6

Backward stochastic differential equations and optimal control

6.1 Introduction

The theory of backward stochastic differential equations (BSDEs) was pioneered by Pardoux and Peng [PaPe90]. It became now very popular, and is an important field of research due to its connections with stochastic control, mathematical finance, and partial differential equations. BSDEs provide a probabilistic representation of nonlinear PDEs, which extends the famous Feynman-Kac formula for linear PDEs. As a consequence, BSDEs can be used for designing numerical algorithms to nonlinear PDEs.

This chapter is an introduction to the theory of BSDEs and its applications to mathematical finance and stochastic optimization. In Section 6.2, we state general results about existence and uniqueness of BSDEs, and useful comparison principles. Section 6.3 develops the connection between BSDEs and viscosity solutions to nonlinear PDEs. We show in Section 6.4 how BSDEs may be used for solving stochastic optimal control. Section 6.5 introduces the notion of reflected BSDEs, and shows how it is related to optimal stopping problems. Finally, Section 6.6 gives some illustrative examples of applications of BSDEs in finance.

6.2 General properties

6.2.1 Existence and uniqueness results

Let $W = (W_t)_{0 \leq t \leq T}$ be a standard d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the natural filtration of W, and T is a fixed finite horizon.

We denote by $\mathbb{S}^2(0,T)$ the set of real-valued progressively measurable processes Y such that

$$E\Big[\sup_{0 \leq t \leq T} |Y_t|^2\Big] < \infty,$$

and by $\mathbb{H}^2(0,T)^d$ the set of $\mathbb{R}^d$-valued progressively measurable processes Z such that

$$E\Big[\int_0^T |Z_t|^2 dt\Big] < \infty.$$

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We are given a pair (ξ, f), called the terminal condition and generator (or driver), satisfying:

- (A) $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$
- (B) $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ s.t.:

  - $f(., t, y, z)$, written for simplicity $f(t, y, z)$, is progressively measurable for all y, z

  - $f(t, 0, 0) \in \mathbb{H}^2(0,T)$

  - f satisfies a uniform Lipschitz condition in (y, z), i.e. there exists a constant C_f such that

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C_f \left(|y_1 - y_2| + |z_1 - z_2| \right), \quad \forall y_1, y_2, \ \forall z_1, z_2, \quad dt \otimes dP \ a.e.$$

We consider the (unidimensional) backward stochastic differential equations (BSDE):

$$-\, dY_t = f(t, Y_t, Z_t)dt - Z_t.dW_t, \quad Y_T = \xi. \tag{6.1}$$

Definition 6.2.1 *A solution to the BSDE (6.1) is a pair* $(Y, Z) \in \mathbb{S}^2(0,T) \times \mathbb{H}^2(0,T)^d$ *satisfying*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s.dW_s, \quad 0 \leq t \leq T.$$

We prove an existence and uniqueness result for the above BSDE.

Theorem 6.2.1 *Given a pair* (ξ, f) *satisfying (A) and (B), there exists a unique solution* (Y, Z) *to the BSDE (6.1).*

Proof. We give a proof based on a fixed point method. Let us consider the function Φ on $\mathbb{S}^2(0,T)^m \times \mathbb{H}^2(0,T)^d$, mapping $(U, V) \in \mathbb{S}^2(0,T) \times \mathbb{H}^2(0,T)^d$ to $(Y, Z) = \Phi(U, V)$ defined by

$$Y_t = \xi + \int_t^T f(s, U_s, V_s)ds - \int_t^T Z_s.dW_s. \tag{6.2}$$

More precisely, the pair (Y, Z) is constructed as follows: we consider the martingale $M_t = E[\xi + \int_0^T f(s, U_s, V_s)ds | \mathcal{F}_t]$, which is square integrable under the assumptions on (ξ, f). We may apply the Itô martingale representation theorem, which gives the existence and uniqueness of $Z \in \mathbb{H}^2(0,T)^d$ such that

$$M_t = M_0 + \int_0^t Z_s.dW_s. \tag{6.3}$$

We then define the process Y by

$$Y_t = E\Big[\xi + \int_t^T f(s, U_s, V_s)ds \Big| \mathcal{F}_t\Big] \ = \ M_t - \int_0^t f(s, U_s, V_s)ds, \quad 0 \leq t \leq T.$$

By using the representation (6.3) of M in the previous relation, and noting that $Y_T = \xi$, we see that Y satisfies (6.2). Observe by Doob's inequality that

$$E\Big[\sup_{0\le t\le T}\Big|\int_t^T Z_s.dW_s\Big|^2\Big] \le 4E\Big[\int_0^T |Z_s|^2 ds\Big] \;<\; \infty.$$

Under the conditions on (ξ, f), we deduce that Y lies in $\mathbb{S}^2(0,T)$. Hence, Φ is a well-defined function from $\mathbb{S}^2(0,T)\times\mathbb{H}^2(0,T)^d$ into itself. We then see that (Y,Z) is a solution to the BSDE (6.1) if and only if it is a fixed point of Φ.

Let (U,V), $(U',V') \in \mathbb{S}^2(0,T)\times\mathbb{H}^2(0,T)^d$ and $(Y,Z) = \Phi(U,V)$, $(Y',Z') = \Phi(U',V')$. We set $(\bar U,\bar V) = (U-U', V-V')$, $(\bar Y,\bar Z) = (Y-Y', Z-Z')$ and $\bar f_t = f(t,U_t,V_t) - f(t,U'_t,V'_t)$. Take some $\beta > 0$ to be chosen later, and apply Itô's formula to $e^{\beta s}|\bar Y_s|^2$ between $s=0$ and $s=T$:

$$\begin{aligned}|\bar Y_0|^2 = &-\int_0^T e^{\beta s}\left(\beta|\bar Y_s|^2 - 2\bar Y_s.\bar f_s\right) ds\\ &-\int_0^T e^{\beta s}|\bar Z_s|^2 ds - 2\int_0^T e^{\beta s}\bar Y_s' \bar Z_s.dW_s.\end{aligned} \tag{6.4}$$

Observe that

$$E\Big[\Big(\int_0^T e^{2\beta t}|Y_t|^2|Z_t|^2dt\Big)^{\frac12}\Big] \le \frac{e^{\beta T}}{2}E\Big[\sup_{0\le t\le T}|Y_t|^2 + \int_0^T |Z_t|^2dt\Big] \;<\; \infty,$$

which shows that the local martingale $\int_0^t e^{\beta s}\bar Y_s'\bar Z_s.dW_s$ is actually a uniformly integrable martingale from the Burkholder-Davis-Gundy inequality. By taking the expectation in (6.4), we get

$$\begin{aligned}E|\bar Y_0|^2 + E\Big[\int_0^T e^{\beta s}\Big(\beta|\bar Y_s|^2 + |\bar Z_s|^2\Big)ds\Big] \;&=\; 2E\Big[\int_0^T e^{\beta s}\bar Y_s.\bar f_s ds\Big]\\ \le 2C_f E\Big[\int_0^T e^{\beta s}|\bar Y_s|(|\bar U_s| + |\bar V_s|)ds\Big]&\\ \le 4C_f^2 E\Big[\int_0^T e^{\beta s}|\bar Y_s|^2ds\Big] \;+\; \frac12 E\Big[\int_0^T e^{\beta s}(|\bar U_s|^2 + |\bar V_s|^2)ds\Big]&\end{aligned}$$

Now, we choose $\beta = 1 + 4C_f^2$, and obtain

$$E\Big[\int_0^T e^{\beta s}\Big(|\bar Y_s|^2 + |\bar Z_s|^2\Big)ds\Big] \le \frac12 E\Big[\int_0^T e^{\beta s}(|\bar U_s|^2 + |\bar V_s|^2)ds\Big].$$

This shows that Φ is a strict contraction on the Banach space $\mathbb{S}^2(0,T)\times\mathbb{H}^2(0,T)^d$ endowed with the norm

$$\|(Y,Z)\|_{\beta} = \Big(E\Big[\int_0^T e^{\beta s}\big(|Y_s|^2 + |Z_s|^2\big)ds\Big]\Big)^{\frac12}.$$

We conclude that Φ admits a unique fixed point, which is the solution to the BSDE (6.1). □

6.2.2 Linear BSDE

We consider the particular case where the generator f is linear in y and z. The linear BSDE is written in the form

$$- dY_t = (A_t Y_t + Z_t.B_t + C_t)\, dt - Z_t.dW_t, \quad Y_T = \xi, \tag{6.5}$$

where A, B are bounded progressively measurable processes valued in $\mathbb{R}$ and $\mathbb{R}^d$, and C is a process in $\mathbb{H}^2(0,T)$. We can solve this BSDE explicitly.

Proposition 6.2.1 *The unique solution* (Y,Z) *to the linear BSDE* (6.5) *is given by*

$$\Gamma_t Y_t = E\Big[\Gamma_T \xi + \int_t^T \Gamma_s C_s ds \Big| \mathcal{F}_t\Big], \tag{6.6}$$

where Γ *is the adjoint (or dual) process, solution to the linear SDE*

$$d\Gamma_t = \Gamma_t \left(A_t dt + B_t.dW_t \right), \quad \Gamma_0 = 1.$$

Proof. By Itô's formula to $\Gamma_t Y_t$, we get

$$d(\Gamma_t Y_t) = -\Gamma_t C_t dt + \Gamma_t (Z_t + Y_t B_t).dW_t,$$

and so

$$\Gamma_t Y_t + \int_0^t \Gamma_s C_s ds = Y_0 + \int_0^t \Gamma_s (Z_s + Y_s B_s).dW_t. \tag{6.7}$$

Since A and B are bounded, we see that $E[\sup_t |\Gamma_t|^2] < \infty$, and by denoting by b_∞ the upper-bound of B, we have

$$\begin{aligned} E\Big[\Big(\int_0^T \Gamma_s^2 |Z_s + Y_s B_s|^2 ds\Big)^{\frac{1}{2}}\Big] &\leq \frac{1}{2} E\Big[\sup_t |\Gamma_t|^2 + 2\int_0^T |Z_t|^2 dt + 2b_\infty^2 \int_0^T |Y_t|^2 dt\Big] \\ &< \infty. \end{aligned}$$

From the Burkholder-Davis-Gundy inequality, this shows that the local martingale in (6.7) is a uniformly integrable martingale. By taking the expectation, we obtain

$$\begin{aligned} \Gamma_t Y_t + \int_0^t \Gamma_s C_s ds &= E\Big[\Gamma_T Y_T + \int_0^T \Gamma_s C_s ds \Big| \mathcal{F}_t\Big] \\ &= E\Big[\Gamma_T \xi + \int_0^T \Gamma_s C_s ds \Big| \mathcal{F}_t\Big], \end{aligned} \tag{6.8}$$

which gives the expression (6.6) for Y. Finally, Z is given via the Itô martingale representation (6.7) of the martingale in (6.8). □

6.2.3 Comparison principles

We state a very useful comparison principle for BSDEs.

Theorem 6.2.2 *Let* (ξ^1, f^1) *and* (ξ^2, f^2) *be two pairs of terminal conditions and generators satisfying conditions (A) and (B), and let* (Y^1, Z^1), (Y^2, Z^2) *be the solutions to their corresponding BSDEs. Suppose that:*

- $\xi^1 \leq \xi^2$ *a.s.*
- $f^1(t, Y_t^1, Z_t^1) \leq f^2(t, Y_t^1, Z_t^1)$ $dt \otimes dP$ *a.e.*

- $f^2(t, Y_t^1, Z_t^1) \in \mathbb{H}^2(0,T)$.

Then $Y_t^1 \leq Y_t^2$ *for all* $0 \leq t \leq T$, *a.s.*

Furthermore, if $Y_0^2 \leq Y_0^1$, *then* $Y_t^1 = Y_t^2$, $0 \leq t \leq T$. *In particular, if* $P(\xi^1 < \xi^2) > 0$ *or* $f^1(t,.,.) < f^2(t,.,.)$ *on a set of strictlly positive measure* $dt \otimes dP$, *then* $Y_0^1 < Y_0^2$.

Proof. To simplify the notation, we assume $d = 1$. We define $\bar{Y} = Y^2 - Y^1$, $\bar{Z} = Z^2 - Z^1$. Then $(\bar{Y}, \bar{Z})$ satisfies the linear BSDE

$$- d\bar{Y}_t = \left(\Delta_t^y \, \bar{Y}_t + \Delta_t^z \, \bar{Z}_t + \bar{f}_t\right) dt - \bar{Z}_t dW_t, \quad \bar{Y}_T = \xi^2 - \xi^1. \tag{6.9}$$

where

$$\begin{aligned}
\Delta_t^y &= \frac{f^2(t, Y_t^2, Z_t^2) - f^2(t, Y_t^1, Z_t^2)}{Y_t^2 - Y_t^1} \, 1_{Y_t^2 - Y_t^1 \neq 0} \\
\Delta_t^z &= \frac{f^2(t, Y_t^1, Z_t^2) - f^2(t, Y_t^1, Z_t^1)}{Z_t^2 - Z_t^1} \, 1_{Z_t^2 - Z_t^1 \neq 0} \\
\bar{f}_t &= f^2(t, Y_t^1, Z_t^1) - f^1(t, Y_t^1, Z_t^1).
\end{aligned}$$

Since the generator f^2 is uniformly Lipschitz in y and z, the processes Δ^y and Δ^z are bounded. Moreover, $\bar{f}_t$ is a process in $\mathbb{H}^2(0,T)$. From 6.2.1, $\bar{Y}$ is then given by

$$\Gamma_t \bar{Y}_t = E\Big[\Gamma_T(\xi^2 - \xi^1) + \int_t^T \Gamma_s \bar{f}_s ds \Big| \mathcal{F}_t\Big],$$

where the adjoint process Γ is strictly positive. We conclude from this expectation formula for Y, and the positivity of $\xi^2 - \xi^1$ and $\bar{f}$. □

Remark 6.2.1 Notice that in the proof of Theorem 6.2.2, it is not necessary to suppose regularity conditions on the generator f_1. The uniform Lipschitz condition is only required for f_2.

