Chapter 2
Continuous-Time Fourier Analysis

Contents

Since the great contribution and revolutionary discovery of Jean Baptiste Joseph Fourier saw the light in 1822 after passing through the long dark tunnel of J.L. Lagrange's stubborn objection and was supported by P.L. Dirichlet's rigorous mathematical proof in 1829, the Fourier series and transform techniques have played very significant role in so many disciplines within the fields of mathematics, science, and engineering.

Joseph Fourier (1768∼1830) was a French mathematician and physicist who initiated the investigation of Fourier series and its application. Born as a son of a tailor, he was orphaned at age 8. In 1784, at only 16 years of age, he became a mathematics teacher at the Ecole Royale Militaire of Auxerre, debarred from entering the army

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on account of his obscurity and poverty. In 1795, Fourier took a faculty position at the Ecole Normale (Polytechnique) in Paris, which is an elite institution training high school teachers, university professors, and researchers. In 1798, he joined Napoleon's army in its expedition to Egypt as scientific advisor to help establish educational facilities there and carry out archaeological explorations.

Eluding the Nelson's British fleet, the Napoleon's Egyption expedition fleet of 300 ships, 30,000 infantry, 2,800 cavalry, and 1,000 cannons started Toulon on May 19,1798, sailing for Alexandria. The great expedition plan plotted by Napoleon attached a library with lots of books, many measurement instruments, various laboratory apparatuses, and about 500 civilians to the army; 150 of them were artists, scientists, scholars, engineers, and technicians. These human and physical resources formed the Institut d'Égypte in Cairo after Egypt was conquered by Napoleon. Napoleon Bonaparte (1769∼1821) not yet 30 years old, a great hero in the human history, and Joseph Fourier, a great scholar of about the same age in their youth were on board the flagship L'Orient of the expedition fleet. What were they thinking of when walking around on the deck and looking up the stars twinkling in the sky above the Mediterranean Sea at several nights in May of 1798? One might have dreamed of Julius Caesar, who conquered Egypt about 1,800 years ago, falling in love with the Queen Cleopatra, or might have paid a tribute to the monumental achievements of the great king Alexander, who conquered one third of the earth, opening the road between the West and the Orient. The other might have refreshed his memory on what he wrote in his diary on his 21st birthday, *Yesterday was my 21st birthday, at that age Newton and Pascal had already acquired many claims to immortality*, arranging his ideas on Fourier series and heat diffusion or recollecting his political ideology which had swept him and made him get very close to guillotine in the vortex of French Revolution.

2.1 Continuous-Time Fourier Series (CTFS) of Periodic Signals

2.1.1 Definition and Convergence Conditions of CTFS Representation

Let a function $x(t)$ be *periodic* with period P in t, that is,

$$x(t) = x(t + P) \quad \forall\, t \tag{2.1.1}$$

where P [s] and $\omega_0 = 2\pi/P$ [rad/s] are referred to as the *fundamental period* and *fundamental* (*angular*) *frequency*, respectively, if P is the smallest positive real number to satisfy Eq. (2.1.1) for periodicity. Suppose $x(t)$ satisfies at least one of the following conditions A and B:

< Condition A >

(**A1**) The periodic function $x(t)$ is square-integrable over the period P, i.e.,

$$\int_P |x(t)|^2 \, dt < \infty \tag{2.1.2a}$$

where $\int_P$ means the integration over any interval of length P. This implies that the signal described by $x(t)$ has finite power.

$< \text{Condition } B : \textit{Dirichlet condition} >$

(**B1**) The periodic function $x(t)$ has only a finite number of extrema and discontinuities in any one period.
(**B2**) These extrema are finite.
(**B3**) The periodic function $x(t)$ is absolutely-integrable over the period P, i.e.,

$$\int_P |x(t)| \, dt < \infty \tag{2.1.2b}$$

Then the periodic function $x(t)$ can be represented by the following forms of *continuous-time Fourier series* (CTFS), each of which is called the *Fourier series representation*:
<Trigonometric form>

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + \sum_{k=1}^{\infty} b_k \sin k\omega_0 t \tag{2.1.3a}$$
$$\text{with } \omega_0 = \frac{2\pi}{P} \ (P : \text{the period of } x(t))$$

where the Fourier coefficients $a_0,\ a_k$, and b_k are

$$\begin{aligned} a_0 &= \frac{1}{P}\int_P x(t)\, dt \ (\text{the integral over one period } P) \\ a_k &= \frac{2}{P}\int_P x(t) \cos k\omega_0 t \, dt \\ b_k &= \frac{2}{P}\int_P x(t) \sin k\omega_0 t \, dt \end{aligned} \tag{2.1.3b}$$

<Magnitude-and-Phase form>

$$x(t) = d_0 + \sum_{k=1}^{\infty} d_k \cos(k\omega_0 t + \phi_k) \tag{2.1.4a}$$

where the Fourier coefficients are

$$d_0 = a_0, \quad d_k = \sqrt{a_k^2 + b_k^2}, \quad \phi_k = \tan^{-1}(-b_k/a_k) \tag{2.1.4b}$$

<Complex Exponential form>

$$x(t) = \frac{1}{P} \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega_0 t} \tag{2.1.5a}$$

where the Fourier coefficients are

$$c_k = \int_P x(t)\, e^{-jk\omega_0 t} \ dt \text{ (the integral over one period } P) \tag{2.1.5b}$$

Here, the k th frequency $k\omega_0$ ($|k| > 1$) with fundamental frequency $\omega_0 = 2\pi/P = 2\pi f_0$ [rad/s](P: period) is referred to as the k th *harmonic*. The above three forms of Fourier series representation are equivalent and their Fourier coefficients are related with each other as follows:

$$c_0 = \int_P x(t)\, dt = P d_0 = P a_0 \tag{2.1.6a}$$

$$\begin{aligned} c_k &= \int_P x(t)\, e^{-jk\omega_0 t}\, dt = \int_P x(t)\,(\cos k\omega_0 t - j \sin k\omega_0 t)\, dt \\ &= \frac{P}{2}(a_k - jb_k) = \frac{P}{2} d_k \angle \phi_k \end{aligned} \tag{2.1.6b}$$

$$\begin{aligned} c_{-k} &= \int_P x(t)\, e^{jk\omega_0 t}\, dt = \int_P x(t)\,(\cos k\omega_0 t + j \sin k\omega_0 t)\, dt \\ &= \frac{P}{2}(a_k + jb_k) = \frac{P}{2} d_k \angle - \phi_k = c_k^* \end{aligned} \tag{2.1.6c}$$

$$a_0 = \frac{c_0}{P}, \quad a_k = \frac{c_k + c_{-k}}{P} = \frac{2\mathrm{Re}\{c_k\}}{P}, \quad b_k = \frac{c_{-k} - c_k}{jP} = -\frac{2\mathrm{Im}\{c_k\}}{P} \tag{2.1.6d}$$

The plot of Fourier coefficients (2.1.4b) or (2.1.5b) against frequency $k\omega_0$ is referred to as the *spectrum*. It can be used to describe the spectral contents of a signal, i.e., depict what frequency components are contained in the signal and how they are distributed over the low/medium/high frequency range. We will mainly use Eqs. (2.1.5a) and (2.1.5b) for spectral analysis.

<Proof of the Complex Exponential Fourier Analysis Formula (2.1.5b)>
To show the validity of Eq. (2.1.5b), we substitute the Fourier synthesis formula (2.1.5a) with the (dummy) summation index k replaced by n into Eq. (2.1.5b) as

$$c_k \overset{?}{=} \int_P \frac{1}{P} \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 t}\, e^{-jk\omega_0 t} \ dt = \sum_{n=-\infty}^{\infty} c_n \, \frac{1}{P} \int_P e^{j(n-k)\frac{2\pi}{P} t} dt \overset{\text{O.K.}}{=} c_k \tag{2.1.7}$$

This equality holds since

$$\frac{1}{P}\int_{-P/2}^{P/2} e^{j(n-k)\frac{2\pi}{P}t}\,dt$$
$$=\begin{cases}\frac{1}{P\cdot j(n-k)2\pi/P}\,e^{j(n-k)\frac{2\pi}{P}t}\Big|_{-P/2}^{P/2}=0 & \text{for}\quad n\neq k\\ \frac{1}{P}\int_{-P/2}^{P/2} dt = 1 & \text{for}\quad n=k\end{cases} = \delta[n-k] \qquad (2.1.8)$$

2.1.2 Examples of CTFS Representation

Example 2.1 Fourier Spectra of a Rectangular (Square) Wave and a Triangular Wave

(a) CTFS Spectrum of a Rectangular (Square) Wave (Fig. 2.1(a1))
Consider an even rectangular wave $x(t)$ with height A, duration D, and period P:

$$x(t) = A\,\tilde{r}_{D/P}(t) \qquad \text{(E2.1.1)}$$

$$\text{where}\quad \tilde{r}_{D/P}(t)=\begin{cases}1 & \text{for } |t-mP|\le D/2 (m:\ \text{an integer})\\ 0 & \text{elsewhere}\end{cases}$$

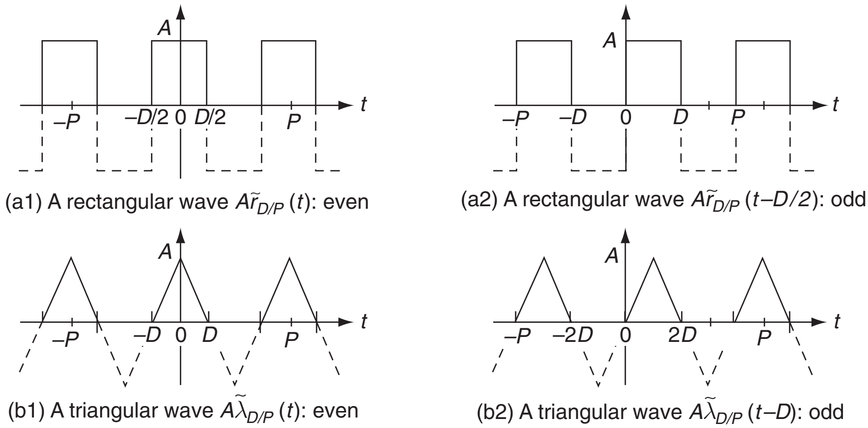


Fig. 2.1 Rectangular waves and triangular waves

We use Eq. (2.1.5b) to obtain the Fourier coefficients as

$$c_k = \int_{-P/2}^{P/2} A\,\tilde{r}_{D/P}(t)\,e^{-jk\omega_0 t}\,dt = A\int_{-D/2}^{D/2} e^{-jk\omega_0 t}\,dt = \frac{A}{-jk\omega_0}e^{-jk\omega_0 t}\Big|_{-D/2}^{D/2}$$

$$= A\frac{e^{jk\omega_0 D/2} - e^{-jk\omega_0 D/2}}{jk\omega_0} = AD\frac{\sin(k\omega_0 D/2)}{k\omega_0 D/2}$$

$$= AD\,\mathrm{sinc}\left(k\frac{D}{P}\right) \text{ with } \omega_0 = \frac{2\pi}{P} \tag{E2.1.2}$$

Now we can use Eq. (2.1.5a) to write the Fourier series representation of the rectangular wave as

$$A\tilde{r}_{D/P}(t) \overset{(2.1.5a)}{=} \frac{1}{P}\sum_{k=-\infty}^{\infty} AD\,\mathrm{sinc}\left(k\frac{D}{P}\right)e^{jk\omega_0 t}$$

$$= \frac{AD}{P} + \sum_{k=1}^{\infty}\frac{2AD}{P}\frac{\sin(k\pi D/P)}{k\pi D/P}\cos k\omega_0 t \tag{E2.1.3}$$

In the case of $D = 1$ and $P = 2D = 2$ as depicted in Fig. 2.2(a1), the (magnitude) spectrum is plotted in Fig. 2.2(b1).
(Q) What about the case of $P = D$, which corresponds to a constant DC (Direct Current) signal?

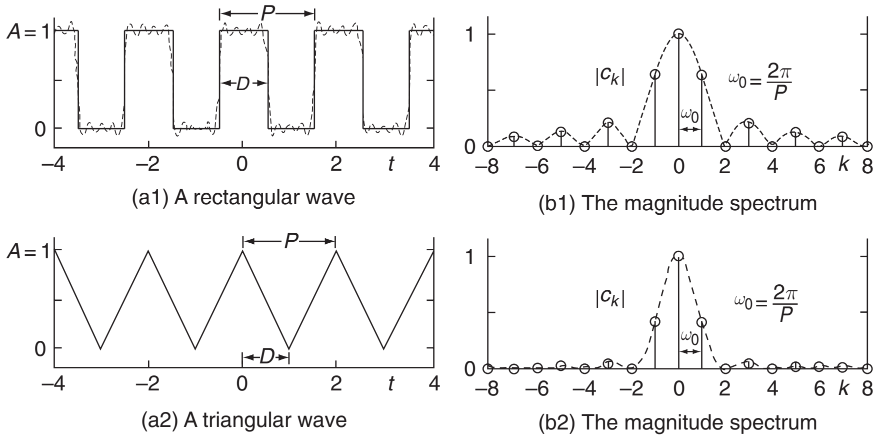


Fig. 2.2 Rectangular/triangular waves and their CTFS magnitude spectra

(b) CTFS Spectrum of a Triangular Wave (Fig. 2.1(b1))
Consider an even triangular wave $x(t)$ with maximum height A, duration $2D$, and period P:

$$x(t) = A\tilde{\lambda}_{D/P}(t) \tag{E2.1.4}$$

$$\text{where } \tilde{\lambda}_{D/P}(t) = \begin{cases} 1 - t/D & \text{for } |t - mP| \le D \ (m : \text{an integer}) \\ 0 & \text{elsewhere} \end{cases}$$

We use Eq. (2.1.5b) to obtain the Fourier coefficients as

$$\begin{aligned} c_k &= \int_{-P/2}^{P/2} A\left(1 - \frac{|t|}{D}\right) e^{-jk\omega_0 t}\, dt = \int_{-D}^{D} A\left(1 - \frac{|t|}{D}\right) \cos(k\omega_0 t)\, dt \\ &= 2\int_0^D A\left(1 - \frac{t}{D}\right)\cos(k\omega_0 t)\, dt = 2A\left(1 - \frac{t}{D}\right)\frac{1}{k\omega_0}\sin(k\omega_0 t)\Big|_0^D \\ &\quad - \int_0^D 2A\left(-\frac{1}{D}\right)\frac{1}{k\omega_0}\sin(k\omega_0 t)\, dt = -2A\frac{1}{(k\omega_0)^2 D}\cos(k\omega_0 t)\Big|_0^D \\ &= 2AD\frac{1 - \cos(k\omega_0 D)}{(k\omega_0 D)^2} = AD\frac{4\sin^2(k\omega_0 D/2)}{(k\omega_0 D/2)^2} \\ &= AD\,\mathrm{sinc}^2\left(k\frac{D}{P}\right) \quad \text{with} \quad \omega_0 = \frac{2\pi}{P} \end{aligned} \tag{E2.1.5}$$

Now we can use Eq. (2.1.5a) to write the Fourier series representation of the triangular wave as

$$A\tilde{\lambda}_{D/P}(t) \overset{(2.1.5a)}{=} \frac{1}{P}\sum_{k=-\infty}^{\infty} AD\,\mathrm{sinc}^2\left(k\frac{D}{P}\right) e^{jk\omega_0 t} \tag{E2.1.6}$$

In the case of $D = 1$ and $P = 2D = 2$ as depicted in Fig. 2.2(a2), the corresponding (magnitude) spectrum is plotted in Fig. 2.2(b2).

(c) MATLAB program to get the Fourier spectra
Once you have defined and saved a periodic function as an M-file, you can use the MATLAB routine "CTFS_exponential ()" to find its complex exponential Fourier series coefficients (c_k's). Interested readers are invited to run the following program "cir02e01.m" to get the Fourier coefficients and plot the spectra for a rectangular wave and a triangular wave as in Fig. 2.2.

```
%sig02e01.m : plot Fig. 2.2 (CTFS spectra of rectangular/triangular waves
clear, clf
global P D
N=8; k= -N:N; % the range of frequency indices
for i=1:2
   if i==1 % true Fourier series coefficients for a rectangular wave
     x = 'rectangular_wave'; P=2; D=1; c_true= D*sinc(k*D/P);
    else % true Fourier series coefficients for a triangular wave
     x = 'triangular_wave'; P=2; D=1; c_true= D*sinc(k*D/P).^2;
   end
    w0=2*pi/P; % fundamental frequency
    tt=[-400:400]*P/200; % time interval
   xt = feval(x,tt); % original signal
   [c,kk] = CTFS_exponential(x,P,N);
   [c; c_true] % to compare with true Fourier series coefficients
   discrepancy_between_numeric_and_analytic=norm(c-c_true)
   jkw0t= j*kk.'*w0*tt;
    xht = real(c/P*exp(jkw0t)); % Eq. (2.1.5a)
   subplot(219+i*2), plot(tt,xt,'k-', tt,xht,'b:')
   axis([tt(1) tt(end) -0.2 1.2]), title('Periodic function x(t)')
   c_mag = abs(c); c_phase = angle(c);
   subplot(220+i*2), stem(kk, c_mag), title('CTFS Spectrum |X(k)|')
end
```

```
function y=rectangular_wave(t)
global P D
tmp=min(abs(mod(t,P)),abs(mod(-t,P)));  y= (tmp<=D/2);
```

```
function y=triangular_wave(t)
global P D
tmp= min(abs(mod(t,P)),abs(mod(-t,P)));  y=(tmp<=D).*(1-tmp/D);
```

```
function [c,kk]=CTFS_exponential(x,P,N)
% Find the complex exponential Fourier coefficients c(k) for k=-N:N
% x: A periodic function with period P
% P: Period,  N: Maximum frequency index to specify the frequency range
w0=2*pi/P; % the fundamental frequency [rad/s]
xexp_jkw0t_= [x '(t).*exp(-j*k*w0*t)'];
xexp_jkw0t= inline(xexpjkw0t_,'t','k','w0');
kk=-N:N; tol=1e-6; % the frequency range tolerance on numerical error
for k=kk
   c(k+N+1)= quadl(xexp_jkw0t,-P/2,P/2,tol,[],k,w0); % Eq. (2.1.5b)
end
```

```
%sig02_01.m : plot Fig. 2.3 (CTFS reconstruction)
clear, clf
global P D
P=2; w0=2*pi/P; D=1; % period, fundamental frequency, and duration
tt=[-400:400]*P/400; % time interval of 4 periods
x = 'rectangular_wave';
xt = feval(x,tt); % original signal
plot(tt,xt,'k:'), hold on
Ns= [1 3 9 19];
for N=Ns
   k= -N:N; jkw0t= j*k.'*w0*tt; % the set of Fourier reconstruction terms
   c= D*sinc(k*D/P);
   xht = real(c/P*exp(jkw0t)); % Eq. (2.1.9)
   plot(tt,xht,'b'), hold on, pause
end
axis([tt(1) tt(end) -0.2 1.2])
```

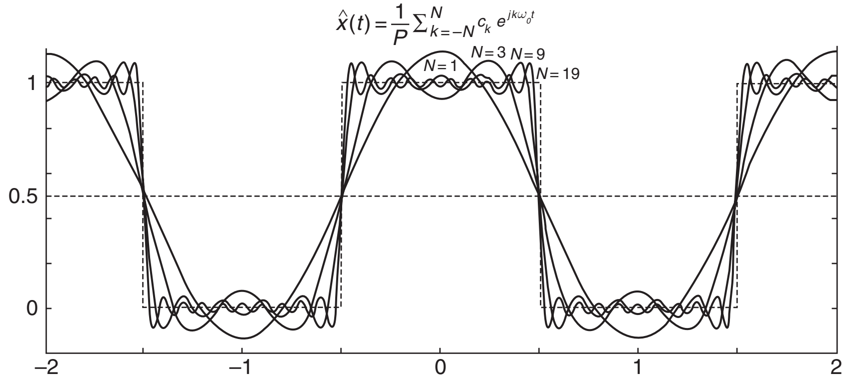


Fig. 2.3 Examples of the approximate Fourier reconstruction for a rectangular pulse

At this point, you may wonder how a rectangular wave with discontinuities can be represented by the sum of trigonometric or complex exponential functions that are continuous for all t. To satisfy your curiosity, let us consider the approximate Fourier series reconstruction formula.

$$\hat{x}_N(t) = \frac{1}{P} \sum_{k=-N}^{N} c_k\, e^{jk\omega_0 t} \tag{2.1.9}$$

This can be used to reconstruct the original time function $x(t)$ from its Fourier series coefficients. We can use the above MATLAB program 'sig02_01.m' to plot the Fourier series reconstructions of a rectangular wave with increasing number of terms $N = 1, 3, 9, 19, \ldots$ as in Fig. 2.3.

