Chapter 2
Symmetric Objects and Functors

Introduction

In this chapter, we recall the definition of the category of Σ_*-objects and we review the relationship between Σ_*-objects and functors. In short, a *Σ_*-object* (in English words, a *symmetric sequence of objects*, or simply a *symmetric object*) is the coefficient sequence of a generalized symmetric functor $\mathrm{S}(M) : X \mapsto \mathrm{S}(M, X)$, defined by a formula of the form

$$\mathrm{S}(M, X) = \bigoplus_{r=0}^{\infty} (M(r) \otimes X^{\otimes r})_{\Sigma_r}.$$

In §2.1, we recall the definition of the tensor product of Σ_*-objects, the operation which reflects the pointwise tensor product of functors and which provides the category of Σ_*-objects with the structure of a symmetric monoidal category over the base category.

Beside the tensor product, the category of Σ_*-objects comes equipped with a composition product that reflects the composition of functors. The definition of this composition structure is recalled in §2.2.

The map $\mathrm{S} : M \mapsto \mathrm{S}(M)$ defines a functor $\mathrm{S} : \mathcal{M} \to \mathcal{F}$, where $\mathcal{M}$ denotes the category of Σ_*-objects and $\mathcal{F}$ denotes the category of functors $F : \mathcal{E} \to \mathcal{E}$ on any symmetric monoidal category $\mathcal{E}$ over the base category $\mathcal{C}$. The adjoint functor theorem implies that this functor has a right adjoint $\Gamma : \mathcal{F} \to \mathcal{M}$. In §2.3 we give an explicit construction of this adjoint functor by using that the symmetric monoidal category $\mathcal{E}$ is enriched over the base category $\mathcal{C}$. In addition, we prove that the map $\mathrm{S} : M \mapsto \mathrm{S}(M)$ defines a faithful functor in the enriched sense as long as the category $\mathcal{E}$ is equipped with a faithful functor $\eta : \mathcal{C} \to \mathcal{E}$. In the case $\mathcal{E} = \mathcal{C} = \Bbbk\,\mathrm{Mod}$, the category of modules over a ring $\Bbbk$, we use the explicit construction of the adjoint functor $\Gamma : G \mapsto \Gamma(G)$ to prove that the functor $\mathrm{S} : M \mapsto \mathrm{S}(M)$ is bijective on object sets under mild conditions on Σ_*-objects or on the ground ring $\Bbbk$.

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In §§2.1-2.3, we deal with global structures of the category of Σ_*-objects. In §2.4, we study the image of colimits under the functor $\mathrm{S}(M) : \mathcal{E} \to \mathcal{E}$ associated to a Σ_*-object M. Explicitly, we record that the functor $\mathrm{S}(M) : \mathcal{E} \to \mathcal{E}$ preserves filtered colimits and reflexive coequalizers (but not all colimits). This verification is required by our conventions on functors (see §0.1) and is also used in §3.3, where we address the construction of colimits in categories of algebras over operads.

2.1 The Symmetric Monoidal Category of Σ_*-Objects and Functors

Formally, a Σ_*-object in a category $\mathcal{C}$ consists of a sequence $M(n)$, $n \in \mathbb{N}$, where $M(n)$ is an object of $\mathcal{C}$ equipped with an action of the symmetric group Σ_n. A morphism of Σ_*-objects $f : M \to N$ consists of a sequence of morphisms $f : M(n) \to N(n)$ that commute with the action of symmetric groups.

Usually, we have a base category $\mathcal{C}$, fixed once and for all, and we deal tacitely with Σ_*-objects in that category $\mathcal{C}$. Otherwise we specify explicitly the category in which we define our Σ_*-object. We may use the notation $\mathcal{E}^{\Sigma_*}$ to refer to the category of Σ_*-objects in a given category $\mathcal{E}$, but we usually adopt the short notation $\mathcal{M}$ for the category of Σ_*-objects in the base category $\mathcal{E} = \mathcal{C}$.

In the introduction of the chapter, we recall that $\mathcal{M}$ forms a symmetric monoidal category over $\mathcal{C}$. In this section, we address the definition and applications of this categorical structure. More specifically, we use the formalism of symmetric monoidal categories over a base category to express the relationship between the tensor product of Σ_*-objects and the pointwise tensor product of functors on a symmetric monoidal category $\mathcal{E}$ over $\mathcal{C}$. Formally, the category $\mathcal{F}$ of functors $F : \mathcal{E} \to \mathcal{E}$ inherits the structure of a symmetric monoidal category over $\mathcal{C}$ and the map $\mathrm{S} : M \mapsto \mathrm{S}(M)$ defines a functor of symmetric monoidal categories over $\mathcal{C}$:

$$(\mathcal{M}, \otimes, \mathtt{1}) \xrightarrow{\mathrm{S}} (\mathcal{F}, \otimes, \mathtt{1}).$$

2.1.1 The Functor Associated to a Σ_*-Object. First of all, we recall the definition of the functor $\mathrm{S}(M) : \mathcal{E} \to \mathcal{E}$ associated to a Σ_*-object M, for $\mathcal{E}$ a symmetric monoidal category over $\mathcal{C}$. The image of an object $X \in \mathcal{E}$ under this functor, denoted by $\mathrm{S}(M, X) \in \mathcal{E}$, is defined by the formula

$$\mathrm{S}(M, X) = \bigoplus_{r=0}^{\infty} (M(r) \otimes X^{\otimes r})_{\Sigma_r},$$

where we consider the coinvariants of the tensor products $M(r) \otimes X^{\otimes r}$ under the action of the symmetric groups Σ_r. We use the internal tensor product of $\mathcal{E}$ to form the tensor power $X^{\otimes r}$, the external tensor product to form the object $M(r) \otimes X^{\otimes r}$ in $\mathcal{E}$, and the existence of colimits in $\mathcal{E}$ to form the coinvariant object $(M(r) \otimes X^{\otimes r})_{\Sigma_r}$ and $\mathrm{S}(M, X)$.

In §2.1.4, we introduce pointwise operations on functors $F : \mathcal{E} \to \mathcal{E}$ that correspond to tensor operations on the target. In light of these structures on functors, we have a functor identity

$$\mathrm{S}(M) = \bigoplus_{r=0}^{\infty} (M(r) \otimes \mathrm{Id}^{\otimes r})_{\Sigma_r},$$

where $\mathrm{Id} : \mathcal{E} \to \mathcal{E}$ denotes the identity functor on $\mathcal{E}$.

The construction $\mathrm{S} : M \mapsto \mathrm{S}(M)$ is clearly functorial in $\mathcal{E}$. Explicitly, for a functor $\rho : \mathcal{D} \to \mathcal{E}$ of symmetric monoidal categories over $\mathcal{C}$, the diagram of functors

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\rho"] \arrow[d, "\mathrm{S}(M)"'] & \mathcal{E} \arrow[d, "\mathrm{S}(M)"] \\ \mathcal{D} \arrow[r, "\rho"'] & \mathcal{E} \end{tikzcd}$$

commutes up to natural isomorphisms. Equivalently, we have a natural functor isomorphism $\mathrm{S}(M) \circ \rho \simeq \rho \circ \mathrm{S}(M)$, for every $M \in \mathcal{M}$.

In the point-set context, the element of $\mathrm{S}(M, V)$ represented by the tensor $\xi \otimes (x_1 \otimes \cdots \otimes x_r) \in M(r) \otimes V^{\otimes r}$ is denoted by $\xi(x_1, \ldots, x_r) \in \mathrm{S}(M, V)$. The coinvariant relations read $\sigma\xi(x_1, \ldots, x_r) = \xi(x_{\sigma(1)}, \ldots, x_{\sigma(r)})$, for $\sigma \in \Sigma_r$.

