Chapter 2
CAP Localization

In this chapter, we will express the π-isotypic Lefschetz numbers of Hecke operators acting on the cohomology of symmetric spaces $S_K(G)$ attached to reductive groups G in terms of so-called elliptic traces T_{ell}, provided the underlying representation π is not a cuspidal representation associated with a parabolic subgroup (CAP representation) of $G(\mathbb{A})$. In the following two chapters we derive from these formulas all the essential information required.

For a connected reductive group G over $\mathbb{Q}$, let K_∞ be a maximal compact subgroup of $G_\infty = G(\mathbb{R})$ and let $A_G(\mathbb{R})^0$ be the topologically connected component of the maximal $\mathbb{Q}$-split component A_G of the center Z_G of G. Then $X_G = G_\infty/\tilde{K}_\infty$ for $\tilde{K}_\infty = K_\infty \cdot A_G(\mathbb{R})^0$ will be called the connected symmetric space attached to G. For a compact open subgroup $K \subseteq G(\mathbb{A}_{fin})$

$$S_K(G) = G(\mathbb{Q}) \setminus \big(X_G \times G(\mathbb{A}_{fin})/K\big) = G(\mathbb{Q}) \setminus \big(X_G \times G(\mathbb{A}_{fin})\big)/K$$

is a disjoint union of arithmetic quotients of X_G.

Example 2.1. For $G = GSp(4,\mathbb{Q})$ we have $X_G = H \cup -H$ for the Siegel upper half-space H of genus 2. Hence, $S_K(G)$ does not coincide with the Shimura variety $S_K(\mathbb{C})$, which is an unramified covering of $S_K(G)$.

Assumption Regarding G_{der}. In this chapter assume that the derived group G_{der} of G is simply connected. This property is inherited by the Levi subgroups L of G.

Proof: $G = G_{der}Z(G)$ and $Z(G) \subseteq L$ implies $L_{der} = (G_{der} \cap L)_{der}$. $L \cap G_{der}$ is a Levi group of G_{der}, since this holds for the Lie algebras by characterizing Levi subgroups as centralizers. So it is enough to consider the semisimple case to see that L_{der} is simply connected. For this case see [99], Lemma 5.3 or Theorem 5.8, p. 208, which proves the claim. Since all groups L_{der} are simply connected implies that the centralizers L_γ of semisimple elements in the Levi groups L are connected reductive groups.

Lefschetz Numbers. An irreducible complex representation of the group $G_\infty = G(\mathbb{R})$ with highest weights λ restricts us to a representation of $G(\mathbb{Q})$, which defines

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a coefficient system[1] V_λ on $S_K(G)$. The cohomology groups $H^\nu(S_K(G), V_\lambda)$ are modules under the Hecke algebra of K-bi-invariant functions on $G(\mathbb{A}_{fin})$ with compact support. Assume $K = \prod_v K_v$. Fix a finite set S of non-Archimedean places such that for all non-Archimedean places $v \notin S$ the group K_v is a special, good maximal compact subgroup of G_v. Let

$$\pi^S = \otimes_{v \notin S} \pi_v$$

be an irreducible spherical automorphic representation of $G(\mathbb{A}_f^S)$. The π^S-isotypic generalized eigenspace of the νth cohomology group

$$H^\nu(S_K(G), V_\lambda)(\pi^S)$$

is a module under the Hecke algebra $\mathcal{H}_{S,K} \subseteq \mathcal{H}_S$, defined by the locally constant K_S-bi-invariant functions on $G(\mathbb{A}_S)$ with compact support. A simple formula for the trace of Hecke operators $f_S \in \mathcal{H}_S = \otimes_{v \in S} \mathcal{H}_v$ in the subspace $\mathcal{H}_{S,K}$ of the Hecke algebra (see Appendices 1 and 2) defined by

$$tr_s(f_S) = \sum_\nu (-1)^\nu tr\Big(f_S, H^\nu(S_K(G), V_\lambda)(\pi^S) \Big)$$

is provided by the topological trace formula of Goresky and MacPherson. Assume that the unramified automorphic spherical representation π^S of $G(\mathbb{A}_f^S)$ is not isomorphic to a subquotient of an induced representation $Ind_{P^S}^{G^S}(\sigma^S)$ for all proper parabolic subgroups $P \neq G$ with Levi component L, and all irreducible automorphic representations σ^S of $L(\mathbb{A}_f^S)$. In this case π^S is cuspidal, and π^S is not a CAP representation in the sense of [69, 97]. With these assumptions, the formula for the trace of f_S is further simplified (Sects. 2.6, 2.8).

Of special interest is the case where G_∞ has discrete series representations (Sect. 2.9). In this case the formula for the trace becomes the following (see Corollary 2.6): If the group $K = \prod_v K_v$ is small and π^S is not CAP, the trace $Tr_s(f_S)$ of f_S is equal to

$$d(G) \cdot \sum_{\gamma \in G(\mathbb{Q})/\sim}{}' \tau(G_\gamma) O_\gamma^{G(\mathbb{A})}(f_S f_{\pi^S} f_\infty).$$

The sum is over all strongly elliptic semisimple conjugacy classes in $G(\mathbb{Q})$ (see page 46); G_γ denotes the centralizer of γ in G, which by our assumptions is a connected reductive group. The coefficients $O_\gamma^{G(\mathbb{A})}$ are adelic orbital integrals. Measures are such that $vol_{dg_f}(K) = 1$ and $vol_{dg_\infty dg_f}\big(G(\mathbb{Q}) \setminus G(\mathbb{A})\big) = \tau(G)$ is the Tamagawa number. The function f_{π^S} is a suitable chosen good π^S-projector depending on the fixed f_S (see Sect. 2.8), and f_∞ is a suitable linear combination of pseudocoefficients of discrete series representations with respect to the measure dg_∞ (see Sect. 2.9). The corresponding L-packet is determined by the representation

[1] In this chapter we consider the dual V_λ of the coefficient system E_λ of Chap. 1.

λ defining the coefficient system V_λ. $d(G)$ denotes the number of discrete series representations in this L-packet.

Remark 2.1. If $\tilde{K}_\infty$ is replaced by a subgroup U of finite index such that $G_\infty/U \to G_\infty/\tilde{K} = X_G$ is a finite unramified covering of degree d, then the trace formula also holds for $G(\mathbb{Q}) \setminus (G_\infty/U \times G(\mathbb{A}_{fin})/K$ except that the formula above has to be multiplied by the degree d of the covering. This applies for Shimura varieties (G,h) attached to a reductive $\mathbb{Q}$-group, for which $Z(G)/A_G$ is $\mathbb{R}$-anisotropic, since in this case the centralizer $Z(h)$ of the structure homomorphism h of the Shimura variety is a subgroup of finite index in $\tilde{K} = K_\infty \cdot A_G(\mathbb{R})^0$. See page 53.

2.1 Standard Parabolic Subgroups

Fix a minimal $\mathbb{Q}$-parabolic subgroup P_0. For a $\mathbb{Q}$-rational parabolic subgroup $P = LN$ containing P_0, and $\gamma \in P(\mathbb{Q})$ let γ_L denote the image of γ under the projection $P(\mathbb{Q}) \to L(\mathbb{Q})$ to the Levi component.

Contractive Elements. A semisimple element $\gamma \in L(\mathbb{Q})$, which is contained in a real torus T of L, which is $\mathbb{R}$-anisotropic modulo A_L is called *P-contractive*, if $|\gamma_L^\sigma|_\infty > 1$ holds for all simple roots σ (over the algebraic closure), which occur in the Lie algebra of the nilpotent radical of P, restricted to the maximal $\mathbb{Q}$-split torus A_L (in the center of L). In fact, it does not matter if we consider the absolute root system or the $\mathbb{Q}$-root system. Since $\gamma_L = a_\infty \cdot x_\infty k_\infty x_\infty^{-1}$ for $a_\infty \in A_L(\mathbb{R})^0, k_\infty \in K_{L,\infty}$, this implies $|\gamma_L^\sigma|_\infty = |a_\infty^\sigma|_\infty$ for all roots σ. Hence, γ_L is P-contractive if and only if the central component a_∞ is P-contractive and this notion depends only on the $L(\mathbb{Q})$-conjugacy class of the element γ. Suppose $P = P_\theta = L_\theta N_\theta$ is a $\mathbb{Q}$-rational standard parabolic subgroup defined by a subset θ of the simple positive $\mathbb{Q}$-roots. Then by definition $|\alpha(\gamma_L)|_\infty = |a_\infty^\alpha|_\infty = 1$ holds for all simple roots $\alpha \in \theta$. Since the roots in $Lie(N_P)$ are the positive roots which are not linear combinations of the roots in θ, the condition defining the notion P-contractive may be replaced by the *stronger condition*: $|\gamma_L^\alpha|_\infty \geq 1$ holds for all positive roots in Φ^+, and $|\gamma_L^\alpha|_\infty = 1$ holds if and only if α is a root which occurs in $Lie(L_P)$, or alternatively this could also be replaced by the condition $|a_\infty^\alpha|_\infty > 1$ for all simple $\mathbb{Q}$-roots $\alpha \notin \theta$.

The Set W'. Let $\Phi_G = \Phi = \Phi^+ \cup \Phi^-$ be the decomposition into the positive and negative roots of the absolute root system. Define W' as a subset of the absolute Weyl group W (considered over the algebraic closure) to consist of the elements $w \in W$ for which $\Phi^+ \cap w\Phi^- \subseteq \Phi(Lie(N_P))$ [33], p. 474, or equivalently $w\Phi^- \cap \Phi_L^+ = \emptyset \Leftrightarrow w^{-1}(\Phi_L^+) \subseteq \Phi_G^+$. Then $W' = W^P$ is the set of all $w \in W$ such that $w^{-1}(\alpha) > 0$ holds for all $\alpha \in \theta$. By a result obtained by Kostant, W is the disjoint union of the cosets $W_L \cdot w$ for representatives $w \in W^P$; hence $|W^{P_\theta}| = |W|/|W_{L_\theta}|$. Here W_L denotes the absolute Weyl group of L, considered as a subgroup of $W = W_G$. The representatives $w \in W^P$ are uniquely characterized as the representatives of minimal length in the W_L left cosets of W_G.

Inductivity. Notice the following inductive property of the sets $W^P \subseteq W$. Let $P = P_{\theta_1} \subseteq Q = P_{\theta_2} \subseteq G$ be standard parabolic subgroups, corresponding to $\theta_1 \subseteq \theta_2$. Let $L = L_{\theta_2}$ be the standard Levi component of P_{θ_2}. Then $P_{\theta_1} \cap L = P'$ is a standard parabolic subgroup of L with Levi component $L' = L_{\theta_1}$. In particular, $W^{P'} \subseteq W_L$ is defined. Then

$$W^{P'} \cdot W^Q = W^P.$$

In fact $w_1 \cdot w_2 = w_1' \cdot w_2'$ with $w_1, w_1' \in W^{P'}$ and $w_2, w_2' \in W^Q$ implies $W_{L_{\theta_2}} w_2 = W_{L_{\theta_2}} w_2'$; hence, $w_2 = w_2'$ and therefore also $w_1 = w_1'$. By the above-mentioned formula for the cardinalities it is enough to show that the product set on the left side is contained in the right side. But this is clear. Every w_1^{-1} for $w_1 \in W^{P'}$ maps $\Phi(L_{\theta_1})^+$ to $\Phi(L_{\theta_2})^+$ and every w_2^{-1} for $w_2 \in W^{P_{\theta_2}}$ maps $\Phi(L_{\theta_2})^+$ to $\Phi^+ = \Phi(G)^+$; thus, $w_1 w_2 \in W^{P_{\theta_1}}$.

Characters. For a dominant weight λ of L let ψ_λ denote the character of the finite-dimensional irreducible complex representation of L with highest weight λ. Let ρ_G denote half of the sum of the roots in Φ^+. Similarly define ρ_L for the reductive group L. Put $\rho_P = \rho_G - \rho_L$ as characters of L. If λ is a dominant weight for G, then $w(\rho_G + \lambda) - \rho_G$ is dominant for L (see [15], Sect. III.1.4 and Sect. III.3.1, and [45]). Using the Coxeter lengths $l(w)$, define

$$\Psi(\gamma, \lambda) = \sum_{w \in W^P} (-1)^{l(w)} \psi_{w(\lambda+\rho_G)-\rho_G}(\gamma_L^{-1}).$$

Since $-\rho_G + \rho_P = -\rho_L$, we have for $\gamma \in L(\mathbb{Q})$

$$|\gamma|_\infty^{-\rho_P} \cdot \Psi(\gamma, \lambda) = \sum_{w \in W^P} (-1)^{l(w)} \psi_{w(\lambda+\rho_G)-\rho_L}(\gamma_L^{-1}).$$

The Function *r*(*γ*). Let $\mathbb{A}$ denote the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_{fin}$ the ring of finite adeles. Let $K = \prod_{v \ finite} K_v$ be a compact open subgroup of $G(\mathbb{A}_{fin})$. For a $\mathbb{Q}$-rational parabolic $P = LN$ and for semisimple $\gamma \in P(\mathbb{Q})$ define $\Gamma = G(\mathbb{Q}) \cap K$, $\Gamma_N = \Gamma \cap N$, $\Gamma' = \Gamma \cap \gamma^{-1}\Gamma\gamma$, $\Gamma_N' = \Gamma' \cap N$. Then

$$r = r(\gamma) = [\Gamma_N : \Gamma_N'] = [\Gamma_N : \Gamma_N \cap \gamma^{-1}\Gamma_N\gamma],$$

$$s = s(\gamma) = [\gamma^{-1}\Gamma_N\gamma : \Gamma_N'] = [\Gamma_N : \gamma\Gamma_N'\gamma^{-1}]$$

satisfy $s(\gamma) = [\Gamma_N : \gamma(\Gamma_N \cap \gamma^{-1}\Gamma_N\gamma)\gamma^{-1}] = [\Gamma_N : \gamma\Gamma_N\gamma^{-1} \cap \Gamma_N] = r(\gamma^{-1})$; hence,

Lemma 2.1. $s(\gamma) = r(\gamma^{-1})$, *which only depends on* γ_L.

Lemma 2.2. $s(\gamma)/r(\gamma) = \gamma^{2\rho_P}$ *or* $|\gamma^{\rho_P}|_\infty r(\gamma) = |\gamma^{-\rho_P}|_\infty r(\gamma^{-1})$.

