Preface

This volume grew out of a series of preprints which were written and circulated between 1993 and 1994. Around the same time, related work was done independently by Harder [40] and Laumon [62]. In writing this text based on a revised version of these preprints that were widely distributed in summer 1995, I finally did not pursue the original plan to completely reorganize the original preprints. After the long delay, one of the reasons was that an overview of the results is now available in [115]. Instead I tried to improve the presentation modestly, in particular by adding cross-references wherever I felt this was necessary. In addition, Chaps. 11 and 12 and Sects. 5.1, 5.4, and 5.5 were added; these were written in 1998.

I will give a more detailed overview of the content of the different chapters below. Before that I should mention that the two main results are the proof of Ramanujan's conjecture for Siegel modular forms of genus 2 for forms which are not cuspidal representations associated with parabolic subgroups (CAP representations), and the study of the endoscopic lift for the group $GSp(4)$. Both topics are formulated and proved in the first five chapters assuming the stabilization of the trace formula. All the remaining technical results, which are necessary to obtain the stabilized trace formula, are presented in the remaining chapters.

Chapter 1 gathers results on the cohomology of Siegel modular threefolds that are used in later chapters, notably in Chap. 3. At the beginning of Chap. 1, important facts from [19] on the Hodge structure and l-adic cohomology of the Siegel modular varieties $S_K(\mathbb{C})$ are reviewed. In the case of genus 2, the Siegel modular varieties $S_K(\mathbb{C})$ define algebraic varieties of dimension 3. They are the Shimura varieties attached to the group of symplectic similitudes $G = GSp(4, \mathbb{Q})$. One can define coefficient systems E_λ on these threefolds associated with irreducible finite-dimensional algebraic representations λ of the group $GSp(4)$, which are defined over $\mathbb{Q}$. The most interesting cohomology groups of these coefficient systems are the cohomology groups $H^i(S_K(\mathbb{C}), E_\lambda)$ in the middle degree $i = 3$. The group $G(\mathbb{A}_{fin})$ acts on the direct limit of the cohomology groups $H^i(S_K(\mathbb{C}), E_\lambda)$, where the limit is over the adelic compact open level groups $K \subseteq G(\mathbb{A}_{fin})$, and this defines an admissible automorphic representation of the group $GSp(4, \mathbb{A})$ for the adele ring $\mathbb{A} = \mathbb{R} \times \mathbb{A}_{fin}$. Since the Siegel moduli spaces are not proper, the cohomology of these varieties is not pure. Besides the interior cohomology, which is the image of the cohomology with compact supports in the ordinary cohomology, there occur representations of Eisenstein type. The automorphic representations of $G(\mathbb{A}_{fin})$ defined by the Eisenstein constituents are by definition isomorphic to constituents of induced representations $Ind_{P(\mathbb{A}_{fin})}^{G(\mathbb{A}_{fin})}(\sigma)$, where σ is an automorphic representation of the Levi group of a $\mathbb{Q}$-rational parabolic subgroup of G. It is well known that the Eisenstein cohomology is rarely pure. But also some part of the cuspidal cohomology, which is a subspace of the interior cohomology, fails to be pure. In fact, some of the irreducible cuspidal automorphic representations behave similarly to Eisenstein representations. These are called CAP representations π. By definition, an irreducible automorphic representation $\pi = \otimes_v \pi_v$ is a CAP representation if there exists some global automorphic representation σ of a Levi group of some proper $\mathbb{Q}$-rational parabolic subgroup of G such that π_v and $Ind_{P_v}^{G_v}(\sigma_v)$ have the same spherical constituent for almost all non-Archimedean places v. For the group $GSp(4)$ the CAP representations were completely classified by Piatetski-Shapiro [69] and Soudry [95], and the Ramanujan conjecture (purity) does not hold for them. The main result obtained in Chap. 1 (Sect. 1.3), Theorem 1.1, states that except for the cohomology degree $i = 3$ all the irreducible automorphic representations, which occur as irreducible constituents of the representations $\lim_K H^i(S_K(\mathbb{C}), E_\lambda)$, are either CAP representations or belong to the Eisenstein cohomology. This, in principle, allows a complete description of the cuspidal part of the cohomology representations in degree $i \neq 3$ by the classification of CAP representations. Even for the degree $i = 3$ the CAP representations occur, indeed those defined by the Saito–Kurokawa lift. Sections 1.3–1.7 contain the proof of Theorem 1.1, which is based on the study of the Lefschetz map and a weak form of the Ramanujan conjecture. This eventually shows that for irreducible automorphic cuspidal representations π, which occur as constituents in degree $i \neq 3$, a certain L-function must have poles at specific points, which forces π to be a CAP representation.