Clearly, the map $\mathrm{S} : M \mapsto \mathrm{S}(M)$ defines a functor $\mathrm{S} : \mathcal{M} \to \mathcal{F}$, where $\mathcal{F} = \mathcal{F}(\mathcal{E}, \mathcal{E})$ denotes the category of functors $F : \mathcal{E} \to \mathcal{E}$. (Because of our conventions on functor categories, we should check that $\mathrm{S}(M) : \mathcal{E} \to \mathcal{E}$ preserves filtered colimits, but we postpone the simple verification of this assertion to §2.4.)

The category $\mathcal{M}$ is equipped with colimits and limits created termwise in $\mathcal{C}$. The category of functors $\mathcal{F} = \mathcal{F}(\mathcal{E}, \mathcal{E})$ is equipped with colimits as well, inherited pointwise from the category $\mathcal{E}$. By interchange of colimits, we obtain immediately that the functor $\mathrm{S} : \mathcal{M} \to \mathcal{F}(\mathcal{E}, \mathcal{E})$ preserves colimits.

2.1.2 Constant Σ_*-Objects and Constant Functors. Recall that a Σ_*-object M is *constant* if we have $M(r) = 0$ for all $r > 0$. The base category $\mathcal{C}$ is isomorphic to the full subcategory of $\mathcal{M}$ formed by constant objects. Explicitly, to any object $C \in \mathcal{C}$, we associate the constant Σ_*-object $\eta(C)$ such that $\eta(C)(0) = C$. This constant Σ_*-object is associated to the constant functor $\mathrm{S}(C, X) \equiv C$.

2.1.3 Connected Σ_*-Objects and Functors. The category embedding $\eta : \mathcal{C} \to \mathcal{M}$ has an obvious left-inverse $\epsilon : \mathcal{M} \to \mathcal{C}$ defined by $\epsilon(M) = M(0)$. The category of *connected Σ_*-objects* $\mathcal{M}^0$ is the full subcategory of $\mathcal{M}$ formed by Σ_*-objects M such that $\epsilon(M) = M(0) = 0$, the initial object of $\mathcal{C}$. Clearly, connected Σ_*-objects are associated to functors $\mathrm{S}(M) : \mathcal{E} \to \mathcal{E}$ such that $\mathrm{S}(M, 0) = 0$.

In the case of a connected Σ_*-object $M \in \mathcal{M}^0$, we can extend the construction of §2.1.1 to reduced symmetric monoidal categories. To be explicit, for objects X in a reduced symmetric monoidal category $\mathcal{E}^0$ over $\mathcal{C}$, we set

$$\mathrm{S}^0(M, X) = \bigoplus_{n=1}^{\infty} (M(n) \otimes X^{\otimes n})_{\Sigma_n}$$

to obtain a functor $\mathrm{S}^0(M) : \mathcal{E}^0 \to \mathcal{E}^0$.

2.1.4 The Symmetric Monoidal Category of Functors. Let $\mathcal{F} = \mathcal{F}(\mathcal{A}, \mathcal{C})$ denote the category of functors $F : \mathcal{A} \to \mathcal{C}$, where $\mathcal{A}$ is any category (see §0.1). Recall that $\mathcal{F} = \mathcal{F}(\mathcal{A}, \mathcal{C})$ has all small colimits and limits, inherited pointwise from the base category $\mathcal{C}$.

Observe that the category $\mathcal{F}$ is equipped with an internal tensor product $\otimes : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ and with an external tensor product $\otimes : \mathcal{C} \otimes \mathcal{F} \to \mathcal{F}$, inherited from the base symmetric monoidal category, so that $\mathcal{F}$ forms a symmetric monoidal category over $\mathcal{C}$. Explicitly: the internal tensor product of functors $F, G : \mathcal{A} \to \mathcal{C}$ is defined pointwise by $(F \otimes G)(X) = F(X) \otimes G(X)$; for all $X \in \mathcal{A}$, the tensor product of a functor $G : \mathcal{A} \to \mathcal{C}$ with an object $C \in \mathcal{C}$ is defined by $(C \otimes F)(X) = C \otimes F(X)$; the constant functor $\mathtt{1}(X) \equiv \mathtt{1}$, where $\mathtt{1}$ is the unit object of $\mathcal{C}$, represents the unit object in the category of functors.

The functor of symmetric monoidal categories

$$\eta : (\mathcal{C}, \otimes, \mathtt{1}) \to (\mathcal{F}, \otimes, \mathtt{1})$$

determined by this structure identifies an object $C \in \mathcal{C}$ with the constant functor $\eta(C)(X) \equiv C$. If $\mathcal{A}$ is equipped with a base object $0 \in \mathcal{A}$, then we have a natural splitting $\mathcal{F} = \mathcal{C} \times \mathcal{F}^0$, where $\mathcal{F}^0$ is the reduced symmetric monoidal category over $\mathcal{C}$ formed by functors F such that $F(0) = 0$, the initial object of $\mathcal{C}$.

Obviously, we can extend the observations of this paragraph to a category of functors $\mathcal{F} = \mathcal{F}(\mathcal{A}, \mathcal{E})$, where $\mathcal{E}$ is a symmetric monoidal category over the base category $\mathcal{C}$. In this case, the category $\mathcal{F} = \mathcal{F}(\mathcal{A}, \mathcal{E})$ forms a symmetric monoidal category over $\mathcal{E}$, and hence over the base category by transitivity.

We have:

2.1.5 Proposition (*cf.* [12, §1.1.3] or [14, §1.2] or [54, Lemma 2.2.4])**.** *The category $\mathcal{M}$ is equipped with the structure of a symmetric monoidal category over $\mathcal{C}$ so that the map $\mathrm{S} : M \mapsto \mathrm{S}(M)$ defines a functor of symmetric monoidal categories over $\mathcal{C}$*

$$\mathrm{S} : (\mathcal{M}, \otimes, \mathtt{1}) \to (\mathcal{F}(\mathcal{E}, \mathcal{E}), \otimes, \mathtt{1}),$$

functorially in $\mathcal{E}$, for every symmetric monoidal category $\mathcal{E}$ over $\mathcal{C}$. □

The functoriality claim asserts explicitly that, for any functor $\rho : \mathcal{D} \to \mathcal{E}$ of symmetric monoidal categories over $\mathcal{C}$, the tensor isomorphisms $\mathrm{S}(M \otimes N) \simeq \mathrm{S}(M) \otimes \mathrm{S}(N)$ and the functoriality isomorphisms $\mathrm{S}(M) \circ \rho \simeq \rho \circ \mathrm{S}(M)$ fit a commutative hexagon

$$\begin{tikzcd}
\mathrm{S}(M \otimes N) \circ \rho \arrow[rr, "\simeq"] \arrow[d, "\simeq"'] & & \rho \circ \mathrm{S}(M \otimes N) \arrow[d, "\simeq"] \\
(\mathrm{S}(M) \otimes \mathrm{S}(N)) \circ \rho \arrow[dr, "="'] & & \rho \circ (\mathrm{S}(M) \otimes \mathrm{S}(N)) \arrow[dl, "\simeq"] \\
\mathrm{S}(M) \circ \rho \otimes \mathrm{S}(N) \circ \rho \arrow[r, "\simeq"'] & \rho \circ \mathrm{S}(M) \otimes \rho \circ \mathrm{S}(N) &
\end{tikzcd}$$

and similarly for the isomorphism $\mathrm{S}(\mathtt{1}) \simeq \mathtt{1}$.