The following remark with Fig. 2.3 will satisfy your curiosity:

Remark 2.1 Convergence of *Fourier Series Reconstruction*

(1) The Fourier series convergence condition A stated in Sect. 2.1.1 guarantees that the Fourier coefficients are finite and the Fourier series reconstruction $\hat{x}_N(t)$ converges to the original time function $x(t)$ in the sense that

$$\int_P |\hat{x}_N(t) - x(t)|^2\, dt \to 0 \text{ as } N \to \infty$$

(2) The Fourier series convergence condition B stated in Sect. 2.1.1 guarantees the following:

- The Fourier coefficients are finite.
- The Fourier series reconstruction $\hat{x}_N(t)$ converges to the original time function $x(t)$ at every t except the discontinuities of $x(t)$ and to the average value of the limits from the left/right at each discontinuity.

(3) Figure 2.3 illustrates that $\hat{x}_N(t)$ has ripples around the discontinuities of $x(t)$, whose magnitude does not decrease as $N \to \infty$. This is called the *Gibbs phenomenon*.

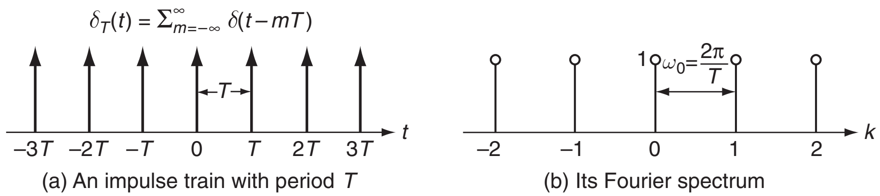


Fig. 2.4 An impulse train and its CTFT spectrum

(4) For practical purposes, we do not need to pay attention to the convergence condition because the "weird" signals that do not satisfy the condition are not important in the study of signals and systems.

Example 2.2 Fourier Spectrum of an Impulse Train

Consider an *impulse train* consisting of infinitely many shifted unit impulses that are equally spaced on the time axis:

$$\delta_T(t) = \sum\nolimits_{m=-\infty}^{\infty} \delta(t - mT) \tag{E2.1.1}$$

We can use Eq. (2.1.5b) with $P = T$ and $\omega_0 = 2\pi/T$ to obtain the Fourier coefficients as

$$c_k = \int_{-T/2}^{T/2} \delta_T(t)\, e^{-jk\omega_0 t}\, dt \overset{\text{(E2.1.1)}}{=} \int_{-T/2}^{T/2} \sum\nolimits_{m=-\infty}^{\infty} \delta(t - mT) e^{-jk\omega_0 t}\, dt$$

(since there is only one impulse $\delta(t)$
within the integration interval $[-T/2, T/2]$)

$$= \int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt \underset{\text{with } t_1=0}{\overset{(1.1.25)}{=}} e^{-jk\omega_0 t}\Big|_{t=0} = 1 \;\forall\, k$$

This means a flat spectrum that is uniformly distributed for every frequency index. Now we can use Eq. (2.1.5a) to write the Fourier series representation of the impulse train as

$$\delta_T(t) \underset{P=T}{\overset{\text{(2.1.5a)}}{=}} \frac{1}{P} \sum\nolimits_{k=-\infty}^{\infty} c_k\, e^{jk\omega_0 t} \overset{P=T}{=} \frac{1}{T} \sum\nolimits_{k=-\infty}^{\infty} e^{jk\omega_0 t} \quad \text{with} \quad \omega_0 = \frac{2\pi}{T} \tag{2.1.10}$$

Fig. 2.4(a) and (b) show an impulse train and its spectrum, respectively.

2.1.3 Physical Meaning of CTFS Coefficients – Spectrum

To understand the physical meaning of spectrum, let us see Fig. 2.5, which shows the major Fourier coefficients c_k of a zero-mean rectangular wave for $k = -3,\ -1,$ 1, and 3 (excluding the DC component c_0) and the corresponding time functions

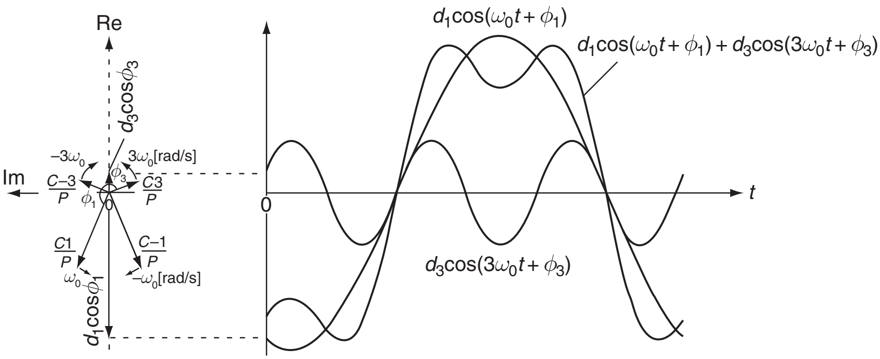


Fig. 2.5 Physical meaning of complex exponential Fourier series coefficients

$$\frac{1}{P}\left(c_{-1}\,e^{-j\omega_0 t}+c_1\,e^{j\omega_0 t}\right)\text{ and }\frac{1}{P}\left(c_{-3}\,e^{-j3\omega_0 t}+c_3\,e^{j3\omega_0 t}\right)$$

The observation of Fig. 2.5 gives us the following interpretations of Fourier spectrum:

Remark 2.2 Physical Meaning of Complex Exponential Fourier Series Coefficients

(1) While the trigonometric or magnitude-and-phase form of Fourier series has only nonnegative frequency components, the complex exponential Fourier series has positive/negative frequency ($\pm k\omega_0$) components that are conjugate-symmetric about $k=0$, i.e.,

$$c_k \overset{(2.1.6b)}{=} \frac{P}{2}d_k e^{j\phi_k}\text{ and } c_{-k} \overset{(2.1.6c)}{=} \frac{P}{2}d_k e^{-j\phi_k} \rightarrow |c_{-k}|=|c_k|\text{ and } \phi_{-k}=-\phi_k$$

as shown in Sect. 2.1.1. This implies that the magnitude spectrum $|c_k|$ is (even) symmetric about the vertical axis $k=0$ and the phase spectrum ϕ_k is odd symmetric about the origin.

(2) As illustrated above, the k th component appears as the sum of the positive and negative frequency components

$$\begin{aligned}\frac{c_k}{P}\,e^{jk\omega_0 t}+\frac{c_{-k}}{P}e^{-jk\omega_0 t} &\overset{(2.1.6)}{=} \frac{1}{2}d_k e^{j\phi_k}e^{jk\omega_0 t}+\frac{1}{2}d_k e^{-j\phi_k}e^{-jk\omega_0 t}\\ &= d_k\cos(k\omega_0 t+\phi_k)\end{aligned}$$

which denote two vectors (phasors) revolving in the opposite direction with positive (counter-clockwise) and negative (clockwise) angular velocities $\pm k\omega_0$ [rad/s] round the origin, respectively.

(3) Figure 2.6 also shows that the spectrum presents the descriptive information of a signal about its frequency components.

To get familiar with Fourier spectrum further, let us see and compare the spectra of the three signals, i.e., a rectangular wave, a triangular wave, and a constant

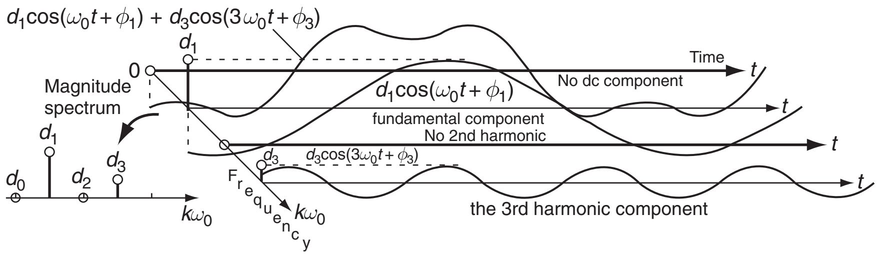


Fig. 2.6 Physical meaning of spectrum – time domain vs. frequency domain

(DC: Direct Current) signal depicted in Fig. 2.7. The observations are stated in the following remarks:

Remark 2.3 Effects of Smoothness and Period on Spectrum

(1) The smoother a time function is, the larger the relative magnitude of low frequency components to high frequency ones is. Compare Fig. 2.7(a1–b1), (a2–b2), and (a3–b3).

(cf.) The CTFS of the unit constant function can be obtained from Eq. (E2.1.3) with $A = 1$ and $P = D$.

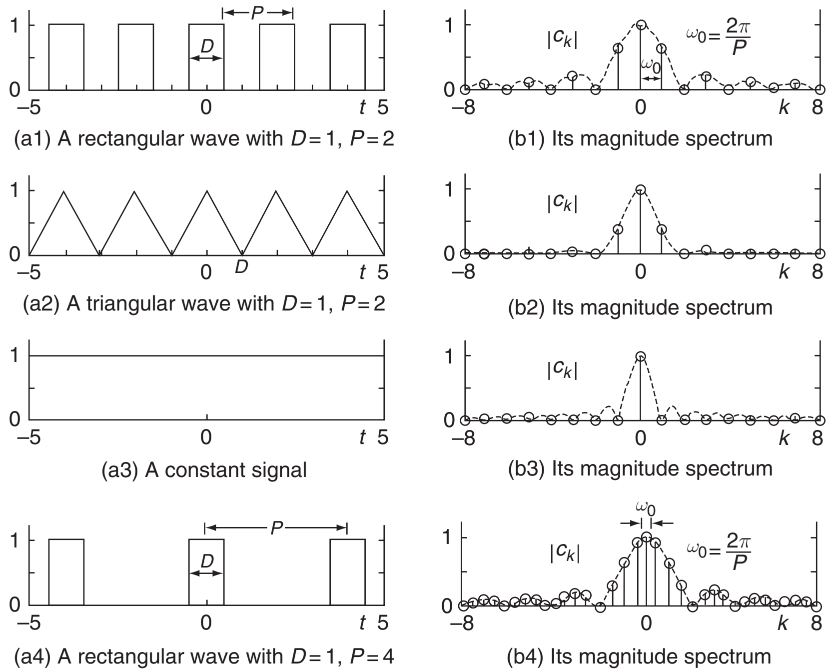


Fig. 2.7 The CTFS spectra of rectangular/triangular waves and a DC signal

(2) The longer the period P is, the lower the fundamental frequency $\omega_0 = 2\pi/P$ becomes and the denser the CTFS spectrum becomes. Compare Fig. 2.7(a4–b4) with (a1–b1).

(cf.) This presages the continuous-time Fourier transform, which will be introduced in the next section.

Now, let us see how the horizontal/vertical translations of a time function $x(t)$ affect the Fourier coefficients.

<Effects of Vertical/Horizontal Translations of $x(t)$ on the Fourier coefficients>
Translating $x(t)$ by $\pm A$ (+: upward, −: downward) along the vertical axis causes only the change of Fourier coefficient $d_0 = a_0$ for $k = 0$ (DC component or average value) by $\pm A$. On the other side, translating $x(t)$ along the horizontal (time) axis by $\pm t_1$ (+: rightward, −: leftward) causes only the change of phases (ϕ_k's) by $\mp k\omega_0 t_1$, not affecting the magnitudes d_k of Eq. (2.1.4b) or $|c_k|$ of Eq. (2.1.5b):

$$\begin{aligned} c'_k &\overset{(2.1.5b)}{=} \int_P x(t-t_1)\, e^{-jk\omega_0 t} dt = \int_P x(t-t_1)\, e^{-jk\omega_0 (t-t_1+t_1)} dt \\ &= e^{-jk\omega_0 t_1} \int_P x(t-t_1)\, e^{-jk\omega_0 (t-t_1)} dt \overset{(2.1.5b)}{=} c_k e^{-jk\omega_0 t_1} = |c_k| \angle(\phi_k - k\omega_0 t_1) \end{aligned} \tag{2.1.11}$$

Note that $x(t - t_1)$ is obtained by translating $x(t)$ by t_1 in the positive (rightward) direction for $t_1 > 0$ and by $-t_1$ in the negative (leftward) direction for $t_1 < 0$ along the horizontal (time) axis. Eq. (2.1.11) implies that horizontal shift of $x(t)$ causes a change not in its magnitude spectrum but in its phase spectrum.

2.2 Continuous-Time Fourier Transform of Aperiodic Signals

In this section we will define the Fourier transform for aperiodic signals. Suppose we have an aperiodic signal $x(t)$ of finite duration $D > 0$ and its periodic extension $\tilde{x}_P(t)$ with period $P > D$ that is obtained by repeating $x(t)$ every P s.

Noting that, as we choose the period P to be longer, $\tilde{x}_P(t)$ appears to be identical to $x(t)$ over a longer interval, we can think of $x(t)$ as the limit of $\tilde{x}_P(t)$ as $P \to \infty$. Since $\tilde{x}_P(t)$ is periodic with period P, it can be represented by the Fourier series of the form

$$\tilde{x}_P(t) = \frac{1}{P} \sum\nolimits_{k=-\infty}^{\infty} c_k\, e^{jk\omega_0 t} = \frac{1}{2\pi} \sum\nolimits_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t} \omega_0 \quad \text{with} \quad \omega_0 = \frac{2\pi}{P}$$

where the CTFS coefficients are

$$X(jk\omega_0) = c_k = \int_P \tilde{x}_P(t)\, e^{-jk\omega_0 t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

Noting that $\tilde{x}_P(t) \to x(t)$ and $\omega_0 = 2\pi/P \to 0$ as $P \to \infty$, we let $\omega_0 = d\omega$ and $k\omega_0 = \omega$ and take the limits of the above two equations as $P \to \infty$ to write the *continuous-time Fourier transform* (CTFT) pair:

$$X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \text{ (Fourier transform/integral)} \tag{2.2.1a}$$

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega \text{ (Inverse Fourier transform)} \tag{2.2.1b}$$

where $X(j\omega)$, called the spectrum of $x(t)$, has values at a continuum of frequencies and is often written as $X(\omega)$ with the constant j omitted. Like the CTFS of periodic signals, the CTFT of aperiodic signals provides us with the information concerning the frequency contents of signals, while the concept of frequency describes rather how rapidly a signal changes than how fast it oscillates.

Note that the sufficient condition for the convergence of CTFT is obtained by replacing the square-integrability condition (2.1.2a) of Condition A with

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty \tag{2.2.2a}$$

or by replacing the absolute-integrability condition (2.1.2b) of Condition B with

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty \tag{2.2.2b}$$

Remark 2.4 Physical Meaning of Fourier Transform – Signal Spectrum and Frequency Response

(1) If a time function $x(t)$ represents a physical signal, its Fourier transform $X(j\omega) = \mathcal{F}\{x(t)\}$ means the signal spectrum, which describes the frequency contents of the signal.
(2) In particular, if a time function $g(t)$ represents the impulse response of a continuous-time LTI (linear time-invariant) system, its Fourier transform $G(j\omega) = \mathcal{F}\{g(t)\}$ means the frequency response, which describes how the system responds to a sinusoidal input of (angular) frequency ω (refer to Sect. 1.2.6 for the definition of frequency response).

Remark 2.5 Frequency Response Existence Condition and Stability Condition of a System

For the impulse response $g(t)$ of a continuous-time LTI system, the absolute-integrability condition (2.2.2b) is identical with the stability condition (1.2.27a). This implies that a stable LTI system has a well-defined system function (frequency response) $G(j\omega) = \mathcal{F}\{g(t)\}$.

Remark 2.6 Fourier Transform and Laplace Transform

For any square-integrable or absolutely-integrable causal function $x(t)$ such that $x(t) = 0\ \forall\, t < 0$, the Fourier transform can be obtained by substituting $s = j\omega$ into the Laplace transform:

$$X(j\omega) \overset{(2.2.1a)}{=} \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \underset{\text{causal signal}}{\overset{x(t)=0 \text{ for } t<0}{=}} \int_{0}^{\infty} x(t)e^{-j\omega t}dt \overset{(A.1)}{=} X(s)|_{s=j\omega} \quad (2.2.3)$$

This argues that for a physical system having causal impulse response $g(t)$, the Fourier transform $G(j\omega)$ of $g(t)$, that is the frequency response, can be obtained by substituting $s = j\omega$ into the system function $G(s)$, which is the Laplace transform of $g(t)$. (See Eq. (1.2.21).)

Example 2.3 CTFT Spectra of a Rectangular (Square) Pulse and a Triangular Pulse

(a) CTFT Spectrum of a Rectangular (Square) Pulse (Fig. 2.8(a))
Consider a single rectangular pulse with height 1 and duration D on the interval $[-D/2,\ D/2]$:

$$Ar_D(t) = A\left(u_s\left(t+\frac{D}{2}\right) - u_s\left(t-\frac{D}{2}\right)\right) = \begin{cases} A & \text{for } -D/2 \le |t| < D/2 \\ 0 & \text{elsewhere} \end{cases} \quad \text{(E2.3.1)}$$

We use Eq. (2.2.1a) to obtain the CTFT coefficients as

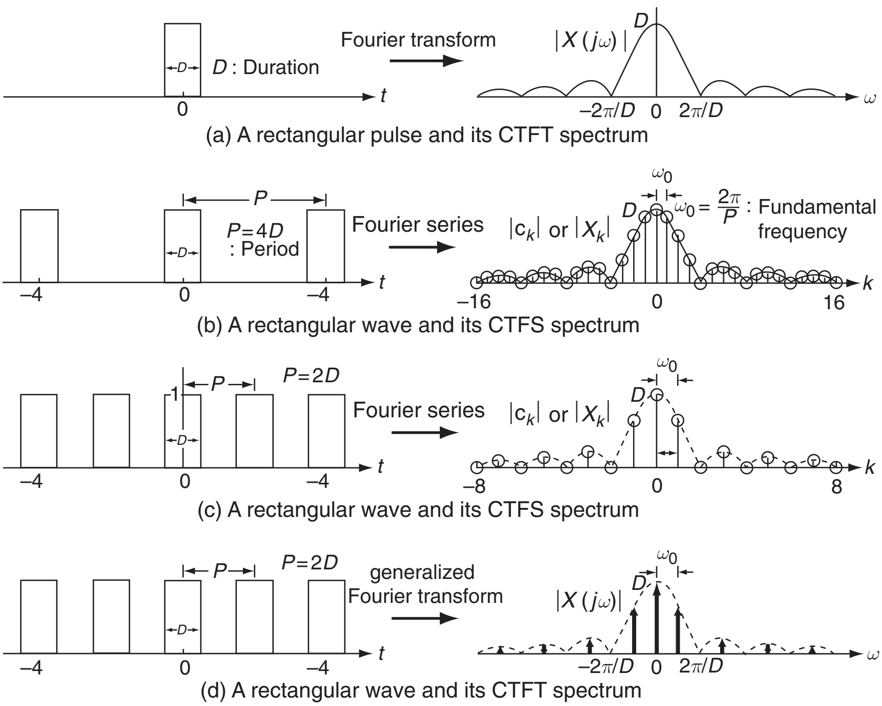


Fig. 2.8 The CTFT or CTFS spectra of rectangular pulses or waves

$$AR_D(j\omega) \overset{(2.2.1a)}{=} \int_{-\infty}^{\infty} Ar_D(t)e^{-j\omega t}\,dt = A\int_{-D/2}^{D/2} e^{-j\omega t}dt = \frac{A}{-j\omega}e^{-j\omega t}\Big|_{-D/2}^{D/2}$$
$$= A\frac{e^{j\omega D/2} - e^{-j\omega D/2}}{j\omega} = AD\frac{\sin(\omega D/2)}{\omega D/2}$$
$$= AD\text{sinc}\left(\frac{\omega D}{2\pi}\right) \qquad \text{(E2.3.2)}$$

This CTFT spectrum is depicted in Fig. 2.8(a). The first zero crossing $B = 2\pi/D$ [rad/s] of the magnitude spectrum is often used as a measure of the frequency spread of a signal and called the *zero-crossing* (*null-to-null*) *bandwidth* of the signal.