In Chap. 2, we consider the topological trace formula of Goresky and MacPherson for general reductive groups G. We do not consider this for the spaces $S_K(\mathbb{C})$ themselves, but for spaces $S_K(G)$ with a slightly better behavior with respect to "parabolic induction." This suffices for our purposes, since $S_K(\mathbb{C})$ is a finite unramified covering of $S_K(G)$:

$$S_K(\mathbb{C}) \twoheadrightarrow S_K(G).$$

For a Hecke operator f the topological trace formula computes the alternating sum $\sum_i (-1)^i tr(f; H^i(S_K(G), E_\lambda)$ of its traces. Its computation is considerably simplified if one discards all contributions from CAP representations and Eisenstein representations, which we abbreviate by the notion "CAP localization." The corresponding simplified formula obtained by CAP localization still allows us to compute the alternating sum of the dimensions of generalized eigenspaces $H^i(S_K(G), E_\lambda)(\pi)$ for an irreducible cuspidal automorphic representation π, which is not CAP (see page 47). To do this for a single fixed π, we construct suitable Hecke operators f

in Sects. 2.7 and 2.8, called good projectors, whose elliptic trace $T^G_{ell}(f)$ computes the alternating sum of these virtual dimensions (Theorem 2.1). For the construction of good projectors, it is essential that all the irreducible representations of $GSp(4, \mathbb{A}_{fin})$, which arise as constituents of the cohomology, are automorphic. This was shown by Schwermer for the group $GSp(4)$ and by Franke in the general case. In Sect. 2.9, we compare the formula thus obtained with Arthur's L^2-trace formula, which has the property that these elliptic traces coincide with the elliptic part of the geometric side of the Selberg trace formula. Now assume π is a cuspidal irreducible representation of $G(\mathbb{A}_{fin})$, which are not CAP. For a prime p, let us denote $\pi^p = \otimes_{v\neq\infty,p}\pi_v$. Then the results obtained in Chap. 2 in the special case $G = GSp(4)$ combined with the results obtained in Chap. 1 give a simple trace formula for the action of Hecke operators on generalized eigenspaces $\lim_K H^3(X, E_\lambda)(\pi^p)$ of the middle cohomology for either $X = S_K(G)$ or $X = S_K(\mathbb{C})$ for large enough primes p. Furthermore, this simple formula also allows us to compute the action of the nth powers of the geometric Frobenius substitution $Frob_p$ at the prime p in terms of certain Hecke operators $h_p = h_p^{(n)}$. See Theorem 2.2 and its applications in Chap. 4.