We have further:

2.1.6 Proposition. *The category $\mathcal{M}^0$ of connected Σ_*-objects forms a reduced symmetric monoidal category over $\mathcal{C}$.*

The category $\mathcal{M}$ admits a splitting $\mathcal{M} = \mathcal{C} \times \mathcal{M}^0$ and is isomorphic to the symmetric monoidal category over $\mathcal{C}$ associated to the reduced category $\mathcal{M}^0$. The functor $\mathrm{S} : M \mapsto \mathrm{S}(M)$ fits a diagram of symmetric monoidal categories over $\mathcal{C}$

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\mathrm{S}"] & \mathcal{F}(\mathcal{E}, \mathcal{E}) \\
\mathcal{C} \times \mathcal{M}^0 \arrow[u, "\simeq"] \arrow[r, "\mathrm{Id} \times \mathrm{S}"'] & \mathcal{C} \times \mathcal{F}(\mathcal{E}, \mathcal{E})^0 \arrow[u, "\simeq"']
\end{tikzcd} . \quad \square$$

We refer to the literature for the proof of the assertions of propositions 2.1.5-2.1.6. For our needs, we recall simply the explicit construction of the tensor product $M \otimes N$. This construction also occurs in the definition of the category of symmetric spectra in stable homotopy (see [30, §2.1]).

2.1.7 The Tensor Product of Σ_*-Objects. The terms of the tensor product of Σ_*-objects are defined explicitly by a formula of the form

$$(M \otimes N)(n) = \bigoplus_{p+q=n} \Sigma_n \otimes_{\Sigma_p \times \Sigma_q} M(p) \otimes N(q),$$

where we use the tensor product over the category of sets, defined explicitly in §1.1.7. In the construction, we use the canonical group embedding $\Sigma_p \times \Sigma_q \subset \Sigma_{p+q}$ which identifies a permutation $\sigma \in \Sigma_p$ (respectively, $\tau \in \Sigma_q$) to a permutation of the subset $\{1, \dots, p\} \subset \{1, \dots, p, p{+}1, \dots, p{+}q\}$ (respectively, $\{p{+}1, \dots, p{+}q\} \subset \{1, \dots, p, p{+}1, \dots, p{+}q\}$). The tensor product $M(p) \otimes N(q)$ forms a $\Sigma_p \times \Sigma_q$-object in $\mathcal{C}$. The group $\Sigma_p \times \Sigma_q$ acts on Σ_n by translations on the right. The quotient in the tensor product makes this right $\Sigma_p \times \Sigma_q$-action agree with the left $\Sigma_p \times \Sigma_q$-action on $M(p) \otimes N(q)$.

The group Σ_n also acts on Σ_n by translation on the left. This left Σ_n-action induces a left Σ_n-action on $(M \otimes N)(n)$ and determines the Σ_*-object structure of the collection $M \otimes N = \{(M \otimes N)(n)\}_{n\in\mathbb{N}}$.

The constant Σ_*-object $\mathtt{1}$ such that

$$\mathtt{1}(n) = \begin{cases} \mathtt{1} \text{ (the unit object of } \mathcal{C}), & \text{if } n = 0, \\ 0, & \text{otherwise,} \end{cases}$$

defines a unit for this tensor product. The associativity of the tensor product of Σ_*-objects is inherited from the base category. Let $\tau(p,q) \in \Sigma_n$ be the permutation such that:

$$\tau(p,q)(i) = p + i, \text{ for } i = 1,\dots,q,$$
$$\tau(p,q)(q + i) = i, \text{ for } i = 1,\dots,p.$$

The symmetry isomorphism $\tau(M,N) : M \otimes N \to N \otimes M$ is induced componentwise by morphisms of the form

$$\Sigma_n \otimes M(p) \otimes N(q) \xrightarrow{\tau(p,q)^* \otimes \tau} \Sigma_n \otimes N(q) \otimes M(p)$$

where we use the symmetry isomorphism $\tau : M(p) \otimes N(q) \to N(q) \otimes M(p)$ of the category $\mathcal{C}$ and a translation of the right by the block transposition $\tau(p,q)$ on the symmetric group Σ_n.

The functor $\eta : \mathcal{C} \to \mathcal{M}$ which identifies the objects of $\mathcal{C}$ to constant Σ_*-objects defines a functor of symmetric monoidal categories

$$\eta : (\mathcal{C}, \otimes, \mathtt{1}) \to (\mathcal{M}, \otimes, \mathtt{1})$$

and makes $(\mathcal{M}, \otimes, \mathtt{1})$ into a symmetric monoidal category over $\mathcal{C}$. By an immediate inspection of definitions, we obtain that the external tensor product of a Σ_*-object M with an object $C \in \mathcal{C}$ is given by the obvious formula $(C \otimes M)(r) = C \otimes M(r)$.

2.1.8 Tensor Powers. For the needs of §3.2, we make explicit the structure of tensor powers $M^{\otimes r}$ in the category of Σ_*-objects.

For all $n \in \mathbb{N}$, we have obviously:

$$M^{\otimes r}(n) = \bigoplus_{n_1+\dots+n_r=n} \Sigma_n \otimes_{\Sigma_{n_1}\times\dots\times\Sigma_{n_r}} (M(n_1) \otimes \dots \otimes M(n_r)).$$

In this formula, we use the canonical group embedding $\Sigma_{n_1} \times \dots \times \Sigma_{n_r} \hookrightarrow \Sigma_n$ which identifies a permutation of Σ_{n_i} to a permutation of the subset $\{n_1+\dots+n_{i-1}+1,\dots,n_1+\dots+n_{i-1}+n_i\} \subset \{1,\dots,n\}$. Again the quotient in the tensor product makes agree the internal Σ_{n_i}-action on $M(n_i)$ with the action of Σ_{n_i} by right translations on Σ_n.

The tensor power $M^{\otimes r}$ is equipped with a Σ_r-action, deduced from the symmetric structure of the tensor product of Σ_*-objects. Let $w \in \Sigma_r$ be any permutation. For any partition $n = n_1 + \dots + n_r$, we form the block permutation $w(n_1, \dots, n_r) \in \Sigma_n$ such that:

$$w(n_1, \dots, n_r)(n_{w(1)} + \dots + n_{w(i-1)} + k) = n_1 + \dots + n_{w(i)-1} + k, \\ \text{for } k = 1, \dots, n_{w(i)},\ i = 1, \dots, r.$$

The tensor permutation $w^* : M^{\otimes r} \to M^{\otimes r}$ is induced componentwise by morphisms of the form

$$\Sigma_n \otimes M(n_1) \otimes \dots \otimes M(n_r) \xrightarrow{w(n_1,\dots,n_r)\otimes w^*} \Sigma_n \otimes M(n_{w(1)}) \otimes \dots \otimes M(n_{w(r)})$$

where we use the tensor permutation $w^* : M(n_1) \otimes \dots \otimes M(n_r) \to M(n_{w(1)}) \otimes \dots \otimes M(n_{w(r)})$ within the category $\mathcal{C}$ and a left translation by the block permutation $w(n_1, \dots, n_r)$ on the symmetric group Σ_n. This formula extends obviously the definition of §2.1.7 in the case $r = 2$. To prove the general formula, check the definition of associativity isomorphisms for the tensor product of Σ_*-objects and observe that composites of block permutations are still block permutations to determine composites of symmetry isomorphisms.

2.1.9 The Pointwise Representation of Tensors in Σ_*-Objects. In the point-set context, we use the notation $w \cdot x \otimes y \in M \otimes N$ to represent the element defined by $w \otimes x \otimes y \in \Sigma_n \otimes M(p) \otimes N(q)$ in the tensor product of Σ_*-objects

$$M \otimes N(n) = \bigoplus_{p+q=n} \Sigma_n \otimes_{\Sigma_p \times \Sigma_q} M(p) \otimes N(q),$$

and the notation $x \otimes y \in M \otimes N$ in the case where $w = \mathrm{id}$ is the identity permutation.