Proof. The quotient $[\Gamma_N : \Gamma_N \cap \gamma\Gamma_N\gamma^{-1}]/[\Gamma_N : \Gamma_N \cap \gamma^{-1}\Gamma_N\gamma]$ is the virtual index

$$[\Gamma_N \cap \gamma^{-1}\Gamma_N\gamma : \Gamma_N \cap \gamma\Gamma_N\gamma^{-1}] = [\Gamma_N \cap \gamma^{-1}\Gamma_N\gamma : \gamma(\Gamma_N \cap \gamma^{-1}\Gamma_N\gamma)\gamma^{-1}] = |\gamma^{2\rho_P}|_\infty. \quad \square$$

2.2 The Adelic Reductive Borel–Serre Compactification

As a set, the adelic reductive Borel–Serre compactification is

$$(S_K^G)^+ = G(\mathbb{Q}) \setminus [\bigcup_P X_L \times (G(\mathbb{A}_{fin})/K)] = G(\mathbb{Q}) \setminus [\bigcup_P X_L \times G(\mathbb{A}_{fin})]/K,$$

a disjoint union over all $\mathbb{Q}$-rational parabolic subgroups P of G. $X_L = L_\infty/\tilde{K}_{L,\infty}$ is the connected symmetric domain attached to L, i.e., $\tilde{K}_\infty = K_{L,\infty}A_L(\mathbb{R})^0$, where $K_{L,\infty}$ denotes a maximal compact subgroup of L_∞ and $A_L(\mathbb{R})^0$ the topologically connected component of the maximal $\mathbb{Q}$-split subtorus A_L in the center $Z(L)$ of L. Elements $g \in G(\mathbb{A}_{fin})$ act on the projective limit $(S^G)^+ = lim_K (S_K^G)^+ = G(\mathbb{Q})\setminus [\bigcup_P X_L \times G(\mathbb{A}_{fin})]$ by $x \mapsto xg^{-1}$. This defines a left action of $G(\mathbb{A}_{fin})$ on $(S^G)^+$, which induces a right action on cohomology groups. Now consider

$$T(g^{-1}) :\ G(\mathbb{Q})x_\infty x_{fin} \mapsto G(\mathbb{Q})x_\infty(x_{fin}g^{-1}).$$

Here $x_\infty \in \bigcup_P X_L$ and $x_{fin} \in G(\mathbb{A}_{fin})$. On the quotients $(S_K^G)^+$ this defines Hecke correspondences. Put

$$K' = K \cap g^{-1}Kg.$$

Then the induced Hecke correspondence is given by two maps $c_1 = T(1)$ and $c_2 = T(g^{-1})$ (see Appendix 1)

$$(S_{K'}^G)^+ \underset{c_2}{\overset{c_1}{\rightrightarrows}} (S_K^G)^+ \ .$$

The action of $G(\mathbb{Q})$ on the $\mathbb{Q}$-parabolic subgroups by conjugation is transitive on the minimal $\mathbb{Q}$-parabolic subgroups. Fixing a minimal parabolic P_0, every $\mathbb{Q}$-parabolic is conjugate over $\mathbb{Q}$ to one and only one standard parabolic $\mathbb{Q}$-subgroup P with respect to P_0. Since the stabilizer of P under conjugation with $G(\mathbb{Q})$ is $P(\mathbb{Q})$, $(S_K^G)^+$ is a union over the finitely many standard $\mathbb{Q}$-parabolic subgroups $P = P_\theta$ containing P_0:

$$(S_K^G)^+ = \bigcup_{P_0 \subseteq P} S_K^P, \quad \text{where} \quad S_K^P = P(\mathbb{Q}) \setminus [X_L \times G(\mathbb{A}_{fin})]/K.$$

Goresky and MacPherson [33] deduced a formula for the alternating trace

$$tr_s(T(g^{-1}); H^\bullet\big(S_K(G), V_\lambda)\big)$$

from the Grothendieck–Verdier–Lefschetz fixed-point formula which they applied for the reductive Borel–Serre compactification $(S_K^G)^+$ of $S_K(G)$. They used the property that the cohomology groups $H^\bullet(S_K(G), V_\lambda)$ coincide with the cohomology groups $H^\bullet((S_K^G)^+, i_*V_\lambda)$ of the reductive Borel–Serre compactification $(S_K^G)^+$, where $i : S_K(G) \hookrightarrow (S_K^G)^+$ is the inclusion. As in [33], Theorem (version 0), the Lefschetz fixed-point theorem of Grothendieck, Verdier, and Illusie therefore expresses the Lefschetz number as a sum of "local" contributions $LC(F)$

$$\sum_P \sum_F LC(F)$$

for the connected components F of the intersection of the fixed-point set of the correspondence with the boundary strata S_K^P attached to the rational parabolic group P.

Rational Hecke Correspondences. We say a double coset KgK or the corresponding Hecke correspondence is rational if a representative g can be chosen to be $g = \gamma_{fin}$ for some $\gamma = \gamma_\infty \gamma_{fin} \in G(\mathbb{Q})$. In this case the correspondence $T(g^{-1})$ defined on $(S^G)^+ = G(\mathbb{Q}) \setminus [\bigcup_P X_L \times G(\mathbb{A}_{fin})]$ satisfies $G(\mathbb{Q})x_\infty \mapsto G(\mathbb{Q})x_\infty \gamma_{fin}^{-1} = G(\mathbb{Q})\gamma_\infty x_\infty$; hence, it induces the Hecke correspondence considered in [33], p. 467, defined by $c_1(\Gamma' y) = \Gamma y$ and $c_2(\Gamma' y) = \Gamma \gamma_\infty y$.

2.2.1 Components

First consider the connected components of S_K^P. Since X_L is topologically connected, the topologically connected components h of the stratum S_K^P are the fibers of the map

$$S_K^P = P(\mathbb{Q}) \setminus [X_L \times (G(\mathbb{A}_{fin})/K)] \longrightarrow \pi_0(S_K^P) = P(\mathbb{Q}) \setminus G(\mathbb{A}_{fin})/K.$$

For each component $h = P(\mathbb{Q})x_{fin}K$ in $\pi_0(S_K^P)$ put

$$\Gamma_{P_h} = P(\mathbb{Q}) \cap x_{fin}Kx_{fin}^{-1} \ \text{ and } \ \Gamma_{N_h} = N(\mathbb{Q}) \cap x_{fin}Kx_{fin}^{-1}.$$

For $K_N(h) := N(\mathbb{A}_{fin}) \cap x_{fin}Kx_{fin}^{-1}$ and $K'_N(h) := N(\mathbb{A}_{fin}) \cap x_{fin}K'x_{fin}^{-1}$ then obviously $[K_N(h) : K'_N(h)] = [\Gamma_{N_h} : \Gamma'_{N_h}]$, where $\Gamma'_{N_h} := K'_N(h) \cap N(\mathbb{Q})$.

Fixed Components. Now consider the connected components F of the fixed-point locus of a Hecke correspondence within S_K^P, for fixed P. Then

$$F \subseteq h$$

for some unique component h of S_K^P. If F is fixed, then h is also fixed. So we first determine the fixed components h of the Hecke correspondence, and then the fixed components F in h.

2.2.2 Fixed Components h

The component $h = P(\mathbb{Q})(X_L \times \{x_{fin}\})K$ is fixed

$$T(g^{-1})h = h,$$

if and only if $x_{fin}g^{-1}K = \gamma x_{fin}K$ holds for some $\gamma \in P(\mathbb{Q})$ and some $k \in K$.

Recall $gK'g^{-1} \subseteq K$. Hence, $\gamma^{-1}x_{fin}g^{-1} = x_{fin}k$ implies $\gamma^{-1}x_{fin}K'x_{fin}^{-1}\gamma \subseteq x_{fin}Kx_{fin}^{-1}$. Thus,

$$\gamma^{-1}\Gamma'_{P_h}\gamma := P(\mathbb{Q}) \cap \gamma^{-1}x_{fin}K'(\gamma^{-1}x_{fin})^{-1} \subseteq P(\mathbb{Q}) \cap x_{fin}Kx_{fin}^{-1} =: \Gamma_{P_h}.$$

$x_{fin}K'x_{fin}^{-1}{=}x_{fin}Kx_{fin}^{-1}{\cap}x_{fin}g^{-1}K(x_{fin}g^{-1})^{-1}{=}x_{fin}Kx_{fin}^{-1}{\cap}\gamma(x_{fin}Kx_{fin}^{-1})\gamma^{-1}$ again using $x_{fin}g^{-1}{=}\gamma x_{fin}k$. Hence, the intersection with $P(\mathbb{Q})$ is $\Gamma'_{P_h}{=}\Gamma_{P_h} \cap \gamma\Gamma_{P_h}\gamma^{-1}$. In particular, $\Gamma'_{N_h}{=}(\Gamma_{N_h} \cap \gamma\Gamma_{N_h}\gamma^{-1})$. Hence, the fixed equation $T(g^{-1})h{=}h$ given by $\gamma^{-1}x_{fin}g^{-1}{=}x_{fin}k$ implies.

Lemma 2.3. $[K_N(h) : K'_N(h)] = [\Gamma_{N_h} : \Gamma'_{N_h}] = [\Gamma_{N_h} : (\Gamma_{N_h} \cap \gamma\Gamma_{N_h}\gamma^{-1})] = r(\gamma^{-1})$. *For fixed g, K this number only depends on P and the coset $P(\mathbb{A}_{fin})x_{fin}$.*

Rationality. To simplify the notation we now replace K by $x_{fin}Kx_{fin}^{-1}$, and g^{-1} by $x_{fin}g^{-1}x_{fin}^{-1}$, which allows us to assume $x_{fin} = 1$ without restriction of generality. Then the fixed-component equation becomes $\gamma \in g^{-1}K$. Hence, the coset $g^{-1}K \subseteq Kg^{-1}K$ has a rational point, and the Hecke correspondence defined by $Kg^{-1}K = K\gamma K$ is rational. For a fixed component h one can thus reduce the local computations of the local term $LC(F)$ for F to the classical setting considered in [33].

2.2.3 Another Formulation

The action of $G(\mathbb{Q})$ on the $\mathbb{Q}$-parabolic subgroups by conjugation is transitive on the minimal $\mathbb{Q}$-parabolic subgroups. Hence, choosing a minimal $\mathbb{Q}$-parabolic P_0, every $\mathbb{Q}$-parabolic is conjugate over $\mathbb{Q}$ to one and only one standard parabolic $\mathbb{Q}$-subgroup P with respect to P_0. Since the stabilizer of P under conjugation with $G(\mathbb{Q})$ is $P(\mathbb{Q})$, $(S_K^G)^+$ is a union over the finitely many standard $\mathbb{Q}$-parabolic subgroups $P = P_\theta$ containing P_0,

$$(S_K^G)^+ = \bigcup_{P_0 \subseteq P} P(\mathbb{Q}) \setminus [X_L \times G(\mathbb{A}_{fin})]/K.$$

Since $gKg^{-1} \cap N_P(\mathbb{A}_{fin})$ is open in $N_P(\mathbb{A}_{fin})$ for $P = L_P N_P$, for the strata $S_K^P = P(\mathbb{Q}) \setminus [X_L \times G(\mathbb{A}_{fin})]/K$ of

$$(S_K^G)^+ = \bigcup_{P_0 \subseteq P} S_K^P$$

an easy density argument gives the formula $S_K^P = L_P(\mathbb{Q}) \setminus [X_L \times (N_P(\mathbb{A}_{fin}) \setminus G(\mathbb{A}_{fin})/K)]$ or

$$S_K^P = L_P(\mathbb{Q})N_P(\mathbb{A}_{fin}) \setminus [L_\infty \times G(\mathbb{A}_{fin})]/\tilde{K}_\infty K.$$

Hence, $\overline{x} \in (S_K^G)^+$ is a double coset represented by some $x = x_\infty x_{fin} \in L_\infty \times G(\mathbb{A}_{fin})$. Iwasawa decomposition $G(\mathbb{A}_{fin}) = P(\mathbb{A}_{fin}) \cdot \Omega$ for some maximal compact group Ω containing the group K gives a finite decomposition $G(\mathbb{A}_{fin}) = \bigcup_g P(\mathbb{A}_{fin})gK$. Therefore, the set $\pi_0(S_K^P)$ of the topologically connected components is finite, since by a result obtained by Borel and Harish-Chandra [14], $M(\mathbb{Q}) \setminus M(\mathbb{A}_{fin})/K_M$ is finite for any reductive $\mathbb{Q}$-group M and any compact open group $K_M \subseteq M(\mathbb{A}_{fin})$. Of course we may choose the representatives elements

$$x_{fin} = k \quad \in \Omega.$$

2.2.4 Small Groups

Consider compact open subgroups $K \subseteq G(\mathbb{A}_f)$ and $\tilde{K}_\infty = K_\infty Z^0_{G,\infty}$, where K_∞ is maximal compact in G_∞. $K \subseteq G(\mathbb{A}_{fin})$ will be called *small* if

$$x^{-1} n\gamma x \in K\tilde{K}_\infty Z_{L,\infty}$$

for $x \in G(\mathbb{A}), n \in N(\mathbb{A}), \gamma \in P(\mathbb{Q})$ and any $\mathbb{Q}$-parabolic $P = L \cdot N$ with unipotent radical N implies $\gamma_L \in Z_L(\mathbb{Q})$ (image in the Levi component is contained in the center) and in addition implies $\gamma_L = 1$ if γ_L is a torsion element.

Remark 2.2. Of course it is enough to demand this for all standard parabolic groups containing a fixed P_0.

Remark 2.3. "Small" implies "neat" in the sense that $L(\mathbb{Q})_{tor} \cap (xKx^{-1} \cap P)_L = 1$.

Small-level groups K exist: $G(\mathbb{A})$ is a finite union of cosets $P(\mathbb{A})kKK_\infty$ for $k \in G(\mathbb{A}_{fin})$. This allows us to replace K by some conjugate K_k, and x by some $p \in P(\mathbb{A})$, and gives equations $p^{-1}n\gamma p \in K_k$ for $p \in P(\mathbb{A})$ instead of $x \in G(\mathbb{A})$. Equivalently, $m^{-1}\gamma_L m \in (K_k \cap P)_L$ for $m \in L(\mathbb{A})$, where the index L indicates projection from P to the Levi component L. γ_L is semisimple since modulo the center it is contained in a maximal compact subgroup of L_∞. The groups L and $L_{ad} = L/Z_L$ are connected reductive groups; hence, by embedding L_{ad} into some linear group and using for L_{ad} the argument at the beginning of the proof [44], Proposition 8.2, one can show that only finitely many $L(\overline{\mathbb{Q}})$ conjugacy classes of semisimple elements γ_L in $L_{ad}(\mathbb{Q})$ meet $(K_k \cap P)_L$. Shrinking K leaves us, considering eigenvalues, with the unique $\overline{\mathbb{Q}}$ conjugacy class $\{1\}$. Thus, $\gamma_L \in Z_L(\mathbb{Q})$. Finally, $Z_L(\mathbb{Q})_{tor}$ is finite (consider a splitting field of Z_L). Since it is enough to consider the finitely many standard parabolic groups P and for each finitely many cosets k, shrinking K therefore allows us to assume $Z_L(\mathbb{Q})_{tor} \cap (K_k \cap P)_L = \{1\}$ for the finitely many relevant cases.

2.3 Fixed Points

Now we want to determine the fixed points $\overline{x}$ of the Hecke correspondence $T(g^{-1})$ in the reductive Borel–Serre compactification $(S^G_{K'})^+$. They are described by the equations $c_1(\overline{x}) = T(1)\overline{x}$ and $c_2(\overline{x}) = T(g^{-1})\overline{x}$ in $(S^G_K)^+$. The unique component h containing $\overline{x}$ is necessarily a fixed component. h is contained in a stratum S^P_K. Now fix the standard parabolic $P = L_P N_P$, or $P = LN$ for short.

Suppose $\overline{x} \in S^P_{K'}$ is represented by $x = x_\infty x_{fin} \in L_\infty \times G(\mathbb{A}_{fin})$. Then $\overline{x}$ is a fixed point of $T(g^{-1})$ if and only if $xg^{-1} = \gamma \cdot x \cdot k$ holds for some $\gamma \in P(\mathbb{Q})$ and $k \in \tilde{K}_{L,\infty} K$, or equivalently if and only if

$$x^{-1}\gamma x \in g^{-1} K \tilde{K}_{L,\infty}.$$

We may replace x by another representative $\delta^{-1}x$, $\delta \in P(\mathbb{Q})$. Then instead of γ its conjugate $\delta\gamma\delta^{-1}$ appears in the fixed-point equation. Moreover

Lemma 2.4. *The element γ is semisimple and $\mathbb{R}$-elliptic. For small K the $L(\mathbb{Q})$-conjugacy class of the image γ_L of γ in $L(\mathbb{Q})$ is uniquely determined by the fixed point $\overline{x} \in S^P_{K'}$.*

Proof. The equation $x_\infty^{-1}\gamma_L x_\infty \in \tilde{K}_{L,\infty}$ implies that $\gamma_L \in L(\mathbb{Q})$ is $\mathbb{R}$-elliptic, hence semisimple. Now choose an equivalent representative $\delta n x k'$ for $\overline{x}$ for some $n \in N(\mathbb{A}_{fin}), \delta \in P(\mathbb{Q}), k' \in \tilde{K}_{L,\infty}K'$. Suppose $xg^{-1} = \gamma_1 x k_1$ and $(\delta n x k')g^{-1} = \gamma_2(\delta n x k')k_2$ holds for $\gamma_i \in P(\mathbb{Q})$, $k_i \in \tilde{K}_{L,\infty}K$. Replacing k_2 by $k' k_2 (g k' g^{-1})^{-1}$ allows us to assume $k' = 1$. Replacing γ_2 by $\delta^{-1}\gamma_2\delta$ allows us to assume $\delta = 1$. Hence, $\gamma_1 x k_1 = x g^{-1} = n^{-1}\gamma_2 n x k_2$. Since K is small, this implies $\gamma_L \in Z_L(\mathbb{Q})$ for $\gamma = \gamma_1^{-1}\gamma_2$ and hence γ commutes with x_∞, which then implies $\gamma_L \in K_\infty K_L$, where K_L is the image of $K \cap P(\mathbb{A}_{fin})$ in $L(\mathbb{A}_{fin})$. Thus, $\gamma_L \in Z_L(\mathbb{Q}) \cap \tilde{K}_{L,\infty}K_L$. Looking at the Archimedean place and the non-Archimedean places separately, this forces γ_L to be a torsion element. Therefore, $\gamma_L = 1$, since K is small.