In Chap. 3, the simple localized topological trace formula for $GSp(4, \mathbb{Q})$ is compared with the Grothendieck–Verdier–Lefschetz trace formula for $S_K(\overline{\mathbb{F}}_p)$, which computes the traces of the Frobenius homomorphism attached for certain l-adic sheaves attached to the coefficient systems E_λ. But unlike the CAP localized form of the topological trace formula, this other trace formula, studied by Langlands, Kottwitz, and Milne, is not a stable trace formula. To compare both trace formulas, one has to stabilize it [53, 59]. This requires certain local identities at the non-Archimedean places – the so-called fundamental lemma and certain variants of it. More precisely, since the Grothendieck–Verdier–Lefschetz trace formula can only be explicitly computed for sufficiently high powers $Frob_p^n$ of the Frobenius $Frob_p$, one needs for each such n a twisted version of the fundamental lemma [51]. With use of these local assumptions, which are considered in the later chapters, one obtains without effort the main formula (Corollary 3.4) which expresses the Frobenius traces as a sum of two terms. One of these two terms is the trace of a suitably defined Hecke operator on the cohomology. The other term is the so-called endoscopic term, which is related to an automorphic lift. This lift is implicitly defined by the trace formula; however, it is not yet properly understood at this point of the discussion. This lift will eventually be constructed in several steps by a bootstrap argument using repeated comparison of traces. At this stage of the discussion we are therefore content with the following weak characterization: a cuspidal automorphic representation is a weak endoscopic lift if its L-functions are the L-function of an irreducible automorphic form $\sigma = (\sigma_1, \sigma_2)$ for $Gl(2, \mathbb{A}) \times Gl(2, \mathbb{A})$ at almost all places, provided σ_1 and σ_2 are cuspidal automorphic representations of $Gl(2, \mathbb{A})$ such that they have the same central character. In fact σ can be viewed as an irreducible cuspidal automorphic representation of the nontrivial elliptic endoscopic group $M \cong Gl(2)^2/\mathbb{G}_m$ of G. In the situation of Corollary 3.4, a preliminary condition at the Archimedean place has also been added. This temporarily relevant definition of a weak lift, involving some technical conditions, can be found at the beginning of Chap. 3. At the end of Chap. 3, we prove the Ramanujan conjecture (purity) for the cuspidal representations π (at the spherical places), which are neither CAP nor a weak lift, and which occur as constituents in the cohomology of degree $i = 3$. We then indicate why this should yield four-dimensional $\overline{\mathbb{Q}}_l$-adic representations of the absolute Galois group of $\mathbb{Q}$ attached to such irreducible automorphic representations. In fact, the trace formula stated in Sect. 3.6 finally leads to this [115]. We neglect the discussion of the cohomology in degree different from 3. In fact, only CAP representations contribute in these degrees, and the CAP representations have all been classified for the group $GSp(4)$; hence, it is not difficult to determine their contributions to the Hasse–Weil zeta function by weight considerations. So there is no need to exploit the trace formula in these cases. Nevertheless this is still worthwhile since it gives refined formulas for the local automorphic representations of CAP representations (see the notion of "Arthur packets" in Sect. 4.11), and it can be done using the formulas of Chap. 2. However, we have not included this discussion.

In Chap. 4, we take up the study of weak lifts. For this discussion we fix a cuspidal irreducible representation $\sigma = (\sigma_1, \sigma_2)$ of $M(\mathbb{A})$ for the unique nontrivial elliptic endoscopic group M of $G = GSp(4, \mathbb{Q})$. In Chap. 4, we then consider irreducible cuspidal automorphic representations π, which are weak lifts attached to σ but which are not CAP. This is the general assumption of Chap. 4. Since only the group $G(\mathbb{A}_{fin})$ acts on the cohomology, it is natural to ask for the Archimedean component π_∞ of the automorphic representations $\pi = \pi_\infty \otimes \pi_{fin}$. For this we fix some coefficient system E_λ. Then π_∞ necessarily must belong to the discrete series of $GSp(4, \mathbb{R})$, and π_∞ is almost determined by the condition that π_{fin} defines a nontrivial generalized eigenspace on the direct limit $\lim_K H^3(S_K(\mathbb{C}), E_\lambda)$. More precisely, this means that π_∞ belongs to a local L-packet of discrete series representations in the sense of Shelstad [91]. This L-packet is uniquely determined by the irreducible representation λ, which defines the coefficient system. This L-packet contains two equivalence classes of irreducible representations. One of the representations, $\pi_{-,\infty} = \pi_\infty^H(\lambda)$, of this L-packet belongs to the holomorphic/antiholomorphic discrete series; the other representation, $\pi_{+,\infty} = \pi_\infty^W(\lambda)$, has a Whittaker model. Let $m_1(\pi_{fin})$ and $m_2(\pi_{fin})$ be the automorphic multiplicities of the cuspidal representations $\pi = \pi_{-,\infty} \otimes \pi_{fin}$ and $\pi = \pi_{+,\infty} \otimes \pi_{fin}$, respectively. The multiplicity of π_{fin} in the generalized eigenspace $\lim_K H^3(S_K(\mathbb{C}), E_\lambda)(\pi_{fin})$ is $2m_1(\pi_{fin}) + 2m_2(\pi_{fin})$. In fact, the semisimplification of the $\overline{\mathbb{Q}}_l$-adic representations of the absolute Galois group of $\mathbb{Q}$ on the corresponding eigenspace defined for the l-adic cohomology is $m_1\rho_1 \oplus m_2\rho_2$. Here, ρ_1 and ρ_2 are the two-dimensional irreducible $\overline{\mathbb{Q}}_l$-representations attached to $\sigma_i, i = 1, 2$, by Deligne. Indeed, if some weak lift π of σ contributes nontrivially to the cohomology in degree 3, then the two cuspidal representations σ_1 and σ_2 are irreducible automorphic representations of $Gl(2, \mathbb{A})$, whose Archimedean component again belongs to the discrete series. Such automorphic representations σ_i are related to elliptic holomorphic new forms of weights r_i. The weights are not arbitrary. They must be different, so we can assume $r_1 > r_2$. This being said, there is the finer result (Lemma 4.2)