Corollary 6.2.1 *If the pair* (ξ, f) *satisfies* $\xi \geq 0$ *a.s. and* $f(t,0,0) \geq 0$ $dt \otimes dP$ *a.e., then* $Y_t \geq 0$, $0 \leq t \leq T$ *a.s. Moreover, if* $P[\xi > 0] > 0$ *or* $f(t,0,0) > 0$ $dt \otimes dP$ *a.e., then* $Y_0 > 0$.

Proof. This is an immediate consequence of the comparison theorem 6.2.2 with $(\xi^1, f^1) = (0,0)$, whose solution is obviously $(Y^1, Z^1) = (0,0)$. □

6.3 BSDE, PDE and nonlinear Feynman-Kac formulae

We recall the well-known result (see Section 1.3.3) that the solution to the linear parabolic PDE

$$\begin{gathered}
-\frac{\partial v}{\partial t} - \mathcal{L}v - f(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n, \\
v(T,x) = g(x), \quad x \in \mathbb{R}^n,
\end{gathered}$$

has the probabilistic Feynman-Kac representation

$$v(t,x) = E\Big[\int_t^T f(s,X_s^{t,x})ds + g(X_T^{t,x})\Big], \tag{6.10}$$

where $\{X_s^{t,x}, t \le s \le T\}$ is the solution to the SDE

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s, \quad t \le s \le T,\ X_t = x,$$

and $\mathcal{L}$ is the second-order operator

$$\mathcal{L}v = b(x).D_x v + \frac{1}{2}\mathrm{tr}(\sigma(x)\sigma'(x)D_{xx}^2 v).$$

In the previous chapter, we derived a generalization of this linear Feynman-Kac formula for nonlinear PDEs in the form

$$-\frac{\partial v}{\partial t} - \sup_{a\in A}\big[\mathcal{L}^a v + f(t,x,a)\big] = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n, \tag{6.11}$$

$$v(T,x) = g(x), \quad x \in \mathbb{R}^n, \tag{6.12}$$

where, for any $a \in A$, subset of $\mathbb{R}^m$,

$$\mathcal{L}^a v = b(x,a).D_x v + \frac{1}{2}\mathrm{tr}(\sigma(x,a)\sigma'(x,a)D_{xx}^2 v).$$

The solution (in the viscosity sense) to (6.11)-(6.12) may be represented by means of a stochastic control problem as

$$v(t,x) = \sup_{\alpha\in\mathcal{A}} E\Big[\int_t^T f(s,X_s^{t,x},\alpha_s)ds + g(X_T^{t,x})\Big],$$

where $\mathcal{A}$ is the set of progressively measurable processes α valued in A, and for $\alpha \in \mathcal{A}$, $\{X_s^{t,x}, t \le s \le T\}$ is the controlled diffusion

$$dX_s = b(X_s,\alpha_s)ds + \sigma(X_s,\alpha_s)dW_s, \quad t \le s \le T,\ X_t = x.$$

In this chapter, we study another extension of Feynman-Kac formula for semilinear PDE in the form

$$-\frac{\partial v}{\partial t} - \mathcal{L}v - f(t,x,v,\sigma' D_x v) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n, \tag{6.13}$$

$$v(T,x) = g(x), \quad x \in \mathbb{R}^n. \tag{6.14}$$

We shall represent the solution to this PDE by means of the BSDE

$$-dY_s = f(s,X_s,Y_s,Z_s)ds - Z_s.dW_s, \quad t \le s \le T, \ \ Y_T = g(X_T), \tag{6.15}$$

and the forward SDE valued in $\mathbb{R}^n$:

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s. \tag{6.16}$$

The functions b and σ satisfy a Lipschitz condition on $\mathbb{R}^n$; f is a continuous function on $[0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$ satisfying a linear growth condition in (x,y,z), and a Lipschitz condition in (y,z), uniformly in (t,x). The continuous function g satisfies a linear growth condition. Hence, by a standard estimate on the second moment of X, we see that the terminal condition and the generator of the BSDE (6.15) satisfy the conditions (A) and (B) stated in Section 6.2. By the Markov property of the diffusion X, and uniqueness of a solution (Y, Z, K) to the BSDE (6.15), we notice that $Y_t = v(t, X_t)$, $0 \leq t \leq T$, where

$$v(t,x) := Y_t^{t,x} \tag{6.17}$$

is a deterministic function of (t,x) in $[0,T] \times \mathbb{R}^n$, $\{X_s^{t,x}, t \leq s \leq T\}$ is the solution to (6.16) starting from x at t, and $\{(Y_s^{t,x}, Z_s^{t,x}), t \leq s \leq T\}$ is the solution to the BSDE (6.15) with $X_s = X_s^{t,x}$, $t \leq s \leq T$. We call this framework a Markovian case for the BSDE.

The next verification result for the PDE (6.13) is analogous to the verification theorem for Hamilton-Jacobi-Bellman equations (6.11), and shows that a classical solution to the semilinear PDE provides a solution to the BSDE.

Proposition 6.3.2 *Let $v \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap C^0([0,T] \times \mathbb{R}^n)$ be a classical solution to (6.13)-(6.14), satisfying a linear growth condition and such that for some positive constants C, q, $|D_x v(t,x)| \leq C(1 + |x|^q)$ for all $x \in \mathbb{R}^n$. Then, the pair (Y, Z) defined by*

$$Y_t = v(t, X_t), \;\; Z_t = \sigma'(X_t) D_x v(t, X_t), \quad 0 \leq t \leq T,$$

is the solution to the BSDE (6.15).

Proof. This is an immediate consequence of Itô's formula applied to $v(t, X_t)$, and noting from the growth conditions on v, $D_x v$ that (Y, Z) lie in $\mathbb{S}^2(0,T) \times \mathbb{H}^2(0,T)^d$. □

We now study the converse property by proving that the solution to the BSDE (6.15) provides a solution to the PDE (6.13)-(6.14).

Theorem 6.3.3 *The function $v(t,x) = Y_t^{t,x}$ in (6.17) is a continuous function on $[0,T] \times \mathbb{R}^n$, and is a viscosity solution to (6.13)-(6.14).*

Proof. 1) For (t_1, x_1), $(t_2, x_2) \in [0,T] \times \mathbb{R}^n$, with $t_1 \leq t_2$, we write $X_s^i = X_s^{t_i, x_i}$, $i = 1, 2$, with the convention that $X_s^2 = x_2$ if $t_1 \leq s \leq t_2$, and $(Y_s^i, Z_s^i) = (Y_s^{t_i,x_i}, Z_s^{t_i,x_i})$, $i = 1, 2$, which is then well-defined for $t_1 \leq s \leq T$. By applying Itô's formula to $|Y_s^1 - Y_s^2|^2$ between $s = t \in [t_1, T]$ and $s = T$, we get

$$\begin{aligned} |Y_t^1 - Y_t^2|^2 = |g(X_T^1) - g(X_T^2)|^2 - \int_t^T |Z_s^1 - Z_s^2|^2 ds \\ + 2 \int_t^T (Y_s^1 - Y_s^2).(f(s, X_s^1, Y_s^1, Z_s^1) - f(s, X_s^2, Y_s^2, Z_s^2)) ds \\ - 2 \int_t^T (Y_s^1 - Y_s^2)'(Z_s^1 - Z_s^2) dW_s. \end{aligned}$$

As in the proof of Theorem 6.2.1, the local martingale $\int_t^s (Y_u^1 - Y_u^2)'(Z_u^1 - Z_u^2) dW_u$, $t \leq s \leq T$, is actually uniformly integrable, and so by taking the expectation in the above relation, we derive

$$
\begin{aligned}
& E\Big[|Y_t^1 - Y_t^2|^2\Big] + E\Big[\int_t^T |Z_s^1 - Z_s^2|^2 ds\Big] \\
= \; & E\Big[|g(X_T^1) - g(X_T^2)|^2\Big] \\
& + 2\, E\Big[\int_t^T (Y_s^1 - Y_s^2).(f(s, X_s^1, Y_s^1, Z_s^1) - f(s, X_s^2, Y_s^2, Z_s^2))ds\Big] \\
\leq \; & E\Big[|g(X_T^1) - g(X_T^2)|^2\Big] \\
& + 2\, E\Big[\int_t^T |Y_s^1 - Y_s^2| \; |f(s, X_s^1, Y_s^1, Z_s^1) - f(s, X_s^2, Y_s^1, Z_s^1)|ds\Big] \\
& + 2C_f E\Big[\int_t^T |Y_s^1 - Y_s^2| \left(|Y_s^1 - Y_s^2| + |Z_s^1 - Z_s^2|\right) ds\Big] \\
\leq \; & E\Big[|g(X_T^1) - g(X_T^2)|^2\Big] \\
& + E\Big[\int_t^T |f(s, X_s^1, Y_s^1, Z_s^1) - f(s, X_s^2, Y_s^1, Z_s^1)|^2 ds\Big] \\
& + (1 + 4C_f^2) E\Big[\int_t^T |Y_s^1 - Y_s^2|^2 ds + \frac{1}{2} E \int_t^T |Z_s^1 - Z_s^2|^2 ds\Big],
\end{aligned}
$$

where C_f is the uniform Lipschitz constant of f with respect to y and z. This yields

$$
\begin{aligned}
E\Big[|Y_t^1 - Y_t^2|^2\Big] \leq E\Big[|g(X_T^1) - g(X_T^2)|^2\Big] + E\Big[\int_t^T |f(s, X_s^1, Y_s^1, Z_s^1) - f(s, X_s^2, Y_s^1, Z_s^1)|^2 ds\Big] \\
+ (1 + 4C_f^2) E\Big[\int_t^T |Y_s^1 - Y_s^2|^2 ds\Big]
\end{aligned}
$$

and so, by Gronwall's lemma

$$
\begin{aligned}
E\Big[|Y_t^1 - Y_t^2|^2\Big] \leq C \Big\{ E\Big[|g(X_T^1) - g(X_T^2)|^2\Big] \\
+ E\Big[\int_t^T |f(s, X_s^1, Y_s^1, Z_s^1) - f(s, X_s^2, Y_s^1, Z_s^1)|^2 ds\Big] \Big\}.
\end{aligned}
$$

This last inequality, combined with continuity of f and g in x, continuity of $X^{t,x}$ in (t, x), shows the mean-square continuity of $\{Y_s^{t,x}, x \in \mathbb{R}^n, 0 \leq t \leq s \leq T\}$, and so the continuity of $(t, x) \to v(t, x) = Y_t^{t,x}$. The terminal condition (6.14) is trivially satisfied.

2) We next show that $v(t, x) = Y_t^{t,x}$ is a viscosity solution to (6.13). We check the viscosity subsolution property, the viscosity supersolution property is then proved similarly. Let φ be a smooth test function and $(t, x) \in [0, T) \times \mathbb{R}^n$ such that (t, x) is a local maximum of $v - \varphi$ with $u(t, x) = \varphi(t, x)$. We argue by contradiction by assuming that

$$
-\frac{\partial \varphi}{\partial t}(t, x) - \mathcal{L}\varphi(t, x) - f(t, x, v(t, x), (D_x\varphi)'(t, x)\sigma(x)) > 0.
$$

By continuity of f, φ and its derivatives, there exists h, $\varepsilon > 0$ such that for all $t \leq s \leq t + h$, $|x - y| \leq \varepsilon$,

$$
v(s, y) \leq \varphi(s, y) \tag{6.18}
$$

$$
-\frac{\partial \varphi}{\partial t}(s, y) - \mathcal{L}\varphi(s, y) - f(s, y, v(s, y), (D_x\varphi)'(s, y)\sigma(y)) \; > \; 0. \tag{6.19}
$$

Let $\tau = \inf\{s \geq t : |X_s^{t,x} - x| \geq \varepsilon\} \wedge (t+h)$, and consider the pair

$$(Y_s^1, Z_s^1) = (Y_{s\wedge\tau}^{t,x}, 1_{[0,\tau]}(s)Z_s^{t,x}), \quad t \leq s \leq t+h.$$

By construction, (Y_s^1, Z_s^1) solves the BSDE

$$\begin{aligned} -dY_s^1 &= 1_{[0,\tau]}(s)f(s, X_s^{t,x}, u(s, X_s^{t,x}), Z_s^1)ds - Z_s^1 dW_s, \quad t \leq s \leq t+h, \\ Y_{t+h}^1 &= u(\tau, X_\tau^{t,x}). \end{aligned}$$

On the other hand, the pair

$$(Y_s^2, Z_s^2) = (\varphi(s, X_{s\wedge\tau}^{t,x}), 1_{[0,\tau]}(s)D_x\varphi(s, X_s^{t,x})'\sigma(X_s^{t,x})), \quad t \leq s \leq t+h.$$

satisfies, by Itô's formula, the BSDE

$$\begin{aligned} -dY_s^2 &= -1_{[0,\tau]}(s)(\frac{\partial\varphi}{\partial t} + \mathcal{L}\varphi)(s, X_s^{t,x}) - Z_s^2 dW_s, \quad t \leq s \leq t+h, \\ Y_{t+h}^1 &= \varphi(\tau, X_\tau^{t,x}). \end{aligned}$$

From the inequalities (6.18)-(6.19), and the strict comparison principle in Theorem 6.2.2, we deduce $Y_0^1 < Y_0^2$, i.e. $u(t,x) < \varphi(t,x)$, a contradiction. □

6.4 Control and BSDE

In this section, we show how BSDEs may be used for dealing with stochastic control problems.