This lemma gives a decomposition of the fixed-point set in the stratum $S^P_{K'}$ according to the conjugacy classes $\gamma_L \in L(\mathbb{Q})/\sim$. □

Fixing $\gamma_L/\sim$. We want to determine the set $Fix(\gamma_L)$ of all fixed points $\overline{x} \in S^P_{K'}$ of $T(g^{-1})$, where in the fixed-point equation for some representative an element γ appears whose projection to $L(\mathbb{Q})$ belongs to the fixed conjugacy class $\gamma_L/\sim$. To unburden the notation we also write $Fix(\gamma)$ instead of $Fix(\gamma_L)$,

$$Fix(\gamma) \;\subseteq\; S^P_{K'} \;=\; L_P(\mathbb{Q}) \setminus [(L_\infty/\tilde{K}_{L,\infty}) \times (N(\mathbb{A}_{fin}) \setminus G(\mathbb{A}_{fin})/K')].$$

For $x = x_\infty x_{fin} \in L_\infty \times G(\mathbb{A}_{fin})$ the double coset $\overline{x} = L(\mathbb{Q})N(\mathbb{A}_{fin})x\tilde{K}_{L,\infty}K'$ is in $Fix(\gamma)$ if and only if there exist $n \in N(\mathbb{A}_f), \delta \in P(\mathbb{Q}), k \in \tilde{K}_{L,\infty}K, \gamma' \in N(\mathbb{Q})$ such that $n(\delta\gamma'\gamma\delta^{-1})xk = xg^{-1}$ holds, or equivalently if and only if there exist $n \in N(\mathbb{A}_f), \delta \in P(\mathbb{Q}), k \in \tilde{K}_{L,\infty}K$, such that

$$(*) \qquad x^{-1} n \delta\gamma\delta^{-1} x \in g^{-1}\tilde{K}_{L,\infty}K,$$

since we are free to replace n by $n\delta\gamma'\delta^{-1}$.

By abuse of notation we do not distinguish between global elements γ, δ in $G(\mathbb{Q})$ and their images in G_v or $G(\mathbb{A}_{fin})$. Since x is considered in $S^P_{K'} = P(\mathbb{Q}) \setminus [X_L \times G(\mathbb{A}_{fin})]/K'$, we may replace x by $\delta^{-1}x$ and n by $\delta^{-1}n\delta$, which simplifies the equations for $Fix(\gamma)$. Hence, we get

Lemma 2.5. *We have* $Fix(\gamma) \cong L(\mathbb{Q}) \setminus \Big(L(\mathbb{Q}) \cdot \prod_v Sol_v(\gamma)\Big)$, *where*

$$Sol_v(\gamma) = \{x_v \in N(\mathbb{Q}_v) \setminus G(\mathbb{Q}_v)/K'_v \mid \ x_v^{-1} n\gamma x_v \in g_v^{-1} K_v \text{ holds for some } n \in N_v\}$$

at the non-Archimedean places v *and* $K'_v = K_v \cap g_v^{-1} K_v g_v$, *and where*

$$Sol_\infty(\gamma) = \{x_\infty \in L_\infty(\mathbb{R})/\tilde{K}_{L,\infty} \mid x_\infty^{-1}\gamma_L x_\infty \in \tilde{K}_{L,\infty}\ \}$$

at the Archimedean place ∞.

By abuse of notation we write L_γ for the centralizer L_{γ_L} of the element γ_L in L, which is a connected reductive group by the assumption that G_{der} is simply connected.

Corollary 2.1. *For small* K *we obtain* $Fix(\gamma) = L_\gamma(\mathbb{Q}) \setminus Sol(\gamma)$ *for* $Sol(\gamma) = \prod_v Sol_v(\gamma)$.

Proof. $L(\mathbb{Q})$-equivalent solutions in $Sol(\gamma)$, say, $x^{-1}n_1\gamma x$ and $x^{-1}n_2\delta^{-1}\gamma\delta x$ in $g^{-1}K\tilde{K}_\infty$ for suitable $n_1, n_2 \in N(\mathbb{A}_{fin})$ and $\delta \in P(\mathbb{Q})$, satisfy $x^{-1}n_2\delta^{-1}\gamma\delta\gamma^{-1} n_1 x \in K\tilde{K}_\infty$ for some $n \in N(\mathbb{A})$. We may then assume $n_1 = 1$, and since K is small, this implies $\delta_L^{-1}\gamma_L\delta_L\gamma_L^{-1} \in Z_L(\mathbb{Q})$. Since the commutator $\delta_L^{-1}\gamma_L\delta_L\gamma_L^{-1}$ is in $L_{der}(\mathbb{Q})$, and since $Z_L(\mathbb{Q}) \cap L_{der}(\mathbb{Q})$ is finite, the commutator is a torsion element, and hence is 1 since K is small. This implies $\delta_L \in L_\gamma(\mathbb{Q})$ and completes the proof. $\square$

2.3.1 Archimedean Place

$Sol_\infty(\gamma) \cong L_{\gamma,\infty}/(L_{\gamma,\infty} \cap \tilde{K}_{L,\infty})$ by the corollary in Appendix 2, unless it is empty. If it is nonempty, the Archimedean fixed-point condition shows that γ_L is L_∞-conjugate to a point in $\tilde{K}_{L,\infty}$. To determine $L_{\gamma,\infty}$ we may therefore assume $\gamma_L \in \tilde{K}_{L,\infty}$ without restriction of generality. Hence, the centralizer $L_{\gamma,\infty}$ becomes θ-stable for the Cartan involution θ (see Appendix 2). Therefore, $K_\infty \cap L_{\gamma,\infty}$ is a maximal compact subgroup $K_{L_{\gamma,\infty}}$ of $L_{\gamma,\infty}$. Since $A_L(\mathbb{R})^0 \subseteq L_{\gamma,\infty}$, $Sol_\infty(\gamma) = L_{\gamma,\infty}/(K_{L_{\gamma,\infty}}A_L(\mathbb{R})^0)$ admits a smooth surjective map to the symmetric space $X_{L_\gamma} = L_{\gamma,\infty}/(K_{L_{\gamma,\infty}}A_{L_\gamma}(\mathbb{R})^0)$ of the centralizer L_γ, which defines a trivial fibration by the Euclidean space $A_{L_\gamma}(\mathbb{R})^0/A_L(\mathbb{R})^0$, and hence a homotopy equivalence. See the Remark 2.15 in Appendix 2.

2.3.2 Non-Archimedean Places

Recall that $T(g^{-1})$ and $\gamma/\sim$ are now fixed. Let Ω_v be the stabilizer of a special point in the Bruhat–Tits building. Special points always exist, and Ω_v is a maximal compact subgroup of G_v. We now assume $K_v \subseteq \Omega_v$; hence, $K'_v \subseteq \Omega_v$. Then by the Iwasawa decomposition $G_v = P_v \cdot \Omega_v$ (see [103], Sect. 3.3.2). For $k \in \Omega_v$ put $g_k^{-1} = kg^{-1}k^{-1}$, $K_k = kK_vk^{-1}$, and $K'_k = kK'_vk^{-1}$. Elements $x_v \in Sol_v(\gamma)$ may be written $x_v = p \cdot k$ for $p \in P_v$ and $k \in \Omega_v$. The coset $(P_v \cap \Omega_v)kK'_v$ is uniquely determined by x_v, and $G/K' = \bigcup_{k \in P_v \cap \Omega_v \backslash \Omega_v / K'_v} P_v/(P_v \cap kK'_vk^{-1})$. Therefore,

$$Sol_v(\gamma) = \bigcup_{k \in P_v \cap \Omega_v \backslash \Omega_v / K'_v} Sol_v(\gamma, k).$$

Here $Sol_v(\gamma, k) = \{p \in N_v \setminus P_v/(P_v \cap kK'_vk^{-1}) \mid p_L^{-1}\gamma_L p_L \in (g_k^{-1}K_k \cap P)_L\}$ or

$$Sol_v(\gamma, k) \cong \mathcal{S}_v(\gamma, k)/K'(k)_v$$

for $K'(k)_v := (P_v \cap kK'_vk^{-1})_L$ and

$$\mathcal{S}_v(\gamma, k) \;=\; \left\{ m \in L_v \;\middle|\; m^{-1}\gamma_L m \in (g_k^{-1}K_k \cap P_v)_L \right\} \;=\; \biguplus_{\xi_v} (L_\gamma)_v \cdot \xi_v \cdot K'(k)_v.$$

This is a finite (possibly empty) union over representatives $\xi_v \in L_v$. From [53], Propositions 7.1 and 8.2, there is only one representative $\xi_v = 1$ for almost all v.

2.3.3 Globally

With this notation

$$Sol(\gamma, k) = \mathcal{S}(\gamma, k)/K'(k)_{\mathbb{A}} \quad \text{for} \quad K'_{\mathbb{A}}(k) = \tilde{K}_{L,\infty} \prod_{v\, fin} K'(k)_v,$$

where $\mathcal{S}(\gamma, k) = \left\{ m \in L(\mathbb{A}) \mid m^{-1}\gamma_L m \in \tilde{K}_{L,\infty}\big(g_k^{-1}K_k \cap P(\mathbb{A}_{fin})\big)_L \right\}$. $L_\gamma(\mathbb{A})$ acts on $\mathcal{S}(\gamma, k)$ from the left. Choose a decomposition

$$\mathcal{S}(\gamma, k) = \biguplus_{\xi} L_\gamma(\mathbb{A}) \cdot \xi \cdot K'(k)_{\mathbb{A}}$$

with representatives $\xi \in L(\mathbb{A})$, where representatives $\xi = \prod_v \xi_v$ are chosen to be products of corresponding local non-Archimedean representatives ξ_v for $L_\gamma(\mathbb{Q}_v) \setminus \mathcal{S}_v(\gamma, k)/K'(k)_v$, and $\xi_\infty = 1$. Then

Lemma 2.6. *For small K the contribution of a fixed conjugacy class $\gamma/\sim$ in $L(\mathbb{Q})$ to the fixed-point locus of $T(g^{-1})$ is*

$$Fix(\gamma) \cong \biguplus_{k\in P(\mathbb{A}_{fin})\cap\Omega\backslash\Omega/K'} Fix(\gamma,k),$$

$$Fix(\gamma,k) \ \cong \ \biguplus_{\xi} L_\gamma(\mathbb{Q}) \setminus L_\gamma(\mathbb{A})/(\xi K'(k)_{\mathbb{A}}\xi^{-1}\cap L_\gamma(\mathbb{A})).$$

Of course $L_\gamma(\mathbb{Q}) \setminus L_\gamma(\mathbb{A})/(\xi K'(k)_{\mathbb{A}}\xi^{-1}\cap L_\gamma(\mathbb{A})) = \biguplus F_\nu$ is a finite union of arithmetic quotients F_ν.

2.4 Lefschetz Numbers

The Lefschetz number becomes

$$\sum_P \sum_{\gamma/\sim} \sum_k \sum_\xi \sum_\nu \ LC(F_\nu),$$

where $k \in P\cap\Omega\setminus\Omega/K'$. Put $F = F_\nu$. For the local terms $LC(F)$ Goresky and MacPherson gave an explicit description as a product $\chi(F)r(\gamma_F)\Psi(\gamma_F,\lambda)$ if γ_F is P-contractive, and it vanishes otherwise. See [33], pp. 470–471 and Theorem (version 3a), p. 474. Here $\gamma_F = \gamma^{-1}$ is the characteristic element defined in [33], p. 469, which is the inverse of the element γ defined in Lemma 2.4. Hence, if it is nonvanishing, the local number $LC(F)$ is the product of:

- The Euler characteristic $\chi(F)$
- $|\gamma_F|_\infty^{\rho_P}\cdot r(\gamma_F,k)$
- $|\gamma_F|_\infty^{-\rho_P}\cdot\Psi(\gamma_F,\lambda) = \sum_{w\in W^P}(-1)^{l(w)}\psi_{w(\lambda+\rho_G)-\rho_L}(\gamma_F^{-1})$

2.4.1 Euler Characteristics

We may sum the terms $\sum_\nu \chi(F_\nu)$ for fixed $P,\gamma/\sim,k,\xi$, which gives the Euler characteristic

$$\chi(L_\gamma(\mathbb{Q})\setminus L_\gamma(\mathbb{A})/(\xi K'(k)_{\mathbb{A}}\xi^{-1}\cap L_\gamma(\mathbb{A}))).$$

To compute it we may replace $L_{\gamma,\infty}/\tilde{K}_{L,\infty}$ by $X_{L_\gamma} = L_{\gamma,\infty}/\tilde{K}_{L_\gamma,\infty}$. See page 44. Notice $L_\gamma(\mathbb{Q})\cap\xi\tilde{K}_{L_\gamma,\infty}K'(k)_{\mathbb{A}}\xi^{-1}$ is contained in the center of L_γ, since K is small. Hence, the intersection is discrete and compact, and hence we have a finite group. By our assumption K is small; hence, the intersection is trivial. Thus, we obtain

$$\sum_{\nu} \chi(F_\nu) \;=\; \frac{\chi(L_\gamma, dg_{fin})}{vol_{dg_{fin}}(\xi K'(k)_{\mathbb{A}_{fin}} \xi^{-1} \cap L_\gamma(\mathbb{A}_{fin}))}$$

for a constant $\chi(L_\gamma) = \chi(L_\gamma, dg_{fin})$ depending only on L_γ and on the choice of the Haar measure dg_{fin} on $L_\gamma(\mathbb{A}_{fin})$.

Remark 2.4. Observe that γ is $\mathbb{R}$-elliptic, and hence is $\mathbb{Q}$-elliptic. For the ambiguity of this notion, see [44], p. 392. We show that in our situation this ambiguity does not cause problems, since the Euler characteristic of the corresponding summands in the trace formula vanishes unless both notions agree. Consider the group L or better L/A_L. The center $Z(L_\gamma)$ is $\mathbb{Q}$-anisotropic modulo A_L. If the quotient were not anisotropic over $\mathbb{R}$, the corresponding global quotient space X would be a nontrivial torus fibration, whose Euler characteristic would therefore vanish. Similarly the Euler characteristic vanishes for locally symmetric arithmetic quotients of semisimple groups unless the $\mathbb{R}$-rank of the maximal compact subgroup equals the $\mathbb{R}$-rank of the group. Considering the map $L_{der} \to L$, we can assume that $(L_\gamma/A_L)(\mathbb{R})$ contains an $\mathbb{R}$-anisotropic torus of maximal rank, or otherwise the Euler characteristic vanishes and the corresponding summand does not contribute to the trace formula.

Definition 2.1. Call $\gamma \in L(\mathbb{Q})$ *strongly elliptic* if γ is $L(\mathbb{R})$-conjugate to an element in $K_{L,\infty} \cdot A_L(\mathbb{R})^0$ such that the Euler characteristic $\chi(L_\gamma)$ does not vanish.

Remark 2.5. For connected reductive groups L over $\mathbb{Q}$, for which the connected component of the center modulo A_L is anisotropic over $\mathbb{R}$, one also wants to compare $\chi(L, dg_{fin})$ with the Tamagawa number. At the moment we do not need to carry through this comparsion. When we need it later, it can be obtained directly from a comparison between the topological L^2-trace formula and Arthur's L^2-trace formula. On the other hand, it should not be difficult to obtain it by reduction to the case of semisimple groups (Harder's theorem [37]) adapting the argument of [68], pp. 129–131, with a z-extension $T' \to L^* \to L$ replacing the sequence (V), and $\tilde{L} = (L^*)_{der} \to L^* \to T$ replacing the sequence (H) in [68].