By definition, the action of a permutation w on $M \otimes N$ maps the tensor $x \otimes y$ to $w \cdot x \otimes y$. Accordingly, the tensor product $M \otimes N$ is spanned, as a Σ_*-object, by the tensors $x \otimes y \in M(p) \otimes N(q)$, where $(x, y) \in M(p) \times N(q)$.

In our sense (see §§0.4-0.5), the tensor product of Σ_*-objects inherits a pointwise representation from the base category. To justify our pointwise representation, we also use the next assertion which identifies morphisms $f : M \otimes N \to T$ with actual multilinear maps on the set of generating tensors.

The abstract definition of §2.1.7 implies that the symmetry isomorphism $\tau(M, N) : M \otimes N \xrightarrow{\simeq} N \otimes M$ maps the tensor $x \otimes y \in M \otimes N$ to a tensor of the form $\tau(p, q) \cdot y \otimes x \in N \otimes M$, where $\tau(p, q)$ is a block permutation. Thus the permutation rule of tensors in Σ_*-objects is determined by the mapping $x \otimes y \mapsto \tau(p, q) \cdot y \otimes x$.

2.1.10 Fact. *For any Σ_*-object T, a morphism $f : M \otimes N \to T$ is equivalent to a collection of morphisms*

$$f : M(p) \otimes N(q) \to T(p+q)$$

which commute with the action of the subgroup $\Sigma_p \times \Sigma_q \subset \Sigma_{p+q}$.

This assertion is an obvious consequence of the definition of the tensor product in §2.1.7.

2.1.11 Enriched Category Structures. In §1.1.12, we observe that any symmetric monoidal category over $\mathcal{C}$ that satisfies the convention of §0.1 is naturally enriched over $\mathcal{C}$. An explicit construction of external hom-objects for categories of functors $\mathcal{F} = \mathcal{F}(\mathcal{A}, \mathcal{E})$ and the category of Σ_*-objects $\mathcal{M}$ can be derived from the existence of hom-objects in $\mathcal{E}$ (respectively, $\mathcal{C}$)*.

The external hom of the functor category $\mathcal{F} = \mathcal{F}(\mathcal{A}, \mathcal{E})$ is given by the end

$$\mathit{Hom}_{\mathcal{F}}(F, G) = \int_{X \in \mathcal{A}} \mathit{Hom}_{\mathcal{E}}(F(X), G(X)).$$

The adjunction relation

$$\mathrm{Mor}_{\mathcal{F}}(C \otimes F, G) = \mathrm{Mor}_{\mathcal{C}}(C, \mathit{Hom}_{\mathcal{F}}(F, G)),$$

for $C \in \mathcal{C}$, $F, G \in \mathcal{F}$, is equivalent to the definition of an end.

The external hom of the category of Σ_*-objects is defined by a product of the form

$$\mathit{Hom}_{\mathcal{M}}(M, N) = \prod_{n=0}^{\infty} \mathit{Hom}_{\mathcal{C}}(M(n), N(n))^{\Sigma_n}.$$

The hom-object $\mathit{Hom}_{\mathcal{C}}(M(n), N(n))$ inherits a conjugate action of the symmetric group from the Σ_n-objects $M(n)$ and $N(n)$. The expression $\mathit{Hom}_{\mathcal{C}}(M(n), N(n))^{\Sigma_n}$ refers to the invariant object with respect to this action of Σ_n. The adjunction relation of hom-objects

$$\mathrm{Mor}_{\mathcal{M}}(C \otimes M, N) = \mathrm{Mor}_{\mathcal{C}}(C, \mathit{Hom}_{\mathcal{M}}(F, G))$$

for $C \in \mathcal{C}$, $M, N \in \mathcal{M}$, is immediate.

2.1.12 Generating Σ_*-Objects. The identity functor $\mathrm{Id} : \mathcal{E} \to \mathcal{E}$ is identified with the functor $\mathrm{S}(I) = \mathrm{Id}$ associated to a Σ_*-object I defined by:

$$I(n) = \begin{cases} \mathtt{1}, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

This object I represents the unit of the composition product of Σ_*-objects defined next.

* But serious set-theoretic difficulties occur for the category of functors $\mathcal{F} = \mathcal{F}(\mathcal{A}, \mathcal{E})$ if $\mathcal{A}$ does not satisfy the condition of §0.1, for instance when we take $\mathcal{A} = \mathcal{E} = \mathrm{Top}$, the category of topological spaces.

For $r \in \mathbb{N}$, let $F_r = I^{\otimes r}$ be the rth tensor power of I in $\mathcal{M}$. Since $\mathrm{S}(F_r) = \mathrm{S}(I)^{\otimes r} = \mathrm{Id}^{\otimes r}$, we obtain that $\mathrm{S}(F_r) : \mathcal{E} \to \mathcal{E}$ represents the rth tensor power functor $\mathrm{Id}^{\otimes r} : X \mapsto X^{\otimes r}$.

The definition of the tensor product of Σ_*-objects (see §2.1.7) implies that $F_r = I^{\otimes r}$ satisfies

$$F_r(n) = \begin{cases} \mathtt{1}[\Sigma_r], & \text{if } n = r, \\ 0, & \text{otherwise.} \end{cases}$$

Recall that $\mathtt{1}[\Sigma_r]$ denotes the Σ_r-object in $\mathcal{C}$ formed by the sum over Σ_r of copies of the tensor unit $\mathtt{1} \in \mathcal{C}$.

The symmetric group Σ_r acts on $F_r(r) = \mathtt{1}[\Sigma_r]$ equivariantly by translations on the right, and hence acts on F_r on the right by automorphisms of Σ_*-objects. This symmetric group action corresponds to the action by tensor permutations on tensor powers $I^{\otimes r}$.

The Σ_*-objects F_r, $r \in \mathbb{N}$, are characterized by the following property:

2.1.13 Proposition. *We have a natural Σ_r-equivariant isomorphism*

$$\omega_r(M) : M(r) \xrightarrow{\simeq} \mathit{Hom}_{\mathcal{M}}(F_r, M),$$

for all $M \in \mathcal{M}$.

Proof. Immediate: we have

$$\mathit{Hom}_{\mathcal{M}}(F_r, M) \simeq \mathit{Hom}_{\mathcal{C}}(\mathtt{1}[\Sigma_r], M(r))^{\Sigma_r}$$
$$\text{and} \quad \mathit{Hom}_{\mathcal{C}}(\mathtt{1}[\Sigma_r], M(r))^{\Sigma_r} \simeq \mathit{Hom}_{\mathcal{C}}(\mathtt{1}, M(r)) \simeq M(r).$$

One checks readily that the Σ_r-action by right translations on $\mathtt{1}[\Sigma_r]$ corresponds to the internal Σ_r-action of $M(r)$ under the latter isomorphisms. Hence we obtain a Σ_r-equivariant isomorphism

$$\omega_r(M) : M(r) \xrightarrow{\simeq} \mathit{Hom}_{\mathcal{M}}(F_r, M),$$

as stated. □

2.1.14 Canonical Generating Morphisms. Observe that

$$(M(r) \otimes F_r(n))_{\Sigma_r} \simeq \begin{cases} M(r), & \text{if } n = r, \\ 0, & \text{otherwise.} \end{cases}$$

Accordingly, for a Σ_*-object M, we have obvious morphisms

$$\iota_r(M) : (M(r) \otimes F_r)_{\Sigma_r} \to M$$

that sum up to an isomorphism

$$\iota(M) : \bigoplus_{r=0}^{\infty} (M(r) \otimes F_r)_{\Sigma_r} \xrightarrow{\simeq} M.$$

