrium that the static friction force is directed against the resultant vector of the active forces which are trying to force the body to slide.

A geometric interpretation of the equilibrium of the forces in the case of stiction is given using the friction cone, see Fig. 2.22. If a body is located on a surface, then the full reaction acting on the body from the surface consists of the normal reaction $\vec{F}_N$ and the friction force $\vec{F}_{fr}$. The angle of friction ρ is connected to the coefficient of static friction μ_S and expressed through the value of the maximal static friction force $F_{fr\max} = F_s$ (static friction force):

$$\mu_S = \tan\rho = \frac{F_s}{F_N}\,.$$

The coefficient of static friction μ_S can also be determined experimentally.

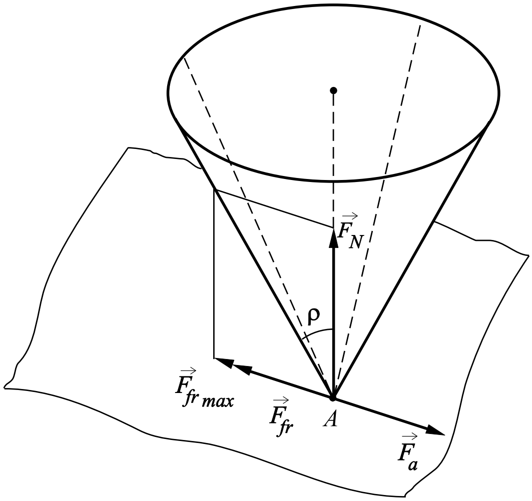

Fig. 2.22 Friction cone

We can construct a cone at the point of contact between the body and the surface. If the cone's axis is directed along the normal of the surface and the angle between the generatrix and the axis is equal to ρ, then the reaction force due to the contact in the state of equilibrium will always be located inside the cone, which is then called the friction cone.

In order to illustrate the meaning of the friction at rest, we can consider a simple experiment. The object is to try to move a body by pulling it with a rope connected to a spring dynamometer, see Fig. 2.23.

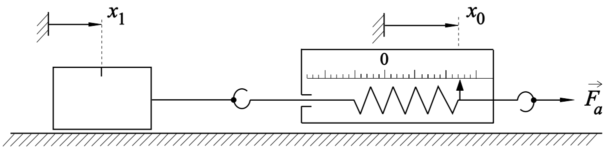

Fig. 2.23 Experiment illustrating friction of rest

The body does not move from a small displacement of the end of the rope. This case means that the friction force fully compensates the applied force. We gradually increase the force $\vec{F}_a$ and, therefore, the displacement of the end of the spring. At a certain moment the body will start moving. The indication of the dynamometer registered at this moment is usually called the (maximum) static friction force. If we continue to slowly pull the rope, the body will move over the surface. It appears that the indications of the dynamometer registered during the motion will not be as high as the reading at the moment of the motion began. Usually, the friction force during sliding is less than the force necessary for to initiate the motion. Thus, the static friction force, generally speaking, differs from the friction force of motion (kinetic friction force).

In Fig. 2.24 the displacement of the force application point x_0 is plotted versus the displacement of the body x_1. Sliding only occurs when a certain force is exceeded, meaning the body will start moving at a certain x_0 (point A).

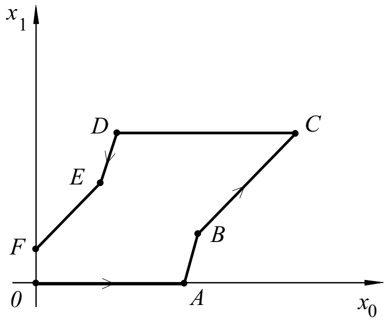

Fig. 2.24 Displacement of the force application point vs. displacement of the mass [118]

In accordance with COULOMB's law, the force necessary to overcome the static friction and to move one body over the surface of another only depends on the normal component F_N of the reaction force.

Now, we consider the one-dimensional case with anisotropic friction, a theme which is discussed extensively in Chapter 6. The motion dynamics of the body under the action of the force $\vec{F}_a$ (the sum of all acting forces except for the dry friction force) and the friction force $\vec{F}_{fr}$ can be described by the differential equation

$$\dot{x} = v\,, \quad m\dot{v} = F_a + F_{fr}(v, F_a)\,. \tag{2.88}$$

The extended COULOMB law (including stiction) takes the form:

$$F_{fr}(v, F_a) = \begin{cases} F_- = \mu_- F_N\,, & v < 0 \\ -F_+ = -\mu_+ F_N\,, & v > 0 \\ -F_a\,, & \\ F_-\,, & v = 0 \quad \text{and} \quad \begin{cases} -F_{s-} \le F_a \le F_{s+}\,, \\ F_a < -F_{s-}\,, \\ F_a > F_{s+}\,. \end{cases} \\ -F_+\,, & \end{cases} \tag{2.89}$$

