2

The Boltzmann Equation and its Formal Hydrodynamic Limits

The kinetic theory, introduced by Boltzmann at the end of the nineteenth century, provides a description of gases at an intermediate level between the hydrodynamic description which does not allow to take into account phenomena far from thermodynamic equilibrium, and the atomistic description which is often too complex. For a detailed presentation of the various models and their derivation from the fundamental laws of physics, we refer to the book of Cercignani, Illner and Pulvirenti [31] or to the survey on the Boltzmann equation by Villani [106]. Here we will just recall some basic facts which are useful for the understanding of the problem of hydrodynamic limits.

Kinetic theory aims at describing a gas (or a plasma), that is a system constituted of a large number N of electrically neutral (or charged) particles from a *microscopic point of view*. The state of the gas is therefore modelled by a distribution function in the particle phase space, which includes both macroscopic variables, i.e. the position x in physical space, and microscopic variables, for instance the velocity v. In the case of a monatomic gas,

$$f \equiv f(t,x,v), \quad t > 0,\, x \in \Omega, v \in \mathbf{R}^3 .$$

meaning that, for all infinitesimal volume $dxdv$ around the point (x,v) of the phase space, $f(t,x,v)dxdv$ represents the number of particles, which at time t, have position x and velocity v.

The function f is of course nonnegative, it is not directly observable but allows to compute all measurable macroscopic quantities which can be expressed in terms of microscopic averages, namely the local density R, the local bulk velocity U or the local temperature T

$$\begin{gathered} R(t,x) = \int f(t,x,v)dv, \quad RU(t,x) = \int f(t,x,v)v dv, \\ R(|U|^2 + 3T)(t,x) = \int f(t,x,v)|v|^2 dv. \end{gathered} \tag{2.1}$$

L. Saint-Raymond, *Hydrodynamic Limits of the Boltzmann Equation*,
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The distribution function f can actually be seen as the one-particle marginal of some probability density $f^{(N)}$ on the space $(\Omega \times \mathbf{R}^3)^N$ of all microscopic configurations. Of course such a *statistical description* makes sense only if the number N of particles is sufficiently large so that the gas can be considered as a continuous medium. Kinetic equations are thus obtained in the thermodynamic limit, i.e. as N tends to infinity.

From Newton's principle we can deduce a linear partial differential equation for $f^{(N)}$, the so-called Liouville equation, and then, if we neglect the interactions between particles, we obtain the following *free transport equation* for f :

$$\partial_t f + v \cdot \nabla_x f = 0, \tag{2.2}$$

meaning that particles travel at constant velocity, along straight lines, and that the density is constant along characteristic lines

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = 0\,.$$

The operator $v \cdot \nabla_x$ is the classical transport operator. Its mathematical properties are much subtler than it would seem at first sight and will be discussed later. Complemented with suitable boundary conditions, equation (2.2) is the right equation for describing a classical gas of noninteracting particles. Many variants are possible. For instance, in the relativistic case, v should be replaced in (2.2) by $p/\sqrt{m^2 + (p/c)^2}$, where c is the speed of light and m is the mass of elementary particles.

Now, if the microscopic interactions between particles are described through a very long-range potential (namely in the case of electromagnetic interactions), it is enough to consider only the global effect on each particle of the interaction forces exerted by all other particles, and we get *mean field models* of the following type

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0, \tag{2.3}$$

where the force F can be computed in terms of the distribution function f. For instance, in the electrostatic approximation, F is proportional to the electric field, which is itself obtained from the density $\rho = \int f dv$ by the Poisson equation.

In the case when microscopic interactions are described by some short-range potential, it is not possible to evaluate the effects of the interacting forces in a global way, using only some averaged quantities. The interactions are indeed very sensitive to the exact positions and velocities of the particles : considering for instance a system of hard spheres, i.e. of particles which collide bounce on each other like billiard balls, it is indeed easy to see that changing slightly the position of one particle may modify strongly the dynamics of the system (see Figure 2.1).

The derivation of *collisional kinetic models* requires therefore very strong assumptions to guarantee some "statistical stability" of the dynamics.

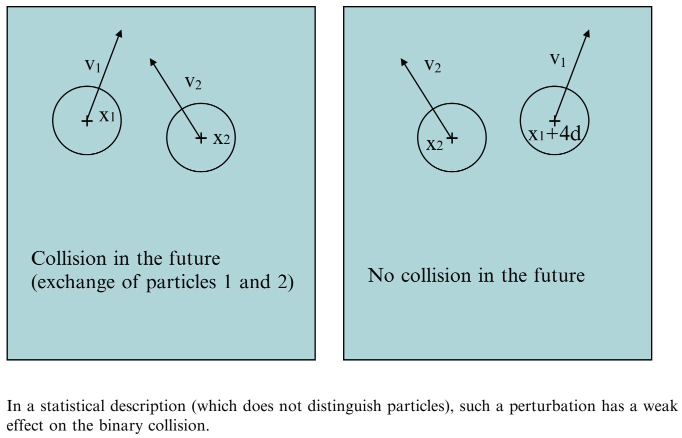

Fig. 2.1. Instability of trajectories

2.1 Formulation and Fundamental Properties of the Boltzmann Equation

2.1.1 The Boltzmann Collision Integral

The Boltzmann equation is obtained in the thermodynamic limit $N \to \infty$ under the following conditions :

- *particles interact via binary collisions*, meaning that the gas is dilute enough that the effect of interactions involving more than two particles can be neglected. Furthermore, collisions are localized both in space and time, meaning that the typical duration and impact parameter of the interacting processes are negligible compared respectively to the typical time and space scales of the description.
  More precisely, the system has to satisfy the scaling assumption, known as *Boltzmann-Grad scaling*
  $$Nd^3 << L^3, \quad Nd^2 = O(L^2),$$
  where d denotes the typical range of microscopic interactions, and L is the typical macroscopic length scale.
- *collisions are elastic*, meaning that momentum and kinetic energy are preserved in the microscopic collision process. Denoting by v', v'_* the velocities before collision, and by v, v_* the velocities after collision, the following equations have to be satisfied
  $$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2, \tag{2.4}$$

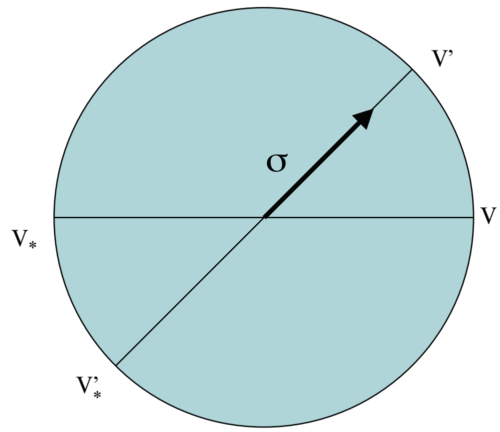

Fig. 2.2. Parametrization of elastic collisions

so that v' and v'_* can be parametrized by $\sigma \in S^2$ as shown in Figure 2.2

$$v' = \frac{v+v_*}{2} + \frac{|v-v^*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v^*|}{2}\sigma \tag{2.5}$$

Note that, as the microscopic dynamics is time-reversible, the probability that (v, v^*) are changed into (v', v'_*) in a collision process is the same as the probability that (v', v'_*) are changed into (v, v_*).

- *collisions involve only uncorrelated particles*, meaning in particular that particles which have already collided are expected not to re-collide in the future. Such a chaos assumption (which implies an asymmetry between the past and the future) allows to consider that the joint distribution of velocities of particles which are about to collide is given by a tensor product (in velocity space) of f with itself.
  It has been proved by Lanford in 1978 [69] that chaos is asymptotically propagated in the Boltzmann-Grad limit (at least for small times), provided that the initial probability density $f_{in}^{(N)}$ is sufficiently close to a tensor product $(f_{in})^{\otimes N}$.

The Boltzmann equation reads therefore

$$\partial_t f + v \cdot \nabla_x f = Q(f, f) \tag{2.6}$$

where Q is a quadratic operator acting only on the v variable (first assumption), and involving tensor products (third assumption).

It is given by

$$Q(f, f) = \int_{\mathbf{R}^3} dv^* \int_{S^2} d\sigma B(v - v_*, \sigma)(f' f'_* - f f_*) \tag{2.7}$$

where we have used the standard abbreviations

$$f' = f(v'), \quad f'_* = f(v'_*), \quad f_* = f(v_*)$$

with (v', v'_*) given by (2.5) (second assumption).

The Boltzmann collision operator can therefore be split into a *gain term* and a *loss term*

$$Q(f, f) = Q^+(f, f) - Q^-(f, f).$$

The loss term counts all collisions in which a given particle of velocity v will encounter another particle, of velocity v^*, and thus will change its velocity leading to a loss of particles of velocity v, whereas the gain term measures the number of particles of velocity v which are created due to some collision between particles of velocities v' and v'_*.

The *collisional cross-section* $B \equiv B(z, \sigma)$ is a nonnegative function depending only on $|z|$ and the scalar product $z \cdot \sigma$ (because of the microreversibility assumption), which measures in some sense the statistical repartition of post-collisional velocities given the pre-collisional velocities. It depends crucially on the nature of the microscopic interactions.

If the particles are assumed to interact via a given potential Φ, the post-collisional velocities and especially the deviation angle θ defined by

$$\cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma$$

can be computed in terms of the impact parameter b and relative velocity $z = v - v_*$ as the result of a classical scattering problem (see [28] for instance) :

$$\theta(b, z) = \pi - 2 \int_0^{b/s_0} \frac{du}{\sqrt{1 - u^2 - \frac{4}{|z|^2}\Phi\left(\frac{b}{u}\right)}},$$

where s_0 is the positive root of

$$1 - \frac{b^2}{s_0^2} - 4\frac{\Phi(s_0)}{|z|^2} = 0\,.$$

Then the cross-section B is implicitly defined by

$$B(|z|, \cos\theta) = \frac{b}{\sin\theta}\frac{db}{d\theta}|z|.$$

It can be made explicit in the case of hard spheres

$$B(|z|, \cos\theta) = a^2|z|,$$

where a is the (scaled) radius of the spheres, and in the case of Coulomb interaction where B is given by Rutherford's formula. In the important model case of inverse-power law potentials, the cross-section cannot be computed explicitly, but one can show that

$$B(|z|,\cos\theta) = b(\cos\theta)|z|^\gamma$$

where γ depends on the power occurring in the potential, and b is a locally smooth function with a nonintegrable singularity at $\theta = 0$. The case of Maxwellian molecules corresponds to the situation when $\gamma = 0$, which is not physically relevant but enables one to do many explicit calculations in agreement with physical observations.