6.4.1 Optimization of a family of BSDEs

Theorem 6.4.4 *Let (ξ, f) and (ξ^α, f^α), $\alpha \in \mathcal{A}$ subset of progressively measurable processes, be a family of the pair terminal condition-generator, and (Y, Z), (Y^α, Z^α) the solutions to their associated BSDEs. Suppose that there exists $\hat\alpha \in \mathcal{A}$ such that*

$$\begin{aligned} f(t, Y_t, Z_t) = \operatorname*{ess\,inf}_\alpha f^\alpha(t, Y_t, Z_t) &= f^{\hat\alpha}(t, Y_t, Z_t), \quad dt \otimes dP \ \ a.e. \\ \xi = \operatorname*{ess\,inf}_\alpha \xi^\alpha &= \xi^{\hat\alpha}. \end{aligned}$$

Then,

$$Y_t = \operatorname*{ess\,inf}_\alpha Y_t^\alpha = Y_t^{\hat\alpha}, \quad 0 \leq t \leq T, \ a.s.$$

Proof. From the comparison theorem 6.2.2, since $\xi \leq \xi^\alpha$ and $f(t, Y_t, Z_t) \leq f^\alpha(t, Y_t, Z_t)$, we have $Y_t \leq Y_t^\alpha$ for all α, and so

$$Y_t \leq \operatorname*{ess\,inf}_\alpha Y_t^\alpha.$$

Moreover, if there exists $\hat\alpha$ such that $\xi = \xi^{\hat\alpha}$ and $f(t, Y_t, Z_t) = f^{\hat\alpha}(t, Y_t, Z_t)$, then (Y, Z) and $(Y^{\hat\alpha}, Z^{\hat\alpha})$ are both solutions to the same BSDE with terminal condition-generator: $(\xi^{\hat\alpha}, f^{\hat\alpha})$. By uniqueness, we deduce that these solutions coincide, and so

$$\operatorname{ess\,inf}_{\alpha} Y_t^{\alpha} \leq \; Y_t^{\hat{\alpha}} \; = \; Y_t \; \leq \operatorname{ess\,inf}_{\alpha} Y_t^{\alpha},$$

which ends the proof. □

By means of the above result, we show how the solution to a BSDE with concave generator may be represented as the value function of a control problem.

Let $f(t,y,z)$ be a generator, concave in (y,z) and (Y,Z) the solution to the BSDE associated to the pair (ξ, f). We consider the Fenchel-Legendre transform of f

$$F(t,b,c) = \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d} \left[f(t,y,z) - yb - z.c \right], \quad (b,c) \in \mathbb{R}\times\mathbb{R}^d. \tag{6.20}$$

Since f is concave, we have the duality relation

$$f(t,y,z) = \inf_{(b,c)\in\mathbb{R}\times\mathbb{R}^d} \left[F(t,b,c) + yb + z.c \right], \quad (y,z) \in \mathbb{R}\times\mathbb{R}^d. \tag{6.21}$$

We denote by $\mathcal{A}$ the set of bounded progressively measurable processes (β,γ), valued in $\mathbb{R}\times\mathbb{R}^d$ such that

$$E\Big[\int_0^T |F(t,\beta_t,\gamma_t)|^2 dt \Big] < \infty.$$

The boundedness condition on $\mathcal{A}$ means that for any $(\beta,\gamma) \in \mathcal{A}$, there exists a constant (dependent of (β,γ)) such that $|\beta_t| + |\gamma_t| \leq C$, $dt\otimes dP$ a.e. Let us consider the family of linear generators

$$f^{\beta,\gamma}(t,y,z) = F(t,\beta_t,\gamma_t) + y\beta_t + z.\gamma_t, \quad (\beta,\gamma) \in \mathcal{A}.$$

Given $(\beta,\gamma) \in \mathcal{A}$, we denote by $(Y^{\beta,\gamma}, Z^{\beta,\gamma})$ the solution to the linear BSDE associated to the pair $(\xi, f^{\beta,\gamma})$.

Theorem 6.4.5 *Y is equal to the value function of the control problem*

$$Y_t = \operatorname{ess}\inf_{\beta,\gamma\in\mathcal{A}} Y_t^{\beta,\gamma}, \quad 0 \leq t \leq T, \ a.s. \tag{6.22}$$

$$Y_t^{\beta,\gamma} = E^{Q^{\gamma}}\Big[\int_t^T e^{\int_t^s \beta_u du} F(s,\beta_s,\gamma_s) ds + e^{\int_t^T \beta_u du} \xi \Big| \mathcal{F}_t \Big],$$

where Q^{γ} is the probability measure with density process

$$dL_t = L_t \gamma_t . dW_t, \quad L_0 \; = \; 1.$$

Proof. (1) Observe from relation (6.21) that $f(t,Y_t,Z_t) \leq f^{\beta,\gamma}(t,Y_t,Z_t)$ for all (β,γ) $\in \mathcal{A}$. Moreover, since F is convex with a linear growth condition, for each (t,ω,y,z), the infimum in the relation (6.21) is attained at $(\hat{b}(t,y,z), \hat{c}(t,y,z))$ belonging to the subdifferential of $-f$, and so is bounded by the Lipschitz constant of f. By a measurable selection theorem (see e.g. Appendix in Chapter III of Dellacherie and Meyer [DM75]), since Y,Z are progressively measurable, we may find a pair of bounded progressively measurable processes $(\hat{\beta},\hat{\gamma})$ such that

$$f(t, Y_t, Z_t) = f^{\hat{\beta},\hat{\gamma}}(t, Y_t, Z_t) \; = \; F(t, \hat{\beta}_t, \hat{\gamma}_t) + Y_t\hat{\beta}_t + Z_t.\hat{\gamma}_t, \quad 0 \leq t \leq T, \; a.s.$$

We then obtain the relation (6.22) by Theorem 6.4.4.

(2) Moreover, by Proposition 6.2.1, the solution $Y^{\beta,\gamma}$ to the linear BSDE associated to the pair $(\xi, f^{\beta,\gamma})$ is explicitly written as

$$\Gamma_t Y_t^{\beta,\gamma} = E\Big[\int_t^T \Gamma_s F(s, \beta_s, \gamma_s)ds + \Gamma_T\xi\Big|\mathcal{F}_t\Big],$$

where Γ is the adjoint (dual) process given by the SDE:

$$d\Gamma_t = \Gamma_t\left(\beta_t dt + \gamma_t.dW_t\right), \quad \Gamma_0 = 1.$$

We conclude by observing that $\Gamma_t = e^{\int_0^t \beta_u du}L_t$, and using the Bayes formula. □

6.4.2 Stochastic maximum principle

In the previous chapter, we studied how to solve a stochastic control problem by the dynamic programming method. We present here an alternative approach, called Pontryagin maximum principle, and based on optimality conditions for controls.

We consider the framework of a stochastic control problem on a finite horizon as defined in Chapter 3: let X be a controlled diffusion on $\mathbb{R}^n$ governed by

$$dX_s = b(X_s, \alpha_s)ds + \sigma(X_s, \alpha_s)dW_s, \tag{6.23}$$

where W is a d-dimensional standard Brownian motion, and $\alpha \in \mathcal{A}$, the control process, is a progressively measurable valued in A. The gain functional to maximize is

$$J(\alpha) = E\Big[\int_0^T f(t, X_t, \alpha_t)dt + g(X_T)\Big],$$

where $f : [0,T] \times \mathbb{R}^n \times A \to \mathbb{R}$ is continuous in (t,x) for all a in A, $g : \mathbb{R}^n \to \mathbb{R}$ is a concave C^1 function, and f, g satisfy a quadratic growth condition in x.

We define the generalized Hamiltonian $\mathcal{H} : [0,T] \times \mathbb{R}^n \times A \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ by

$$\mathcal{H}(t, x, a, y, z) = b(x, a).y + \mathrm{tr}(\sigma'(x, a)z) + f(t, x, a), \tag{6.24}$$

and we assume that $\mathcal{H}$ is differentiable in x with derivative denoted by $D_x\mathcal{H}$. We consider for each $\alpha \in \mathcal{A}$, the BSDE, called the adjoint equation:

$$-\,dY_t = D_x\mathcal{H}(t, X_t, \alpha_t, Y_t, Z_t)dt - Z_t dW_t, \quad Y_T \; = \; D_x g(X_T). \tag{6.25}$$

Theorem 6.4.6 *Let $\hat{\alpha} \in \mathcal{A}$ and $\hat{X}$ the associated controlled diffusion. Suppose that there exists a solution $(\hat{Y}, \hat{Z})$ to the associated BSDE (6.25) such that*

$$\mathcal{H}(t, \hat{X}_t, \hat{\alpha}_t, \hat{Y}_t, \hat{Z}_t) = \max_{a\in A}\mathcal{H}(t, \hat{X}_t, a, \hat{Y}_t, \hat{Z}_t), \quad 0 \leq t \leq T, \;\; a.s. \tag{6.26}$$

and

$$(x, a) \to \mathcal{H}(t, x, a, \hat{Y}_t, \hat{Z}_t) \quad \textit{is a concave function,} \tag{6.27}$$

for all $t \in [0,T]$. Then $\hat{\alpha}$ is an optimal control, i.e.

$$J(\hat{\alpha}) = \sup_{\alpha\in\mathcal{A}} J(\alpha).$$

Proof. For any $\alpha \in \mathcal{A}$, we write

$$J(\hat{\alpha}) - J(\alpha) = E\Big[\int_0^T f(t, \hat{X}_t, \hat{\alpha}_t) - f(t, X_t, \alpha_t) dt + g(\hat{X}_T) - g(X_T) \Big]. \tag{6.28}$$

By concavity of g and Itô's formula, we have

$$\begin{aligned}
& E\left[g(\hat{X}_T) - g(X_T) \right] \;\geq\; E\left[(\hat{X}_T - X_T).D_x g(\hat{X}_T) \right] \;=\; E\left[(\hat{X}_T - X_T).\hat{Y}_T \right] \\
&= E\Big[\int_0^T (\hat{X}_t - X_t).d\hat{Y}_t + \int_0^T \hat{Y}_t.(d\hat{X}_t - dX_t) + \int_0^T \mathrm{tr}\big[(\sigma(\hat{X}_t, \hat{\alpha}_t) - \sigma(X_t, \alpha_t))' \hat{Z}_t \big] dt \Big] \\
&= E\Big[\int_0^T (\hat{X}_t - X_t).(-D_x \mathcal{H}(t, \hat{X}_t, \hat{\alpha}_t, \hat{Y}_t, \hat{Z}_t)) dt + \int_0^T \hat{Y}_t.(b(\hat{X}_t, \hat{\alpha}_t) - b(X_t, \alpha_t)) dt \\
&\qquad + \int_0^T \mathrm{tr}\big[(\sigma(\hat{X}_t, \hat{\alpha}_t) - \sigma(X_t, \alpha_t))' \hat{Z}_t \big] dt \Big].
\end{aligned} \tag{6.29}$$

Moreover, by definition of $\mathcal{H}$, we have

$$\begin{aligned}
E\Big[\int_0^T f(t, \hat{X}_t, \hat{\alpha}_t) - f(t, X_t, \alpha_t) dt \Big] &= E\Big[\int_0^T \mathcal{H}(t, \hat{X}_t, \hat{\alpha}_t, \hat{Y}_t, \hat{Z}_t) - \mathcal{H}(t, X_t, \alpha_t, \hat{Y}_t, \hat{Z}_t) dt \\
&\qquad - \int_0^T (b(\hat{X}_t, \hat{\alpha}_t) - b(X_t, \alpha_t)).\hat{Y}_t \\
&\qquad - \int_0^T \mathrm{tr}\left[(\sigma(\hat{X}_t, \hat{\alpha}_t) - \sigma(X_t, \alpha_t))' \hat{Z}_t \right] dt \Big].
\end{aligned} \tag{6.30}$$

By adding (6.29) and (6.30) into (6.28), we obtain

$$\begin{aligned}
J(\hat{\alpha}) - J(\alpha) \geq E\Big[& \int_0^T \mathcal{H}(t, \hat{X}_t, \hat{\alpha}_t, \hat{Y}_t, \hat{Z}_t) - \mathcal{H}(t, X_t, \alpha_t, \hat{Y}_t, \hat{Z}_t) dt \\
& - \int_0^T (\hat{X}_t - X_t).D_x \mathcal{H}(t, \hat{X}_t, \hat{\alpha}_t, \hat{Y}_t, \hat{Z}_t) dt \Big].
\end{aligned}$$

Under the conditions (6.26) and (6.27), the term between the bracket in the above relation is nonpositive, which ends the proof. □

We shall illustrate in Section 6.6.2 how to use Theorem 6.4.6 for solving a control problem in finance arising in mean-variance hedging.