$$\lim_K H^3(S_K(\mathbb{C}), E_\lambda)(\pi_{fin})^{ss} \;\cong\; m_1 \cdot \rho_1 \;\oplus\; m_2 \cdot (\nu_l^{k_2-2} \otimes \rho_2),$$

where ν_l denotes the cyclotomic character and k_2 is an integer determined by the underlying coefficient system E_λ. The trace formula comparison of Chap. 3 provides a formula which expresses

$$m_1(\pi_{fin}) - m_2(\pi_{fin})$$

in terms of local data, but in which only the local non-Archimedean components π_v of the representations $\pi_{fin} = \prod_{v\neq\infty} \pi_v$ enter. More precisely, this formula is

$$m_1(\pi_{fin}) - m_2(\pi_{fin}) = -\prod_{v\neq\infty} n(\sigma_v, \pi_v),$$

where the coefficients $n(\sigma_v, \pi_v)$ are complex numbers obtained by a distribution formula

$$\chi^G_{\sigma_v} = \sum_{\pi_v} n(\sigma_v, \pi_v)\chi_{\pi_v}.$$

Here, χ_{σ_v} and χ_{π_v} denote characters of admissible irreducible representations, and $\chi^G_{\sigma_v}$ is the endoscopic distribution lift of the character χ_{σ_v} (see page 83). The definition of this local lift for distributions requires the existence of matching functions, where certain transfer factors have been fixed. Transfer factors will be discussed in Chaps. 6 and 7. It should be mentioned that for the group $GSp(4, F_v)$ the existence of matching functions was established by Hales [36]. The existence of a character expansion $\chi^G_{\sigma_v} = \sum_{\pi_v} n(\sigma_v, \pi_v)\chi_{\pi_v}$ is then derived from the trace comparisons studied in Chap. 3 and the first sections of Chap. 4. Most of the content of Sects. 4.5–4.12 are devoted to proving that this sum is finite and that the transfer coefficients $n(\sigma_v, \pi_v)$ are integers. In fact this finally defines the endoscopic lift $r : R_{\mathbb{Z}}[M_v] \to R_{\mathbb{Z}}[G_v]$ between the integral Grothendieck groups of irreducible admissible representations of $G = GSp(4)$ and $M = Gl(2)^2/\mathbb{G}_m$ for non-Archimedean p-adic fields. In the real case such formulas are known in general from the work of Shelstad [90, 91]. The final result is stated in Sect. 4.11. With use of the classification of representations, the results obtained by Moeglin, Rodier, Sally, Shahidi, Soudry, Tadic, Vigneras, and Waldspurger, this is reduced to establish the existence of r for local non-Archimedean admissible irreducible representations σ_v of M_v, which belong to the discrete series. For these representations, it turns out that the local character lift has the form

$$r(\sigma_v) = \pi_+(\sigma_v) - \pi_-(\sigma_v)$$

for two irreducible admissible representations $\pi_\pm(\sigma_v)$ of the group G_v. We furthermore show that $\pi_+(\sigma_v)$ does have a Whittaker model, whereas $\pi_-(\sigma_v)$ does not. Finally, we use global theta series to describe $\pi_\pm(\sigma_v)$ in terms of local theta lifts similarly to the case of the group $Sp(4)$ studied by Howe and Piatetski-Shapiro [41]. In fact this study is continued in Chap. 5. Indeed, some results obtained in Chap. 5 are already used in Sect. 4.12. Besides these local results, studied in Sects. 4.5–4.12, we consider in Sect. 4.4 rationality questions, i.e., questions concerning the field of definition. Using some properties of the endoscopic transfer factor, defined in the later chapters, we can describe the numbers $m_1(\pi_{fin})$ and $m_2(\pi_{fin})$ in terms of Hodge theory. In fact $2m_1(\pi_{fin})$ turns out to be the multiplicity of π_{fin} in the holomorphic/antiholomorphic part and $2m_2(\pi_{fin})$ turns out to be the nonholomorphic contribution.