At the functor level, we have $\mathrm{S}((M(r) \otimes F_r)_{\Sigma_r}) \simeq (M(r) \otimes \mathrm{Id}^{\otimes r})_{\Sigma_r}$ and $\mathrm{S}(\iota_r(M))$ represents the canonical morphism

$$(M(r) \otimes \mathrm{Id}^{\otimes r})_{\Sigma_r} \to \bigoplus_{r=0}^{\infty} (M(r) \otimes \mathrm{Id}^{\otimes r})_{\Sigma_r} = \mathrm{S}(M).$$

The morphism $Hom(F_r, M) \otimes F_r \to M$ induces a natural morphism $(Hom(F_r, M) \otimes F_r)_{\Sigma_r} \to M$. We check readily that the isomorphism of proposition 2.1.13 fits a commutative diagram

$$\begin{tikzcd}
(M(r) \otimes F_r)_{\Sigma_r} \arrow[r, dotted, "\simeq"] \arrow[dr, "\iota_r(M)"'] & (Hom(F_r, M) \otimes F_r)_{\Sigma_r} \arrow[d, "\epsilon"] \\
 & M.
\end{tikzcd}$$

Equivalently, the isomorphism $\omega_r(M)$ corresponds to the morphism $\iota_r(M)$ under the adjunction relation

$$\mathrm{Mor}_{\mathcal{M}}((M(r) \otimes F_r)_{\Sigma_r}, M) \simeq \mathrm{Mor}_{\mathcal{C}}(M(r), Hom_{\mathcal{M}}(F_r, M))^{\Sigma_r}.$$

To conclude, proposition 2.1.13 and the discussion of §2.1.14 imply:

2.1.15 Proposition. *The objects F_r, $r \in \mathbb{N}$, define small projective generators of $\mathcal{M}$ in the sense of enriched categories. Explicitly, the functors*

$$Hom_{\mathcal{M}}(F_r, -) : M \mapsto Hom_{\mathcal{M}}(F_r, M)$$

preserve filtered colimits and coequalizers and the canonical morphism

$$\bigoplus_{r=0}^{\infty} Hom_{\mathcal{M}}(F_r, M) \otimes F_r \to M$$

is a regular epi, for all $M \in \mathcal{M}$. $\square$

Note that the functors $\mathrm{S}(F_r) = \mathrm{Id}^{\otimes r}$ do not generate $\mathcal{F}$ and do not form projective objects in $\mathcal{F}$ in general.

2.1.16 Remark. Since $F_r = I^{\otimes r}$, the isomorphism of §2.1.14 can be identified with an isomorphism

$$S(M, I) = \bigoplus_{r=0}^{\infty} (M(r) \otimes I^{\otimes r})_{\Sigma_r} \simeq M$$

between M and the Σ_*-object $S(M, I) \in \mathcal{M}$ associated to $I \in \mathcal{M}$ by the functor $S(M) : \mathcal{E} \to \mathcal{E}$ for $\mathcal{E} = \mathcal{M}$. This observation can be used to recover a Σ_*-object M from the associated collection of functors $S(M) : \mathcal{E} \to \mathcal{E}$, where $\mathcal{E}$ runs over all monoidal symmetric categories over $\mathcal{C}$.

2.2 Composition of Σ_*-Objects and Functors

The category of functors $\mathcal{F} = \mathcal{F}(\mathcal{E}, \mathcal{E})$ is equipped with another (non-symmetric) monoidal structure $(\mathcal{F}, \circ, \mathrm{Id})$ defined by the composition of functors $F, G \mapsto F \circ G$, together with the identity functor Id as a unit object. The category of Σ_*-objects has a (non-symmetric) monoidal structure that reflects the composition structure of functors. Formally, we have:

2.2.1 Proposition (see [17, 56])**.** *The category of Σ_*-objects $\mathcal{M}$ is equipped with a monoidal structure $(\mathcal{M}, \circ, I)$ so that the map $\mathrm{S} : M \mapsto \mathrm{S}(M)$ defines a functor of monoidal categories*

$$\mathrm{S} : (\mathcal{M}, \circ, I) \to (\mathcal{F}(\mathcal{E}, \mathcal{E}), \circ, \mathrm{Id}),$$

for all symmetric monoidal categories $\mathcal{E}$ over $\mathcal{C}$. □

The composition product of Σ_*-objects refers to the operation $M, N \mapsto M \circ N$ that yields this monoidal structure. For our purposes, we recall the construction of [14, §1.3] which uses the symmetric monoidal structure of the category of Σ_*-objects in the definition of the composition product $M, N \mapsto M \circ N$.

2.2.2 The Monoidal Composition Structure of the Category of Σ_*-Objects. In fact, the composite $M \circ N$ is defined by a generalized symmetric tensor construction formed in the category $\mathcal{E} = \mathcal{M}$:

$$M \circ N = \mathrm{S}(M, N) = \bigoplus_{r=0}^{\infty} (M(r) \otimes N^{\otimes r})_{\Sigma_r}.$$

Since the functor $\mathrm{S} : M \mapsto \mathrm{S}(M)$ preserves colimits and tensor products, we have identities

$$\mathrm{S}(M \circ N) = \bigoplus_{r=0}^{\infty} \mathrm{S}(M(r) \otimes N^{\otimes r})_{\Sigma_r} = \bigoplus_{r=0}^{\infty} (M(r) \otimes \mathrm{S}(N)^{\otimes r})_{\Sigma_r}.$$

Hence, we obtain immediately that this composition product $M \circ N$ satisfies the relation $\mathrm{S}(M \circ N) \simeq \mathrm{S}(M) \circ \mathrm{S}(N)$, asserted by proposition 2.2.1.

The unit of the composition product is the object I, defined in §2.1.12, which corresponds to the identity functor $\mathrm{S}(I) = \mathrm{Id}$. The isomorphism of §2.1.14, identified with

$$\mathrm{S}(M, I) = \bigoplus_{r=0}^{\infty} (M(r) \otimes I^{\otimes r})_{\Sigma_r} \simeq M$$

(see §2.1.16), is equivalent to the right unit relation $M \circ I \simeq M$.

2.2.3 The Distribution Relation Between Tensor and Composition Products. In the category of functors, the tensor product and the composition product satisfy the distribution relation $(F \otimes G) \circ S = (F \circ S) \otimes (G \circ S)$. In the category of Σ_*-modules, we have a natural distribution isomorphism

$$\theta(M, N, P) : (M \otimes N) \circ P \xrightarrow{\simeq} (M \circ P) \otimes (N \circ P)$$

which arises from the relation $\mathrm{S}(M \otimes N, P) \simeq \mathrm{S}(M, P) \otimes \mathrm{S}(N, P)$ yielded by proposition 2.1.5. This distribution isomorphism reflects the distribution relation at the functor level. Formally, we have a commutative hexagon

$$\begin{tikzcd}
 & \mathrm{S}(M \otimes N) \circ \mathrm{S}(P) \arrow[dr, "\simeq"] & \\
\mathrm{S}((M \otimes N) \circ P) \arrow[ur, "\simeq"] \arrow[d, "\simeq"'] & & (\mathrm{S}(M) \otimes \mathrm{S}(N)) \circ \mathrm{S}(P) \arrow[d, "="] \\
\mathrm{S}((M \circ P) \otimes (N \circ P)) \arrow[dr, "\simeq"] & & (\mathrm{S}(M) \circ \mathrm{S}(P)) \otimes (\mathrm{S}(N) \circ \mathrm{S}(P)) \\
 & \mathrm{S}(M \circ P) \otimes \mathrm{S}(N \circ P) \arrow[ur, "\simeq"] &
\end{tikzcd}$$

that connects the distribution isomorphism $\theta(M, N, P)$ to the functor identity $(\mathrm{S}(M) \otimes \mathrm{S}(N)) \circ \mathrm{S}(P) = (\mathrm{S}(M) \circ \mathrm{S}(P)) \otimes (\mathrm{S}(N) \circ \mathrm{S}(P))$.