We conclude this section by providing the connection between maximum principle and dynamic programming. The value function of the stochastic control problem considered above is defined by

$$v(t, x) = \sup_{\alpha \in \mathcal{A}} E\Big[\int_t^T f(s, X_s^{t,x}, \alpha_s) ds + g(X_T^{t,x}) \Big], \tag{6.31}$$

where $\{X_s^{t,x}, t \leq s \leq T\}$ is the solution to (6.23) starting from x at t. Recall that the associated Hamilton-Jacobi-Bellman equation is

$$-\frac{\partial v}{\partial t} - \sup_{a \in A} \left[\mathcal{G}(t, x, a, D_x v, D_x^2 v) \right] = 0, \tag{6.32}$$

where for $(t, x, a, p, M) \in [0,T] \times \mathbb{R}^n \times A \times \mathbb{R}^n \times \mathcal{S}_n$,

$$\mathcal{G}(t, x, a, p, M) = b(x, a).p + \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x, a)M) + f(t, x, a). \tag{6.33}$$

Theorem 6.4.7 *Suppose that $v \in C^{1,3}([0,T) \times \mathbb{R}^n) \cap C^0([0,T] \times \mathbb{R}^n)$, and there exists an optimal control $\hat{\alpha} \in \mathcal{A}$ to (6.31) with associated controlled diffusion $\hat{X}$. Then*

$$\mathcal{G}(t, \hat{X}_t, \hat{\alpha}_t, D_x v(t, \hat{X}_t), D_x^2 v(t, \hat{X}_t)) = \max_{a \in A} \mathcal{G}(t, \hat{X}_t, a, D_x v(t, \hat{X}_t), D_x^2 v(t, \hat{X}_t)), \tag{6.34}$$

and the pair

$$(\hat{Y}_t, \hat{Z}_t) = (D_x v(t, \hat{X}_t)\ ,\ D_x^2 v(t, \hat{X}_t)\ \sigma(\hat{X}_t, \hat{\alpha}_t)), \tag{6.35}$$

is solution to the adjoint BSDE (6.25).

Proof. Since $\hat{\alpha}$ is an optimal control, we have

$$\begin{aligned} v(t, \hat{X}_t) &= E\Big[\int_t^T f(s, \hat{X}_s, \hat{\alpha}_s)ds + g(\hat{X}_T)\Big|\mathcal{F}_t\Big] \\ &= -\int_0^t f(s, \hat{X}_s, \hat{\alpha}_s)ds + M_t, \quad 0 \le t \le T,\ a.s. \end{aligned} \tag{6.36}$$

where M is the martingale $M_t = E\Big[\int_0^T f(s, \hat{X}_s, \hat{\alpha}_s)ds + g(\hat{X}_T)\Big|\mathcal{F}_t\Big]$. By applying Itô's formula to $v(t, \hat{X}_t)$, and identifying the terms in dt in relation (6.36), we get

$$-\frac{\partial v}{\partial t}(t, \hat{X}_t)\ -\ \mathcal{G}(t, \hat{X}_t, \hat{\alpha}_t, D_x v(t, \hat{X}_t), D_x^2 v(t, \hat{X}_t)) = 0. \tag{6.37}$$

Since v is smooth, it satisfies the HJB equation (6.32), which yields (6.34).

From (6.32) and (6.37), we have

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t}(t, \hat{X}_t)\ +\ \mathcal{G}(t, \hat{X}_t, \hat{\alpha}_t, D_x v(t, \hat{X}_t), D_x^2 v(t, \hat{X}_t)) \\ &\ge \frac{\partial v}{\partial t}(t, x)\ +\ \mathcal{G}(t, x, \hat{\alpha}_t, D_x v(t, x), D_x^2 v(t, x)), \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

Since v is $C^{1,3}$, the optimality condition for the above relation implies

$$\frac{\partial}{\partial x}\ \left(\frac{\partial v}{\partial t}(t, x)\ +\ \mathcal{G}(t, x, \hat{\alpha}_t, D_x v(t, x), D_x^2 v(t, x))\right)\Big|_{x = \hat{X}_t} = 0.$$

By recalling the expressions (6.33) and (6.24) of $\mathcal{G}$ and $\mathcal{H}$, the previous equality is written as

$$\begin{aligned} &\frac{\partial^2 v}{\partial t \partial x}(t, \hat{X}_t) + D_x^2 v(t, \hat{X}_t) b(\hat{X}_t, \hat{\alpha}_t) + \frac{1}{2}\mathrm{tr}(\sigma\sigma'(\hat{X}_t, \hat{\alpha}_t) D_x^3 v(t, \hat{X}_t)) \\ &\qquad + D_x \mathcal{H}(t, \hat{X}_t, \hat{\alpha}_t, D_x v(t, \hat{X}_t), D_x^2 v(t, \hat{X}_t)\ \sigma(\hat{X}_t, \hat{\alpha}_t)) = 0. \end{aligned} \tag{6.38}$$

By applying Itô's formula to $D_x v(t, \hat{X}_t)$, and using (6.38), we then get

$$
\begin{aligned}
-dD_x v(t, \hat{X}_t) = & -\left[\frac{\partial^2 v}{\partial t \partial x}(t, \hat{X}_t) + D_x^2 v(t, \hat{X}_t) b(\hat{X}_t, \hat{\alpha}_t) + \frac{1}{2}\mathrm{tr}(\sigma\sigma'(\hat{X}_t, \hat{\alpha}_t) D_x^3 v(t, \hat{X}_t))\right] dt \\
& - D_x^2 v(t, \hat{X}_t)\, \sigma(\hat{X}_t, \hat{\alpha}_t) dW_t \\
= & \, D_x \mathcal{H}(t, \hat{X}_t, \hat{\alpha}_t, D_x v(t, \hat{X}_t), D_x^2 v(t, \hat{X}_t)\, \sigma(\hat{X}_t, \hat{\alpha}_t))\, dt \\
& - \ D_x^2 v(t, \hat{X}_t)\, \sigma(\hat{X}_t, \hat{\alpha}_t) dW_t.
\end{aligned}
$$

Moreover, since $v(T, .) = g(.)$, we have

$$
D_x v(T, \hat{X}_T) = D_x g(\hat{X}_T),
$$

and this proves the result (6.35). □

6.5 Reflected BSDEs and optimal stopping problems

We consider a class of BSDEs where the solution Y is constrained to stay above a given process, called obstacle. An increasing process is introduced for pushing the solution upwards, above the obstacle. This leads to the notion of reflected BSDE, which is formalized as follows.

Let $W = (W_t)_{0 \leq t \leq T}$ be a standard d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the natural filtration of W, and T is a fixed finite horizon. We are given a pair (ξ, f) satisfying conditions (A) and (B), and in addition a continuous process $(L_t)_{0 \leq t \leq T}$, satisfying $\xi \geq L_T$ and

(C) $\qquad L \in \mathbb{S}^2(0, T), \quad$ i.e. $\quad E[\sup_{0 \leq t \leq T} |L_t|^2] < \infty.$

A solution to the reflected BSDE with terminal condition-generator (ξ, f) and obstacle L is a triple (Y, Z, K) of progressively measurable processes valued in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+$ such that $Y \in \mathbb{S}^2(0, T)$, $Z \in \mathbb{H}^2(0, T)^d$, K is a continuous increasing process, $K_0 = 0$, and satisfying

$$
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s . dW_s, \quad 0 \leq t \leq T \tag{6.39}
$$

$$
Y_t \geq L_t, \quad 0 \leq t \leq T, \tag{6.40}
$$

$$
\int_0^T (Y_t - L_t) dK_t = 0. \tag{6.41}
$$

Remark 6.5.2 The condition (6.41) means that the push of the increasing process K is minimal in the sense that it is active only when the constraint is saturated, i.e. when $Y_t = L_t$. There is another formulation of this minimality condition for defining a solution to a reflected BSDE: we say that (Y, Z, K) is a minimal solution to the reflected BSDE, if it satisfies (6.39)-(6.40), and for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{K})$ satisfying (6.39)-(6.40), we have $Y_t \leq \tilde{Y}_t$, $0 \leq t \leq T$ a.s. We shall discuss the equivalence of this formulation in Remark 6.5.3.

In the special case where the generator f does not depend on y, z, the notion of a reflected BSDE is directly related to optimal stopping problems, as stated in the following proposition.

Proposition 6.5.3 *Suppose that f does not depend on y, z, and $f \in \mathbb{H}^2(0,T)$. Then, there exists a unique solution (Y, Z, K) to the reflected BSDE (6.39), (6.40) and (6.41), and Y has the explicit optimal stopping time representation*

$$Y_t = \operatorname{ess}\sup_{\tau \in \mathcal{T}_{t,T}} E\Big[\int_t^\tau f(s)ds + L_\tau 1_{\tau<T} + \xi 1_{\tau=T}\Big|\mathcal{F}_t\Big], \quad 0 \leq t \leq T. \tag{6.42}$$

Proof. Let us consider the process Y defined by (6.42), and observe that $Y_t + \int_0^t f(s)ds$ is the Snell envelope of the process

$$H_t = \int_0^t f(s)ds + L_t 1_{t<T} + \xi 1_{t=T}, \quad 0 \leq t \leq T.$$

From the conditions (A), (B), and (C) on f, ξ and L, and since $\xi \geq L_T$, we notice that the process H is continuous on $[0,T)$, with a positive jump at T, and satisfies $E[\sup_{0\leq t\leq T}|H_t|^2] < \infty$, in particular of class (DL). Hence, by Proposition 1.1.8, the process $Y_t + \int_0^t f(s)ds$ is a continuous supermartingale dominating H, i.e. $Y_t \geq L_t$, $0 \leq t \leq T$, and for any $t \in [0,T]$, the stopping time

$$\tau_t = \inf\{s \geq t : Y_t = L_t\} \wedge T,$$

is optimal, in the sense that

$$Y_t + \int_0^t f(s)ds = E\Big[\int_0^{\tau_t} f(s)ds + L_{\tau_t} 1_{\tau_t<T} + \xi 1_{\tau_t=T}\Big|\mathcal{F}_t\Big]. \tag{6.43}$$

On the other hand, by applying the Doob-Meyer decomposition to the continuous supermartingale $S_t = Y_t + \int_0^t f(s)ds$ of class (DL), we get the existence of a continuous martingale M and an adapted continuous nondecreasing process K, $K_0 = 0$ such that

$$Y_t + \int_0^t f(s)ds = M_t - K_t, \quad 0 \leq t \leq T. \tag{6.44}$$

By observing that $Y_{\tau_t} = L_{\tau_t} 1_{\tau_t<T} + \xi 1_{\tau_t=T}$, and from the optional sampling theorem for the martingale M: $M_t = E[M_{\tau_t}|\mathcal{F}_t]$, we deduce

$$Y_t + \int_0^t f(s)ds = E\Big[\int_0^{\tau_t} f(s)ds + L_{\tau_t} 1_{\tau_t<T} + \xi 1_{\tau_t=T} + K_{\tau_t} - K_t\Big|\mathcal{F}_t\Big].$$

By comparing with (6.43), it follows that $E[K_{\tau_t} - K_t|\mathcal{F}_t] = 0$, and so $K_{\tau_t} = K_t$, or equivalently by definition of τ_t

$$\int_0^T (Y_t - L_t) dK_t = 0.$$

Moreover, since H lies in $\mathbb{S}^2(0,T)$, we easily see that Y also lies in $\mathbb{S}^2(0,T)$. Then, in the decomposition (6.44), the martingale M is square-integrable, and K_T is also square-integrable. By the Itô representation theorem, there exists $Z \in \mathbb{H}^2(0,T)$ such that

$$M_t = M_0 + \int_0^t Z_s.dW_s, \quad 0 \le t \le T.$$

Plugging into (6.44), and recalling that $Y_T = \xi$, we deduce that (Y, Z, K) solves (6.39), (6.40) and (6.41).