The nonintegrable singularity in the angular cross-section b is an effect of the huge amount of *grazing collisions*, i.e. of collisions with a very large impact parameter so that colliding particles are hardly deviated. Such a singularity appears as soon as the forces are of infinite range, no matter how fast they decay at infinity. By the way, it seems strange to allow infinite-range forces, while we assumed interactions to be localized. Anyhow, in all the sequel we shall tame the singularity for grazing collisions and replace the cross-section by a locally integrable one, which is referred to as *cut-off process*. More precisely, following Grad [59], we will assume

$$\begin{gathered}0 < B(|z|,\sigma) \leq C_b(1+|z|)^\beta \text{ a.e. on } \mathbf{R}^3 \times S^2, \text{ with } \beta \in [0,1] \\ \iint_{S^2} B(z,\sigma)d\sigma \geq \frac{1}{C_b}\frac{|z|}{1+|z|} \text{ a.e. on } \mathbf{R}^3\ .\end{gathered} \tag{2.8}$$

In the case of a spatial domain $\Omega \subset \mathbf{R}^3$ with boundaries, the Boltzmann equation has to be supplemented with boundary conditions which model the interaction between the particles and the frontiers of the domain $\partial\Omega$. These boundary conditions have to be prescribed only on incoming trajectories, that is on the set

$$\Sigma_- = \{(t,x,v) \in \mathbf{R}^+ \times \partial\Omega \times \mathbf{R}^3 \ / \ v\cdot n(x) < 0\} \tag{2.9}$$

where $n(x)$ stands for the outward unit normal vector at $x \in \partial\Omega$.

The most natural boundary condition is the *specular reflection*

$$f(t,x,R_x v) = f(t,x,v), \quad R_x v = v - 2(v\cdot n(x))n(x), \quad x \in \partial\Omega. \tag{2.10}$$

Such a condition expresses the fact that particles bounce back on the wall with a post-collisional angle equal to the pre-collisional angle. The wall is therefore considered as a perfect solid with a regular surface whose direction is precisely known. In particular, the atomistic nature of the solid and the fine details of the gas-surface interaction are not taken into account.

An alternative consists in modelling the statistical effects of the boundary irregularities, using a scattering kernel K (see [28] for further details on this topic) :

$$f(t,x,v) = \int_{v'\cdot n(x)>0} K(v',v) f(t,x,v')dv', \quad \text{on } \Sigma_-, \tag{2.11}$$

A particular case is the *Maxwellian reflection*

$$f(t,x,v) = \left(\int_{v' \cdot n(x) > 0} f(t,x,v') dv' \right) M_W(v), \quad \text{on } \Sigma_-, \tag{2.12}$$

where M_W is some fixed normalized gaussian distribution depending on the temperature of the wall. In this model, particles are absorbed and then re-emitted according to the distribution M_W, corresponding to a thermodynamic equilibrium between particles and the wall.

Of course one can combine the above conditions, which leads to more realistic models.

It is important to note that the set of characteristics relying on the singular set

$$\Sigma_0 = \{(t,x,v) \in \mathbf{R}^+ \times \partial\Omega \times \mathbf{R}^3 \ / \ v \cdot n(x) = 0\}$$

is of zero Lebesgue measure, so that it is not necessary to define the distribution function on it. (We refer for instance to the results - based on Sard's Theorem - established by Bardos in [3].)

2.1.2 Local Conservation Laws

The pre-postcollisional change of variable

$$(v', v'_*, \sigma) \mapsto (v, v_*, \sigma)$$

is involutive (since the collisions are assumed to be elastic) and has therefore unit Jacobian. Furthermore, as a consequence of microreversibility, it leaves the cross-section invariant.

Then, if φ is an arbitray continuous function of the velocity v

$$\begin{aligned}
&\int_{\mathbf{R}^3} Q(f,f)\varphi(v)dv \\
&= \int_{\mathbf{R}^3 \times \mathbf{R}^3} dv dv_* \int_{S^2} d\sigma B(v - v_*, \sigma)(f' f'_* - f f_*)\varphi \\
&= \frac{1}{2} \int_{\mathbf{R}^3 \times \mathbf{R}^3} dv dv_* \int_{S^2} d\sigma B(v - v_*, \sigma)(f' f'_* - f f_*)(\varphi + \varphi_*) \\
&= \frac{1}{4} \int_{\mathbf{R}^3 \times \mathbf{R}^3} dv dv_* \int_{S^2} d\sigma B(v - v_*, \sigma)(f' f'_* - f f_*)(\varphi + \varphi_* - \varphi' - \varphi'_*)
\end{aligned}$$

provided that f satisfies convenient integrability conditions.

As an immediate consequence, whenever φ satisfies the functional equation

$$\varphi(v) + \varphi(v_*) = \varphi(v') + \varphi(v'_*) \qquad \forall (v, v_*, \sigma) \in \mathbf{R}^3 \times \mathbf{R}^3 \times S^2 \tag{2.13}$$

then, at least formally

$$\int_{\mathbf{R}^3} Q(f,f)\varphi(v)dv = 0.$$

An important result in the theory of the Boltzmann equation asserts that all measurable a.e. finite functions satisfying (2.13) are linear combinations of the *collision invariants*

$$1, v_1, v_2, v_3, |v|^2.$$

The proof of this result is far from obvious; see for instance [28].

This leads to the formal *conservation laws* for the Boltzmann equation.

Proposition 2.1.1 *Let $f \equiv f(t,x,v)$ be a solution of the Boltzmann equation (2.6) that is locally integrable and rapidly decaying in v for each (t,x). Then the following local conservation laws hold :*

$$\begin{aligned} &\partial_t \int_{\mathbf{R}^3} f dv + \nabla_x \cdot \int_{\mathbf{R}^3} v f dv = 0, \\ &\partial_t \int_{\mathbf{R}^3} v f dv + \nabla_x \cdot \int_{\mathbf{R}^3} v \otimes v f dv = 0, \\ &\partial_t \int_{\mathbf{R}^3} \frac{1}{2}|v|^2 f dv + \nabla_x \cdot \int_{\mathbf{R}^3} \frac{1}{2}|v|^2 v f dv = 0, \end{aligned} \tag{2.14}$$

respectively the local conservation of mass, momentum and energy.

Yet, to this date, no mathematical theory has been able to justify these simple rules at a sufficient level of generality. Even the corresponding global conservation laws in the absence of boundaries are not established. The problem is of course that too little is known about how well behaved are the solutions to the Boltzmann equation.

With the notations of the introduction for the thermodynamic fields, namely the local density R, the local bulk velocity U and the local temperature T

$$R(t,x) = \int f(t,x,v)dv, \quad RU(t,x) = \int f(t,x,v)vdv,$$
$$R(|U|^2 + 3T)(t,x) = \int f(t,x,v)|v|^2 dv,$$

and the following definition of the pressure tensor

$$P(t,x) = \int (v-U)^{\otimes 2} f(t,x,v)dv$$

these continuity equations are

$$\begin{aligned} &\partial_t R + \nabla_x \cdot (RU) = 0, \\ &\partial_t (RU) + \nabla_x \cdot (RU \otimes U + P) = 0, \\ &\partial_t (R|U|^2 + \mathrm{tr}(P)) + \nabla_x \cdot \big(U(R|U|^2 + \mathrm{tr}(P)) + 2P \cdot U\big) \\ &\qquad\qquad = -\nabla_x \cdot \left(\int (v-U)|v-U|^2 f dv \right), \end{aligned}$$

where $\mathrm{tr}(P)$ denotes the trace of the pressure tensor. Note that these equations are very similar to the Euler equations for compressible perfect gases.

2.1.3 Boltzmann's H Theorem

The other very important feature of the Boltzmann equation comes also from the symmetries of the collision operator. Without caring about integrability issues, we plug $\varphi = \log f$ into the symmetrized integral obtained in the previous paragraph, and use the properties of the logarithm, to find

$$\begin{aligned} D(f) &\stackrel{\text{def}}{=} -\int Q(f,f)\log f dv \\ &= \frac{1}{4}\int_{\mathbf{R}^3\times\mathbf{R}^3\times S^2} dv dv_* d\sigma B(v-v_*,\sigma)(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*} \geq 0 \end{aligned} \tag{2.15}$$

The so-defined entropy dissipation is therefore a nonnegative functional, and it can be proved that its minimizers (in the class of locally integrable functions rapidly decaying and such that $\log f$ has at most polynomial growth as $|v| \to \infty$) are Maxwellian densities, i.e. distribution functions of the following form

$$\mathcal{M}_{R,U,T}(v) = \frac{R}{(2\pi T)^{3/2}}\exp\left(-\frac{|v-U|^2}{2T}\right) \tag{2.16}$$

for some $R, T > 0$ and $U \in \mathbf{R}^3$. This result is an easy consequence of the characterization of the collision invariants provided that f is continuous. In the general case, it can be proved by a nice argument due to Perthame (see [16] for instance) using the Fourier transform of the functional equation on f.

This leads to *Boltzmann's H theorem*, also known as second principle of thermodynamics, stating that the entropy is (at least formally) a Lyapunov functional for the Boltzmann equation :

Proposition 2.1.2 *Let $f \equiv f(t,x,v)$ be a solution of the Boltzmann equation (2.6) that is locally integrable and such that f is rapidly decaying in v and* $\log f$ *has at most polynomial growth as $|v| \to \infty$ for each (t,x). Then the following local entropy inequality holds :*

$$\partial_t \int f\log f dv + \nabla_x \cdot \int vf\log f dv = -D(f) \leq 0. \tag{2.17}$$

Again this differential inequality is formally reminiscent of the Lax-Friedrichs criterion that selects admissible solutions of the compressible Euler equations. In particular, it demonstrates that the Boltzmann model has some irreversibility built in. However a considerable difference with the theory of hyperbolic system of conservations laws is that Boltzmann's H theorem provides an expression for the entropy dissipation rate in terms of the distribution function, which is local in (t,x).

2.2 Orders of Magnitude and Qualitative Behaviour of the Boltzmann Equation

The aim of this section is to give an overview of the dynamics associated with the Boltzmann equation, depending on the relative sizes of the various physical parameters. Roughly speaking, the convection phenomena are governed by the transport operator, whereas the diffusion phenomena are ruled by the collision operator. The main features of the macroscopic flow should then depend on the balance between these two terms, and especially of the ratio between the various typical length (or time) scales arising in the system.

2.2.1 Nondimensional Form of the Boltzmann Equation

Choose some observation (macroscopic) length scale l_o and time scale t_o, and a reference temperature T_o. This defines two velocity scales :

- one is the speed at which some macroscopic portion of the gas is transported over a distance l_o in time t_o, i.e.
$$\frac{l_o}{t_o}\,;$$
- the other one is the *thermal speed* of the molecules with energy $\frac{3}{2}kT_o$, where k is the Boltzmann constant; in fact, it is more natural to define this velocity scale as
$$c=\sqrt{\frac{5}{3}\frac{kT_o}{m}}$$
m being the molecular mass, which is the *speed of sound* in a monatomic gas at the temperature T_o.