Only (semisimple) strongly elliptic elements γ contribute to the Lefschetz number. Let χ_P^G be the characteristic function of the P-contractive elements. We obtain for the Lefschetz number the expression

$$\sum_P \sum_{\gamma/\sim} \chi(L_\gamma, dg_{fin}) \chi_P^G(\gamma_\infty^{-1}) \sum_{w \in W^P} (-1)^{l(w)} \psi_{w(\lambda+\rho_G)-\rho_L}(\gamma) \cdot O_\gamma$$

$$= \sum_P \sum_{\gamma/\sim} \chi(L_\gamma, dg_{fin}) \chi_P^G(\gamma_\infty) \sum_{w \in W^P} (-1)^{l(w)} \psi_{w(\lambda+\rho_G)-\rho_L}(\gamma^{-1}) \cdot O_{\gamma^{-1}},$$

where

$$O_\gamma = \sum_k \sum_\xi \frac{|\gamma^{-1}|_\infty^{\rho_P} \cdot r(\gamma^{-1}, k)}{vol_{dg_{fin}}(\xi K'(k)_{\mathbb{A}_{fin}} \xi^{-1} \cap L_\gamma(\mathbb{A}_{fin}))}.$$

Here we used $r(\gamma, k) = r(\delta\gamma\delta^{-1}, k)$ for $\delta \in L(\mathbb{Q})$. To show this, recall $k = x_{fin}$ describes the component h of the P-stratum, which contains F. Recall $r(\gamma, x_{fin}) = r(\delta\gamma\delta^{-1}, \delta x_{fin})$ for $\delta \in P(\mathbb{Q})$. Since $r(\gamma, x_{fin})$ depends only on P and the coset $P(\mathbb{A}_{fin})x_{fin}$ (Lemmas 2.1 and 2.3), replacing γ by a conjugate does not change $r(\gamma, k)$.

2.4.2 Computation of O_γ

Notice $|\gamma^{-1}|_\infty^{\rho_P} r(\gamma^{-1}) = \prod_{v\neq\infty} |\gamma|_v^{\rho_P}[N_v \cap K_k : N_v \cap K_v']$ by Lemma 2.3. Since

$$\sum_{k\in(P_v\cap\Omega_v)\backslash\Omega_v/K_v'} f(k) = \sum_{k\in\Omega_v/K_v'} \frac{f(k)}{[(P_v\cap\Omega_v):(P_v\cap K_k')]},$$

this allows us to write O_γ as a product $\prod_{v\neq\infty} O_{\gamma,v}$ of non-Archimedean local terms

$$O_{\gamma,v} = \sum_{k\in\Omega_v/K_v'} \sum_{\xi_v} \frac{|\gamma|_v^{\rho_P}\cdot[N_v\cap K_k : N_v\cap K_v']}{[(P_v\cap\Omega_v):(P_v\cap K_k')]\cdot vol_{dg_v}(\xi_v K'(k)_v \xi_v^{-1}\cap L_{\gamma,v})}.$$

Since

$$0 \to N_v \cap K_k' \to P_v \cap K_k' \to K'(k)_v \to 0$$

is exact, this gives

$$O_{\gamma,v} = \sum_{k\in\Omega_v/K_v'} \sum_{\xi_v} \frac{|\gamma|_v^{\rho_P}\cdot vol_{N_v}(N_v\cap K_k)\cdot vol_{L_v}(K'(k)_v)}{vol_{dg_v}(\xi_v K'(k)_v \xi_v^{-1}\cap L_{\gamma,v})},$$

where measures are normalized such that $vol(\Omega_v \cap P_v) = 1$ and $vol(\Omega_v \cap N_v) = 1$. In Sect. 2.5 we show that this expresses O_γ as an orbital integral

$$O_\gamma \;=\; O_\gamma^L(\overline{f}^{(P)})$$

of the characteristic function f of the set $Kg^{-1}K$ up to a normalization factor.

2.4.3 Conclusion

The computations in Sects. 2.4.1 and 2.4.2 describe the right action of $1_{KgK}/vol_\Omega(K)$ on the cohomology. Any K-bi-invariant function f is a linear combination of functions f as above. However, we should keep in mind that so far we have used a left action of $G(\mathbb{A}_{fin})$ on S^G, where $g \in G(\mathbb{A}_{fin})$ acts by the formula on page 23; hence, the cohomology becomes a right module under the Hecke algebra.

Theorem 2.1. *Assume the derived group of G is simply connected and K is small. Then the Lefschetz number of the right action of a K-bi-invariant Hecke operator $f \in C_c^\infty(G(\mathbb{A}_{fin}))$ on the cohomology $H^\bullet(S_K(G), V_\lambda)$ is given by*

$$L(f, V_\lambda) = \sum_P \sum_{\gamma \in L(\mathbb{Q})/\sim} \chi(L_\gamma, dg_{fin}) O_\gamma^L(\overline{f}^{(P)}) \cdot |\gamma|_\infty^{-\rho_P} \Psi(\gamma, \lambda).$$

The sum extends over all standard $\mathbb{Q}$-parabolic subgroups $P = LN$ containing the fixed minimal $\mathbb{Q}$-parabolic P_0 and all $L(\mathbb{Q})$-conjugacy classes $\gamma \in L(\mathbb{Q})/\sim$ of semisimple, strongly elliptic elements in $L(\mathbb{Q})$ with P-contractive representatives.

Example 2.2. $G = \mathbb{G}_m$ for the representation $x \mapsto x^r$ of weight $\lambda = r$ on $V = \mathbb{C}$, and K maximal compact. Then S_K is a single point. The element $g = (g_v) \in \mathbb{A}_{fin}^*$, where $g_p = p$ and $g_v = 1$ for $v \neq p$, acts on V_λ by $1 \times 1 \mapsto 1 \times g^{-1} \simeq p^r \times 1$ in $V \times \mathbb{A}_{fin}^*$. It acts on the cohomology via multiplication by p^{-r} (right action on cohomology) or p^r (left action on cohomology).

Remark 2.6. Notice we used

$$O_{\gamma^{-1}}^L(\overline{f}^{(P)}) = O_\gamma^L(\overline{f^-}^{(P)}).$$

For the comparison of trace formulas with those in [64] in Chap. 3, we may turn the right action of the Hecke algebra on the cohomology groups into a left action by the substitution $f(x) \mapsto f^-(x) = f(x^{-1})$. This makes the formula compatible with that in [64], p. 197.

Remark 2.7. The factor $\chi(L_\gamma, dg_{fin}) O_\gamma^L(.)$ does not depend on the choice of the fixed Haar measure dg_{fin} on $L_\gamma(\mathbb{A}_{fin})$; therefore, we do not mention the choice of dg_{fin} in the following.

Remark 2.8. The condition imposed in Theorem 2.1 that $\gamma \in L(\mathbb{Q})/\sim$ contains a P-contractive representative $\gamma \in P(\mathbb{Q})$ can be replaced by the stronger condition that $|\alpha(\gamma)|_\infty \geq 1$ holds for all positive roots α of G and $|\alpha(\gamma)|_\infty = 1$ holds if and only if α is a root from L as explained after the definition of contractiveness. Of course it is enough to consider $\mathbb{Q}$-roots, since G and P are defined over $\mathbb{Q}$ and γ is a $\mathbb{Q}$-rational element. Therefore, the condition in Theorem 2.1 can be replaced by the condition $|\alpha(\gamma)|_{fin} \leq 1$ holds for all positive $\mathbb{Q}$-roots α and $|\alpha(\gamma)|_{fin} = 1$ holds if and only if α is a root from L.

Remark 2.9. For a standard $\mathbb{Q}$-parabolic group $P \supseteq P_0$ with Levi decomposition $P = LN$ let $X^*(P)_\mathbb{Q} = X^*(L)_\mathbb{Q} = Hom_{\mathbb{Q}-alg}(L, \mathbb{G}_m)$ be the group of characters defined over $\mathbb{Q}$. Then $\mathcal{X}_L = Hom(X^*(L)_\mathbb{Q}, \mathbb{R})$ can be canonically identified with the Lie algebra of A_L, and hence with $A_L(\mathbb{R})^0$ by the exponential map. One defines the Harish-Chandra homomorphism

$$H_P : L(\mathbb{A}) \to \mathcal{X}_L$$

by $exp(\langle H_P(l), \chi\rangle) = \|\chi(l)\|$, where $\|.\| : \mathbb{A}^* \to \mathbb{R}^*$ is the idele norm, and $\chi \in X^*(L)_{\mathbb{Q}}$. For the minimal $\mathbb{Q}$-parabolic $P = P_0$ we write $\mathcal{X}_P = \mathcal{X}$. Let Δ be a basis of the simple $\mathbb{Q}$-roots. Then the standard $\mathbb{Q}$-parabolic subgroups $P = P_\theta$ correspond uniquely to the subsets $\theta \subseteq \Delta$. The roots in F are the roots of the Levi component L_θ, and the simple roots in the Lie algebra of the unipotent radical N_θ are the roots in $\Delta \setminus F$. In fact, since γ is strongly elliptic, the condition for $\gamma \in L(\mathbb{Q}) = L_\theta(\mathbb{Q})$, given in Remark 2.8, could also be replaced by the condition

$$|\alpha(H_P(\gamma))|_{fin} < 1$$

for all simple $\mathbb{Q}$-roots α not in θ.

The Decomposition $\mathcal{X} = \mathcal{X}_L \oplus \mathcal{X}_L^\perp$ [1]. Let β_j denote the dual roots such that $\langle \beta_j, \alpha_i\rangle = \delta_{ij}$, both considered as elements of $\mathcal{X}$. Then there exists a natural orthogonal decomposition $\mathcal{X} = \mathcal{X}_L \oplus \mathcal{X}_L^\perp$ such that $\mathcal{X}_L$ is the span $\sum_{j\notin F} \mathbb{R}\beta_j$ and $\mathcal{X}_L^\perp = \sum_{i\in F} \mathbb{R}\alpha_i$. The projection $pr_L : \mathcal{X} \to \mathcal{X}_L$ is $pr(\sum_{j\notin F} x_j\beta_j + \sum_{i\in F} y_i\alpha_i) = \sum_{j\notin F} x_j\beta_j$. The image under pr_L of the open positive Weyl chamber $\mathcal{X}^+ = \sum_{j\in\Delta} \mathbb{R}_{>0}\beta_j \subseteq \mathcal{X}$ defines the open Weyl chamber $\mathcal{X}_M^+$ in $\mathcal{X}_M$; the image of the obtuse Weyl chamber ${}^+\mathcal{X} = \sum_i \mathbb{R}_{>0}\alpha_i \subseteq \mathcal{X}$ defines the obtuse open Weyl chamber in $\mathcal{X}_L$. Obviously ${}^+\mathcal{X}_L = \sum_{j\notin F} \mathbb{R}_{>0}\alpha_j$. Then $\mathcal{X}_L^+ = pr(\mathcal{X}^+) = \sum_{j\notin F} \mathbb{R}_{>0}\beta_j$, since $\langle\alpha_i, \alpha_j\rangle \leq 0$ for $i \neq j$ and $\langle\beta_i, \beta_j\rangle \geq 0$. In fact $\sum_{j\notin F} x_j\beta_j + \sum_{i\in F} y_i\alpha_i \in \sum_{i\in F} \mathbb{R}_{>0}\beta_i$ therefore implies $y_i \geq 0, i \in F$; hence, $x_j > 0, j \notin F$. Also $\mathcal{X}^+ \subseteq {}^+\mathcal{X}$; therefore, $\mathcal{X}_L^+ \subseteq {}^+\mathcal{X}_L$. Finally notice $\mathcal{X}^+ \cap -\overline{{}^+\mathcal{X}} = \{0\}$.

2.5 Computation of an Orbital Integral

We write the terms O_γ in the formula for the Lefschetz numbers as an orbital integral $O_\gamma^L(\overline{f}^{(P)})$. This is done in steps 1–3. The final result is formulated in step 4.

Step 1. Assume measures are normalized by $vol_G(\Omega) = 1$. Recall $K' = K \cap g^{-1}Kg$ and $g \in G(\mathbb{A}_{fin})$ is fixed. The characteristic function $1_{g^{-1}K}(y)$ of the set $g^{-1}K$ is then K'-bi-invariant. Furthermore, $k^{-1}xk \in g^{-1}K \iff x \in kg^{-1}k^{-1}kKk^{-1} =: g_k^{-1}K_k$. Hence,

$$\int_\Omega 1_{g^{-1}K}(k^{-1}xk)dk = [\Omega : K']^{-1} \cdot \sum_{k\in\Omega/K'} 1_{g^{-1}K}(k^{-1}xk)$$
$$= vol_\Omega(K') \cdot \sum_{k\in\Omega/K'} 1_{g_k^{-1}K_k}(x).$$

$\int_K 1_{g^{-1}K}(k^{-1}xk)dk = \int_K 1_{g^{-1}K}(k^{-1}x)dk = vol_\Omega(g^{-1}Kg \cap K)1_{Kg^{-1}K}(x) = vol_\Omega(K')1_{Kg^{-1}K}(x)$ holds for $x \in G(\mathbb{A}_{fin})$. Hence, $\int_\Omega = vol_\Omega(K)^{-1} \int_\Omega \int_K$ implies

$$\int_{\Omega} 1_{g^{-1}K}(k^{-1}xk)dk = vol_{\Omega}(K)^{-1} \int_{\Omega} vol_{\Omega}(K')1_{Kg^{-1}K}(k^{-1}xk)dk.$$

Comparison of the right sides thus gives for the Ω-average

Definition 2.2. $\overline{f}(x) = \int_{\Omega} f(k^{-1}xk)dk, \qquad x \in G(\mathbb{A}_{fin})$, of the normalized characteristic function.

Definition 2.3. $\boxed{f(x) = vol_{\Omega}(K)^{-1}1_{Kg^{-1}K}(x)}$.

Lemma 2.7. $\overline{f}(x) = \sum_{k\in\Omega/K'} 1_{g_k^{-1}K_k}(x)$.

Step 2. For $x \in P(\mathbb{A}_{fin})$ in a standard parabolic subgroup $P = LN$ by Lemma 2.7

$$\int_{N(\mathbb{A}_{fin})} \overline{f}(xn)dn = \sum_{k\in\Omega/K'} \int_{N(\mathbb{A}_{fin})} 1_{g_k^{-1}K_k}(xn)dn.$$

If x_L is not in $(g_k^{-1}K_k \cap P(\mathbb{A}_{fin}))_L$, the corresponding integral on the right side is zero. Otherwise $xn_0 = g_k^{-1}k_0$ holds for some $n_0 \in N(\mathbb{A}_{fin})$ and $k_0 \in K_k$, and in this case the integral becomes $\int_{N(\mathbb{A}_{fin})} 1_{g_k^{-1}K_k}(g_k^{-1}k_0n)dn = vol(N(\mathbb{A}_{fin})\cap K_k)$. Hence, for $x \in L(\mathbb{A}_{fin})$ we get

Lemma 2.8. $\phi(x) := \int_{N(\mathbb{A}_{fin})} \overline{f}(xn)dn = \sum_{k\in\Omega/K'} vol\big(N(\mathbb{A}_{fin}) \cap K_k\big)\cdot 1_{(g_k^{-1}K_k\cap P(\mathbb{A}_{fin}))_L}(x)$.