It remains to check uniqueness. Let (Y, Z, K) and $(\bar{Y}, \bar{Z}, \bar{K})$ be two solutions of (6.39), (6.40) and (6.41), and define $\Delta Y = Y - \bar{Y}$, $\Delta Z = Z - \bar{Z}$, $\Delta K = K - \bar{K}$. Then, $(\Delta Y, \Delta Z, \Delta K)$ satisfies

$$\Delta Y_t = -\int_t^T \Delta Z_s.dW_s + \Delta K_T - \Delta K_t, \quad 0 \le t \le T.$$

Moreover, by (6.40)-(6.41), we have for all $t \in [0, T]$

$$\begin{aligned} \int_t^T \Delta Y_s d\Delta K_s &= \int_t^T (Y_s - L_s + L_s - \bar{Y}_s)(dK_s - d\bar{K}_s) \\ &= -\int_t^T (Y_s - L_s) d\bar{K}_s - \int_t^T (\bar{Y}_s - L_s) dK_s \ \le \ 0. \end{aligned} \tag{6.45}$$

By Itô's formula to $|\Delta Y_t|^2$, we then obtain

$$\begin{aligned} |\Delta Y_t|^2 + \int_t^T |\Delta Z_s|^2 ds &= 2\int_t^T \Delta Y_s d\Delta K_s - 2\int_t^T \Delta Y_s \Delta Z_s.dW_s \\ &\le -2\int_t^T \Delta Y_s \Delta Z_s.dW_s. \end{aligned} \tag{6.46}$$

From the integrability conditions $\Delta Y \in \mathbb{S}^2(0, T)$, $\Delta Z \in \mathbb{H}^2(0, T)^d$, and the Burkholder-Davis-Gundy inequality, we observe that the local martingale $\int_0^t \Delta Y_s \Delta Z_s.dW_s$ is uniformly integrable, thus a martingale. By taking the expectation in (6.46), we conclude that

$$E\Big[|\Delta Y_t|^2 + \int_t^T |\Delta Z_s|^2 ds\Big] \le 0, \quad 0 \le t \le T,$$

which proves that $Y = \bar{Y}$, $Z = \bar{Z}$, and $K = \bar{K}$. □

In the sequel, we consider the general case where f may depend on y, z. We shall prove the existence of a solution to the reflected BSDE, and in the Markovian case, we relate this solution to a variational inequality extending the free boundary problem for optimal stopping problems.

6.5.1 Existence and approximation via penalization

In this section, we prove the existence and uniqueness of a solution to the reflected BSDE (6.39), (6.40) and (6.41), based on approximation via penalization. For each $n \in \mathbb{N}$, we consider the BSDE

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n) ds + n\int_t^T (Y_n^s - L_s)^- ds - \int_t^T Z_s^n.dW_s. \tag{6.47}$$

Notice that the generator $f_n(t,y,z) = f(t,y,z)+n(y-L_t)^-$ satisfies condition (B). From Theorem 6.2.1, there exists for each n, a unique solution (Y^n, Z^n) to the BSDE (6.47). We define

$$K_t^n = n\int_0^t (Y_n^s - L_s)^- ds, \quad 0 \leq t \leq T,$$

which is a continuous nondecreasing process, and is square integrable. Formally, the solution Y^n is penalized (by the factor n) once it falls below the obstacle L. The rest of this section is devoted to the convergence of the sequence $(Y^n, Z^n, K^n)_n$ to the solution to the reflected BSDE.

We first state a priori uniform estimates on the sequence $(Y^n, Z^n, K^n)_n$.

Lemma 6.5.1 *There exists a constant C such that*

$$E\Big[\sup_{0\leq t\leq T}|Y_t^n|^2 + \int_0^T |Z_t^n|^2 dt + |K_T^n|^2\Big] \leq C, \quad \forall n \in \mathbb{N}.$$

Proof. By applying Itô's formula to $|Y_t^n|^2$, we get

$$E\Big[|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds\Big] = E[|\xi|^2] + 2E\Big[\int_t^T f(s, Y_s^n, Z_s^n) Y_n^s ds\Big] + 2E\Big[\int_t^T Y_s^n dK_s^n\Big].$$

Now, by definition of K^n, we have: $\int_t^T Y_s^n dK_s^n \leq \int_t^T L_s dK_s^n \leq \sup_{0\leq t\leq T}|L_t|(K_T^n - K_t^n)$. From the Lipschitz property of f in condition (B), and using the inequality $2ab \leq \frac{1}{\alpha}a^2 + \alpha b^2$ for any constant $\alpha > 0$, we then obtain

$$\begin{aligned}
&E\Big[|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds\Big] \\
&\leq E[|\xi|^2] + 2E\Big[\int_t^T \big(f(s,0,0) + C_f(|Y_s^n| + |Z_s^n|)Y_n^s ds\Big] + 2E\Big[\sup_{0\leq t\leq T}|L_t|(K_T^n - K_t^n)\Big] \\
&\leq C\Big(1 + E\Big[\int_t^T |Y_s^n|^2 ds\Big]\Big) + \frac{1}{2}E\Big[\int_t^T |Z_s^n|^2 ds\Big] + \frac{1}{\alpha}E\big[\sup_{0\leq t\leq T}|L_t|^2\big] + \alpha E|K_T^n - K_t^n|^2,
\end{aligned}$$

and so

$$E\Big[|Y_t^n|^2 + \frac{1}{2}\int_t^T |Z_s^n|^2 ds\Big] \leq C\Big(1 + E\Big[\int_t^T |Y_s^n|^2 ds\Big]\Big) + \alpha E|K_T^n - K_t^n|^2. \quad (6.48)$$

Moreover, from the relation

$$K_T^n - K_t^n = Y_t^n - \xi - \int_t^T f(s, Y_s^n, Z_s^n) ds + \int_t^T Z_s^n.dW_s,$$

and conditions (A) and (B) on ξ and f, there exists a constant C_1 such that

$$E|K_T^n - K_t^n|^2 \leq C_1\Big(1 + E|Y_t^n|^2 + E\Big[\int_t^T |Y_s^n|^2 + |Z_s^n|^2 ds\Big]\Big).$$

By choosing $\alpha = 1/4C_1$, and plugging into (6.48), we get

$$\frac{3}{4}E\Big[|Y_t^n|^2 + \frac{1}{4}\int_t^T |Z_s^n|^2 ds\Big] \leq C\Big(1 + E\Big[\int_t^T |Y_s^n|^2 ds\Big]\Big).$$

By Gronwall's lemma, this implies

$$\sup_{0\leq t\leq T} E|Y_t^n|^2 + E\Big[\int_t^T |Z_s^n|^2 ds\Big] + E|K_T^n|^2 \leq C. \tag{6.49}$$

Finally, by writing from (6.47) that

$$\sup_{0\leq t\leq T} |Y_t^n|^2 \leq C\Big(|\xi|^2 + \int_0^T |f(s,Y_s^n,Z_s^n)|^2 ds + |K_T^n|^2 + \sup_{0\leq t\leq T}\Big|\int_0^T Z_s.dW_s\Big|^2\Big),$$

we obtain the required result from the Burkholder-Davis-Gundy inequality, conditions (A) and (B) on (ξ, f), and estimate (6.49). □

We next focus on the convergence of the sequence $(Y^n)_n$.

Lemma 6.5.2 *The sequence $(Y^n)_n$ converges increasingly to a process $Y \in \mathbb{S}^2(0,T)$, and the convergence also holds in $\mathbb{H}^2(0,T)$, i.e.*

$$\lim_{n\to\infty} E\Big[\int_0^T |Y_t^n - Y_t|^2 dt\Big] = 0. \tag{6.50}$$

Furthermore, $Y_t \geq L_t$, $0 \leq t \leq T$, a.s., and

$$\lim_{n\to\infty} E\big[\sup_{0\leq t\leq T}(Y_t^n - L_t)^-\big] = 0. \tag{6.51}$$

Proof. Since the generator f_n of the BSDE for Y_n is nondecreasing in n: $f_n(t,y,z) \leq f_{n+1}(t,y,z)$, we deduce from the comparison principle (Theorem 6.2.2) that $Y_t^n \leq Y_t^{n+1}$, $0 \leq t \leq T$ a.s. Together with the uniform estimate for $(Y_n)_n$ in $\mathbb{S}^2(0,T)$ in Lemma 6.5.1, this shows that the nondecreasing limit

$$Y_t := \lim_{n\to\infty} Y_t^n, \quad 0 \leq t \leq T,$$

exists a.s., and this defines an adapted process $Y \in \mathcal{S}^2$. From the dominated convergence theorem, we also get the convergence (6.50).

Notice that since the sequence $K_T^n = n\int_0^T (Y_t^n - L_t)^- dt$ is bounded in $L^2(\Omega, \mathcal{F}, P)$ by Lemma 6.5.1, then $E\big[\int_0^T (Y_t - L_t)^- dt\big] = 0$, which implies that $Y_t \geq L_t$ $dt \otimes dP$ a.e. We want to prove the stronger result $Y_t \geq L_t$, $0 \leq t \leq T$, a.s., and so use another argument. Let us consider the solution $(\tilde{Y}^n, \tilde{Z}^n)$ to the linear BSDE

$$\tilde{Y}_t^n = \xi + \int_t^T f(s,Y_s^n,Z_s^n)ds + n\int_t^T (L_s - \tilde{Y}_n^s)ds - \int_t^T \tilde{Z}_s^n.dW_s.$$

The generator $\tilde{f}_n(t,\tilde{y},\tilde{z}) = f(t,Y_t^n,Z_t^n) + n(L_t - \tilde{y})$ of this BSDE satisfies $\tilde{f}_n(t,\tilde{Y}_t^n,\tilde{Z}_t^n) \leq f_n(t,\tilde{Y}_t^n,\tilde{Z}_t^n)$, so that by the comparison principle (Theorem 6.2.2) $\tilde{Y}_t^n \leq Y_t^n$, $0 \leq t \leq T$. Moreover, by Proposition 6.2.1, the solution to this linear BSDE is explicitly given by

$$\tilde{Y}_\tau^n = E\Big[e^{-n(T-\tau)}\xi + \int_\tau^T e^{-n(s-\tau)}(f(Y_s^n,Z_s^n) + nL_s)ds\Big|\mathcal{F}_\tau\Big],$$

for any stopping time τ valued in $[0,T]$. It is not difficult (left to the reader) to check that as n goes to infinity

$$\tilde{Y}^n_\tau \to \xi 1_{\tau=T} + L_\tau 1_{\tau<T} \;\geq\; L_\tau \quad \text{in } L^2(\Omega,\mathcal{F},P).$$

Therefore $Y_\tau \geq L_\tau$ a.s. From that and section theorem (see Theorem 1.1.1), we deduce that $Y_t \geq L_t$, $0 \leq t \leq T$, a.s. This implies $(Y^n_t - L_t)^- \downarrow 0$, $0 \leq t \leq T$ a.s., and this convergence is also uniform in t by Dini's theorem: $\sup_{t\in[0,T]}(Y^n_t - L_t)^- \downarrow 0$ a.s. Finally, we obtain the result (6.51) by the monotone convergence theorem. □

We can finally state the main result of this section.

Theorem 6.5.8 *There exists a unique (Y,Z,K) solution to the reflected BSDE (6.39), (6.40) and (6.41), and this triple (Y,Z,K) is the limit of the sequence $(Y^n,Z^n,Z^n)_n$ in $\mathbb{S}^2(0,T)\times\mathbb{H}^2(0,T)^d\times\mathbb{S}^2(0,T)$, i.e.*

$$\lim_{n\to\infty} E\Big[\sup_{0\leq t\leq T}|Y^n_t - Y_t|^2 + \int_0^T |Z^n_t - Z_t|^2 dt + \sup_{0\leq t\leq T}|K^n_t - K_t|^2\Big] = 0. \tag{6.52}$$

Proof. For any $n,p\in\mathbb{N}$, we apply Itô's formula to $|Y^n_t - Y^p_t|^2$:

$$\begin{aligned}
&|Y^n_t - Y^p_t|^2 + \int_t^T |Z^n_s - Z^p_s|^2 ds \\
&= 2\int_t^T \big(f(s,Y^n_s,Z^n_s) - f(s,Y^p_s,Z^p_s)\big)(Y^n_s - Y^p_s)ds - 2\int_t^T (Y^n_s - Y^p_s)(Z^n_s - Z^p_s).dW_s \\
&\quad + 2\int_t^T (Y^n_s - Y^p_s)d(K^n_s - K^p_s) \\
&\leq C\int_t^T |Y^n_s - Y^p_s|^2 ds + \frac{1}{2}\int_t^T |Z^n_s - Z^p_s|^2 ds - 2\int_t^T (Y^n_s - Y^p_s)(Z^n_s - Z^p_s).dW_s \\
&\quad + 2\int_t^T (Y^n_s - L_s)^- dK^p_s + 2\int_t^T (Y^p_s - L_s)^- dK^n_s,
\end{aligned} \tag{6.53}$$

where we used the Lipschitz condition on f, the inequality $2ab \leq \alpha a^2 + \frac{1}{\alpha}b^2$ (for suitable choice of $\alpha > 0$), and the definitions of K^n, K^p. By taking the expectation, this yields

$$\begin{aligned}
E\Big[\int_t^T |Z^n_s - Z^p_s|^2 ds\Big] &\leq CE\Big[\int_t^T |Y^n_s - Y^p_s|^2 ds\Big] \\
&\quad + 4E\Big[\int_t^T (Y^n_s - L_s)^- dK^p_s + \int_t^T (Y^p_s - L_s)^- dK^n_s\Big].
\end{aligned}$$

From Lemma 6.5.1 and (6.51), we have

$$E\Big[\int_t^T (Y^n_s - L_s)^- dK^p_s + \int_t^T (Y^p_s - L_s)^- dK^n_s\Big] \to 0, \quad \text{as } n,p\to\infty. \tag{6.54}$$