To summarize, we obtain:

2.2.4 Observation. *Let $\mathcal{F} = \mathcal{F}(\mathcal{E}, \mathcal{E})$. For any functor $S \in \mathcal{F}$, the composition product $F \mapsto F \circ S$ defines a functor of symmetric monoidal categories over $\mathcal{C}$*

$$- \circ S : (\mathcal{F}, \otimes, 1) \to (\mathcal{F}, \otimes, 1).$$

For any $N \in \mathcal{M}$, the composition product $M \mapsto M \circ N$ defines a functor of symmetric monoidal categories over $\mathcal{C}$

$$- \circ N : (\mathcal{M}, \otimes, 1) \to (\mathcal{M}, \otimes, 1)$$

and the diagram of functors

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\mathrm{S}"] \arrow[d, "- \circ N"'] & \mathcal{F} \arrow[d, "- \circ \mathrm{S}(N)"] \\
\mathcal{M} \arrow[r, "\mathrm{S}"'] & \mathcal{F}
\end{tikzcd}$$

commutes up to a natural equivalence of symmetric monoidal categories over $\mathcal{C}$.

Besides, we check readily:

2.2.5 Observation. *The distribution isomorphisms $\theta(M,N,P)$ satisfy*

$$\theta(M,N,I) = \mathrm{id}$$

for the unit object $C = I$ and make commute the triangles

$$\begin{tikzcd}
(M \otimes N) \circ P \circ Q \arrow[rr, "\theta(M,N,P\circ Q)"] \arrow[dr, "\theta(M,N,P)\circ Q"'] & & (M \circ P \circ Q) \otimes (N \circ P \circ Q) \\
& ((M \circ P) \otimes (N \circ P)) \circ Q \arrow[ur, "\theta(M\circ P, N\circ P, Q)"'] &
\end{tikzcd},$$

for all $M,N,P,Q \in \mathcal{M}$.

These coherence relations are obvious at the functor level since all isomorphisms are identities in this case.

2.3 Adjunction and Embedding Properties

In the context of a module category $\mathcal{E} = \mathcal{C} = \Bbbk\,\mathrm{Mod}$, where $\Bbbk$ is an infinite field, we recall in [14, §1.2] that the functor $\mathrm{S} : M \mapsto \mathrm{S}(M)$ is full and faithful. To prove this assertion, one can observe that the functor $\mathrm{S} : M \mapsto \mathrm{S}(M)$ has a right adjoint $\Gamma : G \mapsto \Gamma(G)$ so that the adjunction unit $\eta(M) : M \to \Gamma(\mathrm{S}(M))$ forms an isomorphism (see proposition 1.2.5 in *loc. cit.*). In the general case of a module category $\mathcal{E} = \mathcal{C} = \Bbbk\,\mathrm{Mod}$, where $\Bbbk$ is any ground ring, we obtain further that $\eta(M) : M \to \Gamma(\mathrm{S}(M))$ forms an isomorphism if M is a projective Σ_*-module (see proposition 2.3.12).

The aim of this section is to review these properties in the context of a symmetric monoidal category $\mathcal{E}$ over $\mathcal{C}$. For short, we set $\mathcal{F} = \mathcal{F}(\mathcal{E},\mathcal{E})$.

Since we observe that the functor $\mathrm{S} : \mathcal{M} \to \mathcal{F}$ preserves colimits, we obtain that this functor has a right adjoint $\Gamma : \mathcal{F} \to \mathcal{M}$. In a first part, we give an explicit construction of this adjoint functor $\Gamma : G \mapsto \Gamma(G)$. For this purpose, we assume that $\mathcal{C}$ has an internal hom, $\mathcal{E}$ is enriched over $\mathcal{C}$, and we generalize a construction of [14, §1.2]. In a second part, we observe that $\mathrm{S} : M \mapsto \mathrm{S}(M)$ extends to a functor of enriched categories and we prove that this functor $\mathrm{S} : \mathcal{M} \mapsto \mathcal{F}$ is faithful in an enriched sense, at least if the category $\mathcal{E}$ is equipped with a faithful functor $\eta : \mathcal{C} \to \mathcal{E}$. Equivalently, we obtain that the adjunction unit $\eta(M) : M \to \Gamma(\mathrm{S}(M))$ defines a monomorphism.

This account is motivated by the subsequent generalization of §8 in the context of right modules over operad. The results and constructions of this section are not used anywhere else in the book.

2.3.1 The Endomorphism Module of a Pair. Observe first that the functor $M \mapsto \mathrm{S}(M, Y)$, for a fixed object $Y \in \mathcal{E}$, has a right adjoint. For this aim, form, for $X, Y \in \mathcal{E}$, the Σ_*-object $\mathit{End}_{X,Y}$ such that

$$\mathit{End}_{X,Y}(r) = \mathit{Hom}_{\mathcal{E}}(X^{\otimes r}, Y).$$

In §8.1.1, we observe that this Σ_*-object defines naturally a right module over End_X, the endomorphism operad of X, and we call this structure the *endomorphism module of the pair* (X, Y).

For the moment, observe simply:

2.3.2 Proposition (*cf.* [54, Proposition 2.2.7])**.** *We have a natural isomorphism*

$$\mathrm{Mor}_{\mathcal{E}}(\mathrm{S}(M, X), Y) \simeq \mathrm{Mor}_{\mathcal{M}}(M, \mathit{End}_{X,Y})$$

for all $M \in \mathcal{M}$ and $X, Y \in \mathcal{E}$.

Proof. This adjunction relation arises from the canonical isomorphisms:

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{E}}(\bigoplus_{r=0}^{\infty}(M(r) \otimes X^{\otimes r})_{\Sigma_r}, Y) &\simeq \prod_{r=0}^{\infty} \mathrm{Mor}_{\mathcal{E}}((M(r) \otimes X^{\otimes r})_{\Sigma_r}, Y) \\
&\simeq \prod_{r=0}^{\infty} \mathrm{Mor}_{\mathcal{C}}(M(r), \mathrm{Mor}_{\mathcal{E}}(X^{\otimes r}, Y))^{\Sigma_r} \\
&= \mathrm{Mor}_{\mathcal{M}}(M, \mathit{End}_{X,Y}).
\end{aligned}$$

□

2.3.3 Observation. Next (see observation 3.2.15) we observe that the map $\mathrm{S}(N) : X \mapsto \mathrm{S}(N, X)$ defines a functor $\mathrm{S}(N) : \mathcal{E} \to {}_{\mathtt{P}}\mathcal{E}$ to the category ${}_{\mathtt{P}}\mathcal{E}$ of algebras over an operad $\mathtt{P}$ when N is equipped with the structure of a left $\mathtt{P}$-module. One can observe that the endomorphism module $\mathit{End}_{X,Y}$ forms a left module over the endomorphism operad of Y. As a corollary, if $Y = B$ is a $\mathtt{P}$-algebra, then we obtain that $\mathit{End}_{X,B}$ forms a left module over $\mathtt{P}$ by restriction of structures. In the context of $\mathtt{P}$-algebras, we have an adjunction relation

$$\mathrm{Mor}_{{}_{\mathtt{P}}\mathcal{E}}(\mathrm{S}(N, X), B) \simeq \mathrm{Mor}_{{}_{\mathtt{P}}\mathcal{M}}(N, \mathit{End}_{X,B})$$

for all $N \in {}_{\mathtt{P}}\mathcal{M}$, $X \in \mathcal{E}$ and $B \in {}_{\mathtt{P}}\mathcal{E}$, where ${}_{\mathtt{P}}\mathcal{M}$ refers to the category of left $\mathtt{P}$-modules (see §§3.2.9-3.2.10).