Step 3. Next consider the orbital integral of the function ϕ defined on $L(\mathbb{A}_{fin})$

$$O_{\gamma}^{L}(\phi) = \int_{L_{\gamma}(\mathbb{A}_{fin})\backslash L(\mathbb{A}_{fin})} \phi(m^{-1}\gamma m)dm.$$

By the definition of ϕ the value of $O_{\gamma}^{L}(\phi)$ is

$$\sum_{k\in\Omega/K'} vol\big(N(\mathbb{A}_{fin})\cap K_k\big)\cdot \int_{L_{\gamma}(\mathbb{A}_{fin})\backslash L(\mathbb{A}_{fin})}$$
$$char\Big\{m \;\Big|\; m^{-1}\gamma_L m \in \big(g_k^{-1}K_k \cap P(\mathbb{A}_{fin})\big)_L\Big\}dm,$$

or by Sect. 6.16 and the decomposition $\mathcal{S}(\gamma,k) = \bigcup L_{\gamma}(\mathbb{A}_{fin})\cdot\xi_{fin}\cdot K'(k)_{\mathbb{A}_{fin}}$

$$\sum_{k\in\Omega/K'} vol\big(N(\mathbb{A}_{fin})\cap K_k\big)\cdot\sum_{\xi_{fin}} \frac{vol_{L(\mathbb{A}_{fin})}\big(K'(k)_{\mathbb{A}_{fin}}\big)}{vol_{L_{\gamma}(\mathbb{A}_{fin})}\big(\xi_{fin}K'(k)_{\mathbb{A}_{fin}}\xi_{fin}^{-1}\cap L_{\gamma}(\mathbb{A}_{fin})\big)}.$$

Step 4. To put things together. The function

$$\overline{f}(x) = \int_{\Omega} f(k^{-1}xk)dk, \qquad f(x) = vol_{\Omega}(K)^{-1}1_{Kg^{-1}K}(x)$$

is K-bi-invariant on $G(\mathbb{A}_{fin})$. Define

$$\overline{f}^{(P)}(m) = |m|_{fin}^{\rho_P} \int_{N(\mathbb{A}_{fin})} \overline{f}(mn)dn, \qquad m \in L(\mathbb{A}_{fin})$$

$$O_\gamma^L(\overline{f}^{(P)}) = \int_{L_\gamma(\mathbb{A}_{fin})\backslash L(\mathbb{A}_{fin})} \overline{f}^{(P)}(m^{-1}\gamma m)dm,$$

assuming that the integrals are normalized by the conventions $vol_{P(\mathbb{A}_{fin})}\big(\Omega \cap P(\mathbb{A}_{fin})\big) = 1$, $vol(\Omega) = 1$, and $vol_{N(\mathbb{A}_{fin})}\big(\Omega \cap N(\mathbb{A}_{fin})\big) = 1$. See also [16], p. 144. The measure on $L_\gamma(\mathbb{A}_{fin})$ is dg_{fin}. Then the computation above proves that

$$O_\gamma^L(\overline{f}^{(P)}) \;=\; O_\gamma.$$

2.6 Elliptic Traces

Recall G is a connected reductive group over $\mathbb{Q}$ whose derived group is simply connected. Define elliptic "traces"

$$\boxed{T_{ell}^G(f,\tau) = \sum_{\gamma \in G(\mathbb{Q})/\sim} \chi(G_\gamma) O_\gamma^G(f) \cdot tr\big(\tau(\gamma^{-1})\big)}$$

for a finite-dimensional complex representations τ of $G(\mathbb{Q})$ and $f \in C_c^\infty(G(\mathbb{A}_{fin}))$. If τ is an irreducible complex representation defined by a highest weight λ, we also write $T_{ell}^G(f,\lambda)$ instead of $T_{ell}^G(f,\tau)$. Hence, we do not distinguish between representations and their highest weights. The sum defining $T_{ell}^G(f,\tau)$ extends over the $G(\mathbb{Q})$-conjugacy classes of semisimple, strongly elliptic elements in $G(\mathbb{Q})$. The integrals $O_\gamma^G(f)$ in this sum are orbital integrals with respect to the group of finite adeles for functions $f \in C_c^\infty(G(\mathbb{A}_{fin}))$. The same definition defines elliptic traces T_{ell}^L for the Levi subgroups L of all standard $\mathbb{Q}$-parabolic subgroups $P = LN_P$ of G.

Let $\chi_P^G = \tau_P^G \circ H_P$ be defined by the characteristic function τ_P^G of the open positive Weyl chamber of $\mathcal{X}_L = X_*(A_L)_\mathbb{Q} \otimes \mathbb{R}$, lifted to a function on $L(\mathbb{A}_{fin})$ via the Harish-Chandra homomorphism $H_L : L(\mathbb{A}_{fin}) \to \mathcal{X}_L$. Then Theorem 2.1 and the remarks following it imply

Lemma 2.9.

$$L(f, V_\lambda) = \sum_{P_0 \subseteq P \subseteq G} T_{ell}^P(\overline{f}^{(P)}\chi_P^G, \lambda),$$

where

$$T_{ell}^P(h,\lambda) = \sum_{w \in W^P} (-1)^{l(w)} \cdot T_{ell}^L(h, w(\lambda + \rho_G) - \rho_L)$$

for $h \in C_c^\infty(L(\mathbb{A}_{fin}))$, and where $P = LN_P$ runs over the $\mathbb{Q}$-rational standard parabolic subgroups of G.

Let $\hat{\chi}_P^G = \hat{\tau}_P^G \circ H_P$ be the characteristic function $\hat{\tau}_P^G$ of the open obtuse Weyl chamber in $\mathcal{X}_L$, considered as a function on $L(\mathbb{A}_{fin})$. Notice $\chi_P^G \leq \hat{\chi}_P^G$.

Lemma 2.10. *Let the situation be as in Lemma 2.9. Then*

$$T_{ell}^G(f,\lambda) = \sum_{P_0 \subseteq Q \subseteq G} (-1)^{rang(Q)-rang(G)} L^Q(\overline{f}^{(Q)} \hat{\chi}_Q^G, \lambda),$$

where the sum is over the standard $\mathbb{Q}$-parabolic subgroups of G, where $rang(Q)$ denotes the $\mathbb{Q}$-split rank of the Levi subgroup of Q, and the Lefschetz number L^Q for $h \in C_c^\infty(L(\mathbb{A}_{fin}))$ and the parabolic group Q is defined by

$$L^Q(h, V_\lambda) = \sum_{w \in W^Q} (-1)^{l(w)} \cdot L^L(h, V_{w(\lambda+\rho_G)-\rho_L}),$$

and $L^L(h,.)$ is the Lefschetz number attached to coefficient systems for the symmetric space attached to the Levi subgroup L of the (standard) parabolic subgroup Q.

Lemmas 2.9 and 2.10 were expected by Harder [39], pp. 144–145.

Proof of Lemma 2.10. Lemma 2.9 applied to the Levi subgroup L of $Q = LN$ gives

$$L^Q(\overline{f}^{(Q)} \hat{\chi}_Q^G, V_\lambda) = \sum_{w \in W^Q} (-1)^{l(w)} \cdot L^L(\overline{f}^{(Q)} \hat{\chi}_Q^G, V_{w(\lambda+\rho_G)-\rho_L})$$

$$= \sum_{w \in W^Q} (-1)^{l(w)} \sum_{P_0 \cap L \subseteq P' = L'N' \subseteq L} T_{ell}^{P'}\left(\overline{(\overline{f}^{(Q)} \hat{\chi}_Q^G)}^{(P')} \chi_{P'}^L, w(\lambda+\rho_G) - \rho_L \right)$$

$$= \sum_{P_0 \cap L \subseteq P' = L'N' \subseteq L} \; \sum_{w' \in W^{P'}} \; \sum_{w \in W^Q} (-1)^{l(w)+l(w')} \cdot$$
$$T_{ell}^{L'}\left(\overline{(\overline{f}^{(Q)} \hat{\chi}_Q^G)}^{(P')} \chi_{P'}^L, w'w(\lambda+\rho_G) - \rho_{L'} \right).$$

$P \subseteq Q$ induces the parabolic group $P' = P \cap L$ in the Levi component L of Q, and all standard $\mathbb{Q}$ parabolic groups P' are obtained in this way from the standard $\mathbb{Q}$-parabolic subgroups $P \subseteq Q$ such that the Levi components L' of P' and P coincide. Since $sn(w) = (-1)^{l(w)}$ satisfies $sn(w')sn(w) = sn(w'w)$, the inductivity $W^{P'}W^Q = W^P$ and the formula $\overline{f}^{(P)} \hat{\chi}_Q^G = \overline{(\overline{f}^{(Q)} \hat{\chi}_Q^G)}^{(P')}$ implies that the sum simplifies to

$$L^Q(\overline{f}^{(Q)} \hat{\chi}_Q^G, V_\lambda) = \sum_{P_0 \subseteq P = L'N_P \subseteq Q} T_{ell}^P(\overline{f}^{(P)} \hat{\chi}_Q^G \chi_{P'}^L, \lambda).$$

The sum is over all $\mathbb{Q}$-rational standard parabolic subgroups P of G contained in Q. Notice in the formula above $\chi_{P'}^L$ is a function on $\mathcal{X}_{L'}$, whereas $\hat{\chi}_Q^G$, which is defined as a function on $\mathcal{X}_L$, is tacitly considered as a function on $\mathcal{X}_{L'}$ via the canonical projection map $pr : \mathcal{X}_{L'} \to \mathcal{X}_L$. Summing these formulas over the standard parabolic groups Q, with the additional factors $(-1)^{rang(G)-rang(Q)}$,

$$\sum_{P_0 \subseteq Q} (-1)^{rang(Q)-rang(G)} \cdot L^Q(\overline{f}^{(Q)} \hat{\chi}_Q^G, V_\lambda),$$

gives the desired result, by interchanging the order of summation. Fixing P, the sum over all Q with $P \subseteq Q \subseteq G$ gives zero except for $P = G$. Indeed for fixed $P \subseteq G$ the sum $\sum_{P \subseteq Q \subseteq G} (-1)^{rang(G)-rang(Q)} \hat{\chi}_Q^G \chi_{L'}^L$ is zero except for $P = L'N_P = G$, where it is 1 instead. This is a well known result obtained by Arthur [1]. For the convenience of the reader we include the argument. $\square$

Proof. Let $F' \subseteq F \subseteq \Delta$ define $P' \subseteq P \subseteq G$. Since the support $Supp_F$ of the characteristic function $\hat{\tau}_Q^G \tau_{L'}^L$ of the subset $\sum_{i \notin F} \mathbb{R}_{>0}\alpha_i + \sum_{j \in F \setminus F'} \mathbb{R}_{>0}\beta_j$ of $\mathcal{X}_{L'}$ is contained in ${}^+\mathcal{X}_{L'} = \sum_{i \notin F'} \mathbb{R}_{>0}\alpha_i$ (if $F' \neq \Delta$), $Supp_F = {}^+\mathcal{X}_{L'} \cap \bigcap_{j \in F \setminus F'} \{H \mid \alpha_j(H) > 0\}$ follows as an immediate consequence of the inequalities $(\alpha_i, \alpha_j) \leq 0$ for $i \neq j$ and $(\beta_i, \beta_j) \geq 0$. For $H \in \mathcal{X}$ let Δ_H denote the set of $\alpha_i \notin F'$, for which $\langle \alpha_i, H \rangle > 0$. Δ_H is nonempty for $H \in {}^+\mathcal{X}$, since $-\overline{\mathcal{X}^+} \cap {}^+\mathcal{X} = \{0\}$. Hence, $\sum_{P \subseteq Q \subseteq G} (-1)^{rang(G)-rang(Q)} \hat{\tau}_Q^G \tau_{L'}^L(H) = 0$ follows from $\sum_{T \subseteq \Delta_H} (-1)^{|T|} = 0$. $\square$

Corollary 2.2. *The elliptic trace $T_{ell}^G(f, \lambda)$ is*

$$\boxed{\sum_{P_0 \subseteq Q \subseteq G} \sum_{w \in W^Q} (-1)^{rang(Q)-rang(G)+l(w)} \cdot tr_s\big(\overline{f}^{(Q)} \hat{\chi}_Q^G; H^\bullet(S_{L_Q}, V_{w(\lambda+\rho_G)-\rho_L})\big)}.$$

Corollary 2.3. *The Lefschetz number $L(f, V_\lambda)$ is*

$$\boxed{\sum_{P_0 \subseteq P \subseteq G} \sum_{w \in W^P} (-1)^{l(w)} \cdot T_{ell}^L(\overline{f}^{(P)} \chi_P^G, w(\lambda + \rho_G) - \rho_L)}.$$

2.7 The Satake Transform

For a connected reductive group G over a non-Archimedean local field F_v let A be a maximal F_v-split torus in the center of G. Let G^{ab} be the maximal Abelian quotient of G. Write $G_v = G(F_v)$, etc.

***ord*$_G$.** There is a canonical homomorphism $ord_G : G_v \to X_*(G) = Hom_{F_v-alg}(G, \mathbb{G}_m)$ (see [16], p. 134). We also write ord_G for the induced homomorphism $G_v \to \mathcal{X}_{G_v} = X_*(G) \otimes \mathbb{R}$, and ${}^0G$ for the kernel. The homomorphism ord_G is functorial in G and induces the field valuation in the case $G = \mathbb{G}_m$. It factorizes over the quotient G^{ab}, and is trivial on compact subgroups. The kernel of the canonical map $A_v \to G_v^{ab}$ is contained in the maximal compact subgroup ${}^0A_v$. Hence, the quotient group $A_v/{}^0A_v$, which can be identified with the F_v-rational cocharacter lattice $X_*(A)$ of the torus A, is injected into $G_v^{ab}/{}^0G_v^{ab}$ as a subgroup of finite index. Hence, the canonical maps $\mathcal{X}_{A_v} \to \mathcal{X}_{G_v} \to \mathcal{X}_{G_v^{ab}}$ induce isomorphisms, which allows us to identify these vector spaces.

The Map *S*. Now assume $G_v = G(F_v)$ to be quasisplit, and split over a finite unramified extension field of F_v such that the derived group is simply connected. Let Ω_v be a good maximal compact subgroup and $P = MN$ be a minimal F_v-rational parabolic subgroup of G such that $G_v = P_v \cdot \Omega_v$. To be precise, we demand Ω_v to be admissible relative to M_v in the sense of [7], p. 9. The Ω_v-bi-invariant functions on G_v with compact support define the spherical Hecke algebra $\mathcal{H}(G_v, \Omega_v)$ of G_v. Put $\Lambda = {}^0M_v \setminus M_v$. For $f_v \in C_c^\infty(G_v)$ define $\overline{f}_v^{(P_v)}(m) = |m|_v^{\rho_{P_v}} \int_{N_v} \overline{f}(mn)dn$ as on page 36 now locally for F_v. For elements f_v in the spherical Hecke algebra of G_v the Satake transform S is defined by (see [16], p. 146, formula (19))

$$f_v \mapsto S(f_v) = \overline{f}_v^{(P_v)}$$

and defines a function $S(f_v)$ on $M_v(\mathbb{Q}_v)/M_v(\mathbb{Q}_v) \cap \Omega_v = \Lambda$. The group Λ is a lattice, which contains and is commensurable with the cocharacter lattice $X_*(A)$ of the torus A (see [16], p. 135) in $\mathcal{X}_{A_v} = X_*(A) \otimes \mathbb{R}$. The Satake transform defines an isomorphism between the spherical Hecke algebra of the group G_v and the algebra $\mathbb{C}[\Lambda]^W$ (W-invariants in the group ring $\mathbb{C}[\Lambda]$ [16], Theorem 4.1). Furthermore, for γ regular in M_v the Satake transform S is given by the orbital integral up to a normalization factor

$$S(f_v)(\gamma) = D_G(\gamma)^{1/2} O_\gamma^{G_v}(f_v).$$

For an arbitrary function $\chi : \mathcal{X}_{G_v} \to \mathbb{R}$ multiplication by χ determines a $\mathbb{C}$-linear endomorphism $f_v(x) \mapsto \chi(ord_L(x)) f_v(x)$ of the Hecke algebra of G_v, which preserves the spherical Hecke algebra such that for the orbital integral

$$O_\gamma^{G_v}(f_v \chi) = \chi(ord_G(\gamma)) \cdot O_\gamma^{G_v}(f_v)$$

holds, and also for the Satake transform $S(\chi f_v)(m) = \chi(ord_G(\gamma)(m)) S(f_v)(m)$.

Standard F_v-parabolic Groups. Let Q be a F_v-rational standard parabolic subgroup of G with Levi component L. Let A_Q be the maximal F_v-split torus in Q. The natural map $A_v \to L_v \to \mathcal{X}_L$ factorizes over the quotient $A_v/{}^0A_v$, and hence induces a canonical $\mathbb{R}$-linear map

$$pr : \mathcal{X}_{M_v} \to \mathcal{X}_{L_v}.$$

The following two properties characterize the projection pr. Firstly, the embedding $A_{Q_v} \hookrightarrow A_v$ induces a canonical embedding $i : \mathcal{X}_{L_v} = X_*(A_Q) \otimes \mathbb{R} \hookrightarrow \mathcal{X}_{M_v} = X_*(A) \otimes \mathbb{R}$ such that $pr : \mathcal{X}_{M_v} \to \mathcal{X}_{L_v}$ restricts us to the identity map on the subspace $\mathcal{X}_{L_v} \subseteq \mathcal{X}_{M_v}$. Secondly pr is zero on the subspace $X_*(A'_Q) \otimes \mathbb{R} \subseteq \mathcal{X}_{M_v}$, where A'_Q denotes the split torus $L_{der} \cap A$.