We deduce with (6.50) that

$$E\Big[\int_t^T |Y^n_s - Y^p_s|^2 ds + \int_t^T |Z^n_s - Z^p_s|^2 ds\Big] \to 0, \quad \text{as } n,p\to\infty. \tag{6.55}$$

Now, from (6.53) and the Burholder-Davis-Gundy inequality, we get

$$
\begin{aligned}
E\Big[\sup_{0\le t\le T}|Y_t^n-Y_t^p|^2\Big] &\le CE\Big[\int_t^T|Y_s^n-Y_s^p|^2ds+\int_t^T|Z_s^n-Z_s^p|^2ds\Big]\\
&\quad+2E\Big[\int_t^T(Y_s^n-L_s)^-dK_s^p+\int_t^T(Y_s^p-L_s)^-dK_s^n\Big]\\
&\quad+CE\Big[\sup_{0\le t\le T}|Y_s^n-Y_s^p|\Big(\int_t^T|Z_s^n-Z_s^p|^2\Big)^{\frac12}\Big]\\
&\le CE\Big[\int_t^T|Y_s^n-Y_s^p|^2ds+\int_t^T|Z_s^n-Z_s^p|^2ds\Big]\\
&\quad+2E\Big[\int_t^T(Y_s^n-L_s)^-dK_s^p+\int_t^T(Y_s^p-L_s)^-dK_s^n\Big]\\
&\quad+\frac12E\Big[\sup_{0\le t\le T}|Y_t^n-Y_t^p|^2\Big]+CE\Big[\int_t^T|Z_s^n-Z_s^p|^2ds\Big],
\end{aligned}
$$

where we used again the inequality $2ab\le\alpha a^2+\frac1\alpha b^2$. Together with (6.54) and (6.55), this proves that

$$E\Big[\sup_{0\le t\le T}|Y_t^n-Y_t^p|^2\Big]\to0,\quad\text{as}\ \ n,p\to\infty. \tag{6.56}$$

By writing from (6.47) that

$$
\begin{aligned}
K_t^n-K_t^p&=Y_0^n-Y_0^p-(Y_t^n-Y_t^p)-\int_0^t(f(s,Y_s^n,Z_s^n)-f(s,Y_s^p,Z_s^p))ds\\
&\quad+\int_0^t(Z_s^n-Z_s^p).dW_s,
\end{aligned}
$$

we then obtain by the Lipschitz condition on f, (6.55) and (6.56) that

$$E\Big[\sup_{0\le t\le T}|K_t^n-K_t^p|^2\Big]\to0,\quad\text{as}\ \ n,p\to\infty.$$

Consequently, $(Z^n,K^n)_n$ is a Cauchy sequence in the Banach space $\mathbb{H}^2(0,T)^d\times\mathbb{S}^2(0,T)$, and this gives the existence of a $(Z,K)\in\mathbb{H}^2(0,T)^d\times\mathbb{S}^2(0,T)$ such that

$$\lim_{n\to\infty}E\Big[\int_0^T|Z_t^n-Z_t|^2dt+\sup_{0\le t\le T}|K_t^n-K_t|^2\Big]=0.$$

By (6.56), we also know that the convergence of the sequence (Y^n) to the limit Y in Lemma 6.5.2, holds in $\mathbb{S}^2(0,T)$: $E\big[\sup_{0\le t\le T}|Y_t^n-Y_t|^2\big]\to0$. Notice that the limit K of K^n in $\mathbb{S}^2(0,T)$, inherits from K^n the nondecreasing and continuity path properties. We can then pass to the (strong) limit in (6.47), and deduce that (Y,Z,K) solves (6.39)-(6.40). Let us now check the condition (6.41). The convergence of (Y^n,K^n) to (Y,K) in $\mathbb{S}^2(0,T)\times\mathbb{S}^2(0,T)$ implies by the Tchebyshev inequality that the convergence also holds uniformly in t in probability. Then the measure dK^n tends to dK weakly in probability, and so $\int_0^T(Y_t^n-L_t)dK_t^n\to\int_0^T(Y_t-L_t)dK_t$ in probability as n goes to infinity. Moreover, since Y satisfies (6.40), we have $\int_0^T(Y_t-L_t)dK_t\ge0$ a.s. On the other hand, by definition of K^n, we have $\int_0^T (Y_t - L_t)dK_t^n \leq 0$. We conclude that $\int_0^T (Y_t - L_t)dK_t = 0$ a.s., and this shows that (Y, Z, K) is a solution to the reflected BSDE (6.39)-(6.40)-(6.41).

We finally turn to uniqueness. Let (Y, Z, K) and $(\bar{Y}, \bar{Z}, \bar{K})$ be two solutions of (6.39), (6.40) and (6.41), and define $\Delta Y = Y - \bar{Y}$, $\Delta Z = Z - \bar{Z}$, $\Delta K = K - \bar{K}$. Then, $(\Delta Y, \Delta Z, \Delta K)$ satisfies

$$\Delta Y_t = \int_t^T (f(s, Y_s, Z_s) - f(s, \bar{Y}_s, \bar{Z}_s)ds - \int_t^T \Delta Z_s.dW_s + \Delta K_T - \Delta K_t, \quad 0 \leq t \leq T.$$

By applying Itô's formula to $|\Delta Y_t|^2$, and using similar computations as for $|Y_t^n - Y_t^p|^2$ and recalling $\int_t^T \Delta Y_s d\Delta K_s \leq 0$ (see (6.45)), we have

$$E\Big[|\Delta Y_t|^2 + \frac{1}{2}\int_t^T |\Delta Z_s|^2 ds\Big] \leq CE\Big[\int_t^T |\Delta Y_s|^2 ds\Big].$$

By Gronwall's lemma, we conclude that $\Delta Y = 0$, $\Delta Z = 0$ and so $\Delta K = 0$. □

Remark 6.5.3 For any triple $(\tilde{Y}, \tilde{Z}, \tilde{K})$ satisfying (6.39)-(6.40), and for (Y^n, Z^n) solution to the penalized BSDE (6.47) one can prove by a comparison principle that $Y_t^n \leq \tilde{Y}_t$, $0 \leq t \leq T$ a.s. By passing to the limit, this shows that $Y_t \leq \tilde{Y}_t$, $0 \leq t \leq T$ a.s. Therefore, the solution to the reflected BSDE (6.39), (6.40) and (6.41) is also a minimal solution in the sense defined in Remark 6.5.2.

6.5.2 Connection with variational inequalities

We put our reflected BSDE in a Markovian framework in the sense that the terminal condition, the generator and the obstacle are functions of a forward SDE. More precisely, we are given a diffusion on $\mathbb{R}^n$

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s, \tag{6.57}$$

with Lipschitz coefficents b and σ on $\mathbb{R}^n$, and we consider the reflected BSDE

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s.dW_s, \quad 0 \leq t \leq T \tag{6.58}$$

$$Y_t \geq h(X_t), \quad 0 \leq t \leq T, \tag{6.59}$$

$$\int_0^T (Y_t - h(X_t))dK_t = 0, \tag{6.60}$$

where f is a continuous function on $[0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$, satisfying a linear growth condition in (x, y, z), a Lipschitz condition in (y, z) uniformly in (t, x), g is a measurable function on $\mathbb{R}^n$ with a linear growth condition, and h is a continuous function on $\mathbb{R}^n$ with a linear growth condition, and $g \geq h$.

By the Markov property of the diffusion X, and uniqueness of a solution to the reflected BSDE, we see that $Y_t = v(t, X_t)$, $0 \leq t \leq T$, where

$$v(t,x) := Y_t^{t,x} \tag{6.61}$$

is a deterministic function of $(t,x) \in [0,T] \times \mathbb{R}^n$, $\{X_s^{t,x}, t \le s \le T\}$ denotes the solution to (6.57) starting from x at t, and $\{(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x}), t \le s \le T\}$ is the solution to the reflected BSDE (6.58), (6.59) and (6.60) with $X_s = X_s^{t,x}$, $t \le s \le T$. We shall relate this reflected BSDE to the variational inequality

$$\min\big[- \frac{\partial v}{\partial t} - \mathcal{L}v - f(.,v,\sigma' D_x v)\ ,\ v - h\big] = 0, \quad \text{on } [0,T) \times \mathbb{R}^n \tag{6.62}$$

$$v(T,.) = g \quad \text{on } \mathbb{R}^n, \tag{6.63}$$

where $\mathcal{L}$ is the second-order operator associated to the diffusion X:

$$\mathcal{L}v = b(x).D_x v + \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x) D_x^2 v).$$

The following result is the analog of Proposition 6.3.2 for BSDEs, and shows that a classical solution to the variational inequality provides a solution to the reflected BSDE.

Proposition 6.5.4 *Suppose that $v \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap C^0([0,T] \times \mathbb{R}^n)$ is a classical solution to (6.62)-(6.63), satisfying a linear growth condition and such that for some positive constants $C, q > 0$: $|D_x v(t,x)| \le C(1+|x|^q)$, for all $x \in \mathbb{R}^n$. Then, the triple (Y,Z,K) defined by*

$$Y_t = v(t,X_t), \quad Z_t = \sigma'(X_t) D_x v(t,X_t), \quad 0 \le t \le T,$$
$$K_t = \int_0^t \big(- \frac{\partial v}{\partial t}(s,X_s) - \mathcal{L}v(s,X_s) - f(t,X_s,Y_s,Z_s)\big) ds,$$

is the solution to the reflected BSDE (6.58), (6.59) and (6.60).

Proof. By Itô's formula applied to $v(t,X_t)$ and from the terminal condition (6.63), we immediately see that (Y,Z,K) satisfies the relation (6.58). Since v satisfies (6.62), the term in the bracket of K is nonnegative, and so K is nondecreasing. The obstacle constraint (6.59) is also clearly satisfied. Moreover, the minimality condition (6.60) follows from the equality in (6.62). Finally, the integrability conditions on $(Y,Z) \in \mathbb{S}^2(0,T) \times \mathbb{H}^2(0,T)^d$ are direct consequences of the growth conditions on v and $D_x v$. □

We now focus on the converse property, and prove that a solution to the reflected BSDE provides a solution to the variational inequality.

Theorem 6.5.9 *The function $v(t,x) = Y_t^{t,x}$ in (6.61) is continuous on $[0,T] \times \mathbb{R}^n$, and is a viscosity solution to (6.62)-(6.63).*

Proof. The continuity of v is proved similarly as in Theorem 6.3.3, and the terminal condition (6.63) is obviously satisfied from the terminal condition on the BSDE. In order to prove the viscosity property to the variational inequality, we use the approximation by the penalized BSDE. For any $(t,x) \in [0,T] \times \mathbb{R}^n$, $m \in \mathbb{N}$, we denote by $\{(Y^{m,t,x}, Z^{m,t,x}), t \le s \le T\}$ the solution to the penalized BSDE

$$Y_t^m = g(X_T) + \int_t^T f(s,X_s,Y_s^m,Z_s^m) ds + m \int_t^T (Y_s^m - h(X_s))^- ds - \int_t^T Z_s^m.dW_s,$$

with $X_s = X_s^{t,x}$. From Theorem 6.3.3, we know that the function

$$v_m(t,x) := Y_t^{m,t,x}$$

is a continuous viscosity solution to the semilinear PDE

$$-\frac{\partial v_m}{\partial t} - \mathcal{L}v_m - f(.,v_m,\sigma' D_x v_m) - m(v_m - h)^- = 0, \quad \text{on } [0,T) \times \mathbb{R}^n \quad (6.64)$$

$$v_m(T,.) = g, \quad \text{on } \mathbb{R}^n. \quad (6.65)$$

From the convergence result of the penalized BSDEs proved in the previous section, we know that for any $(t,x) \in [0,T] \times \mathbb{R}^n$, $v_m(t,x)$ converges increasingly to $v(t,x)$ as m goes to infinity. Since v_m and v are continuous, this convergence is uniform on compacts of $[0,T] \times \mathbb{R}^n$ by Dini's theorem.

We prove the viscosity solution property of v by using the definition-characterization by super(sub)-jets (see Lemma 4.4.5). We first show the viscosity supersolution property. Let $(t,x) \in [0,T) \times \mathbb{R}^n$, and $(q,p,M) \in \bar{\mathcal{P}}^{2,-}v(t,x)$. From Lemma 6.1 in [CIL92], there exist sequences

$$m_j \to \infty, \;\; (t_j,x_j) \to (t,x), \;\; (q_j,p_j,M_j) \in \mathcal{P}^{2,-}v_{m_j}(t_j,x_j),$$

such that

$$(v_{m_j}(t_j,x_j), q_j, p_j, M_j) \to (v(t,x), q, p, M).$$

From the viscosity supersolution property of v_{m_j} to (6.64), we have

$$\begin{aligned} -q_j - b(x_j).p_j - \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x_j)M_j) - f(t_j,x_j,v_{m_j}(t_j,x_j),\sigma'(x_j)p_j) \\ -m_j(v_{m_j}(t_j,x_j) - h(x_j))^- \geq 0, \end{aligned}$$

and so

$$-q_j - b(x_j).p_j - \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x_j)M_j) - f(t_j,x_j,v_{m_j}(t_j,x_j),\sigma'(x_j)p_j) \geq 0.$$

By sending j to infinity, we then obtain

$$-q - b(x).p - \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x)M) - f(t,x,v(t,x),\sigma'(x)p) \geq 0.$$

Since we already know that $v(t,x) \geq h(x)$ by the obstacle condition on the reflected BSDE, this proves that v is a viscosity supersolution to (6.62).