2.3.4 Definition of the Adjoint Functor $\Gamma : \mathcal{F} \to \mathcal{M}$. We apply the pointwise adjunction relation of proposition 2.3.2 to the category of functors $\mathcal{F}$.

In §2.1.1, we notice that the functor $\mathrm{S}(M)$ satisfies

$$\mathrm{S}(M) = \bigoplus_{r=0}^{\infty}(M(r) \otimes \mathrm{Id}^{\otimes r})_{\Sigma_r} = \mathrm{S}(M, \mathrm{Id}),$$

where Id is the identity functor on $\mathcal{E}$. According to this relation, if we set $\Gamma(G) = \mathit{End}_{\mathrm{Id},G}$ for $G \in \mathcal{F}$, then proposition 2.3.2 returns:

2.3.5 Proposition. *The functor* $\Gamma : \mathcal{F} \to \mathcal{M}$ *defined by the map* $G \mapsto \mathit{End}_{\mathrm{Id},G}$ *is right adjoint to* $\mathrm{S} : \mathcal{M} \to \mathcal{F}$. $\square$

By proposition 1.1.16, we have as well:

2.3.6 Proposition. *The functors* $\mathrm{S} : \mathcal{M} \rightleftarrows \mathcal{F} : \Gamma$ *satisfy an enriched adjunction relation*

$$\mathit{Hom}_{\mathcal{M}}(\mathrm{S}(M), G) \simeq \mathit{Hom}_{\mathcal{F}}(M, \Gamma(G)),$$

where morphism sets are replaced by hom-objects over $\mathcal{C}$. $\square$

Proposition 1.1.15 implies that any functor of symmetric monoidal categories over $\mathcal{C}$, like $\mathrm{S} : \mathcal{M} \to \mathcal{F}$, defines a functor in the enriched sense. Accordingly, the map $f \mapsto \mathrm{S}(f)$, defined for morphisms of Σ_*-objects, extends to a morphism on hom-objects:

$$\mathit{Hom}_{\mathcal{M}}(M, N) \xrightarrow{\mathrm{S}} \mathit{Hom}_{\mathcal{F}}(\mathrm{S}(M), \mathrm{S}(N)).$$

By proposition 1.1.16, we obtain further:

2.3.7 Proposition. *The diagram*

$$\begin{tikzcd}
\mathit{Hom}_{\mathcal{M}}(M,N) \arrow[rr, "\mathrm{S}"] \arrow[dr, "\eta(N)_*"'] & & \mathit{Hom}_{\mathcal{F}}(\mathrm{S}(M),\mathrm{S}(N)) \\
& \mathit{Hom}_{\mathcal{M}}(M,\Gamma(\mathrm{S}(N))) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

commutes. $\square$

According to this assertion, we can use the adjunction unit $\eta(N) : N \to \Gamma(\mathrm{S}(N))$ and the adjunction relation between $\mathrm{S} : \mathcal{M} \to \mathcal{F}$ and $\Gamma : \mathcal{F} \to \mathcal{M}$ to determine $\mathrm{S} : \mathit{Hom}_{\mathcal{M}}(M, N) \to \mathit{Hom}_{\mathcal{F}}(\mathrm{S}(M), \mathrm{S}(N))$. In the converse direction, we can apply the morphism $\mathrm{S} : \mathit{Hom}_{\mathcal{M}}(M, N) \to \mathit{Hom}_{\mathcal{F}}(\mathrm{S}(M), \mathrm{S}(N))$ to the generating Σ_*-objects $M = F_r = I^{\otimes r}$, $r \in \mathbb{N}$, in order to determine the adjunction unit:

2.3.8 Proposition. *The component* $\eta(N) : N(r) \to \mathit{Hom}_{\mathcal{F}}(\mathrm{Id}^{\otimes r}, \mathrm{S}(N))$ *of the adjunction unit* $\eta(N) : N \to \Gamma(\mathrm{S}(N))$ *coincides with the morphism*

$$N(r) \xrightarrow{\simeq} \mathit{Hom}_{\mathcal{M}}(F_r, N) \xrightarrow{\mathrm{S}} \mathit{Hom}_{\mathcal{F}}(\mathrm{S}(F_r), \mathrm{S}(N)) \xrightarrow{\simeq} \mathit{Hom}_{\mathcal{F}}(\mathrm{Id}^{\otimes r}, \mathrm{S}(N))$$

formed by the composite of the isomorphism $\omega_r(N) : N(r) \xrightarrow{\simeq} \mathit{Hom}_{\mathcal{M}}(F_r, N)$ *of proposition 2.1.13, the morphism induced by the functor* $\mathrm{S} : \mathcal{M} \to \mathcal{F}$ *on hom-objects, and the isomorphism induced by the relation* $\mathrm{S}(F_r) \simeq \mathrm{Id}^{\otimes r}$.

Proof. This proposition is a consequence of proposition 2.3.7. In the case $M = F_r$, we obtain a commutative diagram:

$$\begin{tikzcd}
N(r) \arrow[r, "\simeq"] \arrow[d, "\eta(N)"'] & Hom_{\mathcal{M}}(F_r, N) \arrow[d, "\eta(N)_*"'] \arrow[dr, "\mathrm{S}"] & \\
\Gamma(\mathrm{S}(N))(r) \arrow[r, "\simeq"'] & Hom_{\mathcal{M}}(F_r, \Gamma(\mathrm{S}(N))) \arrow[r, "\simeq"'] & Hom_{\mathcal{M}}(\mathrm{S}(F_r), \mathrm{S}(N)).
\end{tikzcd}$$

One proves by a straightforward verification that the composite

$$\begin{tikzcd}
\Gamma(\mathrm{S}(N))(r) \arrow[r, "\simeq"] & Hom_{\mathcal{M}}(F_r, \Gamma(\mathrm{S}(N))) \arrow[r, "\simeq"] & Hom_{\mathcal{M}}(\mathrm{S}(F_r), \mathrm{S}(N)) \arrow[d, "\simeq"] \\
& & Hom_{\mathcal{M}}(\mathrm{Id}^{\otimes r}, N)
\end{tikzcd}$$

is the identity morphism of $\Gamma(\mathrm{S}(N))(r) = Hom_{\mathcal{M}}(\mathrm{Id}^{\otimes r}, \mathrm{S}(N))$ and the proposition follows. □

In the remainder of this section, we check that the morphism $\mathrm{S} : Hom_{\mathcal{M}}(M, N) \to Hom_{\mathcal{F}}(\mathrm{S}(M), \mathrm{S}(N))$ is mono under the assumption that the symmetric monoidal category $\mathcal{E}$ is equipped with a faithful functor $\eta : \mathcal{C} \to \mathcal{E}$. The proof of this observation is based on the next lemma:

2.3.9 Lemma. *Let $\mathbb{1}^{\oplus r} = \mathbb{1}_1 \oplus \cdots \oplus \mathbb{1}_r$ be the sum of r copies of the unit object $\mathbb{1} \in \mathcal{C}$. For $M \in \mathcal{M}$, we have a canonical isomorphism*

$$\mathrm{S}(M, \mathbb{1}^{\oplus r}) \simeq \bigoplus_{n_1 + \cdots + n_r = n} M(n_1 + \cdots + n_r)_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_r}}.$$