Now, we will assume that $v(t) = \dot{x}(t)$ is a piecewise continuously differentiable function with respect to the time. This assumption is sufficient for describing the physically realizable motions. In the one-dimensional case for anisotropic dry friction, the corresponding force can be represented in the following, more general form than formula (2.89):

$$F_{fr}(v,F_a) = \begin{cases} F_-(v)\,, & v<0 \\ -\,F_+(v)\,, & v>0 \\ -\,F_a\,, & \\ F_-\,, & v=0 \quad \text{and} \quad \begin{cases} -F_{s-} \le F_a \le F_{s+}\,, \\ F_a < -F_{s-}\,, \\ F_a > F_{s+}\,. \end{cases} \\ -\,F_+\,, & \end{cases} \tag{2.90}$$

Here, the functions $F_-(v)$ and $F_+(v)$ are continuous and have non-negative values, and it is usually assumed that $F_-(-v) - F_+(v) = const$. The parameters $F_{s\pm}$ are the bounds of the static friction force. Let us pay attention to the limiting values F_- and F_+ of the functions $F_-(v)$ and $F_+(v)$ (i.e., $F_- = \lim_{v\to-0} F_-(v)\,, F_+ = \lim_{v\to+0} F_+(v)$) which can be less than the values of F_{s-} and F_{s+}. In this case the instantaneous decrease of the friction force after the start of motion will be reflected. If the corresponding limits are equal to the values of F_{s-} and F_{s+}, then the friction force will be decreased continuously after the start of motion, but possibly with a high gradient, see Fig. 2.25.

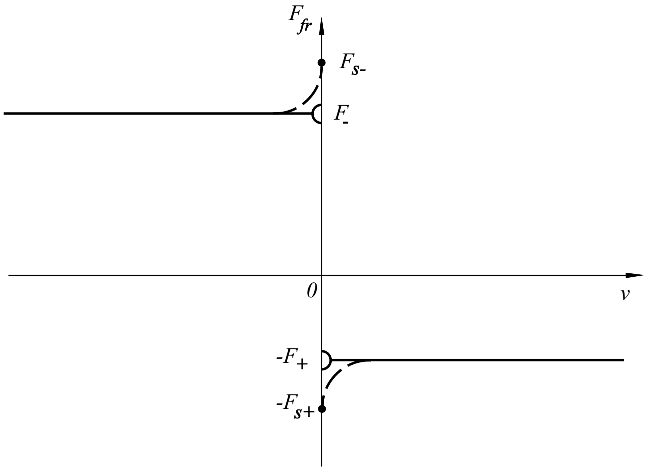

Fig. 2.25 Dry friction force vs. velocity

Exercise 2.6.
What is the fundamental mathematical difference between the dependencies of the friction force $\vec{F}_{fr}$ on the velocity $\vec{v}$ for dry and viscous friction?

Some biological subjects, for example worms, have special attachments to create an asymmetry of friction, see Fig. 2.26.

Fig. 2.26 "Spikes" on an earthworm [Photo courtesy of N. Michiels]

The influence of the dry friction law is essential for the analysis of worm-like locomotion systems, see Chapter 6. Thus, let us finally go more into detail for the conditions of expression (2.90) which correspond to the value of dry friction for $v = 0$.

First, we shall pay attention to the last two conditions. Let us suppose that the velocity during some moment of time t' is $v(t') = 0$ and that one of the two last conditions of expression (2.90) for the friction force is fulfilled. Because of these conditions and because of the equation of motion (2.88) it holds that $\dot{v}(t') \neq 0$. Because of the piecewise continuity $\dot{v}(t) \neq 0$ in a certain neighborhood of the point t' (possibly one-sided), and the sign is preserved in this area, i.e., $v(t)$ decreases (increases) in this neighborhood and the function $v(t)$ is negative (positive) accordingly. Thus, the fulfilment of the condition $v = 0$ with the simultaneous fulfilment of one of the two last conditions (2.90) is possible only at isolated points that *do not affect* the result of the integration of the equation (2.88), that is, the motion of the system. The first condition, corresponding to $v = 0$ in (2.90), is connected either with a full stop or with motion interrupted with stops if, for example, F_a is a periodic function of time. This regime is called "stick-slip", which is characteristic of systems with dry friction when the velocity is zero in a finite interval of time.