We conclude by showing the viscosity subsolution property. Let $(t,x) \in [0,T) \times \mathbb{R}^n$, and $(q,p,M) \in \bar{\mathcal{P}}^{2,+}v(t,x)$ such that $v(t,x) > h(x)$. As above, there exist sequences

$$m_j \to \infty, \;\; (t_j,x_j) \to (t,x), \;\; (q_j,p_j,M_j) \in \mathcal{P}^{2,-}v_{m_j}(t_j,x_j),$$

such that

$$(v_{m_j}(t_j,x_j), q_j, p_j, M_j) \to (v(t,x), q, p, M).$$

From the viscosity subsolution property of v_{m_j} to (6.64), we have

$$- q_j - b(x_j).p_j - \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x_j)M_j) - f(t_j, x_j, v_{m_j}(t_j, x_j), \sigma'(x_j)p_j) \tag{6.66}$$

$$-m_j(v_{m_j}(t_j, x_j) - h(x_j))^- \leq 0. \tag{6.67}$$

Since $v(t,x) > h(x)$, then for j large enough, $v_{m_j}(t_j, x_j) > h(x_j)$ and so $(v_{m_j}(t_j, x_j) - h(x_j))^- = 0$. By sending j to infinity into (6.66), this yields

$$-q - b(x).p - \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x)M) - f(t, x, v(t,x), \sigma'(x)p) \leq 0,$$

which proves the required result. □

6.6 Applications

6.6.1 Exponential utility maximization with option payoff

We consider a financial market with one riskless asset of price $S^0 = 1$ and one risky asset of price process

$$dS_t = S_t(b_t dt + \sigma_t dW_t),$$

where W is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_t, P)$ equipped with the natural filtration $\mathbb{F}$ of W, b and σ are two bounded progressively measurable processes, $\sigma_t \geq \varepsilon$, for all t, a.s. with $\varepsilon > 0$. An agent, starting from a capital x, invests an amount α_t at any time t in the risky asset. His wealth process, controlled by α, is given by

$$X_t^{x,\alpha} = x + \int_0^t \alpha_u \frac{dS_u}{S_u} \;=\; x + \int_0^t \alpha_u(b_u du + \sigma_u dW_u), \quad 0 \leq t \leq T. \tag{6.68}$$

We denote by $\mathcal{A}$ the set of progressively measurable processes α valued in $\mathbb{R}$, such that $\int_0^T |\alpha_t|^2 dt < \infty$ a.s. and $X^{x,\alpha}$ is lower-bounded. The agent must provide at maturity T an option payoff represented by a bounded random variable ξ $\mathcal{F}_T$-measurable. Given his risk aversion characterized by an exponential utility

$$U(x) = -\exp(-\eta x), \quad x \in \mathbb{R}, \quad \eta > 0, \tag{6.69}$$

the objective of the agent is to solve the maximization problem:

$$v(x) = \sup_{\alpha \in \mathcal{A}} E[U(X_T^{x,\alpha} - \xi)]. \tag{6.70}$$

The approach adopted here for determining the value function v and the optimal control $\hat{\alpha}$ is quite general, and is based on the following argument. We construct a family of processes $(J_t^\alpha)_{0 \leq t \leq T}$, $\alpha \in \mathcal{A}$, satisfying the properties:

(i) $J_T^\alpha = U(X_T^{x,\alpha} - \xi)$ for all $\alpha \in \mathcal{A}$

(ii) J_0^α is a constant independent of $\alpha \in \mathcal{A}$

(iii) J^α is a supermartingale for all $\alpha \in \mathcal{A}$, and there exists $\hat\alpha \in \mathcal{A}$ such that $J^{\hat\alpha}$ is a martingale.

Indeed, in this case, for such $\hat\alpha$, we have for any $\alpha \in \mathcal{A}$,

$$E[U(X_T^{x,\alpha} - \xi)] \ = \ E[J_T^\alpha] \leq \ J_0^\alpha \ = J_0^{\hat\alpha} = \ E[J_T^{\hat\alpha}] \ = \ E[U(X_T^{x,\hat\alpha} - \xi)] \ = \ v(x),$$

which proves that $\hat\alpha$ is an optimal control, and $v(x) = J_0^{\hat\alpha}$.

We construct such a family (J_t^α) in the form

$$J_t^\alpha = U(X_t^{x,\alpha} - Y_t), \quad 0 \leq t \leq T, \ \alpha \in \mathcal{A}, \tag{6.71}$$

with (Y, Z) solution to the BSDE

$$Y_t = \xi + \int_t^T f(s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \tag{6.72}$$

where f is a generator to be determined. The conditions *(i)* and *(ii)* are clearly satisfied, and the value function is then given by

$$v(x) = J_0^\alpha \ = \ U(x - Y_0).$$

In order to satisfy the condition *(iii)*, we shall exploit the particular structure of the exponential utility function U. Indeed, by substituting (6.68), (6.72) into (6.71) with U as in (6.69), we obtain

$$J_t^\alpha \ = M_t^\alpha \ C_t^\alpha,$$

where M^α is the (local) martingale given by

$$M_t^\alpha = \exp(-\eta(x - Y_0)) \ \exp\Big(- \int_0^t \eta(\alpha_u\sigma_u - Z_u)dW_u - \frac{1}{2}\int_0^t |\eta(\alpha_u\sigma_u - Z_u)|^2 du\Big),$$

and

$$C_t^\alpha \ = - \exp\Big(\int_0^t \rho(u, \alpha_u, Z_u)du\Big),$$

with

$$\rho(t, a, z) = \eta\Big(\frac{\eta}{2}|a\sigma_t - z|^2 - ab_t - f(t, z)\Big).$$

We are then looking for a generator f such that the process (C_t^α) is nonincreasing for all $\alpha \in \mathcal{A}$, and constant for some $\hat\alpha \in \mathcal{A}$. In other words, the problem is reduced to findng f such that

$$\rho(t, \alpha_t, Z_t) \geq 0, \quad 0 \leq t \leq T, \ \ \forall \alpha \in \mathcal{A} \tag{6.73}$$

and

$$\rho(t, \hat\alpha_t, Z_t) = 0, \quad 0 \leq t \leq T. \tag{6.74}$$

By rewriting ρ in the form

$$\frac{1}{\eta}\rho(t,a,z) = \frac{\eta}{2}\left|a\sigma_t - z - \frac{1}{\eta}\frac{b_t}{\sigma_t}\right|^2 - z\frac{b_t}{\sigma_t} - \frac{1}{2\eta}\left|\frac{b_t}{\sigma_t}\right|^2 - f(t,z),$$

we clearly see that conditions (6.73) and (6.74) will be satisfied with

$$f(t,z) = -z\frac{b_t}{\sigma_t} - \frac{1}{2\eta}\left|\frac{b_t}{\sigma_t}\right|^2, \tag{6.75}$$

and

$$\hat{\alpha}_t = \frac{1}{\sigma_t}\left(Z_t + \frac{1}{\eta}\frac{b_t}{\sigma_t}\right), \quad 0 \le t \le T. \tag{6.76}$$

Theorem 6.6.10 *The value function to problem (6.70) is equal to*

$$v(x) = U(x - Y_0) \;=\; -\exp(-\eta(x - Y_0)),$$

where (Y,Z) is the solution to the BSDE

$$-\,dY_t = f(t,Z_t)dt - Z_t dW_t, \;\; Y_T \;=\; \xi, \tag{6.77}$$

with a generator f given by (6.75). Moreover, the optimal control $\hat{\alpha}$ is given by (6.76).

Proof. In view of the above arguments, it remains to check rigorously the condition (iii) on J^α. Since b/σ and ξ are bounded, we first observe from (6.2.1) that the solution (Y,Z) to the linear BSDE (6.77) is such that Y is bounded. Moreover, for all $\alpha \in \mathcal{A}$, the process M^α is a local martingale, and there exists a sequence of stopping times (τ_n), $\tau_n \to \infty$ a.s., such that $(M^\alpha_{t\wedge\tau_n})$ is a (positive) martingale. With the choice of f in (6.75), the process C^α is nonincreasing, and thus $(J^\alpha_{t\wedge\tau_n}) = (M^\alpha_{t\wedge\tau_n}C^\alpha_{t\wedge\tau_n})$ is a supermartingale. Since $X^{x,\alpha}$ is lower-bounded and Y is bounded, the process J^α, given by (6.71), is also lower-bounded. By Fatou's lemma, we deduce that J^α is a supermartingale.

Finally, with the choice of $\hat{\alpha}$ in (6.76), we have

$$J^{\hat{\alpha}}_t = M^{\hat{\alpha}}_t \;=\; \exp(-\eta(x - Y_0)) \; \exp\Big(-\int_0^t \frac{b_u}{\sigma_u}dW_u - \frac{1}{2}\int_0^t \left|\frac{b_u}{\sigma_u}\right|^2 du\Big).$$

Since b/σ is bounded, we conclude that $J^{\hat{\alpha}}$ is a martingale. □

Remark 6.6.4 The financial model described in this example is a complete market model: any contingent claim ξ, $\mathcal{F}_T$-measurable and bounded, is perfectly replicable by means of a self-financed wealth process. In other words, there exists $\pi \in \mathcal{A}$ such that $\xi = X_T^{x_\xi,\pi}$ where x_ξ is the arbitrage price of ξ given by $x_\xi = E^Q[\xi]$, and Q is the unique probability measure equivalent to P, which makes the price process S a (local) martingale under Q, and called risk-neutral probability. The problem (6.70) may be then formulated as

$$v(x) = \sup_{\alpha\in\mathcal{A}} E[U(X_T^{x - x_\xi, \alpha - \pi})].$$

We are thus reduced to an exponential utility maximization problem without option payoff. Hence, the optimal strategy (6.76) of the initial problem is decomposed into the sum $\alpha_t = \pi_t + \alpha_t^0$ of the hedging strategy $\pi_t = Z_t/\sigma_t$ for the contingent claim ξ and the optimal strategy $\alpha_t^0 = \frac{1}{\eta} b_t/\sigma_t^2$ for the exponential utility maximization without option.

In a more general context of incomplete market, i.e. when the option ξ is not perfectly replicable, the same approach (i), (ii), (iii), can be applied, but leads to a more complex generator f involving a quadratic term in z, see El Karoui and Rouge [ElkR00].

6.6.2 Mean-variance criterion for portfolio selection

We consider a Black-Scholes financial model. There is one riskless asset of price process

$$dS_t^0 = rS_t^0 dt,$$

and one stock of price process

$$dS_t = S_t(bdt + \sigma dW_t),$$

with constants $b > r$ and $\sigma > 0$. An agent invests at any time t an amount α_t in the stock, and his wealth process is governed by

$$\begin{aligned} dX_t &= \alpha_t \frac{dS_t}{S_t} + (X_t - \alpha_t)\frac{dS_t^0}{S_t^0} \\ &= [rX_t + \alpha_t(b-r)]\, dt + \sigma\alpha_t dW_t, \quad X_0 = x. \end{aligned} \tag{6.78}$$

We denote by $\mathcal{A}$ the set of progressively measurable processes α valued in $\mathbb{R}$, such that $E[\int_0^T |\alpha_t|^2 dt] < \infty$.