(cf.) As a by-product, we can apply the inverse CTFT formula (2.2.1b) for Eq. (E2.3.2) to get the integral of a sinc function:

$$r_D(t) \overset{(2.2.1b)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} R_D(j\omega)e^{j\omega t}\,d\omega \overset{(E2.3.2)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} D\frac{\sin(\omega D/2)}{\omega D/2}e^{j\omega t}\,d\omega$$
$$u_s\left(t+\frac{D}{2}\right) - u_s\left(t-\frac{D}{2}\right) \overset{\omega=2w}{=} \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(wD)}{w}e^{j2wt}\,dw$$

Substituting $t = 0$ into this equation yields

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(wD)}{w}dw = u_s\left(\frac{D}{2}\right) - u_s\left(-\frac{D}{2}\right) = \text{sign}(D) = \begin{cases} 1 & \text{for } D>0 \\ 0 & \text{for } D=0 \\ -1 & \text{for } D<0 \end{cases} \qquad \text{(E2.3.3)}$$

(b) CTFT Spectrum of a Triangular Pulse
Like Eq. (E2.1.5), which is the CTFS coefficient of a triangular wave, we can find the CTFT spectrum of a single triangular pulse $x(t) = A\lambda_D(t)$ as

$$X(j\omega) = A\Lambda_D\,(j\omega) = AD\frac{\sin^2(\omega D/2)}{(\omega D/2)^2} = AD\,\text{sinc}^2\left(\frac{\omega D}{2\pi}\right) \qquad \text{(E2.3.4)}$$

Figure 2.8(a) shows a rectangular pulse and its CTFT spectrum, while Fig. 2.8(b) and (c) show two rectangular waves and their CTFS spectra. These figures present us an observation about the relationship between the CTFT of a single pulse $x(t)$ and the CTFS of its periodic repetition $\tilde{x}_P(t)$ with period P, which is summarized in the following remark.

Remark 2.7 Fourier Series and Fourier Transform

(1) We will mainly use the complex exponential Fourier coefficients, but seldom use the trigonometric or magnitude-and-phase form of Fourier series. Thus, from now on, we denote the complex exponential Fourier coefficients of $x(t)$

by X_k instead of c_k, which has been used so far to distinguish it from other Fourier coefficients $a_k,\ b_k$, or d_k.

(2) As can be seen from comparing Eqs. (E2.1.2) and (E2.3.2) or (E2.1.5) and (E2.3.4), the relationship between the CTFT $X(j\omega)$ of $x(t)$ and the CTFS coefficient X_k of $\tilde{x}_P(t)$ (the periodic extension of $x(t)$ with period P) is as follows:

$$X(j\omega)|_{\omega=k\omega_0=2\pi k/P} = X(jk\omega_0) = X_k \tag{2.2.4}$$

As the period P gets longer so that the fundamental frequency or frequency spacing $\omega_0{=}2\pi/P$ decreases, the Fourier coefficients X_k's become more closely spaced samples of the CTFT $X(j\omega)$, implying that the set of CTFS coefficients approaches the CTFT as $P \to \infty$ (see Fig. 2.8(c), (b), and (a)).

(3) Unlike the discrete frequency $k\omega_0$ of CTFS, the continuous frequency ω of CTFT describes how abruptly the signal changes rather than how often it oscillates.

(4) If the CTFT of a single pulse $x(t)$ and the CTFS of the periodic extension $\tilde{x}_P(t)$ were of different shape in spite of the same shape of $x(t)$ and $\tilde{x}_P(t)$ over one period P, it would be so confusing for one who wants the spectral information about a signal without knowing whether it is of finite duration or periodic. In this context, how lucky we are to have the same shape of spectrum (in the sense that CTFS are just samples of CTFT) whether we take the CTFT of $x(t)$ or the CTFS of $\tilde{x}_P(t)$! Furthermore, you will be happier to see that even the CTFT of $\tilde{x}_P(t)$ (Fig. 2.8(d)) is also of the same shape as the CTFS of $\tilde{x}_P(t)$, because one might observe one period of $\tilde{x}_P(t)$ and mistake it for $x(t)$ so that he or she would happen to apply the CTFT for periodic signals. Are you puzzled at the CTFT of a periodic signal? Then rush into the next section.

2.3 (Generalized) Fourier Transform of Periodic Signals

Since a periodic signal can satisfy neither the square-integrability condition (2.2.2a) nor the absolute-integrability condition (2.2.2b), the CTFTs of periodic signals are not only singular but also difficult to compute directly. For example, let us try to compute the CTFT of $x_k(t) = e^{jk\omega_0 t}$ by using Eq. (2.2.1a):

$$\begin{aligned}
\mathcal{F}\{x_k(t)\} &\overset{(2.2.1a)}{=} \int_{-\infty}^{\infty} e^{jk\omega_0 t}\ e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{-j(\omega-k\omega_0)t}\, dt \\
&\overset{(D.33)}{=} \frac{1}{-j(\omega-k\omega_0)}\ e^{-j(\omega-k\omega_0)t}\Big|_{-T}^{T}\Bigg|_{T\to\infty} \\
&= \frac{1}{j(\omega-k\omega_0)}\left(e^{j(\omega-k\omega_0)T} - e^{-j(\omega-k\omega_0)T}\right)\Bigg|_{T=\infty} \\
&= \left(\frac{2\sin(\omega-k\omega_0)T}{(\omega-k\omega_0)}\right)\Bigg|_{T=\infty} =?
\end{aligned} \tag{2.3.1}$$

To get around this mathematical difficulty, let us find the inverse CTFT of

$$X_k(j\omega) = 2\pi\ \delta(\omega - k\omega_0)$$

by applying the inverse Fourier transform (2.2.1b):

$$\mathcal{F}^{-1}\{X_k(j\omega)\} \overset{(2.2.1b)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\ \delta(\omega - k\omega_0)e^{j\omega t}\, d\omega \overset{(1.1.25)}{=} e^{jk\omega_0 t}$$

This implies a CTFT pair as

$$x_k(t) = e^{jk\omega_0 t} \overset{\mathcal{F}}{\leftrightarrow} X_k(j\omega) = 2\pi\ \delta(\omega - k\omega_0) \tag{2.3.2}$$

Based on this relation, we get the Fourier transform of a periodic function $x(t)$ from its Fourier series representation as follows:

$$x(t) \overset{(2.1.5a)}{=} \frac{1}{P}\sum_{k=-\infty}^{\infty} X_k\, e^{jk\omega_0 t} \quad \text{with} \quad \omega_0 = \frac{2\pi}{P}$$

$$\overset{\mathcal{F}}{\leftrightarrow} X(j\omega) \overset{(2.3.2)}{=} \frac{2\pi}{P}\sum_{k=-\infty}^{\infty} X_k\ \delta(\omega - k\omega_0) \tag{2.3.3}$$

This implies that the CTFT of a periodic signal consists of a train of impulses on the frequency axis having the same shape of envelope as the CTFS spectrum. Figure 2.8(d) is an illustration of Eq. (2.3.3) as stated in Remark 2.7(4).

Remark 2.8 Fourier Transform of a Periodic Signal
It would be cumbersome to directly find the CTFT of a periodic function. Thus we had better find the CTFS coefficients first and then use Eq. (2.3.2) as illustrated in Eq. (2.3.3).

2.4 Examples of the Continuous-Time Fourier Transform

Example 2.4 Fourier Transform of an Exponential Function

For an exponential function (Fig. 2.9(a)) with time constant $T > 0$

$$e_1(t) = \frac{1}{T}e^{-t/T}u_s(t) \quad \text{with} \quad T > 0, \tag{E2.4.1}$$

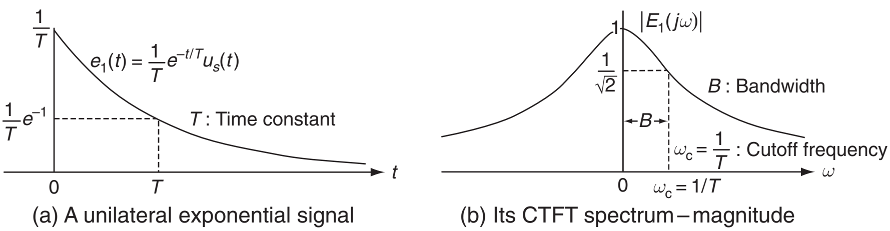


Fig. 2.9 A unilateral exponential signal and its CTFT spectrum

we have the Fourier transform

$$E_1(j\omega) = \mathcal{F}\{e_1(t)\} \overset{(2.2.1a)}{=} \int_{-\infty}^{\infty} \frac{1}{T} e^{-t/T} u_s(t)\, e^{-j\omega t}\, dt = \frac{1}{T}\int_0^{\infty} e^{-t/T} e^{-j\omega t}\, dt$$
$$= \frac{1}{T}\int_0^{\infty} e^{-(1/T+j\omega)t}\, dt \overset{(D.33)}{=} -\frac{1}{T(1/T+j\omega)}\, e^{-(1/T+j\omega)t}\Big|_0^{\infty}$$
$$= \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+(\omega T)^2}} \angle - \tan^{-1}(\omega T) \qquad \text{(E2.4.2)}$$

whose magnitude is depicted in Fig. 2.9(b). From this magnitude spectrum, we see that $\omega_c = 1/T$ is the half-power frequency at which $|E_1(j\omega)|$ is $1/\sqrt{2}$ times the maximum magnitude 1:

$$|E_1(j\omega)| = \frac{1}{\sqrt{1+(\omega\, T)^2}} = \frac{1}{\sqrt{2}};\ 1+(\omega\, T)^2 = 2;\ \omega T = 1;\ \omega_c = \frac{1}{T} \qquad \text{(E2.4.3)}$$

This example makes possible the following interpretations:

Remark 2.9 Signal Bandwidth and System Bandwidth – Uncertainty Principle

(1) In case the function $e_1(t)$ represents a physical signal itself, the above example illustrates the inverse relationship (a kind of duality) between the time and frequency domains that the bandwidth B [rad/s] of the signal is inversely proportional to the time-duration T [s], i.e., BT = constant(= 1). Note that the *bandwidth* of a signal, i.e., the width of the frequency band carrying the major portion of the signal energy, describes how rich frequency contents the signal contains. Such a relationship could also be observed in Example 2.3 and Fig. 2.8 where the time-duration of the rectangular pulse is D and the *signal bandwidth*, defined as the frequency range to the first zero-crossing of the magnitude spectrum, is $2\pi/D$. This observation, called the reciprocal *duration-bandwidth* relationship, is generalized into the *uncertainty principle* that the time-duration and bandwidth of a signal cannot be simultaneously made arbitrarily small, implying that a signal with short/long time-duration must have a wide/narrow bandwidth [S-1, Sect. 4.5.2]. This has got the name from the Heisenberg uncertainty principle in quantum mechanics that the product of the uncertainties in the measurements of the position and momentum of a particle cannot be made arbitrarily small.

(2) In case the function $e_1(t)$ represents the impulse response of a filter such as the RC circuit shown in Fig. 2.10(a), it has another interpretation that the bandwidth of the system behaving as a low-pass filter is inversely proportional to the time constant $T = RC$[s]. Note that the *system bandwidth* of a filter describes the width of frequency band of the signal to be relatively well passed and that it is usually defined as the frequency range to the 3dB-magnitude (half-power) frequency $B = 1/T$ [rad/s]. Also note that, in comparison with the bandwidth as a frequency-domain "capacity" concept, the *time constant* describes how fast the filter produces the response (output) to an applied input

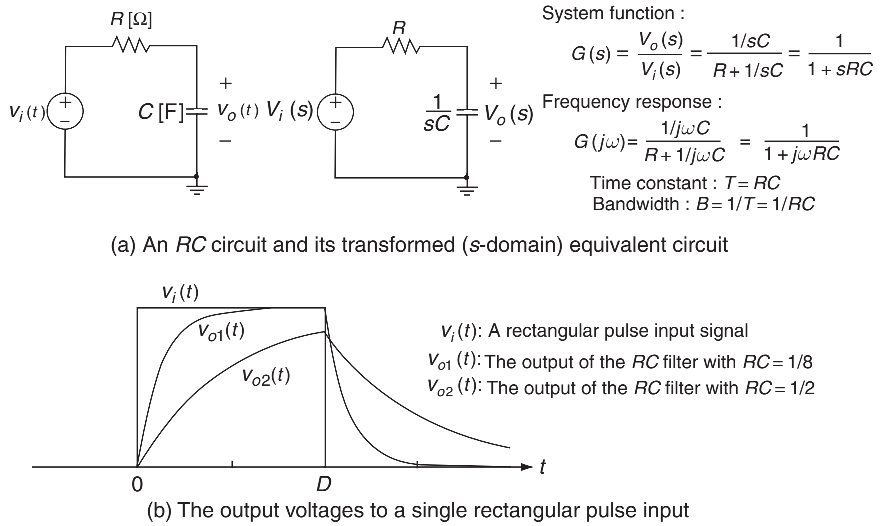


Fig. 2.10 An RC circuit and its outputs to rectangular pulse input signals

signal and that it is defined as the time taken for the output to reach $68.2\%(e^{-1})$ of the final (steady-state) value, or equivalently, the time measured until the (slowest) term of the transient response (converging towards zero) becomes as small as 32.8% of the initial value.

(3) Referring to Fig. 2.10(b), suppose a rectangular pulse of duration D is applied to the RC filter. Then, in order for the low-pass filter to have *hi*gh *fi*delity of reproduction so that the output pulse will appear very much like the input pulses, the system bandwidth (the reciprocal of time constant RC) of the filter had better be greater than the signal bandwidth $2\pi/D$ of the input pulse.

Example 2.5 Fourier Transform of an Even-Symmetric Exponential Function

For an exponential function (Fig. 2.11(a)) with time constant $T > 0$

$$e_2(t) = \frac{1}{2T} e^{-|t|/T} \quad \text{with} \quad T > 0, \tag{E2.5.1}$$

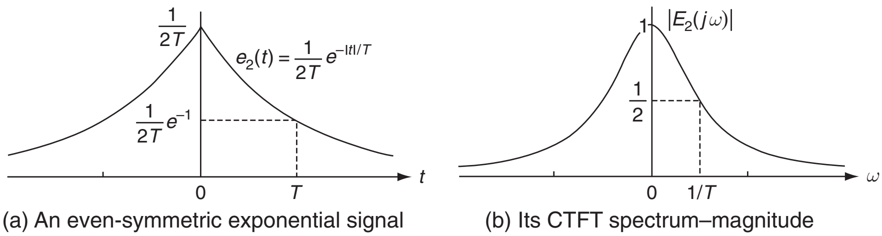


Fig. 2.11 An even-symmetric exponential signal and its CTFT spectrum

we have the Fourier transform

$$
\begin{aligned}
E_2(j\omega) = \mathcal{F}\{e_2(t)\} &\overset{(2.2.1a)}{=} \int_{-\infty}^{\infty} \frac{1}{2T} e^{-|t|/T} e^{-j\omega t}\, dt \\
&= \frac{1}{2T}\left\{\int_{-\infty}^{0} e^{t/T} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-t/T} e^{-j\omega t}\, dt\right\} \\
&= \frac{1}{2T(1/T - j\omega)}\, e^{(1/T - j\omega)t}\Big|_{-\infty}^{0} - \frac{1}{2T(1/T + j\omega)}\, e^{-(1/T + j\omega)t}\Big|_{0}^{\infty} \\
&= \frac{1/2}{1 - j\omega T} + \frac{1/2}{1 + j\omega T} = \frac{1}{1 + (\omega T)^2}
\end{aligned}
\tag{E2.5.2}
$$

whose magnitude is depicted in Fig. 2.11(b).

(Q) Why has the magnitude spectrum shorter bandwidth than that in Fig. 2.9(b)?
(A) It is because the signal in Fig. 2.11(a) is smoother than that in Fig. 2.9(a).

Example 2.6 Fourier Transform of the Unit Impulse (Dirac Delta) Function

We can obtain the Fourier transform of the unit impulse function $\delta(t)$ as

$$
\mathcal{D}(j\omega) = \mathcal{F}\{\delta(t)\} \overset{(2.2.1a)}{=} \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt \overset{(1.1.25)}{=} 1 \;\forall\, \omega \tag{E2.6.1}
$$

This implies that an impulse signal has a flat or white spectrum, which is evenly distributed over all frequencies (see Fig. 2.12).

It is interesting to see that this can be derived by taking the limit of Eq. (E2.3.2) as $D \to 0$ in Example 2.3 (with $A = 1/D$) or Eq. (E2.5.2) as $T \to 0$ in Example 2.5:

$$
\begin{aligned}
\delta(t) &\overset{(1.1.33b)}{=} \lim_{D\to 0} \frac{1}{D} r_D\left(t + \frac{D}{2}\right) \overset{(E2.3.1)}{=} \lim_{D\to 0} \frac{1}{D}\left(u_s\left(t + \frac{D}{2}\right) - u_s\left(t - \frac{D}{2}\right)\right) \\
&\to \lim_{D\to 0} \frac{1}{D} R_D(j\omega) \overset{(E2.3.2)}{=} \lim_{D\to 0} \mathrm{sinc}\left(\frac{\omega D}{2\pi}\right) = 1
\end{aligned}
$$

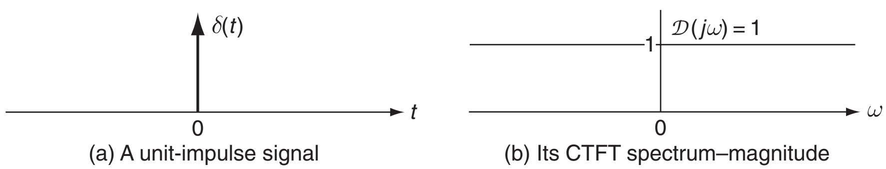


Fig. 2.12 A unit impulse signal and its CTFT spectrum

$$\delta(t) \overset{(1.1.33d)}{=} \lim_{T\to 0} e_2(t) \overset{(E2.5.1)}{=} \lim_{T\to 0} \frac{1}{2T} e^{-|t|/T}$$

$$\to \lim_{T\to 0} E_2(j\omega) \overset{(E2.5.2)}{=} \lim_{T\to 0} \frac{1}{1+(\omega T)^2} = 1$$

As a byproduct, we can apply the inverse Fourier transform (2.2.1b) to obtain an expression of the impulse function as

$$\delta(t) = \mathcal{F}^{-1}\{\mathcal{D}(j\omega)\} \overset{(2.2.1b)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{D}(j\omega) e^{j\omega t}\, d\omega \overset{(E2.6.1)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 e^{j\omega t}\, d\omega \tag{E2.6.2}$$

$$\overset{(D.33)}{=} \lim_{w\to\infty} \frac{1}{2\pi jt} e^{j\omega t}\Big|_{-w}^{w} = \lim_{w\to\infty} \frac{e^{j\omega t} - e^{-j\omega t}}{2\pi jt} \overset{(D.22)}{=} \lim_{w\to\infty} \frac{w}{\pi}\frac{\sin(wt)}{wt}$$

which is identical with Eq. (1.1.33a).

Remark 2.10 An Impulse Signal and Its (White/Flat) Spectrum

(1) Comparing Figs. 2.11 and 2.12, we see that short-duration signals contain stronger high-frequency components than long-duration ones do. This idea supports why a lightning stroke of very short duration produces an observable noise effect over all communication signals from the relatively low frequencies (550∼1600kHz) used in radio system to the relatively higher ones used in television system (60MHz for VHF ∼470MHz for UHF).

(2) We often use the impulse function as a typical input to determine the important characteristics (frequency response or system/transfer function) of linear time-invariant (LTI) systems. One practical reason is because the impulse function has uniform (flat) spectrum, i.e., contains equal power at all frequencies so that applying the impulse signal (or a good approximation to it) as the input to a system would be equivalent to simultaneously exciting the system with every frequency component of equal amplitude and phase.