Define next the dimensionless variables involved in the Boltzmann equation, i.e. the dimensionless time, space and velocity variables as
$$\tilde{t}=\frac{t}{t_o},\quad \tilde{x}=\frac{x}{l_o},\ \text{and}\ \tilde{v}=\frac{v}{c}.$$
Define also the dimensionless number density
$$\tilde{f}(\tilde{t},\tilde{x},\tilde{v})=\frac{l_o^3c^3}{N}f(t,x,v)\stackrel{\text{def}}{=}\frac{c^3}{R_o}f(t,x,v),$$
where N is the total number of gas molecules in a volume l_o^3, meaning that R_o is the average macroscopic density.

Finally, since the Boltzmann kernel B has units of the reciprocal product of density by time, it determines a timescale τ by
$$\int \mathcal{M}_{(R_o,0,T_o)}(v)\mathcal{M}_{(R_o,0,T_o)}(v_*)B(v-v_*,\sigma)d\sigma dv_* dv=\frac{N}{l_o^3\tau}\,.$$

The finiteness of the above integral is ensured by Grad's cutoff assumption (2.8) on B, so that $0 < \tau < +\infty$. This is the scale of the average time that particles in the equilibrium density $\mathcal{M}_{(R_0,0,T_0)}$ spend traveling freely between two collisions, the so-called *mean free time*. It is related to the length scale of the *mean free path* λ

$$\lambda = c\tau.$$

Define the dimensionless Boltzmann kernel $\tilde{B}$ by the relation

$$\tilde{B}(\tilde{v} - \tilde{v}_*, \sigma) = R_0 \tau B(v - v_*, \sigma)$$

and set the corresponding dimensionless collision operator to be

$$\tilde{Q}(\tilde{f}, \tilde{f}) = \iint dv_* d\sigma \tilde{B}(\tilde{v} - \tilde{v}_*, \sigma)(\tilde{f}'_* \tilde{f}' - \tilde{f}_* \tilde{f}).$$

Then, the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f, f),$$

can be reformulated in terms of dimensionless variables

$$\frac{l_o}{ct_o} \partial_{\tilde{t}} \tilde{f} + \tilde{v} \cdot \nabla_{\tilde{x}} \tilde{f} = \frac{l_o}{\lambda} \tilde{Q}(\tilde{f}, \tilde{f}).$$

The factor multiplying the collision integral is the inverse *Knudsen number*

$$\mathrm{Kn} = \frac{\lambda}{l_o},$$

while the factor multiplying the time derivative is the kinetic *Strouhal number*

$$\mathrm{St} = \frac{l_o}{ct_o}$$

(by analogy with the notion of Strouhal number used in the dynamics of vortices). Hence the dimensionless form of the Boltzmann equation is (dropping all tildas)

$$\mathrm{St} \partial_t f + v \cdot \nabla_x f = \frac{1}{\mathrm{Kn}} Q(f, f). \tag{2.18}$$

Before discussing the qualitative behaviour of the solution to the Boltzmann equation in terms of the relative sizes of the parameters Kn and St, let us comment a little bit on the choice of the reference scales, and introduce another dimensionless parameter which allows to compensate the arbitrariness of this choice.

A rather natural thing to do is to choose the length, time and temperature scales l_o, t_o, T_o in a way that is consistent with the geometry of the domain where the gas motion takes place, the time necessary to observe significant gas motion, and the distribution function at the initial instant of time. In this case, the ratio l_o/t_o corresponds to the *bulk velocity* u_o of the flow and the Strouhal number is nothing else than the *Mach number*

$$\mathrm{Ma} = \frac{u_o}{c}.$$

However, considering small fluctuations around some reference flow, it may happen that the bulk velocity u_o to be studied is very small compared to the ratio l_o/t_o (which leads to some "linearized" hydrodynamics), so it makes sense to consider situations such that

$$\mathrm{Ma} << \mathrm{St}.$$

2.2.2 Hydrodynamic Regimes

All hydrodynamic limits of the Boltzmann equation correspond to situations where the Knudsen number Kn satisfies

$$\mathrm{Kn} << 1\,.$$

Indeed, in view of Boltzmann's H theorem, one expects the distribution function to resemble more and more a local Maxwellian when $\mathrm{Kn} \to 0$. In other words, the collision mechanism holds on a time scale which is very small compared to the observation time scale, so that one can consider that local thermodynamic equilibrium is reached almost instantaneously. This means that the Knudsen number Kn governs the transition from kinetic theory to hydrodynamics.

But there is no universal prescription for the Strouhal number in this context; as we shall see below, various hydrodynamic regimes can be derived from the Boltzmann equation by appropriately tuning the Strouhal number St.

The Compressible Euler Limit

is the easiest of all hydrodynamic limits of the Boltzmann equation at the formal level, as can be expected from the previously mentioned analogy between the system of conservation laws (2.14) associated with the Boltzmann equation, and the compressible Euler system. Indeed, as $\mathrm{Kn} \to 0$, solutions of the Boltzmann equation behave as local Maxwellians, namely

$$f(t,x,v) \sim \frac{R(t,x)}{(2\pi T(t,x))^{3/2}} \exp\left(-\frac{|v-U(t,x)|^2}{2T(t,x)}\right)$$

for some $R(t,x), T(t,x) > 0$ and $U(t,x) \in \mathbf{R}^3$.

Therefore, passing to the limit in the local conservation laws (2.14), we get

$$\begin{aligned}
&\mathrm{St}\partial_t R + \nabla_x \cdot (RU) = 0,\\
&\mathrm{St}\partial_t (RU) + \nabla_x \cdot (RU \otimes U + RTId) = 0,\\
&\mathrm{St}\partial_t (R|U|^2 + 3RT) + \nabla_x \cdot \big(U(R|U|^2 + 5RT)\big) = 0,
\end{aligned} \tag{2.19}$$

which are the equations of hydrodynamics for perfect gases, satisfying in particular the *state relation*

$$P = RTId.$$

That there is no excluded volume in this state relation is strongly linked with the Boltzmann-Grad scaling assumption $Nd^3 << l_o^3$, which expresses the fact that the volume occupied by the particles is negligible compared with the volume of the domain.

Furthermore, taking limits in the local entropy inequality (2.17), we obtain

$$\mathrm{St}\partial_t \left(R \log \frac{R}{T^{3/2}} \right) + \nabla_x \cdot \left(RU \log \frac{R}{T^{3/2}} \right) \leq 0, \tag{2.20}$$

which is exactly the Lax admissibility condition, characterizing among the solutions of (2.19) those which are physically relevant, i.e. which satisfied the second principle of thermodynamics.

In other words, we expect the moments of the solution f to the Boltzmann equation to be approximated at order $O(\mathrm{Kn})$ by the solution to the compressible Euler equations.

A natural question is then to determine higher order hydrodynamic corrections to the compressible Euler system.

Higher Order Hydrodynamic Approximations

can be obtained by using asymptotic expansions of the distribution function in terms of the Knudsen number Kn, or in other words by seeking solutions of the scaled Boltzmann equation (2.18) as formal power series in Kn

$$f(t,x,v) = \sum_{k\geq 0} (\mathrm{Kn})^k f_k(t,x,v),$$

with coefficients f_k that are smooth in (t,x,v) and rapidly decaying as $|v| \to \infty$. Of course the leading order approximation f_0 is expected to be the limiting hydrodynamic distribution function, that is the local Maxwellian with thermodynamic fields satisfying the compressible Euler equations (2.19), while the successive corrections f_k account for finite Knudsen effects. Note that, depending on the exact form of the Ansatz, this process will lead to different hierarchies of PDEs.

Hilbert's expansion

$$f(t,x,v) = f_0(t,x,v)\left(1 + \sum_{k\geq 1} (\mathrm{Kn})^k g_k(t,x,v)\right)$$

is historically the older and goes back to Hilbert's fundamental paper [65] on the kinetic theory of gases. Plugging this Ansatz in the scaled Boltzmann equation (2.18), and balancing the resulting coefficients of the successive powers of Kn, one gets, as compatibility conditions to solve the hierarchy, that at each order $k \geq 1$, the hydrodynamic part of g_k satisfies the linearized compressible Euler equations (with source terms depending on g_{k-j}, for $j = 1, ..., n-1$). It seems then natural to collect all the contributions to the local thermodynamic equilibrium at leading order.

Such a variant of Hilbert's expansion was found independently by Chapman and Enskog, and is known today as *Chapman-Enskog's expansion* [33]

$$f(t,x,v) = \mathcal{M}_f(t,x,v)\left(1 + \sum_{k\geq 1} (\mathrm{Kn})^k \tilde{g}_k(t,x,v)\right)$$

where $\mathcal{M}_f$ is the local Maxwellian with same moments as f

$$\begin{gathered}\mathcal{M}_f(t,x,v) = \frac{R(t,x)}{(2\pi T(t,x))^{3/2}} \exp\left(-\frac{|v-U(t,x)|^2}{2T(t,x)}\right),\\ R(t,x) = \int f(t,x,v)dv, \quad RU(t,x) = \int v f(t,x,v)dv,\\ R(|U|^2 + 3T)(t,x) = \int |v|^2 f(t,x,v)dv,\end{gathered} \tag{2.21}$$

and the fluctuations $\tilde{g}_k$ are functions of v depending on (t,x) through $R(t,x)$, $U(t,x)$ and $T(t,x)$ and their partial x-derivatives evaluated at (t,x). Note that, at variance with Hilbert's expansion, Chapman-Enskog's Ansatz requires knowing in advance that the successive corrections to the compressible Euler system (2.19) within any order in Kn are systems of local conservation laws.

The first correction to the compressible Euler equations is then given by

$$\mathrm{St}\partial_t \mathcal{M}_f + v\cdot\nabla_x \mathcal{M}_f = -\mathcal{M}_f \mathcal{L}_{\mathcal{M}_f}(\tilde{g}_1),$$

or equivalently

$$\begin{aligned}\mathrm{St}\partial_t\left(\log R - \frac{3}{2}\log T - \frac{1}{2T}|v-U|^2\right) + \nabla_x\left(\log R - \frac{3}{2}\log T - \frac{1}{2T}|v-U|^2\right)\cdot v\\ = -\mathcal{L}_{\mathcal{M}_f}(\tilde{g}_1),\end{aligned}$$

where $\mathcal{L}_{\mathcal{M}_f}$ denotes the linearization of the collision operator at the local Maxwellian $\mathcal{M}_f$. Then, using the properties of the linearized collision operator $\mathcal{L}_{\mathcal{M}_f}$ (to be studied in the next chapter), namely the fact that it is a

Fredholm operator, one obtains the compressible Navier-Stokes system with $O(\mathrm{Kn})$ dissipation terms :

$$\begin{aligned}
&\mathrm{St}\partial_t R + \nabla_x \cdot (RU) = 0,\\
&\mathrm{St}\partial_t (RU) + \nabla_x \cdot RU \otimes U + RTId) = \mathrm{Kn}\nabla_x \cdot (\mu(R,T)DU) + O(\mathrm{Kn}^2),\\
&\mathrm{St}\partial_t (R|U|^2 + 3RT) + \nabla_x \cdot \big(U(R|U|^2 + 5RT)\big) = \mathrm{Kn}\nabla_x \cdot (\kappa(R,T)\nabla_x T)\\
&\qquad\qquad\qquad + \mathrm{Kn}\nabla_x \cdot (\mu(R,T)DU \cdot U) + O(\mathrm{Kn}^2),
\end{aligned} \tag{2.22}$$

where DU denotes the traceless part of the deformation tensor

$$DU = \frac{1}{2}(\nabla_x U + (\nabla_x U)^T) - \frac{1}{3}(\nabla_x \cdot U)\,\mathrm{Id},$$

and the diffusive coefficients, namely the viscosity $\mu \equiv \mu(R,T)$ and the heat conductivity $\kappa \equiv \kappa(R,T)$, are defined in terms of the linearized collision operator $\mathcal{L}_{\mathcal{M}_f}$.