In Chap. 5, we continue the discussion of Chap. 4, but return to global questions. The main result obtained in this chapter is Theorem 5.2, which is the final version of the preliminary multiplicity formula for weak lifts given in Sects. 4.1–4.3. We obtain the formula $m_1(\pi_{fin}) + m_2(\pi_{fin}) = 1$; hence, one of the global multiplicities $m_i(\pi_{fin})$ is 1 and the other is 0. Which of them does not vanish depends only on the non-Archimedean components of π_{fin}. The essential argument is the principle of exchange, which controls exchange in $\otimes_v \pi_v$ of a representation π_v within its local packet at one specific place v. The final formula is, of course, a special case of Arthur's conjecture [3], which originated from considering the special case of the group $GSp(4,\mathbb{Q})$. However we only consider this formula in the case of weak lifts, which are not CAP. Nevertheless, the multiplicity formula for the cohomology groups in the case of the Saito–Kurokawa lift can be derived along the same line of arguments (although this is not carried through explicitly).

Chapter 5 also contains sections in which the global results on the endoscopic lift are extended to the case $G = Res_{F/\mathbb{Q}}(GSp(4))$ for an arbitrary totally real number field F, and also applies to representations which do not necessarily appear in the cohomology of Shimura varieties. This is contained in Sects. 5.4 and 5.5. The arguments here use Arthur's trace formula instead of the topological trace formula, and they are subtler and more technical than the arguments involving the topological trace formula. The analogous local results, which extend those obtained in Chap. 4 to arbitrary local fields of characteristic 0, are considered in Sect. 5.1.

In Chaps. 6 and 7, the fundamental lemma for the group $GSp(4,F_v)$ over a local non-Archimedean field F_v of residue characteristic different from 2 is proved. This fundamental lemma (Theorems 6.1 and 7.1) is an identity between local orbital integrals $O_\eta^{G_v}(f_v)$ and $O_t^{M_v}(f^{M_v})$ for the groups M_v and G_v. This identity involves a transfer factor $\Delta(\eta,t)$. Here, the elements $\eta \in GSp(4,F_v)$ and $t \in M(F_v)$ are sufficiently regular semisimple elements and η and t are related by a norm mapping. η is an element whose conjugacy class over the algebraic closure is determined by the conjugacy class of t in $M(F_v)$. This does not determine the $G(F_v)$-conjugacy class of η uniquely. In the case under consideration, there are one or two such conjugacy classes in the stable conjugacy class. The κ-orbital integral is the difference $O_\eta^{G_v,\kappa}(f_v) = O_\eta^{G_v}(f_v) - O_{\eta'}^{G_v}(f_v)$ of orbital integrals $O_\eta^{G_v}(f_v) = \int_{G_{\eta,v}\backslash G_v} f_v(g^{-1}\eta g)dg/dg_\eta$ in the case where there are two such classes. Since there is no canonical choice, which might privilege η or η', one has to make a choice for the definition of $O_\eta^{G_v,\kappa}(f_v)$. The dependence on this choice is compensated for by a transfer factor $\Delta(\eta,t)$, which depends on the class of η chosen. Then the fundamental lemma is the statement that there exists a homomorphism $b : f_v \mapsto f_v^{M_v}$ between the spherical Hecke algebras (prescribed by the principles of Langlands functoriality) with the matching condition