The mean-variance criterion for portfolio selection consists in minimizing the variance of the wealth under the constraint that its expectation is equal to a given constant:

$$V(m) = \inf_{\alpha \in \mathcal{A}} \{\mathrm{Var}(X_T) : \ E(X_T) = m\}, \quad m \in \mathbb{R}. \tag{6.79}$$

We shall see in Proposition 6.6.5, by the Lagrangian method, that this problem is reduced to the resolution of an auxiliary control problem

$$\tilde{V}(\lambda) = \inf_{\alpha \in \mathcal{A}} E[X_T - \lambda]^2, \quad \lambda \in \mathbb{R}. \tag{6.80}$$

We shall solve problem (6.80) by the stochastic maximum principle described in Section 6.4.2. In this case, the Hamiltonian in (6.24) takes the form

$$\mathcal{H}(x, a, y, z) = [rx + a(b-r)]\, y + \sigma a z.$$

The adjoint BSDE (6.25) is written for any $\alpha \in \mathcal{A}$ as

$$-\, dY_t = rY_t dt - Z_t dW_t, \quad Y_T \ = \ 2(X_T - \lambda). \tag{6.81}$$

Let $\hat{\alpha} \in \mathcal{A}$ a candidate for the optimal control, and $\hat{X}$, $(\hat{Y}, \hat{Z})$ the corresponding processes. Then,

$$\mathcal{H}(x, a, \hat{Y}_t, \hat{Z}_t) = rx\hat{Y}_t + a\left[(b-r)\hat{Y}_t + \sigma\hat{Z}_t\right].$$

Since this expression is linear in a, we see that conditions (6.26) and (6.27) will be satisfied iff

$$(b-r)\hat{Y}_t + \sigma\hat{Z}_t = 0, \quad 0 \le t \le T, \ a.s. \tag{6.82}$$

We are looking for the $(\hat{Y}, \hat{Z})$ solution to (6.81) in the form

$$\hat{Y}_t = \varphi(t)\hat{X}_t + \psi(t), \tag{6.83}$$

for some deterministic C^1 functions φ and ψ. By substituting in (6.81), and using expression (6.78), we see that φ, ψ and $\hat{\alpha}$ should satisfy

$$\varphi'(t)\hat{X}_t + \varphi(t)(r\hat{X}_t + \hat{\alpha}_t(b-r)) + \psi'(t) = -r(\varphi(t)\hat{X}_t + \psi(t)), \tag{6.84}$$

$$\varphi(t)\sigma\hat{\alpha}_t = \hat{Z}_t, \tag{6.85}$$

together with the terminal conditions

$$\varphi(T) \ = \ 2, \ \ \psi(T) \ = \ -2\lambda. \tag{6.86}$$

By using relations (6.82), (6.83) and (6.85), we obtain the expression of $\hat{\alpha}$:

$$\hat{\alpha}_t = \frac{(r-b)\hat{Y}_t}{\sigma^2\varphi(t)} \ = \ \frac{(r-b)(\varphi(t)\hat{X}_t + \psi(t))}{\sigma^2\varphi(t)}. \tag{6.87}$$

On the other hand, from (6.84), we have

$$\hat{\alpha}_t = \frac{(\varphi'(t) + 2r\varphi(t))\hat{X}_t + \psi'(t) + r\psi(t)}{(r-b)\varphi(t)}. \tag{6.88}$$

By comparing with (6.87), we get the ordinary differential equations satisfied by φ and ψ:

$$\varphi'(t) + \Big(2r - \frac{(b-r)^2}{\sigma^2}\Big)\varphi(t) = 0, \quad \varphi(T) \ = \ 2 \tag{6.89}$$

$$\psi'(t) + \Big(r - \frac{(b-r)^2}{\sigma^2}\Big)\psi(t) = 0, \quad \varphi(T) \ = \ -2\lambda, \tag{6.90}$$

whose explicit solutions are (only $\psi = \psi_\lambda$ depends on λ)

$$\varphi(t) = 2\exp\Big[\Big(2r - \frac{(b-r)^2}{\sigma^2}\Big)(T-t)\Big], \tag{6.91}$$

$$\psi_\lambda(t) = \lambda\psi_1(t) \ = \ -2\lambda\exp\Big[\Big(r - \frac{(b-r)^2}{\sigma^2}\Big)(T-t)\Big]. \tag{6.92}$$

With this choice of φ, ψ_λ, the processes $(\hat{Y}, \hat{Z})$ solve the adjoint BSDE (6.81), and the conditions for the maximum principle in Theorem 6.4.6 are satisfied: the optimal control is given by (6.87), which is written in the Markovian form as

$$\hat{\alpha}_\lambda(t,x) = \frac{(r-b)(\varphi(t)x + \psi_\lambda(t))}{\sigma^2\varphi(t)}. \tag{6.93}$$

To compute the value function $\tilde{V}(\lambda)$, we proceed as follows. For any $\alpha \in \mathcal{A}$, we apply Itô's formula to $\frac{1}{2}\varphi(t)X_t^2 + \psi_\lambda(t)X_t$ between 0 and T, by using the dynamics (6.78) of X and the ODE (6.89)-(6.90) satisfied by φ and ψ_λ. By taking the expectation, we then obtain

$$\begin{aligned} E\left[X_T - \lambda\right]^2 &= \frac{1}{2}\varphi(0)x^2 + \psi_\lambda(0)x + \lambda^2 \\ &\quad + E\Big[\int_0^T \frac{\varphi(t)\sigma^2}{2}\left(\alpha_t - \frac{(r-b)(\varphi(t)X_t + \psi_\lambda(t))}{\sigma^2\varphi(t)}\right)^2 dt\Big] \\ &\quad - \frac{1}{2}\int_0^T \left(\frac{b-r}{\sigma}\right)^2 \frac{\psi_\lambda(t)^2}{\varphi(t)} dt. \end{aligned}$$

This shows again that the optimal control is given by (6.87), and the value function is equal to

$$\tilde{V}(\lambda) = \frac{1}{2}\varphi(0)x^2 + \psi_\lambda(0)x + \lambda^2 - \frac{1}{2}\int_0^T \left(\frac{b-r}{\sigma}\right)^2 \frac{\psi_\lambda(t)^2}{\varphi(t)} dt,$$

and so with the explicit expressions (6.91)-(6.92) of φ and ψ_λ

$$\tilde{V}(\lambda) = e^{-\frac{(b-r)^2}{\sigma^2}T}(\lambda - e^{rT}x)^2, \quad \lambda \in \mathbb{R}. \tag{6.94}$$

We finally show how problems (6.79) and (6.80) are related.

Proposition 6.6.5 *We have the conjugate relations*

$$\tilde{V}(\lambda) = \inf_{m\in\mathbb{R}} \left[V(m) + (m-\lambda)^2\right], \quad \lambda \in \mathbb{R}, \tag{6.95}$$

$$V(m) = \sup_{\lambda\in\mathbb{R}} \left[\tilde{V}(\lambda) - (m-\lambda)^2\right], \quad m \in \mathbb{R}. \tag{6.96}$$

For any m in $\mathbb{R}$, the optimal control of $V(m)$ is equal to $\hat{\alpha}_{\lambda_m}$ given by (6.93) where λ_m attains the maximum in (6.96), i.e.

$$\lambda_m = \frac{m - \exp\left[\left(r - \frac{(b-r)^2}{\sigma^2}\right)T\right]x}{1 - \exp\left[-\frac{(b-r)^2}{\sigma^2}T\right]}. \tag{6.97}$$

Proof. Notice first that for all $\alpha \in \mathcal{A}$, $\lambda \in \mathbb{R}$, we have

$$E[X_T - \lambda]^2 = \mathrm{Var}(X_T) + (E(X_T) - \lambda)^2. \tag{6.98}$$

Fix an arbitrary $m \in \mathbb{R}$. By definition of $V(m)$, for all $\varepsilon > 0$, one can find $\alpha^\varepsilon \in \mathcal{A}$ with controlled diffusion X^ε, such that $E(X_T^\varepsilon) = m$ and $\mathrm{Var}(X_T^\varepsilon) \leq V(m) + \varepsilon$. We deduce with (6.98) that

$$E[X_T^\varepsilon - \lambda]^2 \leq V(m) + (m-\lambda)^2 + \varepsilon,$$

and so

$$\tilde{V}(\lambda) \leq V(m) + (m-\lambda)^2, \quad \forall\, m,\ \lambda \in \mathbb{R}. \tag{6.99}$$

On the other hand, for $\lambda \in \mathbb{R}$, let $\hat{\alpha}_\lambda \in \mathcal{A}$ with controlled diffusion $\hat{X}^\lambda$, an optimal control for $\tilde{V}(\lambda)$. We set $m_\lambda = E(\hat{X}_T^\lambda)$. From (6.98), we then get

$$\begin{aligned}\tilde{V}(\lambda) &= \mathrm{Var}(\hat{X}_T^\lambda) + (m_\lambda - \lambda)^2 \\ &\geq V(m_\lambda) + (m_\lambda - \lambda)^2.\end{aligned}$$

This last inequality, combined with (6.99), proves (6.95):

$$\begin{aligned}\tilde{V}(\lambda) &= \inf_{m \in \mathbb{R}} \left[V(m) + (m-\lambda)^2\right] \\ &= V(m_\lambda) + (m_\lambda - \lambda)^2,\end{aligned}$$

and also that $\hat{\alpha}_\lambda$ is solution to $V(m_\lambda)$.

We easily check that the function V is convex in m. By writing the relation (6.95) under the form $(\lambda^2 - \tilde{V}(\lambda))/2 = \sup_m [m\lambda - (V(m) + m^2)/2]$, we see that the function $\lambda \to (\lambda^2 - \tilde{V}(\lambda))/2$ is the Fenchel-Legendre transform of the convex function $m \to (V(m) + m^2)/2$. We then have the duality relation $(V(m) + m^2)/2 = \sup_\lambda [m\lambda - (\lambda^2 - \tilde{V}(\lambda))/2]$, which gives (6.96).

Finally, for any $m \in \mathbb{R}$, let $\lambda_m \in \mathbb{R}$ be the argument maximum of $V(m)$ in (6.96), which is explicitly given by (6.97) from the expression (6.94) of $\tilde{V}$. Then, m is an argument minimum of $\tilde{V}(\lambda_m)$ in (6.95). Since the function $m \to V(m) + (m-\lambda)^2$ is strictly convex, this argument minimum is unique, and so $m = m_{\lambda_m} = E(\hat{X}_T^{\lambda_m})$. We thus obtain

$$\begin{aligned}V(m) &= \tilde{V}(\lambda_m) + (m - \lambda_m)^2 \\ &= E[\hat{X}_T^{\lambda_m} - \lambda_m]^2 + \Big[E(\hat{X}_T^{\lambda_m}) - \lambda_m\Big]^2 \; = \; \mathrm{Var}(\hat{X}_T^{\lambda_m}),\end{aligned}$$

which proves that $\hat{\alpha}_{\lambda_m}$ is a solution to $V(m)$. □

Remark 6.6.5 There is a financial interpretation of the optimal portfolio strategy (6.93) to problem (6.80). Indeed, observe that it is written also as

$$\hat{\alpha}_t^{(\lambda)} \; := \; \hat{\alpha}_\lambda(t, X_t) = -\frac{b-r}{\sigma^2}(X_t - R_\lambda(t)), \quad 0 \leq t \leq T,$$

where the (deterministic) process $R_\lambda(t) = -\psi_\lambda(t)/\varphi(t)$ is explicitly determined by

$$dR_\lambda(t) = rR_\lambda(t)dt, \qquad R_\lambda(T) = \lambda.$$

R_λ is the wealth process with zero investment in the stock, and replicates perfectly the constant option payoff λ. On the other hand, consider the problem of an investor with self-financed wealth process $\bar{X}_t$, who wants to minimize $E[(\bar{X}_T)^2]$ in this complete market model. His optimal strategy is the Merton portfolio allocation for a quadratic utility function $U(x) = -x^2$, and given by

$$\bar{\alpha}_t = -\frac{b-r}{\sigma^2}\bar{X}_t, \quad 0 \leq t \leq T. \tag{6.100}$$

The optimal strategy for the problem (6.80) is then equal to the stragegy according to (6.100) with wealth process $X_t - R_\lambda(t)$, and could be directly derived with this remark. We illustrated here in this simple example how one may apply the maximum principle for solving the mean-variance criterion. Actually, this approach succeeds for dealing with more complex cases of random coefficients on the price process in incomplete markets, and leads to BSDE for $\varphi(t)$ and $\psi_\lambda(t)$, see e.g. Kohlmann and Zhou [KZ00].

6.7 Bibliographical remarks

BSDEs were introduced in the linear case by Bismut [Bis76] as the adjoint equation associated to the stochastic version of the Pontryagin maximum principle in control theory. The general nonlinear case was studied in the seminal paper by Pardoux and Peng [PaPe90]. Since then, BSDEs generate a very active research area due to their various connections with mathematical finance, stochastic control and partial differential equations. Motivated by applications in mathematical finance, there were several extensions for relaxing the Lipschitz condition on the generator of BSDE. We cite in particular the paper by Kobylanski [Ko00], who shows the existence of a bounded solution when the generator satisfies a quadratic growth condition in z. We also refer to the book edited by El Karoui and Mazliak [ElkM97] or the monograph by Ma and Yong [MY00] for other extensions. The connection between BSDE and PDE is developed in more detail in the survey paper by Pardoux [Pa98].

Applications of BSDEs to control and mathematical finance were first developed in the work by El Karoui, Peng and Quenez [ElkPQ97]. The presentation of Section 6.4.1 is largely inspired by their paper. Other applications of BSDE for control are studied in Hamadène and Lepeltier [HL95]. The sufficient verification theorem for the maximum principle stated in Section 6.4.2 is also dealt with in the book by Yong and Zhou [YZ00].

Reflected BSDEs and their connection with optimal stopping problems were introduced in the reference article by El Karoui et al. [EKPPQ97]. Several extensions of BSDEs with reflections on one or two barriers were studied in the literature motivated especially by applications to real options in finance. We cite among others the papers by Lepeltier, Matoussi and Xu [LMX05], and Hamadène and Jeanblanc [HJ07]. Inspired by applications to hedging problems under portfolio constraints, BSDEs with constraints on the Z component were considered in Cvitanic, Karatzas and Soner [CKS98] and Buckdahn and Hu [BuH98]. We also cite the important paper by Peng [Pe99], who developed nonlinear decomposition theorem of Doob Meyer type, and studied the general case of constraints on (Y, Z).

The use of BSDE for the resolution of the exponential utility maximization problem with option payoff was studied in El Karoui and Rouge [ElkR00], see also the papers by Sekine [Se06] and Hu, Imkeller, Müller [HIM05] for power utility functions. The applications of BSDEs to mean-variance hedging problems, and more generally to control problems with linear state and quadratic costs, were initiated by Bismut [Bis78] and extended in the papers by Kohlmann and Zhou [KZ00], Zhou and Li [ZL00], Kohlmann and Tang [KT02], or Mania [Ma03].

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