In the next subsection we focus on an adequate mathematical model of COULOMB dry friction which explicitly incorporates friction of rest and describes stick-slip motion.

2.4.3.4 An Approach of Mathematical Friction Modeling

In this subsection we go into more details with respect to some aspects concerning the mathematical handling of static and kinetic friction forces in numerical simulations.

Well-known descriptions of friction models were considered in the previous sections. We would now like to emphasize that the modeling makes friction a function of *two arguments*. The dependent variables are the velocity v *and* the sum of all acting forces F_a (except for the force of dry friction). We will give a short introduction to this mathematical friction model and emphasize some figures of 3-D friction graphs that obviously arise in modeling $\vec{F}_{fr} = \vec{F}_{fr}(v, F_a)$.

The friction values, especially at $v = 0$, are only roughly known because of measurement uncertainty and a lack of knowledge of what happens inside the thin layer between contacting surfaces. An adequate mathematical model of friction will be necessary to achieve good stick-slip behavior. In order to achieve a satisfactory handling of the COULOMB rules on the computer, we replace $v = 0$ by $-\Delta < v < \Delta$ with some small Δ (obviously, $\Delta \approx 10^{-12}$ might be seen to model a computer accuracy). This method is based on an idea from KARNOPP to deal with a Δ blow-up interval to numerically accomodate zero velocity.

Summarizing, these facts lead us to a friction model that depends on two variables and consists of a KARNOPP structure. Simulations and experiments do have to be changed, but, for example, with respect to later adaptive tracking problems of artificial worms (ground contact modeled via dry friction with stiction), we have to check some assumptions to get a well-defined solution since $\vec{F}_{fr} = \vec{F}_{fr}(v, F_a)$. Hence, control inputs might be parts of F_a; therefore, we may arrive at more complicated dynamics of the systems.

We again consider a mechanical system with DOF $= 1$, which is described by equation (2.88). Following COULOMB's description $\vec{F}_{fr}$ compensates the force $\vec{F}_a$ if and only if the velocity v is zero and F_a does not exceed certain limit values, whereas $\vec{F}_{fr}$ takes constant values if $v \neq 0$. These values are characteristic of the contacting surfaces. They may depend on the orientation of the motion (anisotropic friction), and often they are supposed to be proportional to the normal force $\vec{F}_N$ acting between the surfaces with constant friction coefficient μ, see equation (2.89). The description above qualifies $\vec{F}_{fr}$ to be a physically given force during motion $v \neq 0$ (kinetic friction force), whereas $\vec{F}_{fr}$ appears as a constraint force with given bounds as long as $v = 0$ (static friction force), according to HAMEL [62].

In connection with the dynamics above, the physical model of friction possesses the form shown in equation (2.91). The model uncertainty mentioned at $v = 0$ might allow the following relaxation of the above model with some small $\Delta > 0$:

$$F_{fr}(v, F_a) = \begin{cases} F_-\,, & v < -\Delta \\ -F_+\,, & v > \Delta \\ -F_a\,, & \\ F_-\,, & |v| \leq \Delta \quad \text{and} \quad \begin{cases} -F_{s-} \leq F_a \leq F_{s+}\,, \\ F_a < -F_{s-}\,, \\ F_a > F_{s+}\,. \end{cases} \\ -F_+\,, & \end{cases} \tag{2.91}$$

Using the HEAVISIDE function h, or more precisely the so-called "boxcar"-function:

$$h(a, b, x) = \begin{cases} 1, \; a \leq x < b\,, \\ 0, \; \text{else}\,, \end{cases} \tag{2.92}$$

and ignoring some inconsistencies at the limit points a and b (with respect to the later smooth approximation this is not a problem) we write h in the form

$$h(a,b,x) = \frac{1}{2}\big(\mathrm{sign}(x-a) + \mathrm{sign}(b-x)\big)\,, \tag{2.93}$$

then F_{fr} can be given in the form (disregarding its values at $v = \pm\Delta$)

$$\left.\begin{aligned} F_{fr}(v,F_a) = {} & -F_a\, h(-\Delta\,,\Delta\,,v)\, h(-F_{s-}\,,F_{s+}\,,F_a) \\ & +F_-\left\{h(-\infty,-\Delta\,,v) + h(-\Delta\,,\Delta\,,v)\, h(-\infty,-F_{s-}\,,F_a)\right\} \\ & -F_+\left\{h(\Delta\,,+\infty,v) + h(-\Delta\,,\Delta\,,v)\, h(F_{s+}\,,+\infty,F_a)\right\}. \end{aligned}\right\} \tag{2.94}$$

In order to avoid computational difficulties caused by the afore-mentioned jumps in the h function, we turn to a smooth mathematical model (in the sense of an approximation). Basically, with respect to later numerical simulations and illustrations, we use a tanh approximation of the sign function

$$\mathrm{sign}(x) \approx \tanh(A\,x)$$

with some sufficiently large $A \gg 1$.