$$\Delta(\eta, t) O_\eta^{G_v,\kappa}(f_v) \;=\; SO_t^{M_v}(f_v^{M_v})$$

for all sufficiently regular t and the corresponding η. Here, SO denotes the stable orbital integral on M_v. A couple of remarks are in order. First, although the fundamental lemma is later used for (G, M)-regular elements t, it is enough to prove the fundamental lemma for sufficiently regular elements by a degeneration argument. Second, it is enough to prove the fundamental lemma for the unit elements of the Hecke algebras, and for almost all primes (see Hales [35] for the case of ordinary endoscopy). See also Chap. 10, where the reduction to the case of the unit element $f_v = 1$ of the spherical Hecke algebra is discussed in the slightly greater generality of twisted base change. Third, for our purposes it is important that the transfer factors have certain nice properties. One of these properties is the product formula (global property)

$$\prod_v \Delta(\eta_v, t_v) = 1$$

for global elements η and t, where the product is over all Archimedean and Nonarchimedean places. For the product formula above, it is essential for us that the formula holds precisely with the Archimedean transfer factor $\Delta(\eta_\infty, t_\infty)$ used by Shelstad. Concerning this, we show in Chap. 8 that for our choice of transfer factor the product formula holds, and that our chosen Archimedean transfer factor is the same as the one defined by Langlands and Shelstad over the field $\mathbb{R}$. Unfortunately, already in the case of the group $GSp(4)$, this amounts to a lengthy and tedious unraveling of the definitions, which are based on the cohomological reciprocity pairings of local class field theory [60]. The proof of the fundamental lemma is done by an explicit calculation. We distinguish two cases dealt with in Chaps. 6 and 7, respectively. In fact, the computation gives the local orbital integrals explicitly, not only the κ-orbital integral. This turned out to be useful for later computations in the twisted case done by Flicker [26]. The explicit calculation of the orbital integrals hinges on an approach which in the case of the group $Gl(2)$ is used in the book by Jaquet and Langlands [42] on $Gl(2)$ and which in an implicit form is based on some double coset computations in the group $G(F_v)$ for the group $G = Gl(2)$. For me, an analogous double coset decomposition for the group $G = GSp(4)$, due to Schröder, suggested this approach. A special case was been carried out by Schröder [81]. It seems that nice representatives for double cosets $H(F_v)\backslash G(F_v)/K$ of this type exist for reductive, hyperspecial maximal compact subgroups K and maximal proper reductive F_v-subgroups H of G quite generally, in the sense that they should define a generalization of the classical theory of genera of quadratic forms. In classical genus theory, H is the orthogonal group contained in the linear group G. The maximal subgroups of reductive groups are well known, and new types of genus theory mainly arise from considering the inclusions $H \hookrightarrow G$ of centralizers $H = G_s$ of semisimple elements s in reductive groups G. In Chap. 12 we consider this situation for the group $G = GSp(2n)$, where we generalize the result obtained by Schröder to the case $n = 2$. Similar computations can be made in the case of classical groups [116]. The case of the exceptional case group G of type G_2 was considered by Weselmann [117].

Chapter 9 considers the fundamental lemma for twisted base change endoscopy. This type of fundamental lemma is needed for the trace comparison theorems in Chap. 3. We show that these twisted endoscopic fundamental lemmas can be reduced to the ordinary fundamental lemmas of standard endoscopy. Such a reduction can be carried out quite generally, except that we consider the global trace formula arguments only in the case of the group $GSp(4)$. However, the argument can be extended to the general case, and this will be considered elsewhere. As a side result, one gets a variant of the fundamental lemma (see Lemma 9.7) where the transfer factors are defined in a slightly different form, which is needed for Chap. 3. This is based on some explicit formulas for the Langlands reciprocity map as given in Kottwitz [48] or Schröder and Weissauer [82]. An entirely local proof was given later by Kaiser [43].

In Chap. 10, we verify that the twisted endoscopic fundamental lemma is a consequence of the special case of the fundamental lemma for unit elements, as one expects from the untwisted case [35], and that it is enough to know it for almost all primes. This reduces the fundamental lemmas needed for Chap. 3 to the statements given in Chaps. 6 and 7. The argument uses the method of Labesse [57]; hence, it is based on elementary functions and can be further generalized [113] to the twisted adjoint cases. For standard endoscopy this reduction was proved in Hales [35] by a different argument. Finally, Chaps. 11 and 12 contain some prerequisite material needed in Chaps. 6–10.

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