Example 2.7 Fourier Transform of a Constant Function

We can obtain the Fourier transform of the unit constant function $c(t) = 1$ (Fig. 2.13(a)) as

$$C(j\omega) = \mathcal{F}\{1\} \overset{(2.2.1a)}{=} \int_{-\infty}^{\infty} 1 e^{-j\omega t}\, dt \underset{\text{with } k=0}{\overset{(2.3.2)}{=}} 2\pi\ \delta(\omega) \tag{E2.7.1}$$

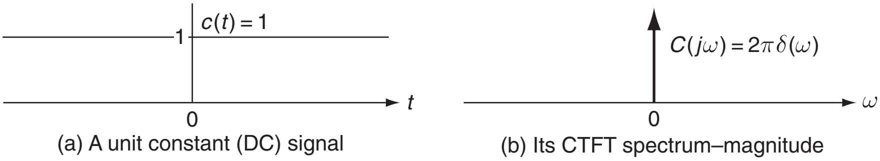


Fig. 2.13 A unit constant (DC) signal and its CTFT spectrum

This spectrum, that is depicted in Fig. 2.13(b), shows that a constant signal has only DC component with zero frequency, i.e., has all of its (infinite) energy at $\omega = 0$.

This can also be obtained by swapping t and ω in (E2.6.2) and verified by using the inverse Fourier transform (2.2.1b) to show that

$$\mathcal{F}^{-1}\{2\pi\delta(\omega)\} \overset{(2.2.1b)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\ \delta(\omega)e^{j\omega t}\ d\omega \overset{(1.1.25)}{=} 1 \tag{E2.7.2}$$

Example 2.8 Fourier Transform of the Unit Step Function

To compute the Fourier transform of the unit step function $u_s(t)$, let us write its even-odd decomposition, i.e., decompose $u_s(t)$ into the sum of an even function and an odd function as

$$u_s(t) = u_e(t) + u_o(t) \tag{E2.8.1}$$

where

$$u_e(t) = \frac{1}{2}(u_s(t) + u_s(-t)) = \begin{cases} 1/2 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases} \tag{E2.8.2}$$

$$u_o(t) = \frac{1}{2}(u_s(t) - u_s(-t)) = \frac{1}{2}\text{sign}(t) = \begin{cases} 1/2 & \text{for } t > 0 \\ 0 & \text{for } t \neq 0 \\ -1/2 & \text{for } t < 0 \end{cases} \tag{E2.8.3}$$

Then, noting that the even part $u_e(t)$ is a constant function of amplitude 1/2 except at $t = 0$, we can use Eq. (E2.7.1) to write its Fourier transform as

$$U_e(j\omega) = \mathcal{F}\{u_e(t)\} = \frac{1}{2}\mathcal{F}\{1\} \overset{(E2.7.1)}{=} \pi\ \delta(\omega) \tag{E2.8.4}$$

On the other side, the Fourier transform of the odd part can be computed as

$$\begin{aligned} U_o(j\omega) &= \mathcal{F}\{u_o(t)\} \overset{(2.2.1a)}{=} \int_{-\infty}^{\infty} u_o(t)e^{-j\omega t}\ dt = \int_{-\infty}^{\infty} \overset{\text{odd}}{u_o(t)}(\overset{\text{even}}{\cos\omega t} - j\ \overset{\text{odd}}{\sin\omega t})\ dt \\ &= -j\int_{-\infty}^{\infty} \overset{\text{odd}\times\text{odd}=\text{even}}{u_o(t)\sin\ \omega t}\ dt = -j2\int_{0}^{\infty} \overset{\text{even}}{u_o(t)\sin\ \omega t}\ dt \\ &\overset{u_o(t)=1/2 \text{ for } t>0}{=} -j\int_{0}^{\infty} \sin\ \omega t\ dt = -j\int_{0}^{\infty} \sin\ \omega t\ e^{-st}\ dt|_{s=0} \\ &\overset{(A.1)}{=} -j\ \mathcal{L}\{\sin\ \omega t\}|_{s=0} \overset{B.8(9)}{=} -j\ \left.\frac{\omega}{s^2+\omega^2}\right|_{s=0} = \frac{1}{j\omega} \end{aligned} \tag{E2.8.5}$$

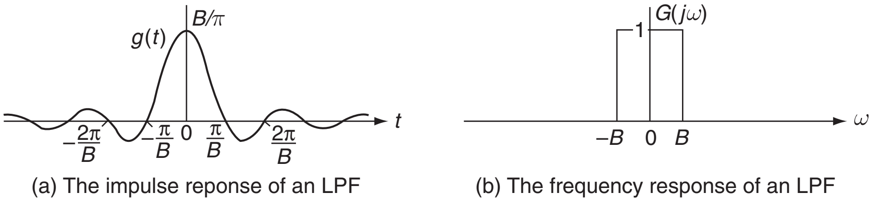


Fig. 2.14 The impulse response and frequency response of an ideal LPF

where we have used the Laplace transform. Now we add these two results to obtain the Fourier transform of the unit step function as

$$\mathcal{F}\{u_s(t)\} = \mathcal{F}\{u_e(t) + u_o(t)\} \overset{(2.5.1)}{=} U_e(j\omega) + U_o(j\omega) \overset{(E2.8.4),(E2.8.5)}{=} \pi\ \delta(\omega) + \frac{1}{j\omega} \tag{E2.8.6}$$

Example 2.9 Inverse Fourier Transform of an Ideal LPF Frequency Response

Let us consider the frequency response of an ideal lowpass filter (LPF) depicted in Fig. 2.14(b):

$$G(j\omega) = \begin{cases} 1 & \text{for } |\omega| \le B \\ 0 & \text{for } |\omega| > B \end{cases} \tag{E2.9.1}$$

Taking the inverse Fourier transform of this LPF frequency response yields the impulse response as

$$\begin{aligned} g(t) &= \mathcal{F}^{-1}\{G(j\omega)\} \overset{(2.2.1b)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-B}^{B} 1e^{-j\omega t}\, d\omega \\ &= \frac{1}{2\pi jt}(e^{jBt} - e^{-jBt}) = \frac{\sin Bt}{\pi t} = \frac{B}{\pi}\text{sinc}\left(\frac{Bt}{\pi}\right) \end{aligned} \tag{E2.9.2}$$

which is depicted in Fig. 2.14(a).

(cf.) It may be interesting to see that a rectangular pulse and a sinc function constitute a Fourier transform pair, i.e., the Fourier transforms of rectangular pulse and sinc function turn out to be the spectra of sinc function and rectangular pulse function form, respectively (see Figs. 2.8(a) and 2.14). This is a direct consequence of the duality relationship between Fourier transform pairs, which will be explained in detail in Sect. 2.5.4.

Remark 2.11 Physical Realizability and Causality Condition

If a system has the frequency response that is strictly bandlimited like $G(j\omega)$ given by Eq. (E2.9.1) and depicted in Fig. 2.14(b), the system is not *physically* *realizable* because it violates the causality condition, i.e., $g(t) \neq 0$ for some $t < 0$ while every physical system must be causal (see Sect. 1.2.9).

Example 2.10 Fourier Transform of an Impulse Train

Let us consider an impulse train

$$\delta_T(t) = \sum\nolimits_{m=-\infty}^{\infty} \delta(t - mT) \tag{E2.10.1}$$

Since this is a periodic function, we first write its Fourier series representation from Eq. (2.1.10) as

$$\delta_T(t) \overset{(2.1.10)}{=} \frac{1}{T} \sum\nolimits_{k=-\infty}^{\infty} e^{jk\omega_s t} \quad \text{with} \quad \omega_s = \frac{2\pi}{T} \tag{E2.10.2}$$

Then we use Eq. (2.3.2) with $\omega_0 = \omega_s = 2\pi/T$ to obtain the Fourier transform as

$$\begin{aligned}\mathcal{D}_T(j\omega) &\underset{(2.3.2)}{\overset{(E2.10.2)}{=}} \frac{2\pi}{T} \sum\nolimits_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \\ &= \frac{2\pi}{T} \sum\nolimits_{k=-\infty}^{\infty} \delta(\omega + k\omega_s) \quad \text{with} \quad \omega_s = \frac{2\pi}{T}\end{aligned} \tag{E2.10.3}$$

(cf.) Applying Eq. (2.2.1b) to take the inverse Fourier transform of Eq. (E2.10.3) will produce Eq. (2.10.2).

(cf.) Note that, as the period T (the interval between the impulses in time) increases, the fundamental frequency $\omega_s = 2\pi/T$ (the spacing between the impulses in frequency) decreases. This is also a consequence of the duality relationship between Fourier transform pairs.

Example 2.11 Fourier Transform of Cosine/Sine Functions

(a) The Fourier transform of $x(t) = \sin(\omega_1 t) = (e^{j\omega_1 t} - e^{-j\omega_1 t})/j2$ can be obtained as

$$X(j\omega) = \mathcal{F}\{\sin(\omega_1 t)\} = \frac{1}{j2}\mathcal{F}\{e^{j\omega_1 t} - e^{-j\omega_1 t}\} \overset{(2.3.2)}{=} j\pi(\delta(\omega + \omega_1) - \delta(\omega - \omega_1)) \tag{E2.11.1}$$

(b) The Fourier transform of $x(t) = \cos(\omega_1 t) = (e^{j\omega_1 t} + e^{-j\omega_1 t})/2$ can be obtained as

$$X(j\omega) = \mathcal{F}\{\cos(\omega_1 t)\} = \frac{1}{2}\mathcal{F}\{e^{j\omega_1 t} + e^{-j\omega_1 t}\} \overset{(2.3.2)}{=} \pi(\delta(\omega + \omega_1) + \delta(\omega - \omega_1)) \tag{E2.11.2}$$

2.5 Properties of the Continuous-Time Fourier Transform

In this section we are about to discuss some basic properties of continuous-time Fourier transform (CTFT), which will provide us with an insight into the Fourier transform and the capability of taking easy ways to get the Fourier transforms or inverse Fourier transforms.

(cf.) From now on, we will use $X(\omega)$ instead of $X(j\omega)$ to denote the Fourier transform of $x(t)$.

2.5.1 Linearity

With $\mathcal{F}\{x(t)\} = X(\omega)$ and $\mathcal{F}\{y(t)\} = Y(\omega)$, we have

$$a\, x(t) + b\, y(t) \overset{\mathcal{F}}{\leftrightarrow} a\, X(\omega) + b\, Y(\omega), \tag{2.5.1}$$

which implies that the Fourier transform of a linear combination of many functions is the same linear combination of the individual transforms.

2.5.2 (Conjugate) Symmetry

In general, Fourier transform has the time reversal property:

$$\begin{aligned}\mathcal{F}\{x(-t)\} &\overset{(2.2.1a)}{=} \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}\, dt \overset{-t=\tau}{=} \int_{\infty}^{-\infty} x(\tau)e^{j\omega\tau}(-d\tau) \overset{\tau=t}{=} \int_{-\infty}^{\infty} x(t)e^{j\omega t}\, dt \\ &\underset{\omega\to-\omega}{\overset{(2.2.1a)}{=}} X(-\omega);\ x(-t) \overset{\mathcal{F}}{\leftrightarrow} X(-\omega)\end{aligned} \tag{2.5.2}$$

In case $x(t)$ is a real-valued function, we have

$$\begin{aligned}X(-\omega) &\underset{\omega\to-\omega}{\overset{(2.2.1a)}{=}} \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}\, dt = \int_{-\infty}^{\infty} x(t)e^{-(-j)\omega t} dt \\ &\underset{j\to-j}{\overset{(2.2.1a)}{=}} X^*(\omega)\ \text{(complex conjugate of } X\ (\omega))\end{aligned}$$

or equivalently,

$$\begin{aligned}\mathrm{Re}\{X(-\omega)\} + j\mathrm{Im}\{X(-\omega)\} &= \mathrm{Re}\{X(\omega)\} - j\mathrm{Im}\{X(\omega)\} \\ |X(-\omega)|\angle X(-\omega) &= |X(\omega)|\angle - X(\omega)\end{aligned} \tag{2.5.3}$$

This implies that the magnitude/phase of the CTFT of a real-valued function is an even/odd function of frequency ω. Thus, when obtaining the Fourier transform of a real-valued time function, we need to compute it only for $\omega \geq 0$ since we can use the conjugate symmetry to generate the values for $\omega < 0$ from those for $\omega > 0$. In other words, for a real-valued time function, its magnitude and phase spectra are symmetrical about the vertical axis and the origin, respectively.

For an even and real-valued function $x_e(t)$ such that $x_e(-t) = x_e(t)$, its Fourier transform is also an even and real-valued function in frequency ω:

$$\begin{aligned} X_e(-\omega) &\underset{\omega=-\omega}{\overset{(2.2.1a)}{=}} \int_{-\infty}^{\infty} x_e(t)\, e^{-j(-\omega)t}\, dt \overset{t=-\tau}{=} \int_{-\infty}^{\infty} x_e(-\tau)\, e^{-j\omega\tau} d\tau \\ &\underset{\text{even}}{\overset{x_e(-\tau)=x_e(\tau)}{=}} \int_{-\infty}^{\infty} x_e(\tau)\, e^{-j\omega\tau}\, d\tau \overset{(2.2.1a)}{=} X_e(\omega) \end{aligned} \tag{2.5.4a}$$

Also for an odd and real-valued function $x_o(t)$ such that $x_o(-t) = -x_o(t)$, its Fourier transform is an odd and imaginary-valued function in frequency ω:

$$\begin{aligned} X_o(-\omega) &\underset{\omega=-\omega}{\overset{(2.2.1a)}{=}} \int_{-\infty}^{\infty} x_o(t)\, e^{-j(-\omega)t}\, dt \overset{t=-\tau}{=} \int_{-\infty}^{\infty} x_o(-\tau)\, e^{-j\omega\tau} d\tau \\ &\underset{\text{odd}}{\overset{x_o(-\tau)=-x_o(\tau)}{=}} -\int_{-\infty}^{\infty} x_o(\tau)\, e^{-j\omega\tau}\, d\tau \overset{(2.2.1a)}{=} -X_o(\omega) \end{aligned} \tag{2.5.4b}$$

Note that any real-valued function $x(t)$ can be expressed as the sum of an even function and an odd one:

$$x(t) = x_e(t) + x_o(t)$$

where

$$x_e(t) = \frac{1}{2}(x(t) + x(-t)) \quad \text{and} \quad x_o(t) = \frac{1}{2}(x(t) - x(-t))$$

Thus we can get the relations

$$\begin{aligned} X(\omega) &= \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\} = X_e(\omega) + X_o(\omega) \\ \mathrm{Re}\{X(\omega)\} &+ j\mathrm{Im}\{X(\omega)\} = X_e(\omega) + X_o(\omega) \end{aligned}$$

which implies

$$\text{even and real-valued } x_e(t) \overset{\mathcal{F}}{\leftrightarrow} \mathrm{Re}\{X(\omega)\} \text{ even and real-valued} \tag{2.5.5a}$$

$$\text{odd and real-valued } x_o(t) \overset{\mathcal{F}}{\leftrightarrow} j\mathrm{Im}\{X(\omega)\} \text{ odd and imaginary-valued} \tag{2.5.5b}$$

2.5.3 Time/Frequency Shifting (Real/Complex Translation)

For a shifted time function $x_1(t) = x(t - t_1)$, we have its Fourier transform

$$\mathcal{F}\{x(t-t_1)\} \stackrel{(2.2.1a)}{=} \int_{-\infty}^{\infty} x(t-t_1)\, e^{-j\omega t}\, dt \stackrel{t-t_1 \to t}{=} \int_{-\infty}^{\infty} x(t)\, e^{-j\omega(t+t_1)} dt$$
$$\stackrel{(2.2.1a)}{=} \mathcal{F}\{x(t)\}\, e^{-j\omega\, t_1};\ x(t-t_1) \stackrel{\mathcal{F}}{\leftrightarrow} X(\omega)\, e^{-j\omega t_1} \tag{2.5.6}$$

This implies that real translation (time shifting) of $x(t)$ by t_1 along the t -axis will affect the Fourier transform on its phase by $-\omega t_1$, but not on its magnitude. This is similar to Eq. (2.1.11), which is the time shifting property of CTFS.

In duality with the time shifting property (2.5.6), the complex translation (frequency shifting) property holds

$$x(t)\, e^{j\omega_1 t} \stackrel{\mathcal{F}}{\leftrightarrow} X(\,\omega - \omega_1\,) \tag{2.5.7}$$

2.5.4 Duality

As exhibited by some examples and properties, there exists a definite symmetry between Fourier transform pairs, stemming from a general property of *duality* between time and frequency. It can be proved by considering the following integral expression

$$f(u) = \int_{-\infty}^{\infty} g(v)\, e^{-juv} dv \stackrel{v \to -v}{=} \int_{-\infty}^{\infty} g(-v)\, e^{juv} dv \tag{2.5.8}$$

This, with $(\omega,\ t)$ or $(\pm t, \mp\omega)$ for $(u,\ v)$, yields the Fourier transform or inverse transform relation, respectively:

$$f(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega\, t}\, dt \stackrel{(2.2.1a)}{=} \mathcal{F}\{g(t)\}: \quad g(t) \stackrel{\mathcal{F}}{\leftrightarrow} f(\omega) \tag{2.5.9a}$$

$$f(\pm\, t) = \int_{-\infty}^{\infty} g(\mp\omega)\, e^{j\omega\, t}\, d\omega \stackrel{(2.2.1b)}{=} 2\pi \mathcal{F}^{-1}\{g(\mp\omega)\}: \quad f(\pm\, t) \stackrel{\mathcal{F}}{\leftrightarrow} 2\pi\ g(\mp\omega) \tag{2.5.9b}$$

It is implied that, if one Fourier transform relation holds, the substitution of $(\pm t, \mp\omega)$ for $(\omega,\ t)$ yields the other one, which is also valid. This property of duality can be used to recall or show Fourier series relations. We have the following examples:

(Ex 0) $$g(t) = x(t) \underset{(2.5.9a)}{\stackrel{\mathcal{F}}{\leftrightarrow}} f(\omega) = X(\omega)$$

$$\Leftrightarrow \frac{1}{2\pi} f(t) = \frac{1}{2\pi} X(t) \underset{(2.5.9b)}{\stackrel{\mathcal{F}}{\leftrightarrow}} g(\omega) = x(\omega)$$

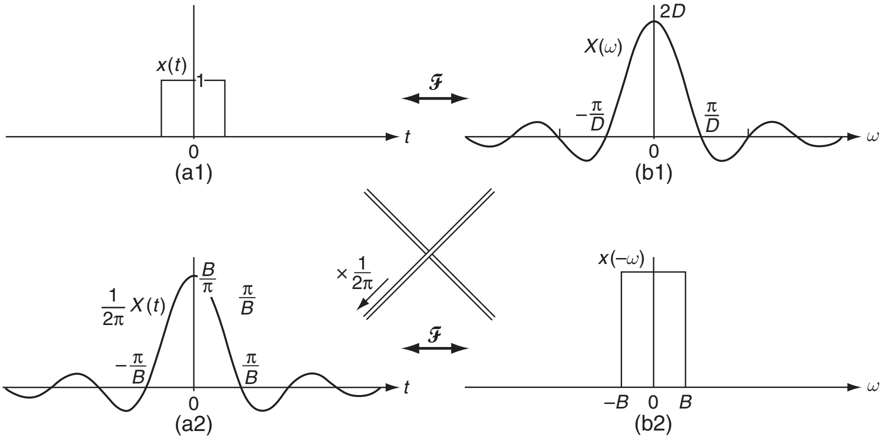


Fig. 2.15 Dual relationship between the Fourier transform pairs

(Ex 1) $\quad x(t) = A\delta(t) \underset{(2.5.9a)}{\overset{\mathcal{F}}{\leftrightarrow}} X(\omega) \overset{(E2.6.1)}{=} A$

$$\Leftrightarrow X(t) = A \underset{(2.5.9b)}{\overset{\mathcal{F}}{\leftrightarrow}} 2\pi\, x(-\omega) = 2\pi A\, \delta(-\omega) \qquad \text{(See Example 2.7.)}$$