We then deduce formally that the solution to the Navier-Stokes equations is close to the moments of the solution f to the Boltzmann equation at order $O(\mathrm{Kn}^2)$.

Such a process can be iterated in order to get further corrections to the Navier-Stokes system, which leads to a hierarchy of hydrodynamic models (note however that their well-posedness requires a convenient truncation algorithm, as that proposed recently by Bobylev and Levermore [13]).

The Main Qualitative Features of the Hydrodynamic Flows

governed by the Boltzman equation can therefore be characterized in terms of the nondimensional parameters introduced at the beginning of this section, namely the Knudsen, Strouhal and Mach numbers Kn, St and Ma.

The previous results are summarized in Figure 2.3.

2.2.3 Corrections to Hydrodynamic Approximations

Furthermore we are also able to estimate by how much the solutions to the scaled Boltzmann equation deviate from their hydrodynamic approximations, at least inside the domain Ω (see Figure 2.4).

The *adiabaticity* of the gas is indeed measured in terms of the Knudsen number Kn. In a gas close to local thermodynamic equilibrium, the deviation from the hydrodynamic approximation is given by an entropic relaxation on a time scale of order Kn.

The *compressibility* of the fluid is then measured in terms of the Mach number Ma. In a weakly compressible fluid, the deviation from the incompressible approximation is given by compression/decompression waves, also called acoustic waves, oscillating on a period of order Ma.

The *viscosity* of a perfect gas is measured in terms of the Reynolds number $\mathrm{Re} = \mathrm{Ma}/\mathrm{Kn}$. In a weakly viscous fluid, the deviation from the hyperbolic approximation is given by a small diffusion which smoothes the shock profiles on length scales of order $1/\sqrt{\mathrm{Re}}$.

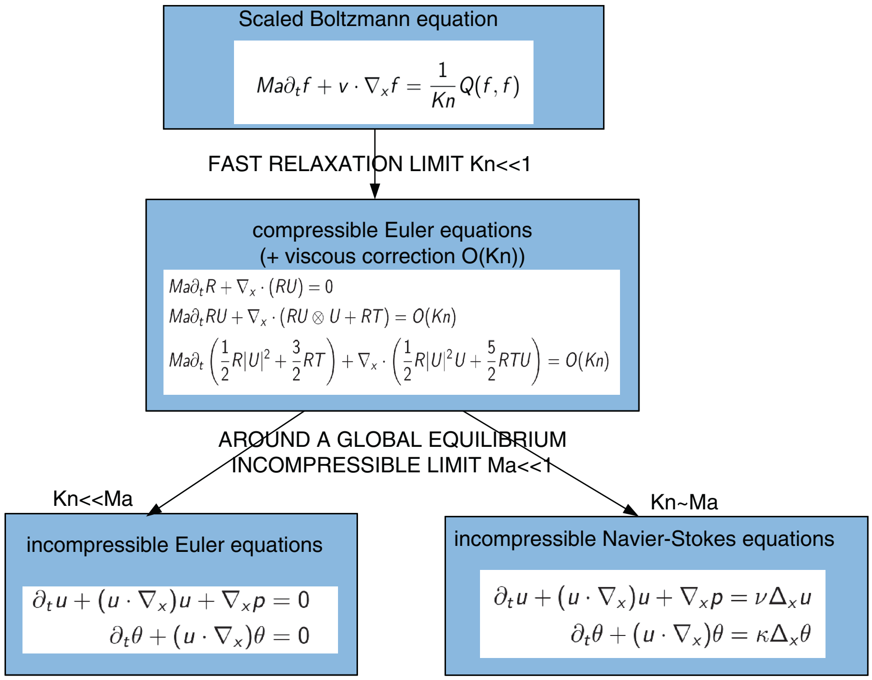

Fig. 2.3. Hydrodynamic models for rarefied gases

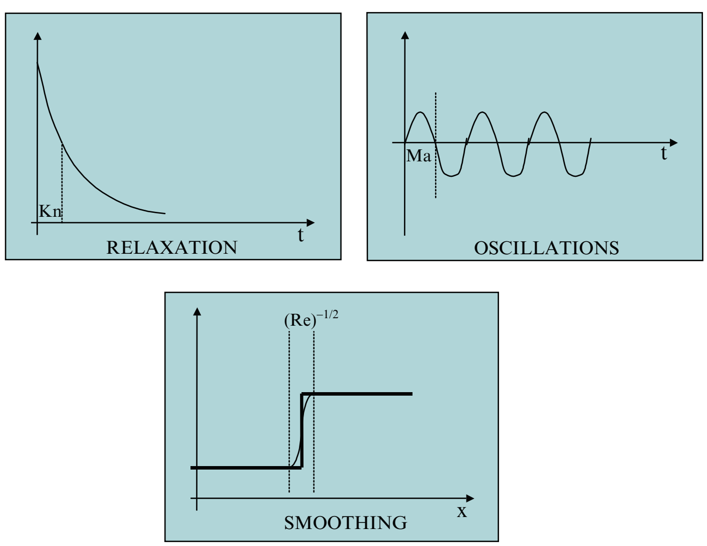

Fig. 2.4. Corrections to hydrodynamic approximations

2.2.4 Taking into Account the Boundary

It remains then to understand what happens in the vicinity of the boundaries $\partial\Omega$, which can be either exterior boundaries or obstacles.

Let us first recall that, at the microscopic level, the interaction between the gas and the boundaries is modelized phenomenologically by Maxwell's condition. If the boundary is perfectly smooth, the reflection is specular. If the boundary is rough, one further introduces some diffusion by a scattering operator, which is a relevant approximation when considering large length scales compared to the boundary irregularities. The roughness of the boundary is then measured by a supplementary non-dimensional parameter $\alpha \in [0,1]$, called the *accommodation coefficient*. More precisely the balance between the outgoing and incoming part of the trace of f states

$$f_{|\Sigma_-} = (1-\alpha)Lf_{|\Sigma_+} + \alpha K(f_{|\Sigma_+}) \quad \text{on } \Sigma_- \tag{2.23}$$

where we recall that the outgoing/incoming sets Σ_+ and Σ_- at the boundary $\partial\Omega$ are defined by

$$\Sigma_\pm = \{(x,v) \in \partial\Omega \times \mathbf{R}^3, \quad \pm n(x)\cdot v \ > 0\} \tag{2.24}$$

denoting by n the outward normal on $\partial\Omega$.

The local reflection operator L is given by

$$Lf(x,v) = f(x, R_x v) \tag{2.25}$$

where $R_x v = v - 2(v\cdot n(x))n(x)$ is the velocity before the collision with the wall. The diffuse reflection operator K is given by

$$Kf(x,v) = M_W(v)\int_{v'.n(x)>0} f(x,v')\,(v'\cdot n(x))dv' \tag{2.26}$$

where M_W is some Maxwellian distribution characterizing the state of the wall and such that

$$\int_{v.n(x)>0}(v\cdot n(x))M_W(v)dv = \int_{v.n(x)<0}|v\cdot n(x)|M_W(v)\ dv = 1,$$

which expresses the conservation of mass at the boundary.

At the macroscopic level, one can obtain two types of behaviours at the boundary : either a braking (represented by the *Dirichlet boundary condition*) or a slipping (represented by the *Navier boundary condition*), or a combination of these two phenomena (expressed by some *mixed Robin boundary condition*). This behaviour will depend of course on the nature of the boundary, but also on the viscosity of the fluid.

If the fluid is viscous, one can characterize the fluid/boundary interaction in terms of the ratio α/Ma (full braking if $\alpha/\mathrm{Ma} \to +\infty$, perfect slipping if $\alpha/\mathrm{Ma} \to 0$).

If the fluid is inviscid, the braking condition is not mathematically admissible. This means that the flow inside the domain will not depend (at least formally) on the nature of the boundary. The fluid/boundary interaction appears only on a thin layer (of size $1/\sqrt{\mathrm{Re}}$), called Prandtl layer. Nevertheless this layer is generally unstable (see [61] for instance) and may give rise to turbulent effects (reflected back inside the domain).

2.3 Mathematical Theories for the Boltzmann Equation

In this section, we will introduce the main existing mathematical frameworks dealing with the Cauchy problem for the Boltzmann equation, which can be useful for the study of hydrodynamic limits. In particular, we will discuss neither the numerous results concerning the spatially homogeneous Boltzmann equation, nor the local existence results.

Let us first describe briefly the most apparent problems in trying to construct a general, good theory. In the full, general situation, known a priori estimates for the Boltzmann equation are only those which are associated to the basic physical laws, namely the formal conservation of mass and energy, and the bounds on entropy and entropy dissipation. Note that, when the physical space is unbounded, the dispersive properties of the free transport operator allow to further expect some control on the moments with respect to x-variables. Yet the Boltzmann collision integral is a quadratic operator that is purely local in the position and time variables, meaning that it acts as a convolution in the v variable, but as a pointwise multiplication in the t and x variables : thus, with the only a priori estimates which seem to hold in full generality, the collision integral is even not a well-defined distribution with respect to x-variables. This major obstruction is one of the reasons why the Cauchy problem for the Boltzmann equation is so tricky, another reason being the intricate nature of the Boltzmann operator.

2.3.1 Perturbative Framework : Global Existence of Smooth Solutions

Historically the first global existence result for the spatially inhomogeneous Boltzmann equation is due to Ukai [103], who considered initial data that are fluctuations around a global equilibrium, for instance around the reduced centered Gaussian M :

$$f_{in} = M(1 + g_{in}).$$

He proved the global existence of a solution to the Cauchy problem for (2.6) under the assumption that this initial perturbation g_{in} is smooth and small enough in a norm that involves derivatives and weights so as to ensure decay for large v.