The smooth mathematical model is now

$$\begin{aligned} F_{fr}(v,F_a) = {} & -F_a\, H(-\Delta\,,\Delta\,,v)\, H(-F_{s-}\,,F_{s+}\,,F_a) \\ & +F_-\left\{H(-\infty,-\Delta\,,v) + H(-\Delta\,,\Delta\,,v)\, H(-\infty,-F_{s-}\,,F_a)\right\} \\ & -F_+\left\{H(\Delta\,,+\infty,v) + H(-\Delta\,,\Delta\,,v)\, H(F_{s+}\,,+\infty,F_a)\right\}, \end{aligned} \tag{2.95}$$

where

$$H(a,b,x) = \frac{1}{2}\Big\{\tanh\big(A\,(x-a)\big) + \tanh\big(A\,(b-x)\big)\Big\} \tag{2.96}$$

is the smooth approximation of $h(a,b,x)$.

The following Fig. 2.27 sketches the blow-up and approximation procedure:

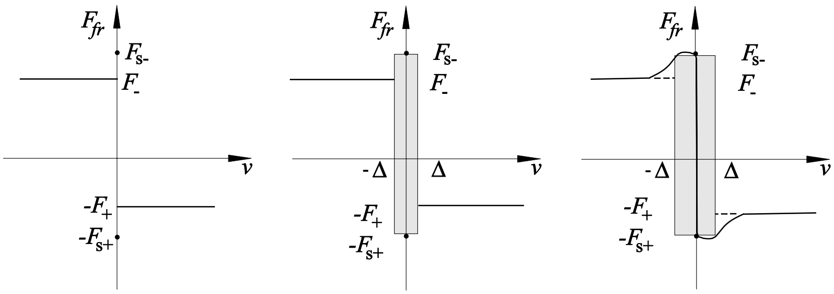

Fig. 2.27 Blow-up and approximation procedure.

a) the original dry (COULOMB) friction with stiction, b) the Δ-blow-up interval around zero (non-smooth), and c) the smooth approximation with the HEAVISIDE function H. In fact, Fig. 2.27 shows the projection of the graph of F_{fr} to a plane $F_a = const$. This graph (for some data) is given in Fig. 2.28, somewhat distorted by the coarse coordinate grid.

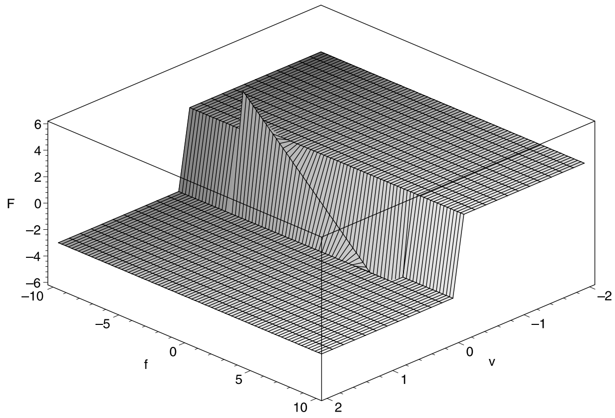

Fig. 2.28 Graph of friction force $F_{fr}(v, F_a)$

Now, we present two examples and simulations of applying this friction law to a DOF = 1 mechanical system. We notice that, for the illustration of the method, we choose arbitrary parameters of the models. Further, if we investigate real prototypes, we take into account the concrete parameter values of the locomotion systems. For approximating $h(-\Delta,\Delta,v)$ the value of A to be chosen will be strongly influenced by the value of Δ to be used. The combination $\Delta = 0.0005$ and $A = 10^5$ has proved suitable in calculations. The following friction values are taken into account: $F_{s\pm} = 6$, $F_{\pm} = 3$.

Example 2.3

$$\dot{x} = v, \quad m\dot{v} = -bt + F_{fr}\big(v(t), -bt\big), \quad x(0) = 0, \quad v(0) = 0, \qquad (2.97)$$

with $m = 1$ and $b = 1$.
Figure 2.29 shows the solutions $x(t)$ and $v(t)$ on a given time interval. The drive F_a has to grow until the friction F_{fr} reaches F_{s-}, only then does the system begin to slide.