This gives the following formulation in terms of the Killing form. Let $\alpha_i \in \Delta(G_v, A_v)$ denote the simple F_v-roots attached to $P_v \subseteq G_v$, let $\langle\ ,\ \rangle$ denote the Killing form, and let β_j denote the dual basis $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$. Use the Killing form to identify $X_*(A) \otimes \mathbb{R}$ with its dual $X^*(A) \otimes \mathbb{R}$. The F_v-rational standard parabolic subgroups are in one-to-one correspondence with the subsets $F \subseteq \Delta(P_v, A_v)$. For $Q = Q_F$ the space $\mathcal{X}_{L_v} = X_*(A_Q) \otimes \mathbb{R}$ is given in $\mathcal{X}_{M_v} = X_*(A) \otimes \mathbb{R}$ by the equations $\langle\ ., \alpha_i \rangle = 0, \alpha_i \in F$ (or $i \in F$ by abuse of notation) for a subset F of the simple roots. $\mathcal{X}_{M_v}$ splits into the orthogonal direct sum of the two subspaces $\mathcal{X}_{L_v} = \sum_{j \notin F} \mathbb{R}\beta_j$ and the orthocomplement $\sum_{i \in F} \mathbb{R}\alpha_i$. pr is the orthogonal projection defined by $pr(\sum_{j \notin F} x_j \beta_j + \sum_{i \in F} y_i \alpha_i) = \sum_{j \notin F} x_j \beta_j$.

Transitivity. Let $Q = LN$ be an F_v-rational parabolic subgroup of G. Let σ_v be an irreducible admissible representation of L_v. The Hecke algebra $C_c^\infty(G_v)$ of locally constant functions with compact support on G_v acts by convolution on the unitary normalized induced representation $\pi_v = Ind_{Q_v}^{G_v}(\sigma_v)$ such that (for measures suitably normalized) the adjunction formula (see, e.g., [44], Sect. 2, Lemma 1, the slightly different definition involving f_v^* in the pairing in loc. cit. has no effect) holds

$$tr\ Ind_{Q_v}^{G_v}(\sigma_v)(f_v) = tr\ \sigma_v(\overline{f}_v^{(Q)}),$$

where $f_v \in C_c^\infty(G_v)$ and by definition $\overline{f}_v^{(Q)}(m) = |m|_v^{\rho_{Q_v}} \int_{N_v} \overline{f}(mn)dn$.

The group $\Omega_v \cap L_v = (\Omega_v \cap Q_v)_{L_v}$ is a good maximal compact subgroup of L_v, i.e., admissible with respect to M_v (see [7], p. 9). L_v is again quasisplit and splits over a unramified extension field. Hence, the spherical Hecke algebra $\mathcal{H}(L_v, \Omega_v \cap L_v)$ is defined. For $f_v \in \mathcal{H}(G_v, \Omega_v)$ the function $S_L^G(f_v) = \overline{f}_v^{(Q)}$ is bi-invariant under $\Omega_v \cap L_v$, and hence the partial Satake transform $S = S_M^G : \mathcal{H}(G_v, \Omega_v) \to \mathcal{H}(M_v, {}^0M_v)$ factorizes over the spherical Hecke algebra $\mathcal{H}(L_v, \Omega_v \cap L_v)$

$$S = S_M^G = S_M^L \circ S_L^G.$$

Absolute Support. In the following, a cone C in Euclidean space is understood to be an open submonoid stable under multiplication by $\mathbb{R}_{>0}$ which does not contain a real line.

Lemma 2.11. *Fix an arbitrary nonempty open cone $C \subseteq \mathcal{X}_M$, which is contained in the positive Weyl chamber attached to P_v. Let $\pi_v = Ind_{P_v}^{G_v}(\sigma_v)$ be an unramified induced representation attached to an unramified character σ_v of M_v with spherical constituent π_v^0. Choose $x_0 \in C$. Then there exist spherical Hecke operators f_v with the properties:*

1. *$tr\ \pi_v(f_v) = tr\ \pi_v^0(f_v) = 1$.*
2. *The support of the Satake transform $S(f_v)$ of f_v is contained in the Weyl group orbit $\bigcup_{w \in W} w(x_0 + C)$ of the translated cone $x_0 + C$.*

Proof. It suffices to find $f_v \in \mathcal{H}(G_v, \Omega_v)$ with $tr\ \pi_v(f_v) \neq 0$ such that (2) holds. $tr\ \pi_v$, considered as a functional on the spherical Hecke algebra $\mathbb{C}[\Lambda]^W$, is a finite sum of characters on the group Λ. Up to a twist by $\delta^{1/2}(x)$ these characters are in the W-orbit of the character σ_v. This character sum is conjugation-invariant, and hence W-invariant. If the assertions of the lemma were false, there would exist finitely many different characters $\chi_i, i = 1, \ldots, r$, of Λ and $n_i \in \mathbb{C}$ such that

$$\sum_{i=1}^{r} n_i \chi_i(x) = 0, \qquad (n_1, \cdots, n_r) \neq 0$$

holds for all $x \in \Lambda \cap (x_0 + C)$. To see that this is impossible we can assume $x_0 = 0$, changing the coefficients n_i to $n_i\chi_i(x_0)$, and then use induction on r. Since $x, y \in C$ implies $x + y \in C$ we can lower the length r of such a nontrivial character relation on C by considering $\sum_i n_i(\chi_i(y) - \chi_1(y))\chi_i(x) = 0$, provided there exists $y \in C$ with $\chi_r(y) \neq \chi_1(y)$ if, say, $n_r \neq 0$. Because $\chi = \chi_r/\chi_1$ is a nontrivial character on Λ, such a y exists, since otherwise χ vanishes on $C \cap \Lambda$, and hence on the generated group $(C \cap \Lambda) - (C \cap \Lambda)$. However, $(C \cap \Lambda) - (C \cap \Lambda) = \Lambda$ holds for any nonempty open cone of $\mathcal{X}$. This proves the lemma. $\square$

Relative Support. For $f_v \in C_c^\infty(G_v)$ consider the support Σ of the orbital integral $O_\gamma^L(\overline{f}_v^{(Q)})$ as a function of $\gamma \in L_v$. Notice the support of $\overline{f}_v^{(Q)}$ itself is contained in Σ. The image of Σ in $\mathcal{X}_L$ of the regular, semisimple subset of this support under $ord_L : L_v \to \mathcal{X}_{L_v}$ will be called the *relative support* of f_v with respect to Q_v. The relative support contains the image of the support of $\overline{f}_v^{(Q)}$ in $\mathcal{X}_{L_v}$ under the map ord_L. Since the regular semisimple elements are dense in Σ, and since the maximal compact subgroup of L_v is in the kernel of ord_L, one could replace the support Σ by the regular, semisimple support of $O_\gamma^L(\overline{f}_v^{(Q)})$ for the definition of relative support above.

The relative support of f_v with respect to Q_v is a finite subset of the vector space $\mathcal{X}_L$. Notice that $\overline{f}_v^{(Q)}$ has compact support on L_v, and ord_L is invariant under conjugation. Hence, the image $ord_L(\Sigma)$ is relatively compact in $\mathcal{X}_{L_v}$. On the other hand $ord_L(L_v)$ is contained in a sublattice of $\mathcal{X}_{L_v}$.

Lemma 2.12. *Let $f_v \in \mathcal{H}(G_v, \Omega_v)$ be a spherical function. Let $Q = LN_Q$ be an F_v-rational standard parabolic subgroup of G containing the minimal F_v-parabolic subgroup $P = MN$. Then $x \in \mathcal{X}_{L_v}$ is in the relative support of $O_\gamma^L(\overline{f}_v^{(Q)})$ if and only if x is in the image of the support of the Satake transform $S(f_v) \in \mathcal{H}(M_v, {}^0M_v)$ under the map $pr \circ ord_M$, where $pr : \mathcal{X}_{M_v} \to \mathcal{X}_{L_v}$ is the canonical projection.*

Proof. Let $\chi_x(\lambda)$ be the function on $\mathcal{X}_L$, which is not zero for $\lambda = x$ and is zero otherwise. Then by definition the following statements are equivalent. By abuse of notation we consider χ_x as a function on L_v using the map ord_L. Then $x \in \mathcal{X}_L$ is in the relative support of f_v if and only if

$$\chi_x(\gamma) O_\gamma^{L_v}(\overline{f}_v^{(Q)}) = O_\gamma^{L_v}(\chi_x \cdot \overline{f}_v^{(Q)})$$

does not vanish identically for all semisimple, regular elements $\gamma \in L_v$. Since f_v is a spherical function on G_v, $\overline{f}_v^{(Q)} = S_L^G(f_v)$ is spherical on L_v; hence, $\chi_x \cdot \overline{f}_v^{(Q)} = \chi_x \cdot S_L^G(f_v)$ is again spherical on L_v. If $O_\gamma^{L_v}(\chi_x \overline{f}_v^{(Q)})$ does not vanish identically for all semisimple, regular elements $\gamma \in L_v$, then $\chi_x \overline{f}_v^{(Q)}$ does not vanish identically on L_v. Since $\chi_x \overline{f}_v^{(Q)}$ is spherical, this implies $S_M^L(\chi_x \overline{f}_v^{(Q)}) \neq 0$; hence, $O_\gamma^{L_v}(\chi_x \overline{f}_v^{(Q)})$ does not vanish identically for all semisimple, regular elements $\gamma \in M_v \subseteq L_v$. In other words, $x \in \mathcal{X}_L$ is in the relative support of f_v if and only if $S_{M_v}^{L_v}(\chi_x \overline{f}_v^{(Q)}) \neq 0$. Obviously $S_{M_v}^{L_v}(\chi_x \overline{f}_v^{(Q)}) = \chi_x S_{M_v}^{L_v}(\overline{f}^{(Q)}) = (\chi_x \circ pr \circ ord_M) \cdot S_M^L(S_L^G(f_v)) = (\chi_x \circ pr \circ ord_M) \cdot S(f_v)$. This does not vanish identically if and only if x is in the image of the support of $S(f_v)$ in $\mathcal{X}_{M_v}$ under pr. This proves the lemma. $\square$

2.7.1 Subdivision of the Weyl Chambers

Suppose $Q = LN_Q$ is an F_v-rational standard parabolic subgroup $Q = Q_F$ defined by $F \subseteq \Delta(G_v, A_v)$, containing the minimal F_v-parabolic group $P = MN$. Then an element $x = \sum_{i \in \Delta(G_v, A_v)} x_i \alpha_i$ in $\mathcal{X}_{M_v}$ is contained in the support of the function

$$\hat{\chi}_{Q_F}^G = \hat{\tau}_{Q_F}^G \circ pr \circ ord_M$$

if and only if its projection $pr(x) = \sum_{i \notin F} x_i \alpha_i \in \mathcal{X}_{L_v}$ is in the obtuse Weyl chamber ${}^+\mathcal{X}_{L_v} = \sum_{i \notin F} \mathbb{R}_{>0} \alpha_i$, which means $x_i = \langle x, \beta_i \rangle > 0$ for all $i \notin F$.

The equations $\alpha_i(x) = 0$ and $\beta_i(x) = 0$ for $\alpha_i \in \Delta(G_v, A_v)$ define hyperplanes in $\mathcal{X}_{M_v}$. The images of these hyperplanes under the action of the Weyl group on $\mathcal{X}_{M_v}$ define finitely many hyperplanes. The complement of these hyperplanes in $\mathcal{X}_{M_v}$ is a union of open connected cones. Each of these cones is the image under the Weyl group of a subcone of the open Weyl chamber $\mathcal{X}_{M_v}^+$. Pick one of these cones C.

Example 2.3. For $G_v = Sl(3, F_v)$ the positive Weyl chamber contains two such cones.

Support Conditions. Suppose f_v is a spherical function on G_v such that its Satake transform is contained in the W-orbit of $x_0 + C \subseteq \mathcal{X}_{M_v}$ for some $x_0 \in C$, as in Lemma 2.11. Then a regular semisimple element γ is in the support of $O_\gamma^L(\overline{f}_v^{(Q)} \hat{\chi}_Q^G)$ if and only if $x = ord_L(\gamma)$ is in $pr(\bigcup_{w \in W}(x_0 + C))$. If this is the case then $x_i = \beta_i(x) > 0$ for all $i \notin F$. But then moreover, by our specific choice of the cone, we even get $x_i > const(x_0) > 0$ for all $i \notin F$. Similarly, if γ is not in the support of $O_\gamma^L(\overline{f}_v^{(Q)} \hat{\chi}_Q^G)$, then $x_i < -const(x_0)$ holds for at least one $i \notin F$. The constant $const(x_0)$ which appears in these formulas of course depends on the choice of $x_0 \in C$. By a suitable choice of x_0 it can be made arbitrarily large. A similar statement holds for the condition that $x = ord(\gamma) \in \mathcal{X}_{L'_v}$ is in the support of $O^{L'}(\overline{f}^{(P)} \hat{\chi}_Q^G \chi_{L'}^L)$ for $L' \subseteq L$, $P = L'N_P$, and $Q = LN_Q$. In fact all values

$$\alpha_i(w(x))\ ,\ \beta_j(w(x)) \qquad w \in W,\ i,j \in \Delta(G_v, A_v)$$

are different from zero, and either $> const(x_0)$ or $< -const(x_0)$.

Preferred Places S'. These facts can now be used in the global context to concentrate the effect of the adelic cutoff functions $\hat{\chi}_Q^G$, as they appear in the formula of Corollary 2.2, to a finite set S' of "preferred" local non-Archimedean places in the sense that

$$tr_s(\overline{f}^{(Q)}\hat{\chi}_Q^G; H^\bullet(S_{L_Q}, V)) \ = \ tr_s(\overline{f}^{(Q)}(\hat{\chi}_Q^G)_{S'}; H^\bullet(S_{L_Q}, V))$$

holds (in a suitable context). For this it would suffice to know that

$$T_{ell}^{L'}\big(\overline{f}^{(P)}\hat{\chi}_Q^G\chi_{P'}^L, .\big) = T_{ell}^{L'}\big(\overline{f}^{(P)}(\hat{\chi}_Q^G\chi_{P'}^L)_{S'}, .\big)$$

holds for all $L' \subseteq L$, where L' is a Levi component of $P = L'N_P \subseteq Q = LN_Q$ (Corollary 2.2). Alternatively (Corollary 2.3) it would be enough to know that

$$O_\gamma^{L'}\big(\overline{f}^{(P)}\hat{\chi}_Q^G\chi_{P'}^L\big) = O_\gamma^{L'}\big(\overline{f}^{(P)}(\hat{\chi}_Q^G\chi_{P'}^L)_{S'}\big).$$

Before we explain under which conditions this holds, we first recall certain definitions.

2.7.2 Global Situation

For $\mathbb{Q}$-rational parabolic subgroups P and Q of T the global cutoff function $\hat{\chi}_Q^G\chi_{P'}^L$ on $L'(\mathbb{Q})$, which occurs in Corollary 2.3, was defined for $P = L'N_P$ using the Harish-Chandra map H_P via

$$L'(\mathbb{Q}) \hookrightarrow L'(\mathbb{A}) \xrightarrow{H_P} \mathcal{X}_{L'}\ .$$

In fact, by the product formula $H_P(\gamma) = log|\gamma_\infty|_\infty - \sum_{v\neq\infty} q_v \cdot ord_{L'}(\gamma_v)$, the global cutoff condition can be written as the condition on the point

$$\sum_{v\neq\infty} q_v \cdot ord_{L'}(\gamma_v) \ \ \in\ \mathcal{X}_{L'}$$

to lie in the support of $\hat{\tau}_Q^G\tau_{P'}^L$.