The convenient functional space to be considered is indeed

$$H_{l,k} = \{g \equiv g(x,v) \mid \|g\|_{l,k} = \sup_v (1+|v|^k)\|M^{1/2}g(\cdot,v)\|_{H^l_x} < +\infty\}\,.$$

Theorem 2.3.1 *Assume that the collision kernel satisfies Grad's cutoff assumption (2.8) for some* $\beta \in [0,1]$. *Let* $g_{in} \in H_{l,k}$ *for* $l > 3/2$ *and* $k > 5/2$ *such that*

$$\|g_{in}\|_{l,k} \le a_0 \tag{2.27}$$

for some a_0 *sufficiently small.*

Then, there exists a unique global solution $f = M(1+g)$ *with* $g \in L^\infty(\mathbf{R}^+, H_{l,k}) \cap C(\mathbf{R}^+, H_{l,k})$ *to the Boltzmann equation (2.6) with initial data*

$$g_{|t=0} = g_{in}\,.$$

Remark 2.3.2 *The classical theory of the Boltzmann equation close to equilibrium, started with the works of Ukai, has been developed in the framework of hard potentials. Many such existence results, based on linearization and spectral estimates, have been proved, considering initial data which are small and very smooth perturbations of a global (Maxwellian) equilibrium.*

Using some "nonlinear energy method" instead of the spectral study of the linearized problem, and the decomposition of the solution into a "hydrodynamic" part and a "purely kinetic" part, Guo [62] was then able to extend the theory of Boltzmann's equation close to equilibrium, to cover basically all the physically meaningful range of decays of the cross-section.

Sketch of proof of Theorem 2.3.1. Such a global existence result is based on Duhamel's formula and on Picard's fixed point theorem. It requires a very precise study of the *linearized collision operator* $\mathcal{L}_M$ defined by

$$\mathcal{L}_M g = -\frac{2}{M} Q(M, Mg),$$

and more precisely of the semi-group U generated by

$$\frac{1}{\mathrm{St}} v \cdot \nabla_x + \frac{1}{\mathrm{StKn}} \mathcal{L}_M\,.$$

• The first step consists actually in reducing the Boltzmann equation to the integral equation

$$g = N[g], \tag{2.28}$$

where the functional N is defined by

$$\begin{aligned} N[g](t) &= U(t)g_{in} + \psi[g,g](t)\,,\\ \psi[g,g](t) &= \frac{1}{\mathrm{Kn}} \int_0^t U(t-s)\frac{1}{M} Q(Mg, Mg)(s)ds\,. \end{aligned} \tag{2.29}$$

The global well-posedness of the Cauchy problem for (2.6) will then be established by proving that N is a contraction in a ball of $L^\infty(\mathbf{R}^+, H_{l,k}) \cap C(\mathbf{R}^+, H_{l,k})$.

• The second step is to prove the continuity of the linear semi-group U. Using its spectral representation and spectral estimates due to Ellis and Pinsky [45], one obtains the continuity of U in $H^l_x(L^2(Mdv))$.

In order to obtain refined estimates, and especially to gain integrability with respect to the v-variable, one has to use more about the structure of the linearized collision operator, namely the following decomposition due to Hilbert [65] (see also section 2 in Chapter 3)

$$\mathcal{L}_M = \nu - \mathcal{K}$$

where the frequency part satisfies the lower bound

$$\nu(|v|) \geq \nu_- > 0\,,$$

and the integral part $\mathcal{K}$ improves integrability in the v variable (as proved by Caflisch [23]) :

$$\mathcal{K}:\ H^l_x(L^2_v) \to H_{l,0}\,,\ \text{ and } \mathcal{K}:\ H_{l,k} \to H_{l,k+1}\,.$$

From the explicit formula for the semi-group $\bar{U}$ generated by

$$\frac{1}{\mathrm{St}} v \cdot \nabla_x + \frac{1}{\mathrm{StKn}} \nu$$

and Duhamel's formula

$$U(t) = \bar{U}(t) + \frac{1}{\mathrm{StKn}} \int_0^t \bar{U}(t-s) \mathcal{K} U(s) ds\,,$$

we deduce that

$$\|U(t)g\|_Y \leq \exp\Big(-\frac{\nu_-}{\mathrm{StKn}} t\Big)\, \|g\|_Y + C_{X\to Y} \int_0^t \exp\Big(-\frac{\nu_-}{\mathrm{StKn}} (t-s)\Big) \|U(s)u\|_X ds$$

where $\mathcal{K}$ maps X into Y.

Iterating the process shows that, if $k > \frac{3}{2}$, there exists a nonnegative constant C_1 (depending on l and k) such that

$$\|U(t)g\|_{l,k} \leq C_1 \|g\|_{l,k}\,.$$

• The continuity of the bilinear operator ψ is obtained in a very similar way.

Standard continuity estimates for Q shows that

$$\left\|\nu^{-1} \frac{1}{M} Q(Mg, Mh)\right\|_{l,k} + \left\|\frac{1}{M} Q(Mg, Mh)\right\|_{L^1(dx,(L^2(Mdv))} \leq C \|g\|_{l,k} \|h\|_{l,k}\,.$$

for $k > 3/2$, $l > 3/2$.

Then, starting from the spectral estimates on U and using Hilbert's decomposition to gain integrability in the v variable as previously, we obtain the expected continuity property, namely

$$\|\psi[g,h]\|_{l,k} \leq C_2 \|g\|_{l,k} \|h\|_{l,k} \,,$$

where C_2 is a nonnegative constant depending on l and k, provided that $H_{l,k} \subset \nu H^l_x(L^2(Mdv))$, or equivalently $k > 5/2$.

• Equipped with these preliminary results, we get immediately the global existence of a unique solution to (2.6). Indeed, we have

$$\|N[g]\|_{l,k} \leq C_1 \|g_{in}\|_{l,k} + C_2 \|g\|^2_{l,k}$$

and

$$\|N[g] - N[h]\|_{l,k} \leq C_2 \left(\|g\|_{l,k} + \|h\|_{l,k}\right) \|g-h\|_{l,k}$$

Choosing a_0 and a_1 such that

$$2C_2 a_1 < 1 \text{ and } C_1 a_0 + C_2 a_1^2 \leq a_1 \,,$$

we get that N is a contraction on the ball of radius a_1 as soon as

$$\|g_{in}\|_{l,k} \leq a_0 \,,$$

We then conclude by Picard's fixed point theorem. □

The first disadvantage inherent to that strategy is the need for a deep result of spectral theory. In particular, this approach fails to provide a real understanding of the coupling between relaxation and hydrodynamic modes in the full nonlinear Boltzmann equation.

For the purpose of deriving incompressible hydrodynamic limits, it would seem that Ukai's result is exactly what is needed. The difficulty is that it cannot be used as a black box, because of the potential lack of uniformity with respect to the Knudsen number Kn on the critical size of the initial perturbation that guarantees global existence. Let us mention however that Bardos and Ukai [7] have obtained the first mathematical derivation of the incompressible Navier-Stokes equations in that framework.

Nevertheless one cannot expect to extend such a result to classes of initial data with less regularity.

2.3.2 Physical Framework : Global Existence of Renormalized Solutions

For those reasons, we will use a global existence theory for the Boltzmann equation that holds for physically admissible initial data of arbitrary sizes. This theory goes back to the late 80s and is due to DiPerna and Lions [44]. For the sake of completeness, we shall sketch here the main arguments leading to that result, most of which will be detailed in the next chapter since they are also fundamental tools to study hydrodynamic limits.

Our presentation of the subject incorporates later developments of the theory of renormalized solutions :

- we will indeed consider solutions of the Boltzmann equation that converge at infinity to some uniform Maxwellian, for instance the reduced centered Gaussian M (following Lions in [73]);

- we will further present a simplification of the original proof based on compactness properties of the gain term in the collision operator (established by Lions in [72]);

- we will give moreover a weak version of the global conservation of energy and of the local conservation of momentum, involving some defect measure which characterizes the possible loss of energy at large velocities in the approximation scheme (introduced by Lions and Masmoudi in [75]);

- we will also take into account the boundary effects (modelized by Maxwell's boundary condition) (using some refined results of functional analysis due to Mischler [84][85]).

The DiPerna-Lions theory does not yield solutions that are known to solve the Boltzmann equation in the usual weak sense. Rather, it gives the existence of a global weak solution to a class of formally equivalent initial-value problems.

Definition 2.3.3 *A renormalized solution of the Boltzmann equation(2.6) (2.23) relatively to the global equilibrium M is a function*

$$f \in C(\mathbf{R}^+, L^1_{loc}(\Omega \times \mathbf{R}^3))$$

which satisfies in the sense of distributions

$$\begin{aligned}
&M\big(\mathrm{St}\partial_t + v\cdot\nabla_x\big)\Gamma\left(\frac{f}{M}\right) = \frac{1}{\mathrm{Kn}}\Gamma'\left(\frac{f}{M}\right)Q(f,f) \quad \text{on } \mathbf{R}^+\times\Omega\times\mathbf{R}^3\,,\\
&f_{|t=0} = f_{in} \geq 0 \quad \text{on } \Omega\times\mathbf{R}^3,
\end{aligned} \tag{2.30}$$

for any $\Gamma \in C^1(\mathbf{R}^+)$ such that $|\Gamma'(z)| \leq C/\sqrt{1+z}$.

We further require that for every $\varphi \in C^1_c(\bar{\Omega}\times\mathbf{R}^3)$ and every $[t_1,t_2]\subset\mathbf{R}^+$, we have

$$\begin{aligned}
&\mathrm{St}\int_\Omega\int M\varphi\Gamma\left(\frac{f}{M}\right)(t_2,x,v)dvdx - \mathrm{St}\int_\Omega\int M\varphi\Gamma\left(\frac{f}{M}\right)(t_1,x,v)dvdx\\
&\quad -\int_{t_1}^{t_2}\int_\Omega\int M(v\cdot\nabla_x\varphi)\Gamma\left(\frac{f}{M}\right)(t,x,v)dvdxdt\\
&\quad +\int_{t_1}^{t_2}\int_{\partial\Omega}\int M\varphi\Gamma\left(\frac{f}{M}\right)(t,x,v)(v.n(x))dvd\sigma_x dt\\
&= \frac{1}{\mathrm{Kn}}\int_{t_1}^{t_2}\int_\Omega\int\varphi\Gamma'\left(\frac{f}{M}\right)Q(f,f)(t,x,v)dvdxdt
\end{aligned} \tag{2.31}$$

with the renormalized boundary condition

$$\Gamma\left(\frac{f_{|\Sigma_-}}{M}\right) = \Gamma\left(\frac{(1-\alpha)L(f_{|\Sigma_+}) + \alpha\sqrt{2\pi}M\int f_{|\Sigma_+}(v\cdot n(x))_+dv}{M}\right) \quad \text{on } \Sigma_-. \tag{2.32}$$

With the above definition of renormalized solution relatively to M, the following existence result holds :

Theorem 2.3.4 *Assume that the collision kernel satisfies Grad's cutoff assumption (2.8) for some $\beta \in [0,1]$. Given any initial data f_{in} satisfying*

$$H(f_{in}|M) \stackrel{def}{=} \int_\Omega\int \left(f_{in}\log\frac{f_{in}}{M} - f_{in} + M\right)(x,v)\; dv\, dx < +\infty, \tag{2.33}$$

there exists a renormalized solution $f \in C(\mathbf{R}^+, L^1_{loc}(\Omega\times\mathbf{R}^3))$ relatively to M to the Boltzmann equation (2.6)(2.23) with initial data f_{in}.