Proof. We have Σ_n-equivariant isomorphisms

$$\begin{aligned}
(\mathbb{1}_1 \oplus \cdots \oplus \mathbb{1}_r)^{\otimes n} &\simeq \bigoplus_{(i_1, \ldots, i_n)} \mathbb{1}_{i_1} \otimes \cdots \otimes \mathbb{1}_{i_n} \\
&\simeq \bigoplus_{(i_1, \ldots, i_n)} \mathbb{1}
\end{aligned}$$

where the symmetric group Σ_n acts on n-tuples $(i_1, \ldots, i_n)$ by permutations of terms. We have an identification

$$\bigoplus_{(i_1, \ldots, i_n)} \mathbb{1} = \bigoplus_{n_1 + \cdots + n_r = n} \mathbb{1}[\Sigma_{n_1} \times \cdots \times \Sigma_{n_r} \backslash \Sigma_n],$$

from which we deduce the splitting

$$\begin{aligned}(M(n) \otimes (\mathtt{1}_1 \oplus \cdots \oplus \mathtt{1}_r)^{\otimes n})_{\Sigma_n} &\simeq \bigoplus_{n_1+\cdots+n_r=n} (M(n) \otimes \mathtt{1}[\Sigma_{n_1} \times \cdots \times \Sigma_{n_r} \backslash \Sigma_n])_{\Sigma_n} \\ &\simeq \bigoplus_{n_1+\cdots+n_r=n} M(n_1 + \cdots + n_r)_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_r}}\end{aligned}$$

and the lemma follows. □

We deduce from lemma 2.3.9:

2.3.10 Proposition. *The functor* $\mathrm{S} : \mathcal{M} \to \mathcal{F}(\mathcal{E}, \mathcal{E})$ *is faithful for all symmetric monoidal categories over* $\mathcal{C}$ *equipped with a faithful functor* $\eta : \mathcal{C} \to \mathcal{E}$*.*

Moreover, the functor $\mathrm{S} : \mathcal{M} \to \mathcal{F}(\mathcal{E}, \mathcal{E})$ *is faithful in an enriched sense. Explicitly, the morphism induced by* S *on hom-objects*

$$\mathit{Hom}_{\mathcal{M}}(M, N) \xrightarrow{\mathrm{S}} \mathit{Hom}_{\mathcal{F}}(\mathrm{S}(M), \mathrm{S}(N))$$

is mono in $\mathcal{C}$*, for all* $M, N \in \mathcal{M}$*.*

Proof. The object $M(r)$ is isomorphic to the component $n_1 = \cdots = n_r = 1$ in the decomposition of lemma 2.3.9. As a byproduct, lemma 2.3.9 implies the existence of a natural monomorphism $\sigma(M) : M(r) \to \mathrm{S}(M, \mathtt{1}^{\oplus r})$, for all $M \in \mathcal{M}$. From this assertion we deduce that S induces an injection on hom-sets

$$\mathrm{Mor}_{\mathcal{M}}(M, N) \xrightarrow{\mathrm{S}} \mathrm{Mor}_{\mathcal{C}}(\mathrm{S}(M, \mathtt{1}^{\oplus r}), \mathrm{S}(N, \mathtt{1}^{\oplus r})),$$

for all $M, N \in \mathcal{M}$. If $\mathcal{E}$ is a symmetric monoidal category equipped with a faithful functor $\eta : \mathcal{C} \to \mathcal{E}$, then the map

$$\mathrm{Mor}_{\mathcal{C}}(\mathrm{S}(M, \mathtt{1}^{\oplus r}), \mathrm{S}(N, \mathtt{1}^{\oplus r})) \to \mathrm{Mor}_{\mathcal{E}}(\mathrm{S}(M, \mathtt{1}^{\oplus r}), \mathrm{S}(N, \mathtt{1}^{\oplus r}))$$

is injective as well. Hence we conclude readily that S induces an injection on hom-sets

$$\mathrm{Mor}_{\mathcal{M}}(M, N) \xrightarrow{\mathrm{S}} \int_{X \in \mathcal{E}} \mathrm{Mor}_{\mathcal{C}}(\mathrm{S}(M, X), \mathrm{S}(N, X)) = \mathrm{Mor}_{\mathcal{F}}(\mathrm{S}(M), \mathrm{S}(N)),$$

for all $M, N \in \mathcal{M}$, and defines a faithful functor $\mathrm{S} : \mathcal{M} \to \mathcal{F}$.

In the context of enriched categories, we obtain that the map on hom-sets

$$\mathrm{Mor}_{\mathcal{M}}(C \otimes M, N) \xrightarrow{\mathrm{S}} \mathrm{Mor}_{\mathcal{F}}(\mathrm{S}(C \otimes M), \mathrm{S}(N)) \simeq \mathrm{Mor}_{\mathcal{F}}(C \otimes \mathrm{S}(M), \mathrm{S}(N))$$

is injective for all $C \in \mathcal{C}$, $M, N \in \mathcal{M}$. By adjunction, we conclude immediately that

$$\mathit{Hom}_{\mathcal{M}}(M, N) \xrightarrow{\mathrm{S}} \mathit{Hom}_{\mathcal{F}}(\mathrm{S}(M), \mathrm{S}(N))$$

is mono. □

By proposition 2.3.7 and proposition 2.3.8, we have equivalently:

2.3.11 Proposition. *The adjunction unit*

$$\eta(N) : N \to \Gamma(\mathrm{S}(N))$$

is mono in $\mathcal{M}$, for all $N \in \mathcal{M}$. □

We record stronger results in the case $\mathcal{E} = \mathcal{C} = \Bbbk\,\mathrm{Mod}$:

2.3.12 Proposition. *In the case $\mathcal{E} = \mathcal{C} = \Bbbk\,\mathrm{Mod}$, the category of modules over a ring $\Bbbk$, the adjunction unit $\eta(M) : M \to \Gamma(\mathrm{S}(M))$ is an isomorphism as long as M is a projective Σ_*-module or the ground ring is an infinite field.*

Proof. The case of an infinite ground field, recalled in the introduction of this section, is stated explicitly in [14, Proposition 1.2.5]. In the other case, one can check directly that the adjunction unit $\eta(M) : M \to \Gamma(\mathrm{S}(M))$ forms an isomorphism for the generating projective Σ_*-modules $M = F_r$, $r \in \mathbb{N}$. This implies that $\eta(M) : M \to \Gamma(\mathrm{S}(M))$ forms an isomorphism if M is a projective Σ_*-module. □

2.4 Colimits

In §2.1.1, we observe that the functor $\mathrm{S} : M \mapsto \mathrm{S}(M)$ preserves colimits. Since colimits in functor categories are obtained pointwise, we obtain equivalently that the bifunctor $(M, X) \mapsto \mathrm{S}(M, X)$ preserves colimits in M, for any fixed object $X \in \mathcal{E}$.

In contrast, one can observe that the functor $\mathrm{S}(M) : \mathcal{E} \to \mathcal{E}$ associated to a fixed Σ_*-object does not preserve all colimits. Equivalently, the bifunctor $(M, X) \mapsto \mathrm{S}(M, X)$ does not preserve colimits in X in general.

Nevertheless:

2.4.1 Proposition (*cf.* [54, Lemma 2.3.3])**.** *The functor $\mathrm{S}(M) : \mathcal{E} \to \mathcal{E}$ associated to a Σ_*-object $M \in \mathcal{M}$ preserves filtered colimits and reflexive coequalizers.*

Proof. In proposition 1.2.1 we observe that the tensor power functors $\mathrm{Id}^{\otimes r} : X \mapsto X^{\otimes r}$ preserves filtered colimits and reflexive coequalizers. By assumption, the external tensor products $Y \mapsto M(r) \otimes Y$ preserves these colimits. By interchange of colimits, we deduce readily from these assertions that the functor $\mathrm{S}(M, X) = \bigoplus_{r=0}^{\infty} (M(r) \otimes X^{\otimes r})_{\Sigma_r}$ preserves filtered colimits and reflexive coequalizers as well. □

As regards reflexive coequalizers, a first occurrence of proposition 2.4.1 appears in [51, §B.3] in the particular case of the symmetric algebra $V \mapsto \mathrm{S}(V)$ on dg-modules.

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