Notation: $\gamma = (\gamma_v)_v \in L(\mathbb{A}_{fin})$. q_v denotes the cardinality of the residue field, and $ord_{L'}(\gamma_v)$ the image of the local element $ord_{L'}(\gamma_v) \in \mathcal{X}_{L'_v}$ in $\mathcal{X}_{L'}$ under the natural projection map $\mathcal{X}_{L'_v} \to \mathcal{X}_{L'}$ (notice that locally the maximal F_v-split torus may be larger than the maximal $\mathbb{Q}$-split torus $A_{L'}$).

Assumptions. To be more specific about the concentration at specific places, let us assume $f = \prod_{v \neq \infty} f_v$. Furthermore, suppose there are two finite disjoint sets S and S' of non-Archimedean places such that f_v is the unit element of the spherical Hecke algebra for all $v \notin S \cup S'$. Suppose $f_S = \prod_{v \in S} f_v$ has support in a fixed compact subset of $G(\mathbb{A}_S)$. Finally, suppose that all f_v for $v \in S'$ are spherical such that the Satake transform $S(f_v)$ has the following property.

Property $(*)$. For all roots α and all dual roots β in the set of $\mathbb{Q}$-rational simple roots of (G, P_0) and all elements $w \in W$ the absolute value of the linear forms $\alpha \circ w$ and $\beta \circ w$ on

$$\sum_{v \in S'} q_v \cdot ord_{L'}(\gamma_v) \quad \in \mathcal{X}_{L'}$$

is larger than a fixed constant $c > 0$.

If c is sufficiently large compared with the support of f_S, we obviously get

Lemma 2.13. *Under the assumptions above, if the constant c is large enough depending only on the support of f_S, the truncation condition concentrates on the places in S'*

$$O_\gamma^{L'}(\overline{f}^{(P)} \hat{\chi}_Q^G \chi_{P'}^L) = O_\gamma^{L'}(\overline{f}^{(P)} (\hat{\chi}_Q^G \chi_{P'}^L)_{S'}).$$

Notation. Let $\mathcal{E}_\nu$ denote the set of irreducible constituents $\rho = \rho_{S'} \otimes \rho^{S'} \in \mathcal{E}_\nu$ of the admissible representation of $G(\mathbb{A}_{fin})$ on the cohomology group $H^\nu(S_L, V)$.

Corollary 2.4. *Let the situation be as in Lemma 2.13. Then the truncated Lefschetz number $tr_s(\overline{f}^{(Q)} \hat{\chi}_Q^G; H^\bullet(S_L, V))$ is given by $tr_s(\overline{f}_S^{(Q)} (\hat{\chi}_Q^G)_{S'}; H^\bullet(S_L, V))$, or alternatively by a sum*

$$\sum_\nu (-1)^\nu \cdot \sum_{\rho \in \mathcal{E}_\nu} tr\Big(f^{S'}; Ind_{L(\mathbb{A}_S)}^{G(\mathbb{A}_S)}(\rho_S)\Big) \cdot tr\Big(\overline{f}_{S'}^{(Q)} (\hat{\chi}_Q^G)_{S'}; \rho_{S'}\Big),$$

where now $f^{S'} = f_S \prod_{w \notin S', w \neq \infty} 1_w$.

Proof. The first statement follows from Corollary 2.3 together with Lemma 2.13, which implies $T_{ell}^L(f \hat{\chi}_Q^G, \tau) = T_{ell}^L(f_S \cdot (f^S (\hat{\chi}_Q^G)_{S'}), \tau)$. The second formula then follows from the first assertion via the adjunction formula. □

2.8 Automorphic Representations

Fix λ and a compact open subgroup $K = \prod_{v \neq \infty} K_v \subseteq \Omega$ of $G(\mathbb{A}_{fin})$, which defines the "level," the level group. The $G(\mathbb{A}_{fin})$-module given by the limit $H^\bullet(S(G), V_\lambda)$ is an admissible representation of $G(\mathbb{A}_{fin})$. Only finitely many irreducible constituents π with the property $\pi^K \neq 0$ occur. The same holds for the finitely many Levi subgroups L, the induced level groups $K_L = (K \cap P(\mathbb{A}_{fin})_L$, and the induced coefficient systems attached to the highest weights $\lambda' = w(\lambda + \rho_G) - \rho_L$. Thus, the admissible representation

$$\Pi(\lambda) = \bigoplus_{P_0 \subseteq Q \subseteq G} \bigoplus_{w \in W^Q} \bigoplus_i Ind_Q(\mathbb{A}_{fin})^{G(\mathbb{A}_{fin})}(H^i(S(L), V_{w(\lambda+\rho_G)-\rho_L})),$$

the "halo" of the $G(\mathbb{A}_{fin})$-module $H^\bullet(S(G), V_\lambda)$, again contains only finitely many irreducible $G(\mathbb{A}_{fin})$-constituents π with the property $\pi^K \neq 0$. Let $\mathcal{P}$ be the set of equivalence classes of these representations of level K.

Remark 2.10. $\Pi(\lambda)$ should be considered as a superspace whose grading is induced by the sign defined by the parity of the sum of the number $rank(G) - rank(Q)$, the length $l(w)$ for $w \in W^P$, and the degree i.

Let S_0 be the set of places for which $K_v \neq \Omega_v$ (level primes). Outside S_0 representations in $\mathcal{P}$ are unramified. Fix a prime $p \notin S_0$, the "Frobenius" prime. For π in $\mathcal{P}$ consider the representation π^p of $G(\mathbb{A}^p_{fin})$ defined by $\pi = \pi^p \otimes \pi_p$. The set of places S_0 can be enlarged to a finite set S of places not containing p such that $\pi_1^p \cong \pi_2^p \iff (\pi_1)_S \cong (\pi_2)_S$. There exists $f_S \in C_c^\infty(G(\mathbb{A}_S))$, so $tr\ \pi_S(f_S) = 0$ holds for all representations π' in $\mathcal{P}$ for which $(\pi')^p$ is not isomorphic to π^p, where π is some fixed representation in $\mathcal{P}$. Furthermore, we can assume $tr\ \pi_S(f_S) = 1$. For a suitable choice of K (in a cofinal system, where K^S is a product of special good maximal compact open subgroups), one can assume in addition that f_S is K_S-bi-invariant (see the Remark 4.3 on page 79). Now fix the π^p-projector f_S. For a non-Archimedean place $v \notin S$ consider functions

$$f = f_S \cdot h_p \cdot f_v \cdot \prod_{w \neq \infty\ else} 1_w$$

in $C_c^\infty(G(\mathbb{A}_{fin}))$, where h_p and f_v are suitable functions in the spherical Hecke algebra $\mathcal{H}(G_p, \Omega_p)$, respectively, $\mathcal{H}(G_v, \Omega_v)$. f_v is chosen subject to the conditions:

- Property $(*)$ (see the assumptions preceding Lemma 2.13) holds for $S' = \{v\}$ with respect to the fixed function f_S or more precisely its fixed support in $G(\mathbb{A}_S)$.
- $tr\pi_v(f_v) = 1$ holds for the unramified component π_v of our fixed representation $\pi^p = \otimes_{w \neq p, \infty} \pi_w$.

Such functions f_v exist, as explained on page 42, as a consequence of Lemma 2.11 choosing x_0 in the cone C to be sufficiently large. The function h_p is chosen to be either:

- $h_p = 1_p$ (unit element of $\mathcal{H}(G_p, \Omega_p)$) or
- $h_p^{(n)} = b(\phi_n)$ (the local cyclic base change of the Kottwitz function ϕ_n on $G(E_p)$ of [51] under the unramified base change map homomorphism b of spherical

Hecke algebras for some unramified local field extension $E_p/\mathbb{Q}_p$ of degree $[E_p : \mathbb{Q}_p] = n$) in the context where G is attached to a Shimura variety as in [51] with reflex field $\mathbb{Q}$ (for simplicity)

We claim that either for $h_p = 1$, $S' = \{v\}$, or for $h_p = h_p^{(n)}$, $S' = \{p, v\}$, and for sufficiently large $n \gg 0$, the assumptions preceding Lemma 2.13 are satisfied. For $h_p = 1$ this has already been explained. The case $h_p = h_p^{(n)}$ and $n \gg 0$ can be reduced to the ensuing Lemma 2.14. We leave this as an exercise. So taking this for granted, now assume $n \gg 0$ or $h_p = 1$.

Then we get from Corollary 2.4 an expression for the truncated Lefschetz numbers

$$tr_s(\overline{f}^{(Q)} \hat{\chi}_Q^G, H^\bullet(S_L, V))$$

in terms of

$$\sum_{\nu} \sum_{\rho \in \mathcal{E}_\nu} (-1)^\nu \cdot tr\Big(f^{S'}, Ind_{L(\mathbb{A}_S)}^{G(\mathbb{A}_S)}(\rho^{S'})\Big) \cdot tr\Big(\overline{f}_{S'}^{(Q)}(\hat{\chi}_Q^G)_{S'}, \rho_{S'}\Big).$$

This allows us to apply a theorem of Franke [27] which states that all irreducible representations ρ of $L(\mathbb{A}_{fin})$ which occur in $\mathcal{E}_\nu$ as constituents of the cohomology group $H^\nu(S_L, V)$ are automorphic representations of $L(\mathbb{A}_{fin})$. Hence, all induced representations in

$$Ind_{L(\mathbb{A}_S)}^{G(\mathbb{A}_S)}(\rho^{S'})$$

are automorphic representations of $G(\mathbb{A}^{S'})$, and are Eisenstein representations for $L \neq G$.

Therefore, if the fixed representation $\pi \in \mathcal{P}$ is cuspidal and not CAP, π^p does not occur as a constituent in $\mathcal{P}$ from these induced representations in the case $L \neq G$. Since f and $\overline{f}$ are K_S-bi-invariant, the trace of f on $\Pi(\lambda)$ involves only constituents in $\mathcal{P}$, i.e., for the fixed level K. Since f_S is a projector for π^p among the representations in $\mathcal{P}$, this implies $tr\big(f_S, Ind_{L(\mathbb{A}_S)}^{G(\mathbb{A}_S)}(H^\nu(S_L, V))\big) = 0$. Hence, the truncated Lefschetz numbers

$$tr_s\Big(\overline{f}^{(Q)} \hat{\chi}_Q^G, H^\bullet(S_L, V)\Big)$$

all vanish except for the case $G = Q$, where the truncated Lefschetz number is the trace $tr_s(\overline{f}, H^\bullet(S_L, V))$ of $\overline{f}$ on the cohomology $H^\bullet(S_L, V)$. Notice $\overline{f} = \overline{f}^{(G)} \neq f$ in general. However, f and $\overline{f}$ have the same trace on every irreducible admissible representation. This follows from $O^G(\overline{f}) = O^G(f)$, since $vol(\Omega) = 1$. But then we can replace $\overline{f}$ by f. Then, since f is K_S-bi-invariant, the remaining Lefschetz number is the trace of f on the finite-dimensional space $H^\bullet(S_K(G), V)$ for fixed level K, and it only involves the representations in $\mathcal{P}$. Since f_S is a π^p-projector, the trace of $h_p f_v$ on this space is the trace of $h_p f_v$ on the generalized π^p-eigenspace of the cuspidal cohomology. Since $tr\ \pi_v(f_v) = 1$, this simplifies the formula for $T_{ell}^G(f_S f_v h_p, \lambda)$ of Corollary 2.2, and leaves only the term for $Q = G$ and $w = 1$. This proves

Theorem 2.2. *Suppose π is an irreducible cuspidal representation of $G(\mathbb{A}_{fin})$ and not CAP (see [69, 97]). Then for $f = f_S f_v h_p \prod_{w \neq \infty \ else} 1_w$, where f_S, f_v, and $h_p = 1$ or $h_p = h_p^{(n)}$ and $n \gg 0$ is chosen as above, we get for the trace on the π^p-constituents*

$$tr_s\Big(h_p, H^\bullet(S_K(G), V_\lambda)(\pi^p)\Big) = T_{ell}^G\Big(f_S f_v h_p, \lambda\Big).$$

This theorem will be used in Chap. 3 for the "Frobenius" prime p.

Notice that $f_S f_v$ again is a projector on π^p among the representations in $\mathcal{P}$. We write $f_S f_v = f_{\pi^p}$, and call it a "*good π^p-projector*."

Lemma 2.14. *Let $C \subseteq V$ be a cone in Euclidean space, W a finite group acting on V, and $L_1, \ldots, L_r$ a W-stable set of linear forms in V^* nonvanishing on C. For $x_0 \in C$ and $x \in V$ and a bounded set $M \subseteq V$, there exists a integer m depending on M and an integer N depending on m and M such that the following holds. Suppose $v = v_1 + v_2 + v_3$ for $v_1 \in \bigcup_{w \in W}\{n \cdot w(v) \mid n \geq N\}$, $v_2 \in \bigcup_{w \in W} w(m \cdot x_0 + C)$, and $v_3 \in M$. Then $L_i(v) > 0$ for some $i = 1, \ldots, r$ holds if and only if $L_i(v_1 + v_2) > 0$ holds.*

Proof. Obvious. $\square$

Remark 2.11. In the case $h_p = 1$, we may also omit the auxiliary prime p or choose p to be large so that the formula in Theorem 2.2 becomes

$$\sum_\nu (-1)^\nu dim_{\mathbb{C}}(H^\nu(S_K(G), V_\lambda)(\pi)) = T_{ell}^G(f_\pi, \lambda)$$

for a good π-projector $f_\pi \in C_c^\infty(G(\mathbb{A}_{fin}))$.

Remark 2.12. In the Hermitian symmetric case there exists a formula analogous to Theorem 2.2 for the L^2-cohomology instead of the Betti cohomology. In this case the L^2-cohomology is finite-dimensional, so one can define the traces of Hecke operators on the L^2-cohomology. Using the results in [33], one obtains a formula for the L^2-Lefschetz numbers analogous to the one of Corollary 2.3. The relevant change in this case amounts to a subtler substitute of W^P, which in the case of L^2-cohomology also depends on the elements γ. In fact one obtains the following formula for the L^2-Lefschetz number:

$$\sum_{P_0 \subseteq P = LN \subseteq G} \ \sum_{w \in W^P} (-1)^{\#I(w)} \cdot T_{ell}^L(\overline{f}^{(P)} \chi_P^G(w), w(\lambda + \rho_G) - \rho_L),$$

where the cutoff functions $\chi_P^G(w)$ now depend on $w \in W^P$. They are defined as follows: $\chi_P^G(w)$ is the characteristic function of the set of all $\gamma \in L(\mathbb{Q})$, which satisfy $I(\gamma) = I(w)$, for certain finite sets $I(w)$ depending only on w, P, G, and λ

(see [33], p. 474), and where $I(\gamma)$ is the set of simple roots α of A_P in N_P such that $|\alpha(\gamma)|_f^{-1} = |\alpha(\gamma)|_\infty \leq 1$ (see [33], p. 471). Let $\chi_{P,i}$ denote the characteristic function of the set of all $\gamma \in L(\mathbb{R})$ such that $|\alpha_i(\gamma)|_\infty > 1$ for the simple root α_i. Then the characteristic function $\chi_P^G(w)$ can be expressed in the form $\prod_{i \notin I} \chi_{P,i} \prod_{j \in I}(1 - \chi_{P,j})$ for $I = I(w)$. It can therefore be expanded into a finite linear combination of the functions $\chi_K = \prod_{i \in K} \chi_{P,i}$ for $K \subseteq \Delta(G, A)$. For these finitely many global cutoff functions χ_K, which now appear in the L^2-Lefschetz formula, the effect of the cutoff can now be concentrated at some preferred non-Archimedean places $v \in S'$ by the choice of a suitable good π^S-projector, modified at a single place $S' = \{v\}$ as in the discussion above using a variant of Lemma 2.13. This implies

Corollary 2.5. *Suppose G_∞ is of Hermitian symmetric type. Suppose π is an irreducible cuspidal representation of $G(\mathbb{A}_{fin})$ and not CAP. Then there exists a good π-projector f_{π^S} such that the L^2-Lefschetz number of the π-constituents is*

$$tr_s(H^\bullet_{(2)}(S(G), V_\lambda)(\pi)) = T^G_{ell}(f_\pi, \lambda).$$

In particular, the alternating sums of the π-multiplicities on the cohomology and the L^2-cohomology coincide.