Moreover, f satisfies

- the continuity equation

$$\mathrm{St}\partial_t\int f dv + \nabla_x\cdot\int f v dv = 0; \tag{2.34}$$

- the momentum equation with defect measure

$$\mathrm{St}\partial_t\int f v dv + \nabla_x\cdot\int f v\otimes v dv + \nabla_x\cdot m = 0 \tag{2.35}$$

where m is a Radon measure on $\mathbf{R}^+\times\Omega$ with values in the nonnegative symmetric matrices;

- the entropy inequality

$$\begin{aligned} H(f|M)(t) + \int \mathrm{tr}(m)(t) + \frac{1}{\mathrm{StKn}}\int_0^t\int_\Omega D(f)(s,x)dsdx \\ +\frac{\alpha}{\mathrm{St}}\int_0^t\int_{\partial\Omega} E(f|M)(s,x)dsd\sigma_x \le H(f_{in}|M) \end{aligned} \tag{2.36}$$

where $\mathrm{tr}(m)$ is the trace of the nonnegative symmetric matrix m, the entropy dissipation $D(f)$ is defined by (2.15) and the boundary term $E(f|M)$, referred to as the Darrozès-Guiraud information is defined by

$$\begin{aligned} E(f|M)(s,x) &= \int_{v\cdot n(x)>0}\left(f\log\frac{f}{M} - f + M\right)(v\cdot n(x))dv \\ &-\left(\int f(x,v)\,(v\cdot n(x))_+dv\right)\log\left(\int f(x,v)\,\sqrt{2\pi}(v\cdot n(x))_+dv\right) \\ &+\left(\int f(x,v)\,(v\cdot n(x))_+dv\right) - \frac{1}{\sqrt{2\pi}} \end{aligned} \tag{2.37}$$

Sketch of proof of Theorem 2.3.4. We recall here the main arguments leading to that existence result, following the presentation of Golse and the author in [58] for the convergence of the approximation scheme inside the domain Ω, and the proof of Mischler in [85] for the convergence at the boundary.

Because our goal is to point out similarities between these arguments and those used in the framework of hydrodynamic limits, we focus on the weak stability of sequences (f_n) of renormalized solutions to (2.6), and do not present the underlying approximation scheme. Note that, in any case, the parameters Kn and St are fixed.

Step 1 : weak compactness results.

We have first to obtain some *weak compactness* on (f_n) using the (physical) a priori bounds.

From the **uniform bound on the relative entropy**

$$\sup_{t\in\mathbf{R}^+} H(f_n|M)(t) \leq C,$$

we deduce by Young's inequality (see (3.4) in Chapter 3) and pointwise estimates that

$$\begin{aligned} &\left(\frac{f_n}{M}\right) \text{ is bounded in } L^\infty(dt, L^1_{loc}(dx, L^1(M(1+|v|^2)dv))), \\ &\left(\frac{f_n}{M}\right) \text{ is weakly compact in } L^1_{loc}(dtdxdv) \end{aligned} \tag{2.38}$$

(see Lemma 3.1.2 in Chapter 3 for a detailed proof of that statement), and

$$\frac{f_n}{M} - \frac{1}{\delta}\log\left(1+\delta\frac{f_n}{M}\right) \to 0 \text{ in } L^\infty(\mathbf{R}^+, L^1_{loc}(dx, L^1(Mdv))) \text{ uniformly in } n \tag{2.39}$$

as $\delta \to 0$. In particular, for fixed $\delta > 0$,

$$\left(\frac{Q^-(f_n, f_n)}{1+\delta f_n/M}\right) \text{ is weakly compact in } L^1_{loc}(dtdxdv).$$

Then, from the **uniform bound on the entropy dissipation**

$$\int_0^{+\infty}\int_\Omega D(f_n)(t,x)dxdt \leq C,$$

we deduce, using a convenient splitting of the integral according to the tail of $(f_n f_{n*})/(f'_n f'_{n*})$, that for fixed $\delta > 0$,

$$\left(\frac{Q(f_n, f_n)}{1+\delta f_n/M}\right) \text{ is weakly compact in } L^1_{loc}(dtdxdv). \tag{2.40}$$

In particular, the sequence $\frac{1}{\delta}\log(1+\delta\frac{f_n}{M})$ (which is uniformly bounded in $L^\infty(\mathbf{R}^+, L^2_{loc}(dx, L^2(M(1+|v|)dv)))$ by the relative entropy bound) satisfies

$$M(\mathrm{St}\partial_t + v \cdot \nabla_x)\frac{1}{\delta}\log\left(1+\delta\frac{f_n}{M}\right) = \frac{1}{\mathrm{Kn}}\frac{Q(f_n,f_n)}{1+\delta f_n/M} = O(1)_{L^1_{loc}(dtdxdv)}. \tag{2.41}$$

By interpolation (see [74] for instance), we eventually arrive at

$$\left(\frac{1}{\delta}\log(1+\delta\frac{f_n}{M})\right) \text{ is relatively compact in } C([0,T], w - L^2_{loc}(dxMdv)),$$

which, coupled with (2.38) and (2.39), leads to

$$f_n \rightharpoonup f \text{ weakly in } L^1_{loc}(dx, L^1(dv)) \text{ locally uniformly in } t \text{ as } n \to \infty \tag{2.42}$$

modulo extraction of a subsequence.

Step 2 : strong compactness results.

In order to take limits in the renormalized Boltzmann equation, we have further to obtain some *strong compactness*, which is the matter of the second step. The crucial idea here is to use the **velocity averaging lemma** due to Golse, Lions, Perthame and Sentis [53] (and detailed in the third section of Chapter 3), stating that the moments in v of the solution to some transport equation are more regular than the function itself.

From the uniform bound on $\frac{1}{\delta}\log(1+\delta\frac{f_n}{M})$ and the estimate (2.41) on the transport, we deduce in particular that, for all $\varphi \in C^1(\mathbf{R}^3)$ with subquadratic growth at infinity,

$$\frac{1}{\delta}\int M\log\left(1+\delta\frac{f_n}{M}\right)\varphi(v)dv \text{ is strongly relatively compact in } L^1_{loc}(dtdx),$$

and thus by (2.39) that

$$\int f_n\varphi(v)dv \text{ is strongly relatively compact in } L^p_{loc}(dt, L^1_{loc}(dx)). \tag{2.43}$$

This convergence statement allows to take limits in the Boltzmann collision integral, once it is renormalized by some convenient macroscopic quantity. This average renormalization is here only to guarantee that all the quantities considered are at least locally integrable. Using a variant of Egorov's Theorem (namely the **Product Limit theorem** established in [44] and recalled in Appendix A), we are actually able to establish that, modulo extraction of a subsequence, for all $\phi \in C_c(\mathbf{R}^+ \times \Omega \times \mathbf{R}^3)$

$$\int \frac{Q^\pm(f_n,f_n)}{1+\int f_n dv}\phi dv \to \int \frac{Q^\pm(f,f)}{1+\int f dv}\phi dv \text{ in } L^1_{loc}(dtdx). \tag{2.44}$$

Step 3 : limiting macroscopic equations.

From the previous steps, one can easily obtain the *entropy inequality* and the *variants of the conservation laws* satisfied by f.

By (2.38), we have

$$\left.\begin{array}{l}\int f_n dv \rightharpoonup \int f dv\\ \int f_n v dv \rightharpoonup \int f v dv\end{array}\right. \text{ weakly in } L^1_{loc}(dtdx)$$

which allows to take limits in the local conservation of mass. Furthermore, by the Banach-Alaoglu theorem, up to extraction of a subsequence, for each i, j

$$\int f_n v_i v_j dv \rightharpoonup \mu_{ij} \text{ weakly-* in } L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega)). \tag{2.45}$$

By monotone convergence, one can then prove that

$$\mu_{ij} = \int f v_i v_j dv + m_{ij},$$

where m_{ij} is a nonnegative symmetric element of $L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega, M_3(\mathbf{R})))$. Taking limits in the local conservation of momentum leads then to (2.35).

By weak limits

$$f_n \rightharpoonup f \text{ in } L^1_{loc}(dtdx, L^1((1+|v|)dv)$$

$$\int f_n |v|^2 dv \rightharpoonup \int f|v|^2 dv + \mathrm{tr}(m) \text{ in } L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega))$$

and

$$\frac{f_n f_{n*}}{1+\delta \int f_n dv} \rightharpoonup \frac{f f_*}{1+\delta \int f dv} \text{ in } L^1_{loc}(dtdx, L^1(B dv dv_* d\sigma))$$

$$\frac{f'_n f'_{n*}}{1+\delta \int f_n dv} \rightharpoonup \frac{f' f'_*}{1+\delta \int f dv} \text{ in } L^1_{loc}(dtdx, L^1(B dv dv_* d\sigma))$$

(obtained similarly as (2.44)), using the **convexity** of the functionals defining the relative entropy and the entropy dissipation, we get

$$\begin{aligned} &H(f|M)(t) + \int \mathrm{tr}(m)(t) \leq \liminf_{n\to\infty} H(f_n|M)(t),\\ &\int_0^t \int D(f)(s,x) dx ds \leq \liminf_{n\to\infty} \int_0^t \int D(f_n)(s,x) dx ds \end{aligned} \tag{2.46}$$

thus passing to the limit in the entropy inequality leads to (2.36) in the absence of boundary.

Step 4 : limiting renormalized kinetic equation.

The most technical step of the proof is then to take limits in *the renormalized equation* (2.30). With the information at our disposal, and although the previous step provides useful information on the nonlinear term, this convergence is not trivial, in particular because the only source of compactness in the problem, i.e. velocity averaging, does not give any information on the distribution f_n itself.