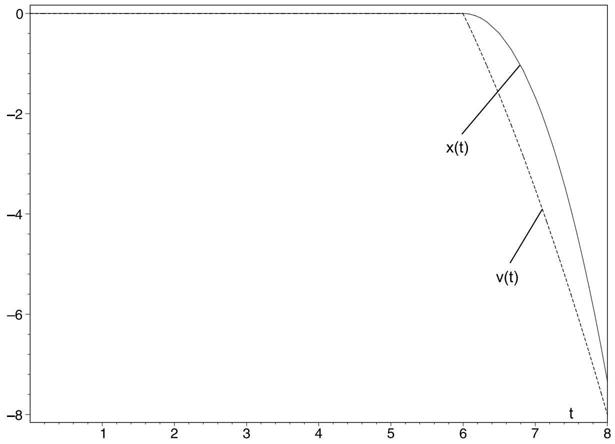

Fig. 2.29 The solutions $x(t)$ and $v(t)$

The corresponding space curve $t \mapsto \Big(v(t)\,, F_a(t)\,, F_{fr}\big(v(t)\,, F_a(t)\big)\Big)$ lies on the graph of the friction function, see Fig. 2.30.

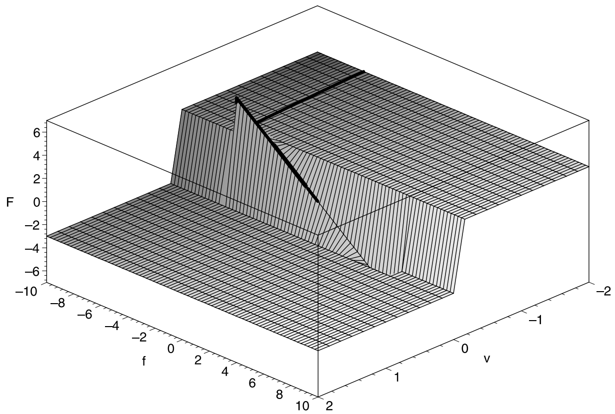

Fig. 2.30 The space curve (thick) on the friction graph

Example 2.4

$$\dot{x}(t) = v(t),\; m\dot{v} = -d\,x(t) + 4\,b\,t + F_{fr}\big(v(t), -d\,x(t) + 4\,b\,t\big) \\ x(0) = 0\,,\; v(0) = 0, \tag{2.98}$$

with $m = 1$, $b = 1$ and $d = 2$.
The stick-slip effect can clearly be seen in Fig. 2.31.

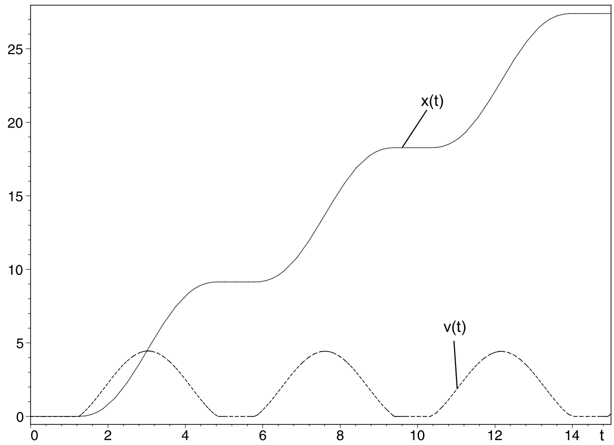

Fig. 2.31 The solutions $x(t)$ and $v(t)$

The corresponding space curve $t \mapsto \Big(v(t), F_a(t), F_{fr}\big(F_a(t), v(t)\big)\Big)$ lies on the graph of the friction function, see Fig. 2.32.

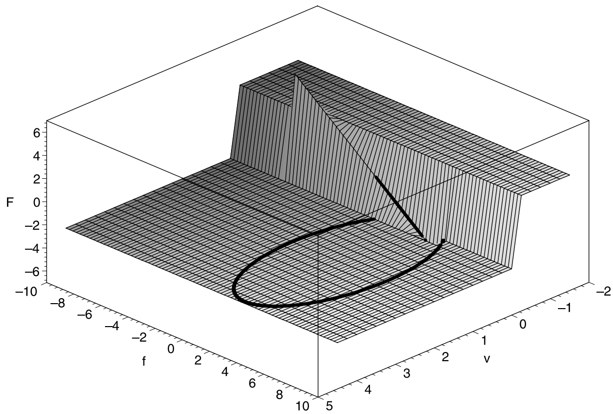

Fig. 2.32 The space curve (thick) on the friction graph

In using the afore-mentioned friction law in the later adaptive tracking problem, we must mind the fact that this function not only depends on the velocity but also on the visco-elastic forces (and the control inputs).

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