2.9 The Discrete Series Case

This is the case considered in [4]. Suppose G is a connected reductive group over $\mathbb{Q}$, G_{der} is simply connected, and G contains a maximal $\mathbb{R}$-torus B, for which $B(\mathbb{R})/A_G(\mathbb{R})^0$ is compact (see [4], p. 262).

Notation. Let $2q(G)$ denote the real dimension of the symmetric domain attached to G_∞ and $d(G)$ the cardinality of the packets of discrete series representations of G_∞. Let τ be an irreducible complex representation of $G(\mathbb{Q})$ defined by the highest weight $\lambda \in X^*(B)_\mathbb{C}$. λ defines a representation of $G(\mathbb{C})$, and hence of the compact inner form $\overline{G}$ of G over the field $\mathbb{R}$. Let τ^* denote the contragredient representation. Attached to the representation τ of $\overline{G}$ is a packet $\Pi_{disc}(\tau)$ of discrete series representations π_∞. Let π^*_∞ denote the contragredient. Attached to τ and λ is the function

$$f_\infty = \frac{f_\lambda}{d(G)},$$

where $f_\lambda \in \mathcal{H}_{ac}(G_\infty, \xi_\lambda^{-1})$ (in the notation in [4], Lemma 3.1) is the stable cuspidal function (i.e., supported in discrete series, see [4], Sect. 4) defined by Clozel and Delorme. f_∞ is compactly supported modulo $A_G(\mathbb{R})^0$ and is $\tilde{K}_\infty$-invariant. Then using the notation in [4], p. 271, formula (4.3), $tr\ \rho^*(f_\infty) = tr\ \rho^*(f_\infty) = (-1)^{q(G_\infty)} tr\ \pi^*_\infty(\frac{f_\lambda}{d(G)})$ for $\pi_\infty \in \Pi_{disc}(\rho)$ becomes $d(G)^{-1}$ if $\pi^*_\infty \in \Pi_{disc}(\tau)$ and is zero otherwise (see [4], Lemma 3.1). Notice $\pi^*_\infty \in \Pi_{disc}(\tau)$ if and only if $\rho^* \cong \tau$. Hence,

$$tr\ \rho^*(f_\infty) = d(G)^{-1}$$

if $\rho \cong \tau^*$, and $tr\ \rho^*(f_\infty) = 0$ otherwise.

The orbital integral $O_\gamma^G(f) = \int_{G_\gamma(\mathbb{R})\backslash G(\mathbb{R})} f(x^{-1}gx)dx$, considered for fixed $\gamma \in G(\mathbb{R})$ as a distribution on $\mathcal{H}_{ac}(G(\mathbb{R}), \xi_\lambda^{-1})$, is denoted $\Phi_G(\gamma, f)$ in [4], p. 269, and in [5], p. 325. Theorem 5.1 in [4] gives a formula valid for all $\gamma \in G(\mathbb{R})$ which expresses the orbital integral of *stable cuspidal* functions $f_\infty \in \mathcal{H}_{ac}(G_\infty, \xi_\lambda^{-1})$ in terms of the distributions $\rho^*(f)$ discussed above,

$$O_\gamma^G(f_\infty) = (-1)^{q(G)} d(G_\gamma) vol(\overline{G}_{\gamma,\infty}/A_G(\mathbb{R})^0)^{-1} \sum_\rho \Phi_G(\gamma, \rho) \cdot tr\ \rho^*(f_\infty),$$

for certain coefficients $\Phi_G(\gamma, \rho)$. The sum runs over irreducible representations ρ of $\overline{G}(\mathbb{R})$ in $\Pi(\overline{G}(\mathbb{R}), \xi_\lambda)$. In particular, $O_\gamma^G(f)$ is zero unless γ is semisimple and $\gamma \in T(\mathbb{R})$ for some maximal $\mathbb{R}$-torus of G such that $(\mathbb{R})/A_G(\mathbb{R})^0$ is compact. Notice $T(\mathbb{R}) \cong B(\mathbb{R})$. On the regular part $T_{reg}(\mathbb{R})$ the function is $\Phi_G(\gamma, \rho) = tr\ \rho(f)$ (see [4], p. 271). Since $\Phi_G(\gamma, \rho)$ extends to a continuous function on $T(\mathbb{R})$ (see [4], Lemma 4.2), this holds for all $\gamma \in T(\mathbb{R})$. Hence, if $O_\gamma^G(f)$ does not vanish a priori, one has $\gamma \in T(\mathbb{R})$, where T is a maximal $\mathbb{R}$-torus in G such that $T(R)/A_G(\mathbb{R})$ is compact. And for all $\gamma \in T(\mathbb{R})$ one has the formula

$$O_\gamma^G(f_\infty) = (-1)^{q(G)} d(G_\gamma) vol(\overline{G}_{\gamma,\infty}/A_G(\mathbb{R})^0)^{-1} tr\ \tau^*(\gamma) d(G)^{-1},$$

since only $\rho \cong \tau^*$ contributes to the sum over all ρ. Notice $tr\ \tau(\gamma^{-1}) = tr\ \tau^*(\gamma)$ for the contragredient representation. Next, from the formula for the Euler numbers (see, e.g., [4], p. 281, formula (6.3), and also p. 282)

$$\chi(G, dg_f) = (-1)^{q(G)} \cdot d(G) \cdot vol(G(\mathbb{Q})A_G(\mathbb{R})^0 \backslash G(\mathbb{A})) \cdot vol(\overline{G}(\mathbb{R})/A_G(\mathbb{R})^0)^{-1},$$

one obtains for $f_{fin} \in C_c^\infty(G(\mathbb{A}_{fin}))$

$$\begin{aligned} &\chi(G_\gamma) \cdot tr\ \tau(\gamma^{-1}) \cdot O_\gamma^G(f_{fin}) \\ &= (-1)^{q(G_\gamma)} d(G_\gamma) vol(\overline{G}_\gamma(\mathbb{R})/A_G(\mathbb{R})^0)^{-1} \tau(G_\gamma) \cdot tr\ \tau^*(\gamma) \cdot O_\gamma^G(f_{fin}) \\ &= d(G)\tau(G_\gamma) O_\gamma^{G_\infty}(f_\infty) O_\gamma^G(f_{fin}) \\ &= d(G)\tau(G_\gamma) O_\gamma^{G(\mathbb{A})}(f_{fin} f_\infty), \end{aligned}$$

provided the measure dg_∞ is chosen such that $dg_\infty dg_{fin}$ is the Tamagawa measure on $G(\mathbb{A})$. Hence, from the definition of $T_{ell}^G(f_{fin})$ we obtain

Lemma 2.15.

$$T_{ell}^G(f_{fin}, \tau) = d(G) \sideset{}{'}\sum_{\gamma \in G(\mathbb{Q})/\sim} \tau(G_\gamma) O_\gamma^{G(\mathbb{A})}(f_{fin} f_\infty).$$

The summation is over all semisimple, strongly elliptic conjugacy classes of $G(\mathbb{Q})$. *Here* $\tau(G_\gamma)$ *is the Tamagawa number* $vol(G_\gamma A_G(\mathbb{R})^0 \backslash G_\gamma(\mathbb{A}))$, *where the measure* dg_∞ *is chosen such that* $dg_\infty dg_{fin}$ *is the Tamagawa measure on* $G(\mathbb{A})$.

Corollary 2.6. *With the assumptions and the notation used in Theorem 2.2 we get for* $f_{fin} = h_p f_{\pi^p}$

$$tr(h_p, H^\bullet(S_K(G), V_\lambda)(\pi^p)) = d(G) \sum_{\gamma \in G(\mathbb{Q})/\sim}{}' \tau(G_\gamma) O_\gamma^{G(\mathbb{A})}(h_p f_{\pi^p} f_\infty).$$

The summation is over all semisimple, strongly elliptic conjugacy classes of $G(\mathbb{Q})$. *The measures defining the orbital integrals are assumed to be Tamagawa measures on* $G(\mathbb{A})$ *and* $G_\gamma(\mathbb{A})$.

Remark 2.13. The term $O_\gamma^G(f_S f_{\pi^S} f_\infty)$ is independent of the chosen measures dg_f and dg_∞ provided $dg_\infty dg_f$ is the Tamagawa measure on $G(\mathbb{A})$. This follows from the definition of f_{fin} and f_∞. Hence, in applications we are now free to normalize the measures dg_f and dg_∞, e.g., such that $vol_{dg_f}(K) = 1$ following the convention of [51].

Remark 2.14. Assume that Z_G/A_G is anisotropic over $\mathbb{R}$. If one considers a Shimura variety attached to G (as in [51]) one replaces $S_K(G) = G(\mathbb{Q}) \setminus G(\mathbb{A})/\tilde{K}_\infty K$ by $G(\mathbb{Q}) \setminus G(\mathbb{A})/Zentr(h)_\infty K$, where h is the underlying structure homomorphism of the Shimura variety. For small K this multiplies the trace by the index $[\tilde{K} : Zentr(h)_\infty]$. See also the remark on page 21 In fact $\gamma\epsilon_\infty \in KZentr(h)_\infty$ for $\epsilon_\infty \in \tilde{K}_\infty$, and $\gamma \in G(\mathbb{Q})$ implies $\gamma \in Z_G(\mathbb{Q})$ (K is small) and $\gamma \in K\tilde{K}_\infty$. Hence, γ is finite, and hence is 1 (K is small). Therefore, $\epsilon_\infty \in Zentr(h)_\infty$.

Appendix 1

Let G be a reductive connected group over $\mathbb{Q}$. Let $K \subseteq G(\mathbb{A}_{fin})$ be a compact open subgroup. For $g \in G(\mathbb{A}_{fin})$ put $K' = K_g = g^{-1}Kg \cap K \subseteq K$. Consider $M = G(\mathbb{Q})\backslash G(\mathbb{A})$, or some compactification, with continuous $G(\mathbb{A}_{fin})$ left action $m \mapsto mg^{-1}, g \in G(\mathbb{A}_{fin})$ together with the maps $p(m) = m$ and $p'(m) = mg^{-1}$

$$p : M/K' \to M/K$$

$$p' : M/K' \to M/K.$$

The map p (or the map p') is equivariant with respect to the map q (or the map q') from $K' = K_g$ to K, defined by $k \mapsto k$ or $k \mapsto gkg^{-1}$. Two points mK and $m'K$ in M/K are related by the correspondence underlying p, p' if there exists a point $m''K' \in M/K'$ such that

$$p(m''K') = mK \text{ and } p'(m''K') = m'K \text{ in } M/K.$$

This means that there exist $k, k' \in K, k'' \in K'$ such that $mkk'' = m''$ and $m''g^{-1} = m'(k')^{-1}$ holds. Hence, $mkk''g^{-1}k' = m'$. Stated in other terms, $m' = mx^{-1}$ for some $x \in KgK$. There exists a finite decomposition $KgK = \biguplus_j Kg_j$. Hence,

$$m'K = mg_j^{-1}K$$

for some j. Conversely, suppose $m'K = mkg^{-1}K$ for some $k \in K$. Then for $m'' := mkK_g$, we tget $p(m''K_g) = mkK = mK$ and $p'(m''K_g) = mkg^{-1}K = m'K$.

Put $\Gamma = K \cap G(\mathbb{Q})$. In general for $\gamma \in G(\mathbb{Q})$ the double coset $\Gamma\gamma\Gamma = \biguplus_i \Gamma\gamma_i$ decomposition gives $K\gamma\Gamma = \biguplus_i K\gamma_i$, again a disjoint union. Since $k_1\gamma_1 = k_2\gamma_2$ implies $k_2^{-1}k_1 = \gamma_2\gamma_1^{-1} \in G(\mathbb{Q}) \cap K = \Gamma$, we get $\Gamma\gamma_1 = \Gamma\gamma_2$. Passing to the closure defines the subset $K\gamma\overline{\Gamma} = \biguplus_i K\gamma_i$ of $K\gamma K = \biguplus_j Kg_j$, which might be smaller than KgK if $\overline{\Gamma} \neq K$. Therefore, to relate fixed points of the adelic correspondence to its classical analogue, one has to ensure that fixed points belong to cosets gK of the form γK for some $\gamma \in G(\mathbb{Q})$ and in particular $KgK = K\gamma K$. However, this is the case (see page 24). Only rational cosets γK contribute to the fixed points of the Goresky–MacPherson trace formula for the Lefschetz numbers.

Appendix 2

Let G_∞ be the group of real points of a reductive group over $\mathbb{R}$. Let K_∞ be a maximal compact group, and let $V_1 \subseteq Z_\infty$ be a vector group in the center Z_∞.

Claim 2.1. Then for every $y \in K_\infty \cdot V_1$, the set $\mathcal{S}$ of all $x \in G_\infty$, such that $x^{-1}yx \in K_\infty \cdot V_1$, is either empty or

$$\mathcal{S} = G_{y,\infty} \cdot K_\infty.$$

Here $G_{y,\infty}$ denotes the centralizer of y in G_∞.

Proof. The proof of this assertion is easily reduced to the case $V_1 = 1$. In fact, $G_\infty = {}^0G \cdot V$, where V is the maximal vector group in the center of G_∞ and ${}^0G$ is the normal subgroup of G_∞ with ${}^0G \cap V = \{e\}$ chosen as in [98], p. 19. Notice $K_\infty \subseteq {}^0G_\infty$.

This allows us to reduce the proof to the case where $y \in K_\infty$ and x satisfies the equation $x^{-1}yx \in K_\infty$. In fact, if $x_0^{-1}yx_0 = k \cdot v_1$ holds for some $x = x_0$ and $k \in K_\infty$, $v \in V_1$, we simply replace y by $y_1 = x_0yv_1^{-1}x_0^{-1} \in K_\infty$ and x by $x_1 = x_0^{-1}x$. Then $x_1^{-1}y_1x_1 \in K_\infty V_1$ is equivalent to $x^{-1}yx \in K_\infty \cdot V_1$. However $x_1^{-1}y_1x_1 \in K_\infty V_1$ if and only if $x_1^{-1}y_1x_1 \in {}^0G_\infty \cap (K_\infty V_1) = K_\infty$. So we assume $y \in K_\infty$ and $x^{-1}yx \in K_\infty$.

Choose a Cartan involution θ of G_∞ such that $g \in K_\infty$ if and only if $\theta(g) = g$ (see [98], Proposition 5). For x as above, the element $z = \theta(x)x^{-1}$ is in $G_{y,\infty}$, and satisfies $\theta(z) = z^{-1}$. One can write $x = s \cdot \kappa$ for $\kappa \in K_\infty$ and $s = exp(\sigma)$ and $\theta(\sigma) = -\sigma \in Lie(G_\infty)$ (follows from [98], Proposition 5). Then $z = exp(-2\sigma) \in G_{y,\infty}$. Since $y \in K_\infty$, y and hence also $G_{y,\infty}$ is θ-stable. Therefore, there exists a symmetric one-parameter subgroup in $G_{y,\infty}$ passing through z. See, e.g., [98], p. 20. In other words we find a symmetric root $r = exp(-\sigma) \in G_{y,\infty}, \theta(r) = r^{-1}$ of $z = r^2$ for $\theta(\sigma) = -\sigma \in Lie(G_{y,\infty})$. We conclude $1 = r^{-1}\theta(x)x^{-1}r^{-1} = \theta(rx)(rx)^{-1}$. Thus, $rx = k \in K_\infty$ and $x = r^{-1}k \in G_{y,\infty} \cdot K_\infty$, which proves the claim. $\square$

Corollary 2.7. $\mathcal{S}/(K_\infty \cdot V_1)$ *is either empty or* $G_{y,\infty}/(G_{y,\infty} \cap K_\infty)V_1$*, where* $G_{y,\infty} \cap K_\infty$.

Proof. Notice $V_1 \subseteq G_{y,\infty}$. $\square$

Remark 2.15. Finally, there exists a diffeomorphism $G_\infty/(K_\infty \cdot V_1) \cong ({}^0G_\infty/K_\infty) \times V/V_1$. In particular for $V_1 \subseteq A_G(\mathbb{R})^0$, we see that $G_\infty/(K_\infty \cdot V_1)$ is homotopic to $X_G = G_\infty/(K_\infty \cdot A_G(\mathbb{R})^0)$.

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