(Ex 2) $\quad x(t) = \begin{cases} 1 & \text{for } |t| \le D \\ 0 & \text{for } |t| > D \end{cases} \underset{(2.5.9a)}{\overset{F}{\leftrightarrow}}$

$$X(\omega) \overset{(E2.3.1)}{=} 2D \operatorname{sinc}\left(\frac{D\omega}{\pi}\right) \qquad \text{(See Example 2.9 and Fig.2.15)}$$

$$\Leftrightarrow \frac{1}{2\pi} X(t) = \frac{B}{\pi} \operatorname{sinc}\left(\frac{Bt}{\pi}\right) \underset{(2.5.9b)}{\overset{\mathcal{F}}{\leftrightarrow}} x(-\omega) = \begin{cases} 1 & \text{for } |\omega| \le B \\ 0 & \text{for } |\omega| > B \end{cases}$$

(Ex 3) $\quad x(t) = A\delta(t - t_1) \underset{(2.5.9a)}{\overset{\mathcal{F}}{\leftrightarrow}} X(\omega) \overset{(E2.6.1)\&(2.5.6)}{=} A\, e^{-j\omega\, t_1}$

$$\Leftrightarrow X(-t) = A\, e^{j\omega_1\, t} \underset{(2.5.9b)}{\overset{\mathcal{F}}{\leftrightarrow}} 2\pi x(\omega) = 2\pi A\delta(\omega - \omega_1) \quad \text{(See Eq.(2.3.2))}$$

2.5.5 Real Convolution

For the (linear) convolution of two functions $x(t)$ and $g(t)$

$$y(t) = x(t) * g(t) = \int_{-\infty}^{\infty} x(\tau) g(t-\tau)\, d\tau = \int_{-\infty}^{\infty} g(\tau) x(t-\tau)\, d\tau = g(t) * x(t), \tag{2.5.10}$$

we can get its Fourier transform as

$$Y(\omega) = \mathcal{F}\{y(t)\} \overset{(2.2.1a)}{\underset{(2.5.10)}{=}} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(\tau)g(t-\tau)\,d\tau \right\} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau} \left\{ \int_{-\infty}^{\infty} g(t-\tau)e^{-j\omega\,(t-\tau)}\,dt \right\} d\tau$$

$$\overset{t-\tau\to t}{\underset{(2.2.1a)}{=}} \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\, G(\omega)\,d\tau = G(\omega)\int_{-\infty}^{\infty} x\,(\tau)e^{-j\omega\tau} d\tau \overset{(2.2.1a)}{=} G(\omega)X(\omega)$$

$$y(t) = x(t) * g(t) \overset{\mathcal{F}}{\leftrightarrow} Y(\omega) = X(\omega)\,G(\omega) \tag{2.5.11}$$

where $Y(\omega) = \mathcal{F}\{y(t)\}$, $X(\omega) = \mathcal{F}\{x(t)\}$, and $G(\omega) = \mathcal{F}\{g(t)\}$. This is the frequency-domain input-output relationship of a linear time-invariant (LTI) system with the input $x(t)$, the output $y(t)$, and the impulse response $g(t)$ where $G(\omega)$ is called the frequency response of the system.

On the other hand, if two functions $\tilde{x}_P(t)$ and $\tilde{g}_P(t)$ are periodic with common period P, their linear convolution does not converge so that we need another definition of convolution, which is the *periodic* or *circular convolution* with the integration performed over one period:

$$\begin{aligned}\tilde{y}_P(t) = \tilde{x}_P(t) \underset{P}{*} \tilde{g}_P(t) &= \int_P \tilde{x}_P(\tau)\tilde{g}_P(t-\tau)\,d\tau \\ &= \int_P \tilde{g}_P(\tau)\tilde{x}_P(t-\tau)\,d\tau = \tilde{g}_P(t) \underset{P}{*} \tilde{x}_P(t)\end{aligned} \tag{2.5.12}$$

where $\underset{P}{*}$ denotes the circular convolution with period P. Like the Fourier transform of a linear convolution, the Fourier series coefficient of the periodic (circular) convolution turns out to be the multiplication of the Fourier series coefficients of two periodic functions $\tilde{x}_P(t)$ and $\tilde{g}_P(t)$ (see Problem 2.8(a)):

$$\tilde{y}_P(t) = \tilde{x}_P(t) \underset{P}{*} \tilde{g}_P(t) \overset{\text{Fourier series}}{\leftrightarrow} Y_k = X_k\,G_k \tag{2.5.13}$$

where Y_k, X_k, and G_k are the Fourier coefficients of $\tilde{y}_P(t)$, $\tilde{x}_P(t)$, and $\tilde{g}_P(t)$, respectively.

2.5.6 Complex Convolution (Modulation/Windowing)

In duality with the convolution property (2.5.11) that convolution in the time domain corresponds to multiplication in the frequency domain, we may expect the *modulation property* that multiplication in the time domain corresponds to convolution in the frequency domain:

$$y(t) = x(t)m(t) \overset{\mathcal{F}}{\leftrightarrow} Y(\omega) = \frac{1}{2\pi} X(\omega) * M(\omega) \tag{2.5.14}$$

where $Y(\omega) = \mathcal{F}\{y(t)\}$, $X(\omega) = \mathcal{F}\{x(t)\}$, and $M(\omega) = \mathcal{F}\{m(t)\}$.

On the other hand, if two functions $\tilde{x}_P(t)$ and $\tilde{m}_P(t)$ are periodic with common period P, then their multiplication is also periodic and its Fourier series coefficient can be obtained from the convolution sum (see Problem 2.8(b)):

$$\tilde{y}_P(t) = \tilde{x}_P(t)\tilde{m}_P(t) \overset{\text{Fourier series}}{\leftrightarrow} Y_k = \frac{1}{P} \sum\nolimits_{n=-\infty}^{\infty} X_n \, M_{k-n} \tag{2.5.15}$$

where Y_k, X_k, and M_k are the Fourier coefficients of $\tilde{y}_P(t)$, $\tilde{x}_P(t)$, and $\tilde{m}_P(t)$, respectively.

Example 2.12 Sinusoidal Amplitude Modulation and Demodulation

(a) Sinusoidal Amplitude Modulation
Consider a sinusoidal amplitude-modulated (AM) signal

$$x_m(t) = x(t)m(t) = x(t)\cos(\omega_c t) \tag{E2.12.1}$$

Noting that the Fourier transform of the carrier signal $m(t) = \cos(\omega_c\, t)$ is

$$M(\omega) = \mathcal{F}\{m(t)\} = \mathcal{F}\{\cos(\omega_c t)\} \overset{\text{(E2.11.2)}}{=} \pi(\delta(\omega + \omega_c) + \delta(\omega - \omega_c)) \tag{E2.12.2}$$

we can use the modulation property (2.5.14) to get the Fourier transform of the AM signal as

$$\begin{aligned} X_m(\omega) = \mathcal{F}\{x_m(t)\} = \mathcal{F}\{x(t)\cos\omega_c\, t\} &\overset{(2.5.14)}{=} \frac{1}{2\pi} X(\omega) * M(\omega) \\ &\overset{\text{(E2.12.2)}}{=} \frac{1}{2\pi} X(\omega) * \pi(\delta(\omega + \omega_c) + \delta(\omega - \omega_c)) \\ &\overset{(1.1.22)}{=} \frac{1}{2}(X(\omega + \omega_c) + X(\omega - \omega_c)) \end{aligned} \tag{E2.12.3}$$

This implies that the spectrum of the AM signal $x(t)\cos(\omega_c t)$ consists of the sum of two shifted and scaled versions of $X(\omega) = \mathcal{F}\{x(t)\}$. Note that this result can also be obtained by applying the frequency shifting property (2.5.7) to the following expression:

$$x_m(t) = x(t)\cos(\omega_c t) \overset{\text{(D.21)}}{=} \frac{1}{2}(x(t)e^{j\omega_c t} + x(t)\, e^{-j\omega_c t}) \tag{E2.12.4}$$

(b) Sinusoidal Amplitude Demodulation
In an AM communication system, the receiver demodulates the modulated signal $x_m(t)$ by multiplying the carrier signal as is done at the transmitter:

$$x_d(t) = 2x_m(t)\cos(\omega_c t) \tag{E2.12.5}$$

We can use the modulation property (2.5.14) together with Eqs. (E2.12.3) and (E2.12.2) to express the Fourier transform of the demodulated signal in terms of the signal spectrum $X(\omega)$ as

$$\begin{aligned} X_d(\omega) &= \mathcal{F}\{x_d(t)\} = \mathcal{F}\{x_m(t)2\cos\omega_c\, t\} \overset{(2.5.14)}{=} \frac{1}{2\pi} X_m(\omega) * 2\, M(\omega) \\ &\underset{(E2.12.3)}{\overset{(E2.12.2)}{=}} \frac{1}{2\pi}\frac{1}{2}(X(\omega+\omega_c) + X(\omega-\omega_c)) *\ 2\pi(\delta(\omega+\omega_c) + \delta(\omega-\omega_c)) \\ &\overset{(1.1.22)}{=} \frac{1}{2}(X(\omega+2\omega_c) + X(\omega) + X(\omega) + X(\omega-2\omega_c)) \\ &= \frac{1}{2}X(\omega+2\omega_c) + X(\omega) + \frac{1}{2}X(\omega-2\omega_c) \end{aligned} \tag{E2.12.6}$$

Example 2.13 Ideal (Impulse or Instant) Sampler and Finite Pulsewidth Sampler

(a) Ideal Sampling
We can describe the output of the *ideal* (or *impulse* or *instant*) *sampler* to a given input $x(t)$ as

$$x_*(t) = x(t)\,\delta_T(t) \left(\delta_T(t) = \sum\nolimits_{m=-\infty}^{\infty} \delta(t-mT) : \text{the impulse train}\right) \tag{E2.13.1}$$

This is illustrated in Fig. 2.16(a1). Here, the switching function has been modeled as an impulse train with period T and its Fourier transform is given by Eq. (E2.10.3) as

$$\mathcal{D}_T(\omega) \overset{(E2.10.3)}{=} \frac{2\pi}{T} \sum\nolimits_{k=-\infty}^{\infty} \delta(\omega + k\omega_s) \text{ with } \omega_s = \frac{2\pi}{T} \tag{E2.13.2}$$

which is shown in Fig. 2.16(c1). We can use the modulation property (2.5.14) together with Eq. (E2.13.2) to express the Fourier transform of the ideal sampler output in terms of the input spectrum $X(\omega)$ as

$$\begin{aligned} X_*(\omega) &\overset{(2.5.14)}{=} \frac{1}{2\pi} X(\omega) * \mathcal{D}_T(\omega) \overset{(E2.13.2)}{=} \frac{1}{T} \sum\nolimits_{k=-\infty}^{\infty} X(\omega) * \delta(\omega + k\omega_s) \\ &\overset{(1.1.22)}{=} \frac{1}{T} \sum\nolimits_{k=-\infty}^{\infty} X(\omega + k\omega_s) \end{aligned} \tag{E2.13.3}$$

which is depicted in Fig. 2.16(d1).

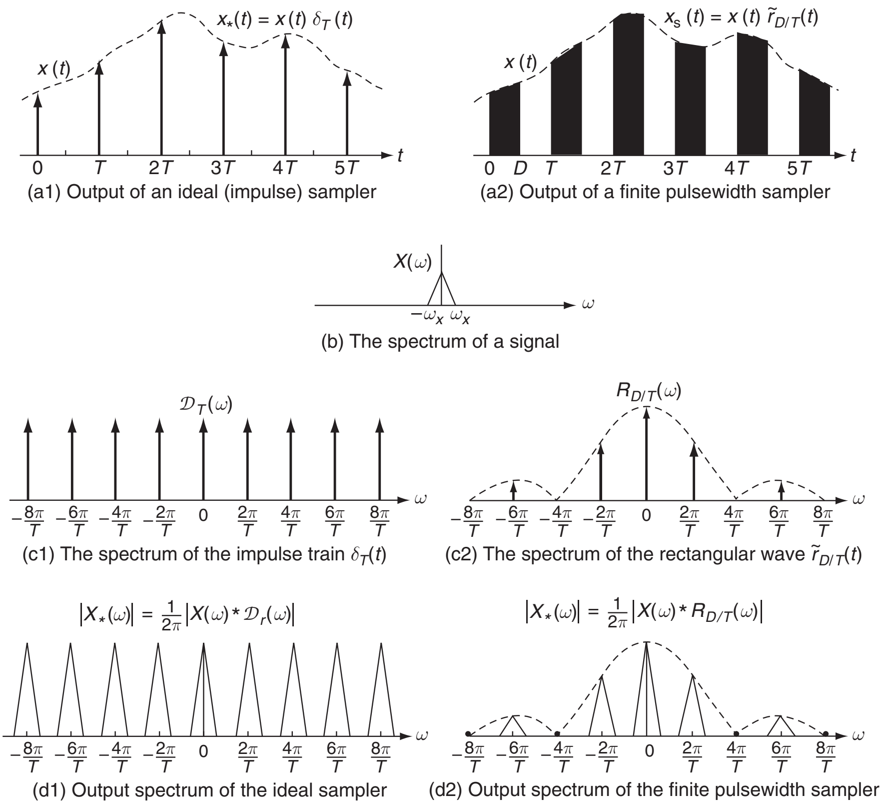


Fig. 2.16 Ideal sampler and finite pulsewidth sampler

(b) Finite Pulsewidth Sampling
We can describe the output of a *finite pulsewidth sampler* with period T to a given input $x(t)$ as

$$x_s(t) = x(t)\, \tilde{r}_{D/T}(t) \tag{E2.13.4}$$

$$(\tilde{r}_{D/T}(t) : \text{a rectangular wave with duration } D \text{ and period } T)$$

This is illustrated in Fig. 2.16(a2). Here, the switching function has been modeled as a rectangular wave, $\tilde{r}_{D/P}(t)$, with duration D and period T. Noting that from Eq. (E2.1.3), the Fourier series representation of this rectangular wave is

$$\tilde{r}_{D/P}(t) \underset{A=1,P=T}{\overset{(\text{E2.1.3})}{=}} \frac{1}{T} \sum\nolimits_{k=-\infty}^{\infty} D \,\text{sinc}\left(k\frac{D}{T}\right) e^{jk\omega_s t} \text{ with } \omega_s = \frac{2\pi}{T} \tag{E2.13.5}$$

we can use Eq. (2.3.3) to write its Fourier transform as

$$R_{D/P}(\omega) \underset{P=T}{\overset{(2.3.2)}{=}} \frac{2\pi}{T} \sum\nolimits_{k=-\infty}^{\infty} c_k \, \delta(\omega - k\omega_s) \text{ with } c_k = D \text{ sinc}\left(k\frac{D}{T}\right) \tag{E2.13.6}$$

which is shown in Fig. 2.16(c2). Now we can apply the modulation property (2.5.14) together with Eq. (E2.13.6) to express the Fourier transform of the finite pulsewidth sampler output in terms of the input spectrum $X(\omega)$ as

$$X_s(\omega) \overset{(2.5.14)}{=} \frac{1}{2\pi} X(\omega) * R_{D/P}(\omega) \overset{(E2.13.6)}{=} \frac{1}{T} \sum\nolimits_{k=-\infty}^{\infty} X(\omega) * \ c_k \, \delta(\omega - k\omega_s)$$
$$\overset{(1.1.22)}{=} \frac{1}{T} \sum\nolimits_{k=-\infty}^{\infty} c_k \ X(\omega - k\omega_s) \tag{E2.13.7}$$

which is depicted in Fig. 2.16(d2).

2.5.7 Time Differential/Integration – Frequency Multiplication/Division

By differentiating both sides of the inverse Fourier transform formula (2.2.1b) w.r.t. t, we obtain

$$\frac{dx(t)}{dt} \overset{(2.2.1b)}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left(\frac{d}{dt} e^{j\omega t} \right) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega X(\omega)) e^{j\omega t} d\omega \tag{2.5.16}$$

which yields the time-differentiation property of the Fourier transform as

$$\frac{dx(t)}{dt} \overset{F}{\longleftrightarrow} j\omega X(\omega) \tag{2.5.17}$$

This means that differentiation in the time domain results in multiplication by $j\omega$ in the frequency domain.

On the other hand, the time-integration property is obtained by expressing the integration of $x(t)$ in the form of the convolution of $x(t)$ and the unit step function $u_s(t)$ and then applying the convolution property (2.5.11) together with Eq. (E2.8.6) as

$$\int_{-\infty}^{t} x(\tau) \, d\tau = \int_{-\infty}^{\infty} x(\tau) \, u_s(t - \tau) \, d\tau = x(t) \ * \ u_s(t) \underset{(2.5.11)}{\overset{\mathcal{F}}{\leftrightarrow}}$$
$$\mathcal{F}\{x(t)\}\mathcal{F}\{u_s(t)\} \overset{(E2.8.6)}{=} \pi \ X(\omega)\delta(\omega) + \frac{1}{j\omega} X(\omega) = \pi \ X(0)\delta(\omega) + \frac{1}{j\omega} X(\omega) \tag{2.5.18}$$

where the additional impulse term $\pi X(0)\delta(\omega)$ on the RHS reflects the DC value resulting from the integration. This equation is slightly above our intuition that integration/differentiation in the time domain results in division/multiplication by $j\omega$ in the frequency domain.

The differentiation/integration properties (2.5.17)/(2.5.18) imply that differentiating/integrating a signal increases the high/low frequency components, respectively because the magnitude spectrum is multiplied by $|j\omega| = \omega$ (proportional to frequency ω) or $|1/j\omega| = 1/\omega$ (inversely proportional to frequency ω). That is why a differentiating filter on the image frame is used to highlight the edge at which the brightness changes rapidly, while an integrating filter is used to remove impulse-like noises. Note also that a differentiator type filter tends to amplify high-frequency noise components and an integrator type filter would blur the image.

2.5.8 Frequency Differentiation – Time Multiplication

By differentiating both sides of the Fourier transform formula (2.2.1a) w.r.t. ω, we obtain

$$\frac{dX(\omega)}{d\omega} \overset{(2.2.1a)}{=} \int_{-\infty}^{\infty} x(t)\left(\frac{d}{d\omega}e^{-j\omega t}\right)dt = -j\int_{-\infty}^{\infty}(tx(t))e^{-j\omega t}dt \tag{2.5.19}$$

which yields the frequency-differentiation property of the Fourier transform as

$$t\,x(t) \overset{\mathcal{F}}{\leftrightarrow} j\frac{dX(\omega)}{d\omega} \tag{2.5.20}$$

This means that multiplication by t in the time domain results in differentiation w.r.t. ω and multiplication by j in the frequency domain.

2.5.9 Time and Frequency Scaling

The Fourier transform of $x(at)$ scaled along the time axis can be obtained as

$$\begin{aligned} F\{x(at)\} &= \int_{-\infty}^{\infty} x(at)\,e^{-j\omega t}dt \\ &\overset{t=\tau/a}{=} \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau/a}d\tau \overset{(2.2.1a)}{=} \frac{1}{|a|}X\left(\frac{\omega}{a}\right) \end{aligned}$$

which yields the time and frequency scaling property of Fourier transform:

$$x(at) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{\omega}{a}\right) \tag{2.5.21}$$

This is another example of the dual (inverse) relationship between time and frequency. A common illustration of this property is the effect on frequency contents of playing back an audio tape at different speeds. If the playback speed is higher/slower than the recording speed, corresponding to compression($a>1$)/expansion($a<1$) in time, then the playback sounds get higher/lower, corresponding to expansion/compression in frequency.

2.5.10 Parseval's Relation (Rayleigh Theorem)

If $x(t)$ has finite energy and $\mathcal{F}\{x(t)\} = X(\omega)$, then we have

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega \tag{2.5.22}$$

(Proof)

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2\, dt &= \int_{-\infty}^{\infty} x(t)x^*(t)\, dt \\
&\overset{(2.2.1b)}{=} \int_{-\infty}^{\infty} x(t) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t}\, d\omega \right\} dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left\{ \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right\} d\omega \\
&\overset{(2.2.1a)}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) X(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega
\end{aligned}$$

This implies that the total energy in the signal $x(t)$ can be determined either by integrating $|x(t)|^2$ over all time or by integrating $|X(\omega)|^2/2\pi$ over all frequencies. For this reason, $|X(\omega)|^2$ is called the *energy-density spectrum* of the signal $x(t)$.