• Using pointwise estimates on $\Gamma_\delta(z) = \frac{z}{1+\delta z}$ and on its derivative, we deduce from the weak compactness statements established in Step 1 that, for all $\delta > 0$,

$$\begin{aligned} &M\Gamma_\delta\left(\frac{f_n}{M}\right) \rightharpoonup f_\delta \text{ weakly-* in } L^\infty(\mathbf{R}^+ \times \Omega \times \mathbf{R}^3), \\ &\Gamma'_\delta\left(\frac{f_n}{M}\right) Q^\pm(f_n, f_n) \rightharpoonup Q^\pm_\delta \text{ weakly in } L^1_{loc}(\mathbf{R}^+ \times \Omega \times \mathbf{R}^3). \end{aligned} \tag{2.47}$$

Furthermore, from the relative entropy bound and the uniform convergence (2.42) we deduce that

$$\frac{f_n}{M}(t) - \Gamma_\delta\left(\frac{f_n}{M}\right)(t) \to 0 \text{ in } L^1_{loc}(dx, L^1(dv)) \text{ uniformly in } t, n \text{ as } \delta \to 0,$$

and thus that

$$\begin{aligned} f_\delta \to f \text{ as } \delta \to 0 \;\; &\text{in } L^1_{loc}(dx, L^1(dv)) \text{ uniformly in } t, \\ &\text{and a.e. on } \mathbf{R}^+ \times \Omega \times \mathbf{R}^3 . \end{aligned} \tag{2.48}$$

The idea is then to take limits in

$$\begin{aligned} &M(\mathrm{St}\partial_t + v \cdot \nabla_x) \log\left(1 + \frac{f_\delta}{M}\right) = \frac{1}{\mathrm{Kn}} \frac{Q^+_\delta - Q^-_\delta}{1 + f_\delta/M}, \\ &f_{\delta|t=0} = \Gamma_\delta(f_{in}). \end{aligned}$$

• By the strong compactness statements (2.43) established in Step 2, and the Product Limit theorem, we have, for all $\delta > 0$,

$$\begin{aligned} \frac{Q^-(f_n, f_n)}{(1 + \delta f_n/M)^2} &= \frac{f_n}{(1 + \delta f_n/M)^2} \iint f_{n*} B dv_* d\sigma \\ &\rightharpoonup \tilde{f}_\delta \iint f_* B dv_* d\sigma \text{ in } L^1_{loc}(\mathbf{R}^+ \times \Omega \times \mathbf{R}^3) \end{aligned}$$

with $\tilde{f}_\delta \le f_\delta$ and

$$\begin{aligned} \tilde{f}_\delta \to f \text{ as } \delta \to 0 \;\; &\text{in } L^1_{loc}(dx, L^1(dv)) \text{ uniformly in } t, \\ &\text{and a.e. on } \mathbf{R}^+ \times \Omega \times \mathbf{R}^3, \end{aligned}$$

using the same arguments as for (2.48). We then obtain the *convergence of the loss term* (up to extraction of a subsequence)

$$\frac{Q^-_\delta}{1 + f_\delta/M} \to \frac{f}{1 + f/M} \iint f_* B dv_* d\sigma \text{ as } \delta \to 0 \text{ a.e. on } \mathbf{R}^+ \times \Omega \times \mathbf{R}^3 \tag{2.49}$$

and thus in $L^1_{loc}(\mathbf{R}^+ \times \Omega \times \mathbf{R}^3)$ by Lebesgue's theorem.

• The *convergence of the gain term* is more complicated to establish. Starting from

$$\Gamma'_\delta\left(\frac{f_n}{M}\right)\frac{Q^+(f_n,f_n)}{1+\int f_n dv} \leq \left(\frac{f_n}{M}\right)\frac{Q^+(f_n,f_n)}{1+\int f_n dv}$$

then integrating against some $\phi = \phi(v) \geq 0$ and taking limits as $n \to \infty$, we get

$$Q^+_\delta \leq Q^+(f,f) \text{ a.e. on } \mathbf{R}^+ \times \Omega \times \mathbf{R}^3$$

using the convergence (2.44) obtained in Step 2, and the Product Limit theorem.

Then, introducing some suitable decomposition according to the tail of $(f'_n f'_{n*})/(f_n f_{n*})$, and using the convergence (2.44) and the Product Limit theorem, we establish that, for all $\lambda > 0$,

$$\frac{Q^+(f_n,f_n)}{1+\lambda\int f_{n*} dv_*} \rightharpoonup \frac{Q^+(f,f)}{1+\lambda\int f_* dv_*} \text{ weakly in } L^1_{loc}(\mathbf{R}^+ \times \Omega \times \mathbf{R}^3).$$

Starting from a refined decomposition, and using the convergence of the entropy dissipation in the vague sense of measures, we then obtain that, for all $\lambda > 0$,

$$\frac{Q^+(f,f)}{1+\lambda\int f_* dv_*} \leq \liminf_{\delta\to 0} Q^+_\delta.$$

Finally, we get

$$\frac{Q^+_\delta}{1+f_\delta/M} \to \frac{Q^+(f,f)}{1+f/M} \text{ as } \delta \to 0 \text{ a.e. on } \mathbf{R}^+ \times \Omega \times \mathbf{R}^3 \tag{2.50}$$

and thus in $L^1_{loc}(\mathbf{R}^+ \times \Omega \times \mathbf{R}^3)$ by Lebesgue's theorem.

• Combining all results leads to

$$M(\mathrm{St}\partial_t + v\cdot\nabla_x)\log\left(1+\frac{f}{M}\right) = \frac{1}{\mathrm{Kn}}\frac{Q^+(f,f)-Q^-(f,f)}{1+f/M},$$

with initial condition $f_{|t=0} = f_{in}$ (since the convergence is uniform in t). It remains then to check that the same identity holds for any admissible renormalization Γ

$$M(\mathrm{St}\partial_t + v\cdot\nabla_x)\Gamma\left(\frac{f}{M}\right) = \frac{1}{\mathrm{Kn}}\Gamma'\left(\frac{f}{M}\right)(Q^+(f,f)-Q^-(f,f)), \tag{2.51}$$

which is done by composition if $|\Gamma'(z)| \leq C(1+z)^{-1}$, and else by approximation, using the fact that $Q(f,f)/\sqrt{1+f/M}$ is controlled by the entropy dissipation and the relative entropy (see the proof of Proposition 4.3.1 in Chapter 4 for an analogous result).

Step 5 : limiting boundary conditions.

In the case of a spatial domain with boundary, it remains then to take limits in *Maxwell's boundary condition*. This requires powerful tools of functional analysis, which are consequences of **Chacon's Biting Lemma** and are stated in Appendix C.

Let us first note that the boundary term obtained formally in the entropy inequality

$$\int_0^t \int_{\partial\Omega} \int \left(f_{n|\Sigma^+} \log \frac{f_{n|\Sigma^+}}{M} - f_{n|\Sigma^+} + M \right) (s,x,v)(v\cdot n(x))_+ dv d\sigma_x ds$$
$$- \int_0^t \int_{\partial\Omega} \int \left(f_{n|\Sigma^-} \log \frac{f_{n|\Sigma^-}}{M} - f_{n|\Sigma^-} + M \right) (s,x,R_x v)(v\cdot n(x))_+ dv d\sigma_x ds$$

controls the Darrozès-Guiraud information

$$\alpha \int_0^t \int_{\partial\Omega} E(f_n)(s,x) d\sigma_x ds$$

defined by (2.37) (by a simple convexity argument), and so we start from a sequence (f_n) such that the Darrozès-Guiraud information $E(f_n)$ is uniformly bounded in $L^1(\mathbf{R}^+ \times \partial\Omega)$.

The trace is then defined by some Green's formula written on the renormalized equation. The main difficulty to take limits in the renormalized form (2.32) of Maxwell's boundary condition, is therefore the lack of an a priori bound on the trace, giving in particular some local equi-integrability in v.

• We first establish the following *renormalized convergence* (see Appendix C for a precise definition of this notion)

$$f_{n|\partial\Omega} \to f_{\partial\Omega} \text{ in renormalized sense on } \mathbf{R}^+ \times \partial\Omega \times \mathbf{R}^3, \tag{2.52}$$

using the a priori estimates coming from the inside, and the weak formulation (2.31) of the renormalized Boltzmann equation.

Starting from (2.31) with

$$\varphi(x,v) = \frac{v\cdot n(x)}{1+|v|^2}\chi(x)$$

where $\chi \in C_c^\infty(\mathbf{R}^3, \mathbf{R}^+)$ and n denotes some vector field of $W^{1,\infty}(\bar{\Omega})$ which coincides with the outward unit normal vector at the boundary, we get

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\partial\Omega}\int M \frac{(v\cdot n(x))^2\chi(x)}{1+|v|^2}\Gamma\left(\frac{f_{n|\partial\Omega}}{M}\right)(t,x,v)dvd\sigma_x dt\\
&= \mathrm{St}\int_\Omega\int M\frac{(v\cdot n(x))\chi(x)}{1+|v|^2}\Gamma\left(\frac{f_n}{M}\right)(t_1,x,v)dvdx\\
&-\mathrm{St}\int_\Omega\int M\frac{(v\cdot n(x))\chi(x)}{1+|v|^2}\Gamma\left(\frac{f_n}{M}\right)(t_2,x,v)dvdx\\
&+\int_{t_1}^{t_2}\int_\Omega\int M(v\cdot\nabla_x)\left(\frac{(v\cdot n(x))\chi(x)}{1+|v|^2}\right)\Gamma\left(\frac{f_n}{M}\right)(t,x,v)dvdxdt\\
&+\frac{1}{\mathrm{Kn}}\int_{t_1}^{t_2}\int_\Omega\int \frac{(v\cdot n(x))\chi(x)}{1+|v|^2}\Gamma'\left(\frac{f_n}{M}\right)Q(f_n,f_n)(t,x,v)dvdxdt.
\end{aligned} \tag{2.53}$$

Thus from the uniform bounds obtained in Step 1 and Cauchy-Schwarz inequality, we deduce that

$$M\Gamma\left(\frac{f_{n|\partial\Omega}}{M}\right) \text{ is weakly compact in } L^1_{loc}(dtd\sigma_x, L^1(|v\cdot n(x)|dv))\,.$$

Using the convergence results stated in Step 4, we can take limits as $n\to\infty$ in the right-hand side of (2.53), and then identify the limit writing Green's formula for the limiting kinetic equation (2.51). We thus obtain

$$M\Gamma\left(\frac{f_{n|\partial\Omega}}{M}\right) \rightharpoonup M\Gamma\left(\frac{f_{|\partial\Omega}}{M}\right) \text{ weakly in } L^1_{loc}(dtd\sigma_x, L^1(|v\cdot n(x)|dv)).$$

which implies the renormalized convergence (2.52). Note that, up to extraction of a subsequence, we also get the pointwise convergence

$$f_{n|\partial\Omega}\to f_{|\partial\Omega} \text{ a.e. on } \mathbf{R}^+\times\partial\Omega\times\mathbf{R}^3\,.$$

• Then, using the uniform bound on the *Darrozès-Guiraud information*, we prove that

$$\int f_{n|\partial\Omega}(v\cdot n(x))_+dv \to \tilde f_{|\partial\Omega} \text{ in renormalized sense on } \mathbf{R}^+\times\partial\Omega, \tag{2.54}$$

for some measurable, almost everywhere finite function $\tilde f_{|\partial\Omega}$.

Indeed, remarking that

$$\begin{aligned}(z\log z - z+1)-(y\log y-y+1)-(z-y)\log y &= \int_0^1 \frac{|z-y|^2}{\tau x+(1-\tau)y}d\tau\\ &\geq \left(\sqrt{z}-\sqrt{y}\right)^2\end{aligned}$$

and that

$$\int\left(f_n-\sqrt{2\pi}M\int f_n(v\cdot n(x))_+dv\right)\log\left(\int f_n(v\cdot n(x))_+dv\right)(v\cdot n(x))_+dv=0$$

we get

$$\int\left(\sqrt{\frac{f_n}{M}}-\sqrt{\int f_n\sqrt{2\pi}(v\cdot n(x))_+dv}\right)^2 M(v\cdot n(x))_+dv\leq 2E(f_n|M)$$

which, coupled with the uniform bound on the Darrozès-Guiraud information, shows that

$$\sqrt{\frac{f_n}{M}}-\sqrt{\int f_n\sqrt{2\pi}(v\cdot n(x))_+dv} \text{ is weakly compact in } L^2(dt(v\cdot n(x))_+d\sigma_x M dv), \tag{2.55}$$

and thus converges a.e. on Σ_+ up to extraction of a subsequence.