On the other hand, if $\tilde{x}_P(t)$ is periodic with period P and its Fourier series coefficients are X_k's, then we have an analogous relation

$$\int_{-\infty}^{\infty} |\tilde{x}_P(t)|^2\, dt = \frac{1}{P} \sum\nolimits_{k=-\infty}^{\infty} |X_k|^2 \tag{2.5.23}$$

where $|X_k|^2/P$ is called the *power-density spectrum* of the periodic signal $\tilde{x}_P(t)$.

2.6 Polar Representation and Graphical Plot of CTFT

Noting that a signal $x(t)$ can be completely recovered from its Fourier transform $X(\omega)$ via the inverse Fourier transform formula (2.3.1b), we may say that $X(\omega)$ contains all the information in $x(t)$. In this section we consider the polar or magnitude-phase representation of $X(\omega)$ to gain more insight to the (generally complex-valued) Fourier transform. We can write it as

$$X(\omega) = |X(\omega)| \angle X(\omega)$$

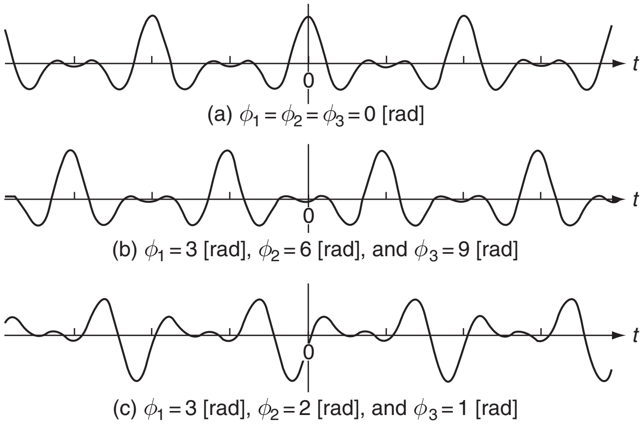


Fig. 2.17 Plots of $x(t) = 0.5\cos(2\pi t - \phi_1) + \cos(4\pi t - \phi_2) + (2/3)\cos(6\pi t - \phi_3)$ with different phases

where $|X(\omega)|$ and $\angle X(\omega)$ give us the information about the magnitudes and phases of the complex exponentials making up $x(t)$. Notice that if $x(t)$ is real, $|X(\omega)|$ is an even function of ω and $\angle X(\omega)$ is an odd function of ω and thus we need to plot the spectrum for $\omega \geq 0$ only (see Sect. 2.5.2).

The signals having the same magnitude spectrum may look very different depending on their phase spectra, which is illustrated in Fig. 2.17. Therefore, in some instances phase distortion may be serious.

2.6.1 Linear Phase

There is a particular type of phase characteristic, called *linear phase*, that the phase shift at frequency ω is a linear function of ω. Specifically, the Fourier transform of $x(t)$ changed in the phase by $-\alpha\omega$, by the time shifting property (2.5.6), simply results in a time-shifted version $x(t-\alpha)$:

$$X_1(\omega) = X(\omega)\angle - \alpha\omega = X(\omega)e^{-j\alpha\omega} \overset{\mathcal{F}}{\leftrightarrow} x_1(t) = x(t-\alpha) \tag{2.6.1}$$

For example, Fig. 2.17(b) illustrates how the linear phase shift affects the shape of $x(t)$.

2.6.2 Bode Plot

To show a Fourier transform $X(\omega)$, we often use a graphical representation consisting of the plots of the magnitude $|X(\omega)|$ and phase $\angle X(\omega)$ as functions of frequency ω. Although this is useful and will be used extensively in this book, we introduce another representation called the *Bode plot*, which is composed of two graphs, i.e., magnitude curve of log-magnitude $20 \log_{10} |X(\omega)|$ [dB] and the phase curve of $\angle X(\omega)$ [degree] plotted against the log frequency $\log_{10} \omega$. Such a representation using the logarithm facilitates the graphical manipulations performed in analyzing LTI systems since the product and division factors in $X(\omega)$ become additions and subtractions, respectively. For example, let us consider a physical system whose system or transfer function is

$$G(s) = \frac{K(1 + T_1 s)(1 + T_2 s)}{s(1 + T_a s)(1 + 2\zeta T_b s + (T_b s)^2)} \tag{2.6.2}$$

As explained in Sect. 1.2.6, its frequency response, that is the Fourier transform of the impulse response, is obtained by substituting $s = j\omega$ into $G(s)$:

$$G(j\omega) = G(s)|_{s=j\omega} = \frac{K(1 + j\omega\, T_1)(1 + j\omega\, T_2)}{j\omega(1 + j\omega\, T_a)(1 + j\omega 2\zeta T_b - (\omega T_b)^2)} \tag{2.6.3}$$

The magnitude of $G(j\omega)$ in decibels is obtained by taking the logarithm on the base 10 and then multiplying by 20 as follows:

$$\begin{aligned}
|G(j\omega)| &= 20 \log_{10} |G(j\omega)|[\text{dB}] \\
&= 20 \log_{10} |K| + 20 \log_{10} |1 + j\omega\, T_1| + 20 \log_{10} |1 + j\omega\, T_2| \\
&\quad - 20 \log_{10} |j\omega| - 20 \log_{10} |1 + j\omega\, T_a| \\
&\quad - 20 \log_{10} |1 + j\omega 2\zeta T_b - (\omega T_b)^2|
\end{aligned} \tag{2.6.4a}$$

The phase of $G(j\omega)$ can be written as

$$\begin{aligned}
\angle G(\omega) = \angle K + \angle(1 + j\omega\, T_1) + \angle(1 + j\omega\, T_2) - \angle j\omega - \angle(1 + j\omega\, T_a) \\
- \angle(1 + j\omega 2\zeta T_b - (\omega T_b)^2)
\end{aligned} \tag{2.6.4b}$$

The convenience of analyzing the effect of each factor on the frequency response explains why Bode plots are widely used in the analysis and design of linear time-invariant (LTI) systems.
(cf.) The MATLAB function "`bode(n,d,..)`" can be used to plot Bode plots.

2.7 Summary

In this chapter we have studied the CTFS (continuous-time Fourier series) and CTFT (continuous-time Fourier transform) as tools for analyzing the frequency characteristic of continuous-time signals and systems. One of the primary motivations for the use of Fourier analysis is the fact that we can use the Fourier series or transform to represent most signals in terms of complex exponential signals which are eigenfunctions of LTI systems (see Problem 2.14). The Fourier transform possesses a wide variety of important properties. For example, the convolution property allows us to describe an LTI system in terms of its frequency response and the modulation property provides the basis for the frequency-domain analysis of modulation, sampling, and windowing techniques.

Problems

2.1 Fourier Series Analysis and Synthesis of Several Periodic Functions

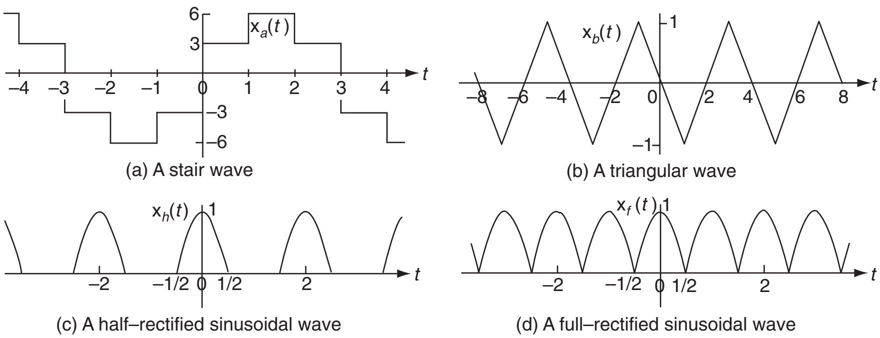


Fig. P2.1

(a) Noting that the stair wave in Fig. P2.1(a) can be regarded as the sum of three scaled/time-shifted rectangular (square) waves with common period 6[s] and different durations

$$x_a(t) = 6\tilde{r}_{3/6}(t - 1.5) - 3 + 3\tilde{r}_{1/6}(t - 1.5) - 3\tilde{r}_{1/6}(t - 4.5), \quad \text{(P2.1.1)}$$

use Eq. (E2.1.3) together with (2.1.9) to find its complex exponential Fourier series coefficients. Then complete the following MATLAB program "cir02p_01a.m" and run it to plot the Fourier series representation (2.1.7) to see if it becomes close to $x_a(t)$ as the number of terms in the summation increases, say, from 5 to 20. Also, compare the Fourier coefficients with those obtained by using "CtFS_exponential()".

```
%cir02p_01a.m
clear, clf
P= 6; w0= 2*pi/P; % Period, Fundamental frequency
tt=[-400:400]*P/400; % time interval of 4 periods
x = 'x_a'; N=20; k= -N:N; % the range of frequency indices
c= 6*3*sinc(k*3/P).*exp(-j*1.5*k*w0)+3*sinc(k/P).*exp(-j*1.5*k*w0) ...
    - ?????????????????????????????;
c(N+1) = c(N+1) - ???;
% [c_n,k] = CtFS_exponential(x,P,N);
xt = feval(x,tt); % original signal
jkw0t= j*k.'*w0*tt; xht = real((c/P)*exp(jkw0t)); % Eq. (2.1.5a)
subplot(221), plot(tt,xt,'k-', tt,xht,'r:')
c_mag = abs(c); c_phase = angle(c);
subplot(222), stem(k, c_mag)
function y=x_a(t)
P=6; t= mod(t,P);
y= 3*(0<=t&t<1) +6*(1<=t&t<2) +3*(2<=t&t<3) ...
    -3*(3<=t&t<4) -6*(4<=t&t<5) -3*(5<=t&t<6);
```

(b) Use Eqs. (E.2.1.6) together with (2.1.9) to find the complex exponential Fourier series coefficients for the triangular wave in Fig. P2.1(b). Then compose a MATLAB program, say, "cir02p01b.m" to plot the Fourier series representation (2.1.7) and run it to see if the Fourier series representation becomes close to $x_b(t)$ as the number of terms in the summation increases, say, from 3 to 6. Also, compare the Fourier coefficients with those obtained by using "CtFS_exponential()".

(c) Consider a half-wave rectified cosine wave $x_h(t) = \max(\cos\omega_0 t, 0)$, which is depicted in Fig. P2.1(c). We can use Eq. (2.1.5b) to obtain the Fourier coefficients as

$$
\begin{aligned}
c_k &= \int_{-P/4}^{P/4} \cos\omega_0 t\, e^{-jk\omega_0 t} dt \\
&= \frac{1}{2}\int_{-P/4}^{P/4} (e^{j\omega_0 t} + e^{-j\omega_0 t}) e^{-jk\omega_0 t}\, dt \;\text{ with }\; \omega_0 = \frac{2\pi}{P} = \pi \\
&\overset{k\neq 1\text{or}-1}{=} \frac{1}{2}\left(\frac{1}{-j(k-1)\omega_0} e^{-j(k-1)\omega_0 t} + \frac{1}{-j(k+1)\omega_0} e^{-j(k+1)\omega_0 t}\right)\Bigg|_{-1/2}^{1/2} \\
&= \frac{1}{(k-1)\pi}\sin(k-1)\frac{\pi}{2} + \frac{1}{(k+1)\pi}\sin(k+1)\frac{\pi}{2} \\
&\overset{k=2m(\text{even})}{=} (-1)^m \frac{1}{\pi}\left(\frac{1}{k+1} - \frac{1}{k-1}\right) = \frac{(-1)^{m+1}2}{(k^2-1)\pi} \qquad \text{(P2.1.2)} \\
c_k &\overset{k=1\text{or}-1}{=} \int_{-P/4}^{P/4} (e^{j\omega_0 t} + e^{-j\omega_0 t}) e^{\pm j\omega_0 t} dt \\
&= \frac{1}{2}\int_{-P/4}^{P/4} (1 + e^{\pm j2\omega_0 t}) dt = \frac{P}{4} = \frac{1}{2} \qquad \text{(P2.1.3)}
\end{aligned}
$$

Thus compose a MATLAB program, say, "cir02p01c.m" to plot the Fourier series representation (2.1.7) and run it to see if the Fourier series representation becomes close to $x_h(t)$ as the number of terms in the summation increases, say, from 3 to 6. Also, compare the Fourier coefficients with those obtained by using "CtFS_exponential()".

(d) Consider a full-wave rectified cosine wave $x_f(t) = |\cos\omega_0 t|$, which is depicted in Fig. P2.1(d). Noting that it can be regarded as the sum of a half-wave rectified cosine wave $x_h(t)$ and its $P/2$-shifted version $x_h(t - P/2)$, compose a MATLAB program, say, "cir02p01d.m" to plot the Fourier series representation (2.1.7) and run it to see if the Fourier series representation becomes close to $x_f(t)$ as the number of terms in the summation increases, say, from 3 to 6. Also, compare the Fourier coefficients with those obtained by using "CtFS_exponential()".

(cf.) In fact, the fundamental frequency of a full-wave rectified cosine wave is $2\omega_0 = 4\pi/P$, which is two times that of the original cosine wave.

2.2 Fourier Analysis of *RC* Circuit Excited by a Square Wave Voltage Source
Figure P2.2(a) shows the PSpice schematic of an *RC* circuit excited by a rectangular (square) wave voltage source of height $\pm V_m = \pm\pi$, period $P = 2$[s], and duration (pulsewidth) $D = 1$[s], where the voltage across the capacitor is taken as the output. Figure P2.2(b) shows the input and output waveforms obtained from the PSpice simulation. Figure P2.2(c) shows the Fourier spectra of the input and output obtained by clicking the FFT button on the toolbar in the PSpice A/D (Probe) window. Figure P2.2(d) shows how to fill in the Simulation Settings dialog box to get the Fourier analysis results (for chosen variables) printed in the output file. Figure P2.2(e) shows the output file that can be viewed by clicking View/Output_File on the top menu bar and pulldown menu in the Probe window.

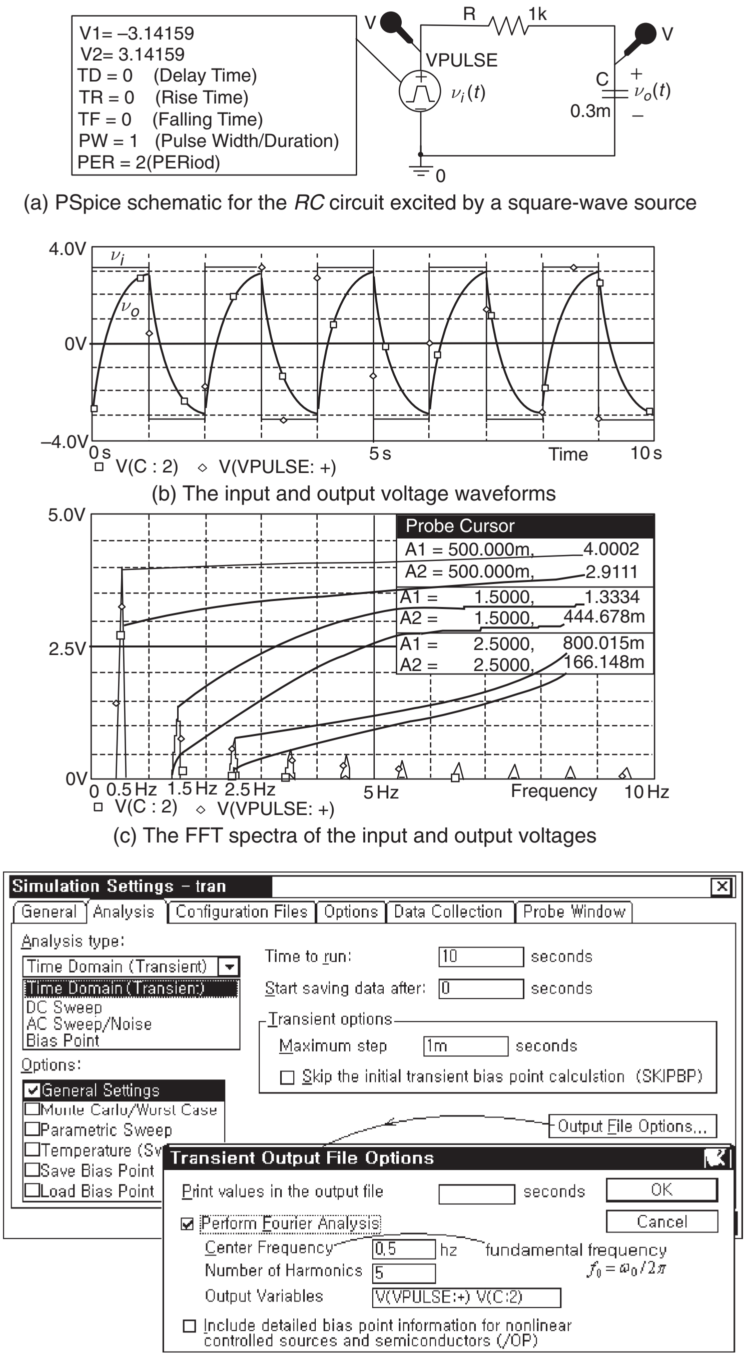
(a) Let us find the three leading frequency components of the input $v_i(t)$ and output $v_o(t)$. To this end, we first write the Fourier series representation of the rectangular wave input $v_i(t)$ by using Eq. (E2.1.3) as follows:

$$\begin{aligned} v_i(t) = 2\pi\ \tilde{r}_{1/2}(t-0.5) - \pi &\overset{\text{(E2.1.3)}}{=} \sum_{k=1}^{\infty} 2\pi \frac{\sin(k\pi/2)}{k\pi/2} \cos k\pi(t-0.5) \\ &= \sum_{k=\text{odd}}^{\infty} \frac{4}{k} \sin k\pi t = 4\sin\pi t + \frac{4}{3}\sin 3\pi t + \frac{4}{5}\sin 5\pi t + \cdots \end{aligned} \tag{P2.2.1}$$

Since the *RC* circuit has the system (transfer) function and frequency response

$$G(s) = \frac{1/sC}{R + 1/sC} = \frac{1}{1 + s\ RC};$$

$$G(j\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{\sqrt{1 + (\omega RC)^2}} \angle -\tan^{-1}\omega RC, \tag{P2.2.2}$$



(a) PSpice schematic for the *RC* circuit excited by a square-wave source

(b) The input and output voltage waveforms

(c) The FFT spectra of the input and output voltages

(d) Simulation Settings window and Transient Output File Options dialog box

Fig. P2.2 (continued)

```
** Profile: "SCHEMATIC1-tran" [C:\ORCAD\sig02p02-pspicefiles\tran.sim]
*Analysis directives:
.TRAN  0 10 0 1m
.FOUR 0.5 5 V([N00183]) V([N00070])

****     FOURIER ANALYSIS                    TEMPERATURE =   27.000 DEG C
FOURIER COMPONENTS OF TRANSIENT RESPONSE V(N00183)
 DC COMPONENT =   3.140322E-03(d'0)
 HARMONIC   FREQUENCY      FOURIER     NORMALIZED    PHASE       NORMALIZED
    NO(k)       (HZ)       COMPONENT    COMPONENT     (DEG)       PHASE (DEG)

     1     5.000E-01(f0)  4.000E+00(d'1)  1.000E+00  -1.800E-01(φ'1)  0.000E+00
     2     1.000E+00(2f0) 6.281E-03(d'2)  1.570E-03   8.964E+01(φ'2)  9.000E+01
     3     1.500E+00(3f0) 1.333E+00(d'3)  3.333E-01  -5.401E-01(φ'3) -7.176E-10
     4     2.000E+00(4f0) 6.281E-03(d'4)  1.570E-03   8.928E+01(φ'4)  9.000E+01
     5     2.500E+00(5f0) 8.000E-01(d'5)  2.000E-01  -9.002E-01(φ'5) -3.588E-09

     TOTAL HARMONIC DISTORTION =   3.887326E+01 PERCENT

FOURIER COMPONENTS OF TRANSIENT RESPONSE V(N00070)
 DC COMPONENT =   3.141583E-03(d'0)
 HARMONIC   FREQUENCY      FOURIER     NORMALIZED    PHASE       NORMALIZED
    NO(k)       (HZ)       COMPONENT    COMPONENT     (DEG)       PHASE (DEG)

     1     5.000E-01(f0)  2.911E+00(d'1)  1.000E+00  -4.348E+01(φ'1)  0.000E+00
     2     1.000E+00(2f0) 2.945E-03(d'2)  1.012E-03   2.759E+01(φ'2)  1.146E+02
     3     1.500E+00(3f0) 4.446E-01(d'3)  1.527E-01  -7.106E+01(φ'3)  5.939E+01
     4     2.000E+00(4f0) 1.611E-03(d'4)  5.534E-04   1.414E+01(φ'4)  1.881E+02
     5     2.500E+00(5f0) 1.661E-01(d'5)  5.705E-02  -7.892E+01(φ'5)  1.385E+02

     TOTAL HARMONIC DISTORTION =   1.630388E+01 PERCENT

          JOB CONCLUDED
```

(e) The PSpice output file with FFT analysis

Fig. P2.2 PSpice simulation and analysis for Problem 2.2

its output to the input $v_i(t)$ (described by Eq. (P2.2.1)) can be written as

$$v_o(t) = \sum_{k=2m+1}^{\infty} \frac{4}{k\sqrt{1+(k\omega_0 RC)^2}} \sin\left(k\,\omega_0 t - \tan^{-1} k\omega_0 RC\right)$$

$$\text{with } \omega_0 = \frac{2\pi}{P} = \pi \tag{P2.2.3}$$

Show that the three leading frequency components of the output phasor are

$$\mathbf{V}_o^{(1)} \overset{k=1}{=} G(j\omega_0)\mathbf{V}_i^{(1)} = \frac{4}{\sqrt{1+(\omega_0 RC)^2}} \angle -\tan^{-1}\omega_0 RC$$

$$\overset{RC=0.3}{=} \frac{4}{\sqrt{1+(0.3\pi)^2}} \angle -\tan^{-1} 0.3\pi = 2.91\angle -43.3^{\circ} \tag{P2.2.4}$$

$$\mathbf{V}_o^{(3)} \overset{k=3}{=} G(j3\omega_0)\mathbf{V}_i^{(3)} = \frac{4}{3\sqrt{1+(3\omega_0 RC)^2}} \angle -\tan^{-1} 3\omega_0 RC$$

$$\overset{RC=0.3}{=} \frac{4}{3\sqrt{1+(0.9\pi)^2}} \angle -\tan^{-1} 0.9\pi = 0.4446\angle -70.5^{\circ} \tag{P2.2.5}$$

$$\mathbf{V}_o^{(5)} \overset{k=5}{=} G(j5\omega_0)\mathbf{V}_i^{(5)} = \frac{4}{5\sqrt{1+(5\omega_0 RC)^2}}\angle -\tan^{-1} 5\omega_0 RC$$

$$\overset{RC=0.3}{=} \frac{4}{5\sqrt{1+(1.5\pi)^2}}\angle -\tan^{-1} 1.5\pi = 0.166\angle -78^{\circ} \quad \text{(P2.2.6)}$$

where the phases are taken with that of $\sin\omega t$ as a reference in the sine-and-phase form of Fourier series representation in order to match this Fourier analysis result with that seen in the PSpice simulation output file (Fig. P2.2(e)).

(cf.) Note that PSpice uses the sine-and-phase form of Fourier series representation for spectrum analysis as below:

$$x(t) = d'_0 + \sum\nolimits_{k=1}^{\infty} d'_k \sin(k\omega_0 t + \phi'_k) \quad \text{(P2.2.7a)}$$

where the Fourier coefficients are

$$d'_0 = a_0 = d_0, \quad d'_k = \sqrt{a_k^2 + b_k^2} = d_k, \quad \phi'_k = \tan^{-1}(a_k/b_k) = \phi_k + \pi/2 \quad \text{(P2.2.7b)}$$

The magnitude ratios among the leading three frequency components of the input and output are

$$\text{input}: \quad |\mathbf{V}_i^{(1)}| : |\mathbf{V}_i^{(3)}| : |\mathbf{V}_i^{(5)}| = 4 : \frac{4}{3} : \frac{5}{3} = 15 : 5 : 3$$

$$\text{output}: |\mathbf{V}_o^{(1)}| : |\mathbf{V}_o^{(3)}| : |\mathbf{V}_o^{(5)}| = 2.91 : 0.4446 : 0.166 = 15 : 2.3 : 0.86$$

This implies that the relative magnitudes of high frequency components to low ones become smaller after the input signal passes through the filter. This is a piece of evidence that the *RC* circuit with the capacitor voltage taken as the output functions as a lowpass filter.

(b) Find the Fourier analysis results on $\mathbf{V}_i^{(k)}$'s and $\mathbf{V}_o^{(k)}$'s for $k = 0 : 5$ (and compare them with those obtained from PSpice simulation in (a)) by completing and running the following MATLAB program "sig02p_02.m", which uses the MATLAB function "Fourier_analysis()" declared as

```
function [yt,Y,X]= Fourier_analysis(n,d,x,P,N)
```

This function takes the numerator (n) and denominator (d) of the transfer function $G(s)$, the (periodic) input function (x) defined for at least one period $[-P/2, +P/2]$, the period (P), and the order (N) of Fourier analysis as the input arguments and produces the output (yt) for one period and the sine-and-phase form of Fourier coefficients Y and X of the output and input (for $k = 0, \ldots, N$).

```
%sig02p_02.m : to solve Problem 2.2
% Perform Fourier analysis to solve the RC circuit excited by a square wave
clear, clf
global P D Vm
P=2; w0=2*pi/P; D=1; Vm=pi; % Period, Frequency, Duration, Amplitude
N=5; kk=0:N; % Frequency indices to analyze using Fourier analysis
tt=[-300:300]*P/200; % Time interval of 3 periods
vi= 'f_sig02p02'; % Bipolar square wave input function defined in an M-file
RC=0.3; % The parameter values of the RC circuit
n=1; d=[?? 1]; % Numerator/Denominator of transfer function (P2.2.2)
% Fourier analysis of the input & output spectra of a system [n/d]
[Vo,Vi]= ???????_analysis(n,d,vi,P,N);
% Magnitude_and_phase_of_input_output_spectrum
disp(' frequency X_magnitude X_phase Y_magnitude Y_phase')
[kk; abs(Vi); angle(Vi)*180/pi; abs(Vo); angle(Vo)*180/pi].'
vit = feval(vi,tt); % the input signal for tt
vot= Vo(1); % DC component of the output signal
for k=1:N % Fourier series representation of the output signal
   % PSpice dialect of Eq.(2.1.4a)
   vot = vot + abs(Vo(k+1))*sin(k*w0*tt + angle(Vo(k+1))); % Eq.(P2.2.7a)
end
subplot(221), plot(tt,vit, tt,vot,'r') % plot input/output signal waveform
subplot(222), stem(kk,abs(Vi)) % input spectrum
hold on, stem(kk,abs(Vo),'r') % output spectrum
```

```
function y=f_sig02p02(t)
% defines a bipolar square wave with period P, duration D, and amplitude Vm
global P D Vm
t= mod(t,P); y= ((t<=D) - (t>D))*Vm;
```

```
function [Y,X]= Fourier_analysis(n,d,x,P,N)
%Input:  n= Numerator polynomial of system function G(s)
%        d= Denominator polynomial of system function G(s)
%        x= Input periodic function
%        P= Period of the input function
%        N= highest frequency index of Fourier analysis
%Output: Y= Fourier coefficients [Y0,Y1,Y2,...] of the output
%        X= Fourier coefficients [X0,X1,X2,...] of the input
% Copyleft: Won Y. Yang, wyyang53@hanmail.net, CAU for academic use only
if nargin<5, N=5; end
w0=2*pi/P; % Fundamental frequency
kk=0:N; % Frequency index vector
c= CtFS_??????????(x,P,N); % complex exponential Fourier coefficients
Xmag=[c(N+1) 2*abs(c(N+2:end))]/P; % d(k) for k=0:N by Eq.(P2.2.7b)
Xph=[0 angle(c(N+2:end))+pi/2]; % phi'(k) for k=0:N by Eq.(P2.2.7b)
X= Xmag.*exp(j*Xph); % Input spectrum
Gw= freqs(n,d,kk*w0); % Frequency response
Y= X.*Gw; % Output spectrum
```

2.3 CTFT (Continuous-Time Fourier Transform) of a Periodic Function

In Eq. (2.3.1), we gave up computing the Fourier transform of a periodic function $e^{jk\omega_0 t}$:

$$\mathcal{F}\{x_k(t)\} \overset{(2.2.1a)}{=} \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-j(\omega - k\omega_0)t} dt$$

$$= \frac{1}{-j(\omega - k\omega_0)} e^{-j(\omega-k\omega_0)t}\Big|_{-T}^{T} = \frac{1}{-j(\omega - k\omega_0)} \left(e^{j(\omega-k\omega_0)T} - e^{-j(\omega-k\omega_0)T} \right)\Big|_{T=\infty}$$

$$= \left(\frac{2\sin(\omega - k\omega_0)T}{-(\omega - k\omega_0)} \right)\Big|_{T=\infty} = ? \tag{P2.3.1}$$

Instead, we took a roundabout way of finding the inverse Fourier transform of $2\pi\delta(\omega - k\omega_0)$ in expectation of that it will be the solution and showed the validity of the inverse relationship. Now, referring to Eq. (E2.6.2) or using Eq. (1.1.33a), retry to finish the derivation of Eq. (2.3.2) from Eq. (P2.3.1).

2.4 Two-Dimensional Fourier Transform of a Plane Impulse Train
The two-dimensional (2-D) Fourier transform of a 2-D signal $f(x, y)$ on the x-y plane such as an image frame is defined as

$$F(u, v) = \mathcal{F}_2\{f(x, y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-j(ux+vy)}dx\, dy$$
$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x, y)e^{-jux}dx\right)e^{-jvy}\, dy \qquad \text{(P2.4.1)}$$

where x and y are the spatial cordinates and u and v are the spatial angular frequencies [rad/m] representing how abruptly or smoothly $f(x, y)$ changes w.r.t. the spatial shift along the x and y -axes, respectively.

(a) Show that the 2-D Fourier transform of $f(x, y) = \delta(y - \alpha x)$ is

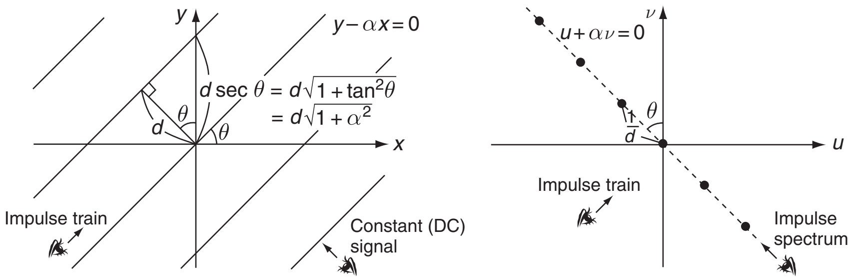
$$f(x, y) = \delta(y - \alpha x) \quad \overset{\mathcal{F}_2}{\leftrightarrow} \quad F(u, v) = 2\pi\ \delta(u + \alpha v) \qquad \text{(P2.4.2)}$$

(b) Show that the 2-D Fourier transform of $f(x, y) = \sum_{n=-\infty}^{\infty}\delta(y - \alpha x - nd\sqrt{1+\alpha^2})$ (Fig. P2.4(a)) is

$$F(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(y - \alpha x - nd\sqrt{1+\alpha^2})e^{-j(ux+vy)}dx\, dy$$
$$\overset{\text{(D.33)}}{=} \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j(ux+v(\alpha x+nd\sqrt{1+\alpha^2}))}\, dx$$
$$= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j(u+\alpha v)x}\, dx e^{-jvnd\sqrt{1+\alpha^2}}$$
$$\underset{\text{with } \omega=u+\alpha v,\ t=x,\ k=0}{\overset{\text{(2.3.2)}}{=}} \sum_{n=-\infty}^{\infty} 2\pi\delta(u + \alpha v)e^{-jvnd\sqrt{1+\alpha^2}}$$
$$\underset{\text{with } \Omega=vd\sqrt{1+\alpha^2},\Omega_0=0}{\overset{\text{(3.1.5)}}{=}} 2\pi\delta(u + \alpha v)\sum_{i=-\infty}^{\infty} 2\pi\delta(vd\sqrt{1+\alpha^2} - 2\pi\ i)$$
$$\overset{\text{(1.1.34)}}{=} \frac{(2\pi)^2}{d\sqrt{1+\alpha^2}}\sum_{i=-\infty}^{\infty}\delta(u + \frac{2\pi i\alpha}{d\sqrt{1+\alpha^2}},\ v - \frac{2\pi i}{d\sqrt{1+\alpha^2}})$$
$$= \frac{(2\pi)^2}{d\sqrt{1+\alpha^2}}\sum_{i=-\infty}^{\infty}\delta(u + \frac{2\pi i}{d}\sin\theta,\ v - \frac{2\pi i}{d}\cos\theta) \qquad \text{(P2.4.3)}$$

as depicted in Fig. P2.4(b).

2.5 ICTFT (Inverse Continuous-Time Fourier Transform) of $U_o(j\omega) = 1/j\omega$
Using Eq. (E2.3.3), show that the ICTFT of $U_o(j\omega) = 1/j\omega$ is the odd component of the unit step function:



(a) A 2–D (plane) impulse wave in the spatial domain (b) The 2–D Fourier transform depicted in the spatial frequency domain

Fig. P2.4 An example of 2-D Fourier transform

$$\mathcal{F}^{-1}\{U_o(j\omega)\} \overset{(2.2.1b)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{j\omega}e^{j\omega t}\,d\omega = \frac{1}{2}\mathrm{sign}(t) = u_o(t) \tag{P2.5.1}$$

which agrees with Eq. (E2.8.5).

2.6 Applications and Derivations of Several Fourier Transform Properties

(a) Applying the frequency shifting property (2.5.7) of Fourier transform to Eq. (E2.7.1), derive Eq. (2.3.2).

(b) Applying the duality relation (2.5.9) to the time shifting property (2.5.6) of Fourier transform, derive the frequency shifting property (2.5.7).

(c) From the time-differentiation property (2.5.7) of Fourier tansform that differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain, one might conjecture that integration in the time domain corresponds to division by $j\omega$ in the frequency domain. But, Eq. (2.5.18) shows that the Fourier transform of $\int_{-\infty}^{t} x(\tau)d\tau$ has an additional impulse term reflecting the DC or average value of $x(t)$. Can you apply the time-differentiation property to Eq. (2.5.18) to derive the original Fourier relation $\mathcal{F}\{x(t)\} = X(\omega)$?

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\leftrightarrow} \pi X(0)\delta(\omega) + \frac{1}{j\omega}X(\omega) \tag{2.5.18}$$

(d) Applying the duality relation (2.5.9) to the convolution property (2.5.11) of Fourier transform, derive the modulation property (2.5.14).

(e) Apply the time-differentiation property to the time-integration property (2.5.18) to derive the original Fourier relation $\mathcal{F}\{x(\tau)\} = X(\omega)$.

(f) Applying the time-differentiation property (2.5.7) to the time-domain input-output relationship $y(t) = dx(t)/dt$ of a differentiator, find the frequency response of the differentiator.

(g) Applying the time-integration property (2.5.18) to the time-domain input-output relationship $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ of an integrator, find the frequency response of the integrator on the assumption that the input has zero DC value.

(h) Find the time-domain and frequency-domain input-output relationships of a system whose impulse response is $g(t) = \delta(t - t_0)$.

2.7 Fourier Transform of a Triangular Pulse
Applying the convolution property (2.5.11), find the Fourier transform of a triangular pulse $D\lambda_D(t)$ (Eq. (1.2.25a)) obtained from the convolution of two rectangular pulses of height 1 and duration D.

2.8 Convolution/Modulation Properties of Fourier Series
Let $\tilde{x}_P(t)$ and $\tilde{y}_P(t)$ be periodic functions with common period P that have the following Fourier series representations, respectively:

$$\tilde{x}_P(t) = \frac{1}{P}\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \quad \text{and} \quad \tilde{y}_P(t) = \frac{1}{P}\sum_{k=-\infty}^{\infty} Y_k e^{jk\omega_0 t} \tag{P2.8.1}$$

(a) The periodic or circular convolution of two periodic functions is defined as

$$\begin{aligned}\tilde{z}_P(t) = \tilde{x}_P(t) * \tilde{y}_P(t) &\overset{(2.5.12)}{=} \int_P \tilde{x}_P(\tau)\tilde{y}_P(t-\tau)d\tau \\ &= \int_P \tilde{y}_P(\tau)\tilde{x}_P(t-\tau)d\tau = \tilde{y}_P(t) * \tilde{x}_P(t)\end{aligned} \tag{P2.8.2}$$

The product of the two Fourier series coefficients X_k and Y_k is the Fourier series coefficients of this convolution so that

$$\tilde{z}_P(t) = \frac{1}{P}\sum_{k=-\infty}^{\infty} Z_k e^{jk\omega_0 t} = \frac{1}{P}\sum_{k=-\infty}^{\infty} X_k Y_k e^{jk\omega_0 t} \tag{P2.8.3}$$

Applying the frequency shifting property (2.5.7) of Fourier transform to Eq. (E2.7.1), derive Eq. (2.3.2).

(b) Show that the Fourier series coefficients of $\tilde{w}_P(t) = \tilde{x}_P(t)\tilde{y}_P(t)$ are

$$\tilde{w}_P(t) = \tilde{x}_P(t)\tilde{y}_P(t) \underset{(2.5.15)}{\overset{\text{Fourier series}}{\leftrightarrow}} W_k = \frac{1}{P}\sum_{n=-\infty}^{\infty} X_n\, Y_{k-n} \tag{P2.8.4}$$

so that we have

$$\tilde{w}_P(t) = \frac{1}{P}\sum_{k=-\infty}^{\infty} W_k e^{jk\omega_0 t} \tag{P2.8.5}$$

2.9 Overall Input-Output Relationship of Cascaded System
Consider the systems of Fig. P2.9 consisting of two subsystems connected in cascade, i.e., in such a way that the output of the previous stage is applied to the input of the next stage. Note that the overall output $y(t)$ can be expressed as the convolution of the input $x(t)$ and the two subsystem's impulse responses

$g_1(t)$ and $g_2(t)$ as

$$y(t) = g_1(t) * g_2(t) * x(t) = g_2(t) * g_1(t) * x(t). \qquad \text{(P2.9.1)}$$

Find the overall input-output relationship in the frequency domain.

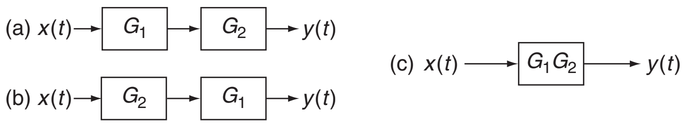


Fig. P2.9

2.10 Hilbert Transform and Analytic Signal

(a) Fourier Transform of Hilbert Transformed Signal
The *Hilbert transform* of a real-valued signal $x(t)$ is defined as the convolution of $x(t)$ and $1/\pi t$ or equivalently, the output of a *Hilbert transformer* (with the impulse response of $h(t) = 1/\pi t$) to the input $x(t)$:

$$\hat{x}(t) = h(t) * x(t) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} x(\tau)\, \frac{1}{t-\tau}\, d\tau \qquad \text{(P2.10.1)}$$

- First, let us find the Fourier transform of $h(t) = 1/\pi t$ by applying the duality (2.5.9) to Eq. (E2.8.5), which states that

$$\frac{1}{2}\text{sign}(t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{j\omega} \overset{\text{duality}}{\Leftrightarrow} \frac{1}{jt} \overset{\mathcal{F}}{\leftrightarrow} (\qquad\qquad\qquad) \qquad \text{(P2.10.2)}$$

- We can multiply both sides by j/π to get the Fourier transform of $h(t) = 1/\pi t$ as

$$\frac{1}{\pi t} \overset{\mathcal{F}}{\leftrightarrow} (\qquad\qquad\qquad) \qquad \text{(P2.10.3)}$$

- Now, we apply the convolution property to get the Fourier transform of $\hat{x}(t)$ as

$$\hat{x}(t) = \frac{1}{\pi t} * x(t) \underset{\text{convolution property}}{\overset{\mathcal{F}}{\leftrightarrow}}$$

$$\mathcal{F}\left(\frac{1}{\pi t}\right) \mathcal{F}(x(t)) = (\qquad\qquad\qquad)\, X(\omega) \qquad \text{(P2.10.4)}$$

This implies that the Hilbert transform has the effect of shifting the phase of positive/negative frequency components of $x(t)$ by $-90^\circ/+90^\circ$, allowing the Hilbert transform a -90° *phase shifter*.