Therefore, from the decomposition

$$\sqrt{\int f_n\sqrt{2\pi}(v\cdot n(x))_+dv} = \sqrt{\int f_n\sqrt{2\pi}(v\cdot n(x))_+dv} - \sqrt{\frac{f_n}{M}} + \sqrt{\frac{f_n}{M}}$$

the renormalized convergence (2.52) and the weak compactness (2.55), we deduce that (2.54) holds up to extraction of a subsequence.

• It remains then to characterize the limit $\tilde{f}_{|\partial\Omega}$ in terms of $f_{|\partial\Omega}$, which requires a variant of Chacon's Biting Lemma giving some partial equiintegrability on $f_{n|\partial\Omega}$ with respect to the v variables.

From (2.54) and the uniform bound on the Darrozès-Guiraud information, we deduce by Proposition C.4 in Appendix that for every $\varepsilon > 0$ and every compact $K \subset \mathbf{R}^+ \times \partial\Omega$, one can find some $A \subset K$ with

$$\int_{K\setminus A} dtd\sigma_x < \varepsilon \text{ and } f_{n|\partial\Omega} \rightharpoonup f_{|\partial\Omega} \text{ weakly in } L^1(A\times\mathbf{R}^3, dt(v\cdot n(x))_+d\sigma_x dv).$$

In particular,

$$\tilde{f}_{|\partial\Omega} = \int f_{|\partial\Omega}(v\cdot n(x))_+dv \text{ on every such } A,$$

and thus a.e. on $\mathbf{R}^+ \times\partial\Omega$.

We are then able to take limits in the renormalized form of Maxwell's boundary condition (2.32), which leads to

$$\Gamma\left(\frac{f_{|\Sigma_-}}{M}\right) = \Gamma\left(\frac{(1-\alpha)L(f_{|\Sigma_+}) + \alpha K(f_{|\Sigma_+})}{M}\right) \quad \text{on } \Sigma_-.$$

Furthermore, using the convexity of the Darrozès-Guiraud information (also established in Proposition C.4), we get

$$\int_0^t\int_{\partial\Omega} E(f|M)(s,x)d\sigma_x ds \le \liminf_{n\to\infty}\int_0^t\int_{\partial\Omega} E(f_n|M)(s,x)d\sigma_x ds,$$

which concludes the proof of the entropy inequality (2.36) studied in Step 3, in the case of a spatial domain with boundary. □

2.3.3 Further Results in One Space Dimension

In the one spatial dimensional case, the previous result has actually been improved by Cercignani [29], who established the global existence of weak solutions to (2.6) satisfying in particular the global conservation of energy.

The key idea of that theory is to introduce the weak form of the collision term, and the corresponding suitable notion of weak solution. For the sake of simplicity, we will restrict our attention to the case of a spatial domain without boundary, for instance the periodic box $\mathbf{T}$.

Definition 2.3.5 *A weak solution to the one-dimensional Boltzmann equation (2.6) is a function*

$$f \in C(\mathbf{R}^+, L^1(\mathbf{T} \times \mathbf{R}^3))$$

such that, for every test function $\varphi \in C^1_c(\mathbf{R}^+ \times \mathbf{T} \times \mathbf{R}^3)$ *which is twice differentiable as a function of* v *with second derivatives uniformly bounded with respect to* x *and* t*, we have*

$$\begin{aligned}&\iiint f(\mathrm{St}\partial_t\varphi + v_1\partial_x\varphi)(t,x,v)dxdvdt + \iint f_{in}\varphi(0,x,v)dxdv\\ &= \frac{1}{2\mathrm{Kn}}\iint\left(\iiint ff_*(\varphi+\varphi_*-\varphi'-\varphi'_*)B(v-v_*,\sigma)dvdv_*d\sigma\right)(t,x)dxdt\end{aligned} \tag{2.56}$$

With the above definition of weak solution, the following existence result holds :

Theorem 2.3.6 *Assume that the collision kernel* B *is bounded and satisfies Grad's cutoff assumption (2.8) as well as*

$$\int_{S^2}(1+\cos\theta)B(v-v_*,\sigma)d\sigma \geq r\int_{S^2}B(v-v_*,\sigma)d\sigma \tag{2.57}$$

for some $r > 0$*. Given any initial data* $f_{in} \in L^1_{loc}(\mathbf{T} \times \mathbf{R}^3)$ *satisfying*

$$H(f_{in}|M) \overset{def}{=} \iint\left(f_{in}\log\frac{f_{in}}{M} - f_{in} + M\right)(x,v)\ dv\,dx < +\infty, \tag{2.58}$$

there exists a weak solution $f \in C(\mathbf{R}^+, L^1(\mathbf{T} \times \mathbf{R}^3))$ *to (2.6) with initial data* f_{in}*.*

Furthermore this solution satisfies the continuity equation (2.34), the momentum equation (2.35) without defect measure and the entropy inequality (2.36) without defect measure, as well as the energy conservation

$$\iint f(t,x,v)|v|^2dvdx = \iint f_{in}(x,v)|v|^2dvdx.$$

Sketch of Proof of Theorem 2.3.6. The idea is to use the knowledge that there is a renormalized solution in the sense of DiPerna-Lions, and to establish estimates which entail that this solution is indeed a weak solution in the sense defined above. As usual these estimates will be obtained by formal computations, which can be justified for approximate solutions to the Boltzmann equation (2.6), and then established for any renormalized solution by passing to the limit.

The crucial tool to establish these estimates, which is specific to the one-dimensional case, is the functional

$$I(f)(t) \stackrel{\text{def}}{=} \iint_{x<y} \iint (v_1 - v_{1*}) f(t,x,v) f(t,y,v_*) dv_* dv dx dy \tag{2.59}$$

which extends the *potential for interaction* introduced by Bony in the one-dimensional discrete velocity context. No functional with similar pleasant properties is known, at this time, in more than one dimension. Note indeed that, because of the bounds on the total mass $\iint f(t,x,v) dv dx$ and on the total momentum $\iint f(t,x,v) v_1 dv dx$ in x-direction, we have the following control over the functional $I(f)(t)$

$$\forall t \in \mathbf{R}, \quad |I(f)(t)| \leq C_{in},$$

where C_{in} is a constant depending only on the initial data.

• The first step of the proof consists then in using that bound to establish the following basic estimates

$$\begin{aligned} &\int_0^t \int \iint (v_1 - u_1(s,x))^2 f(s,x,v) f(s,x,v_*) dv_* dv dx ds \leq C_{in}, \\ &\int_0^t \int \iint |v - v_*|^2 f(s,x,v) f(s,x,v_*) B(v - v_*, \sigma) d\sigma dv_* dv dx ds \leq C_{in} \end{aligned} \tag{2.60}$$

where C_{in} is as previously some constant depending only on the initial data, and u_1 is the bulk velocity defined by

$$u_1(s,x) = \frac{\int v_1 f(s,x,v) dv}{\int f(s,x,v) dv}.$$

A short calculation with proper use of the collision invariants of the Boltzmann collision operator shows that

$$I(f)(t) - I(f)(0) = -\int_0^t \int \iint (v_1 - v_{1*})^2 f(s,x,v) f(s,x,v_*) dv_* dv dx dt.$$

which immediately gives the first estimate in (2.60), remarking that

$$\int (v_1 - v_{1*})^2 f(s,x,v_*) dv_* \geq \int (v_1 - u_1)^2 f(s,x,v_*) dv_* \,.$$

From the weak form of the Boltzmann equation, we deduce using the conservation of mass and momentum that

$$\begin{aligned} &2\mathrm{Kn} \iint f v_1^2 dv dx - 2\mathrm{Kn} \iint f_{in} v_1^2 dv dx = \\ &\int_0^t \iiiint f f_* ((v_1 - u_1)^2 + (v_{1*} - u_1)^2 - (v_1' - u_1)^2 - (v_{1*}' - u_1)^2) B dv dv_* d\sigma dx dt \end{aligned}$$

The loss term is bounded because of the bound on the collision frequency and the first estimate in (2.60), and the left hand side is bounded because of the energy bound. We therefore deduce that the gain term is also bounded. Then by explicit computations based on symmetries and assumption (2.57), we get the second estimate in (2.60).

• Equipped with these preliminary estimates, we are now able to prove that the integral defining the weak form of the collision operator is bounded in terms of constants depending on the initial data, for any test function $\varphi \equiv \varphi(t, x, v)$ which is twice differentiable as a function of v with second derivatives uniformly bounded with respect to x and t.

The result follows from Taylor's formula at second order, remarking that the expression multiplying the first derivatives is zero because of momentum conservation. We indeed have, using the second estimate in (2.60),

$$\begin{aligned}
&\iint \left(\iiint f f_* |\varphi + \varphi_* - \varphi' - \varphi'_*| B(v - v_*, \sigma) dv dv_* d\sigma \right) dx dt \\
&\leq C \iiiint\!\!\int f f_* \left(|v - v_*|^2 + |v - v'|^2 + |v - v'_*|^2 \right) B(v - v_*, \sigma) dv dv_* d\sigma dx dt \\
&\leq 6C \iiiint\!\!\int f f_* |v - v_*|^2 B(v - v_*, \sigma) dv dv_* d\sigma dx dt \leq 6C C_{in}
\end{aligned}$$

which shows that the weak form of $Q(f, f)$ is well-defined. □

Remark 2.3.7 *The present result can actually be extended to slightly more general situations.*

• *Easy modifications, presented for instance in the paper [30] by Cercignani and Illner, allow to deal with the case of different boundary conditions, namely to consider the case of a slab with diffusive boundary conditions.*

• *Let us now discuss the assumptions on the collision kernel B. In principle, solutions for inverse power potentials might be considered without introducing Grad's cutoff (2.8) : this would require considering approximate solutions of the Boltzmann equation without cutoff, and study precisely the convergences, instead of using the knowledge that there is a renormalized solution.*

On the other hand, the present version of the result does not allow a growth for large values of the relative velocity $|v - v_*|$, i.e. excludes hard spheres and potentials harder than the inverse fifth power. This is an important simplification, which perhaps might be removed by much harder work.

This particular structure of the Boltzmann equation in one space dimension is reminiscent of the specificity of the one dimensional hyperbolic systems of conservation laws. In particular the functional referred to as the potential for interaction, and obtained by doubling the space variable, has to be compared with Glimm's functional for systems of conservation laws, which could be a track to investigate the compressible hydrodynamic limits.

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