(b) Inverse Hilbert Transform
Note that multiplying $-H(\omega)$ by the Fourier transform of the Hilbert transformed signal $\hat{x}(t)$ brings the spectrum back into that of the original signal $x(t)$:

$$-H(\omega)H(\omega)\,X(\omega) = -(-j\ \mathrm{sign}(\omega))(-j\ \mathrm{sign}(\omega))\,X(\omega) = X(\omega) \tag{P2.10.5}$$

This implies the *inverse Hilbert transform* as

$$(-h(t)) * \hat{x}(t) = \left(-\frac{1}{\pi t}\right) * \frac{1}{\pi t} * x(t) = x(t) \tag{P2.10.6}$$

(c) Analytic Signal
The *analytic signal*, or analytic representation, of a real-valued signal $x(t)$ is defined as

$$x_a(t) = x(t) + j\hat{x}(t) \tag{P2.10.7}$$

where $\hat{x}(t)$ is the Hilbert transform of $x(t)$ and j is the imaginary unit. Show that the Fourier transform of the analytic signal is

$$X_a(\omega) = \mathcal{F}\{x(t)\} + j\mathcal{F}\{\hat{x}(t)\} \cong 2u_s(\omega)X(\omega) \tag{P2.10.8}$$

where $u_s(\omega)$ is the unit step function in frequency ω. This implies that $x_a(t)$ has only nonnegative frequency component of $x(t)$.

(d) Examples of Hilbert Transform and Analytic Signal
Show that the following Hilbert transform relations hold:

$$\cos(\omega_c\, t) \overset{\mathcal{H}}{\leftrightarrow} \sin(\omega_c\, t) \tag{P2.10.9}$$

$$\sin(\omega_c t) \overset{\mathcal{H}}{\leftrightarrow} -\cos(\omega_c t) \tag{P2.10.10}$$

In general, it would be difficult to get the Hilbert transform of a signal $x(t)$ directly from its definition (P2.10.1) and therefore, you had better take the inverse Fourier transform, $X(\omega)$, of $x(t)$, multiply $-j\ \mathrm{sign}(\omega)$, and then take the inverse Fourier transform as

$$\hat{x}(t) = \mathcal{F}^{-1}\{-j\ \mathrm{sign}(\omega)X(\omega)\} \tag{P2.10.11}$$

Now, let us consider a narrowband signal

$$s(t) = m(t)\cos(\omega_c t + \phi) \text{ with } m(t) : \text{a baseband message signal} \tag{P2.10.12}$$

whose spectrum is (narrowly) bandlimited around the carrier frequency ω_c. Show that the Hilbert transform of $s(t)$ is

$$s(t) = m(t)\cos(\omega_c t + \phi) \overset{\mathcal{H}}{\leftrightarrow} \hat{s}(t) = m(t)\sin(\omega_c t + \phi) \qquad \text{(P2.10.13)}$$

Also, show that the analytic signal for $s(t)$ can be expressed as

$$s_a(t) = s(t) + j\hat{s}(t) = m(t)e^{j(\omega_c t+\phi)} \qquad \text{(P2.10.14)}$$

This implies that we can multiply the analytic signal by $e^{-j\omega_c t}$ to obtain the envelope of the baseband message signal $m(t)$.

(e) Programs for Hilbert Transform

One may wonder how the convolution in the time domain results in just the phase shift. To satisfy your curiosity, try to understand the following program "sig02p_10e.m", which computes the output of a Hilbert transformer to a cosine wave $\cos\omega_c t$ with $\omega_c = \pi/16$ and sampling interval $T_s = 1/16$ [s]. For comparison, it uses the MATLAB built-in function "`hilbert( )`" to generate the analytic signal and take its imaginary part to get $\hat{x}(t)$.

(cf.) "`hilbert( )`" makes the Hilbert transform by using Eq. (P2.10.11).

```
%sig02p_10e.m
% To try the Hilbert transform in the time/frequency domain
clear, clf
Ts=1/16; t=[-149.5:149.5]*Ts; % Sample interval and Duration of h(t)
h= 1/pi./t; % h(t): Impulse response of a Hilbert transformer
Nh=length(t); Nh2=floor(Nh/2); % Sample duration of noncausal part
h_= fliplr(h); % h(-t): Time-reversed version of h(t)
wc= pi/16; % the frequency of an input signal
Nfft=512; Nfft2= Nfft/2; Nbuf=Nfft*2; % buffer size
tt= zeros(1,Nbuf); x_buf= zeros(1,Nbuf); xh_buf= zeros(1,Nbuf);
for n=-Nh:Nbuf-1 % To simulate the Hilbert transformer
  tn=n*Ts; tt=[tt(2:end) tn];
  x_buf = [x_buf(2:end) cos(wc*tn)];
  xh_buf = [xh_buf(2:end) h_*x_buf(end-Nh+1:end).'*Ts];
end
axis_limits= [tt([1 end]) -1.2 1.2];
subplot(321), plot(tt,x_buf), title('x(t)'), axis(axis_limits)
subplot(323), plot(t(1:Nh2),h(1:Nh2), t(Nh2+1:end),h(Nh2+1:end))
title('h(t)'), axis([t([1 end]) -11 11])
% To advance the delayed response of the causal Hilbert transformer
xh = xh_buf(Nh2+1:end); xh_1 = imag(hilbert(x_buf));
subplot(325), plot(tt(1:end-Nh2),xh,'k', tt,xh_1), axis(axis_limits)
subplot(326), plot(tt,xh_buf), axis(axis_limits)
ww= [-Nfft2:Nfft2]*(2*pi/Nfft);
Xw= fftshift(fft(x_buf,Nfft)); Xw= [Xw Xw(1)]; % X(w): spectrum of x(t)
Xhw_1= fftshift(fft(xh_1,Nfft)); Xhw_1= [Xhw_1 Xhw_1(1)]; % Xh(w)
norm(Xhw_1+j*sign(ww).*Xw)
```

In the program "sig02p_10e.m", identify the statements performing the following operations:

- The impulse response of the Hilbert transformer
- Generating an input signal to the Hilbert transformer

- Computing the output expected to be the Hilbert transformed signal

  Note the following:
- The output sequence has been advanced by Nh/2 for comparison with $\hat{x}(t)$ obtained using "`hilbert ( )`" because the real output of the causal Hilbert transformer is delayed by Nh/2, which is the length of the noncausal part of the Hilbert transformer impulse response h(n).

(f) Application of Hilbert Transform and Analytic Signal for Envelope Detection
To see an application of Hilbert transform and analytic signal, consider the following narrowband signal

$$s(t) = m(t)\cos(\omega_c t + \phi)$$
$$= \operatorname{sinc}\left(\frac{B}{\pi}t\right)\cos(\omega_c t + \frac{\pi}{6}) \text{ with } B = 100, \omega_c = 400\pi \quad \text{(P2.10.15)}$$

```
%sig02p_10f.m
% To obtain the lowpass equivalent of Bandpass signal
Ts =0.001; fs =1/Ts; % Sampling Period/Frequency
t=[-511: 512]*Ts; % Duration of signal
fc=200; wc=2*pi*fc; B=100; % Center frequency & bandwidth of signal s(t)
m= sinc(B/pi*t);
s= m.*cos(wc*t+pi/6); % a narrowband (bandpass) signal
sa= hilbert(x);  % Analytic signal sa(t) = s(t) + j s^(t)
sl= sa.*exp(-j*wc*t);  % Lowpass Equivalent (Complex envelope) sl(t)
```

Referring to the above program "sig02p_10f.m", use the MATLAB function "`hilbert ( )`" to make the *analytic* or *pre-envelope* signal $s_a(t)$, multiply it by $e^{-j\omega_c t}$ to get the *lowpass equivalent* or *complex envelope* $s_l(t)$, take its absolute values $|s_l(t)|$ to obtain the envelope of the baseband message signal $m(t)$, and plot it to compare with $m(t) = |\operatorname{sinc}(2Bt)|$.

(g) Hilbert Transform with Real-Valued Signals for Signal Detection
Consider the system of Fig. P2.10 where a narrowband signal $x(t)$ having *in-phase* component $x_c(t)$ and *quadrature* component $x_s(t)$ is applied as an input to the system:

$$x(t) = x_c(t)\cos\omega_c t - x_s(t)\sin\omega_c t \quad \text{(P2.10.16)}$$

Verify that the two outputs of the system are the same as $x_c(t)$ and $x_s(t)$, respectively by showing that their spectra are

$$X_{c\_d}(\omega) = X_c(\omega) \text{ and } X_{s\_d}(\omega) = X_s(\omega) \quad \text{(P2.10.17)}$$

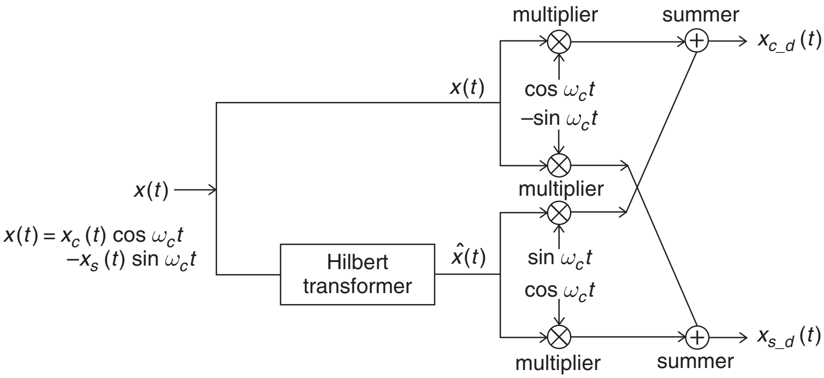


Fig. P2.10

Complete and run the following program "sig02p_10g.m" to check if these results are valid for

$$x_c(t) = \text{sinc}\,(100t)\,,\;\; x_s(t) = \text{sinc}\,(450t)\,,\;\text{and}\; \omega_c = 400\pi$$

Does the simulation result show that $x_{c\_d}(t) = x_c(t)$ and $x_{s\_d}(t) = x_s(t)$? If one of them does not hold, figure out the reason of the discrepancy.

```
%sig02p_10g.m
% To use the Hilbert transform for signal detection
clear, clf
Ts=0.0002; t=[-127:128]*Ts; % Sampling Period and Time interval
fc=200; wc=2*pi*fc; % Center frequency of signal x(t)
B1=100*pi; B2=450*pi; % Bandwidth of signal x(t)
xc= sinc(B1/pi*t); xs= sinc(B2/pi*t);
x= xc.*cos(wc*t) - xs.*sin(wc*t);
xh= imag(hilbert(x));  % imag(x(t)+j x^(t))=x^(t)
xc_d= x.*cos(wc*t) + xh.*?????????; % xc_detected
xs_d= -x.*sin(wc*t) + xh.*?????????; % xc_detected
subplot(221), plot(t,xc, t,xc_d,'r')
subplot(222), plot(t,xs, t,xs_d,'r')
norm(xc-xc_d)/norm(xc), norm(xs-xs_d)/norm(xs)
```

2.11 Spectrum Analysis of Amplitude Modulation (AM) Communication System
Consider the AM communication system of Fig. P2.11(a) where the message signal $x(t)$ is band-limited with maximum frequency ω_x and the spectrum $X(\omega)$ is depicted in Fig. P2.11(b). Assume that the frequency response of the (bandpass) channel is

$$H(\omega) = \begin{cases} 1 & \text{for } |\omega - \omega_c| < B_C/2 \\ 0 & \text{elsewhere} \end{cases} \text{with } B_C \geq 2\omega_x \qquad \text{(P2.11.1)}$$

and that of the LPF (lowpass filter) at the receiver side is

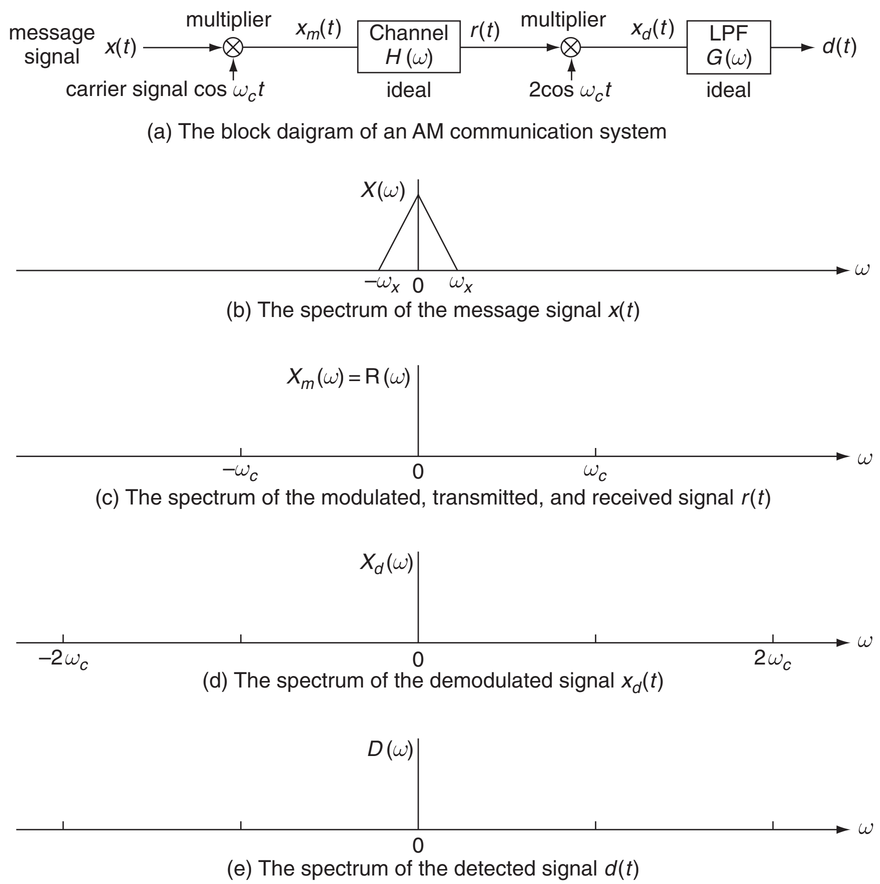


Fig. P2.11 An amplitude–modulation (AM) communication system

$$G(\omega) = \begin{cases} 1 & \text{for } |\omega| < B_R \\ 0 & \text{elsewhere} \end{cases} \text{ with } \omega_x < B_R < 2\omega_c - \omega_x \tag{P2.11.2}$$

(a) Express the spectrum, $X_d(\omega)$, of the demodulated signal $x_d(t)$ in terms of $X(\omega)$.

(b) Draw the spectra of the modulated and received signal $r(t)$, the demodulated signal $x_d(t)$, and the detected signal $d(t)$ on Fig. P2.11(c), (d), and (e).

(c) Complete and run the following program "sig02p_11.m" to simulate the AM communication system depicted in Fig. P2.11(a). Note that the spectrum of each signal is computed by using the MATLAB built-in function "`fft()`" instead of CTFT (Continuous-Time Fourier Transform) since DFT (Discrete Fourier Transform) is applied to any sequence (consisting of sampled values of an arbitrary continuous-time signal), while CTFT can be applied only to continuous-time signals expressed by a linear combination of basic functions. DFT and "`fft()`" will be introduced in Chap. 4.

```
%sig02p_11.m
% AM (amplitude-modulation) communication system
clear, clf
Ts=0.0001; t=[-127:128]*Ts; % Sampling Period and Time interval
fc=1000; wc=2*pi*fc; % Carrier frequency
B=1000; % (Signal) Bandwidth of x(t)
x= sinc(B/pi*t).^2; % a message signal x(t) having triangular spectrum
xm= x.*cos(wc*t);  % AM modulated signal
r= xm;  % received signal
xd= r.*(2*?????????); % demodulated signal
Br=wc/2; % (System) Bandwidth of an ideal LPF
g= Br/pi*sinc(Br/pi*t); % (truncated) Impulse Response of the ideal LPF
d= conv(xd,g)*Ts; % the output of the LPF to the demodulated signal
subplot(421), plot(t,x), title('A message signal x(t)')
Nfft=256; Nfft2=Nff/2; % FFT size and a half of Nfft
w= [-Nfft/2:Nfft/2]*(2*pi/Nfft/Ts); % Frequency vector
X= fftshift(fft(x)); X=[X X(1)]; % Spectrum of x(t)
subplot(422), plot(w,abs(X)), title('The spectrum X(w) of x(t)')
subplot(423), plot(t,xm), title('The modulated signal xm(t)')
Xm= fftshift(fft(xm)); Xm=[Xm Xm(1)]; % Spectrum of modulated signal
subplot(424), plot(w,abs(Xm)), title('The spectrum Xm(w) of xm(t)')
subplot(425), plot(t,xd), title('The demodulated signal xd(t)')
Xd= fftshift(fft(xd)); Xd=[Xd Xd(1)]; % Spectrum of demodulated signal
subplot(426), plot(w,abs(Xd)), title('The spectrum Xd(w) of xd(t)')
d= d(Nfft2+[0:Nfft-1]);
subplot(427), plot(t,d), title('The LPF output signal d(t)')
D= fftshift(fft(d)); D=[D D(1)]; % Spectrum of the LPF output
subplot(428), plot(w,abs(D)), title('The spectrum D(w) of d(t)')
```

2.12 Amplitude-Shift Keying (ASK) and Frequency-Shift Keying (FSK)

(a) Amplitude-Shift Keying (ASK) or On-Off Keying (OOK)
Find the spectrum $X_{ASK}(\omega)$ of a BASK (binary amplitude-shift keying) or OOK modulated signal $x_{ASK}(t) = r_D(t)\cos(\omega_1 t)$ (Fig. P2.12(a)) where $r_D(t)$ is a unit-height rectangular pulse with duration D.

(b) Frequency-Shift Keying (FSK)
Find the spectrum $X_{FSK}(\omega)$ of a BFSK (binary frequency-shift keying) modulated signal $x_{FSK}(t) = r_D(t)\cos(\omega_1 t) + r_D(t-D)\cos(\omega_2(t-D))$ (Fig. P2.12(b)).

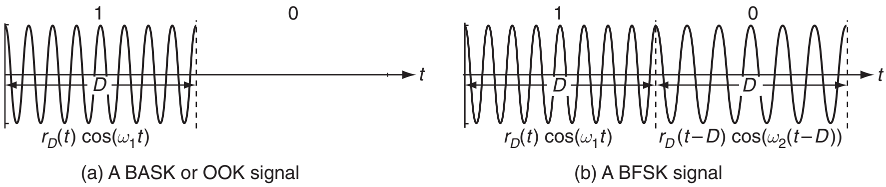


Fig. P2.12

2.13 Autocorrelation/Crosscorrelation Theorem
Suppose we have $X(\omega) = \mathcal{F}\{x(t)\}$, $Y(\omega) = \mathcal{F}\{y(t)\}$, and $G(\omega) = \mathcal{F}\{g(t)\}$, where $x(t)$, $y(t)$, and $g(t)$ are the real-valued input, output, and impulse response of an LTI system G so that

$$y(t) = g(t) * x(t) \overset{\mathcal{F}}{\leftrightarrow} Y(\omega) = G(\omega) * X(\omega) \tag{P2.13.1}$$

(a) Prove the autocorrelation theorem:

$$\Phi_{xx}(\omega) = \mathcal{F}\{\phi_{xx}(t)\} \overset{(1.4.12a)}{=} \mathcal{F}\{x(t) * x(-t)\} = |X(\omega)|^2$$
$$\text{(Energy Density Spectrum)} \tag{P2.13.2}$$

(b) Prove the crosscorrelation theorem:

$$\Phi_{xy}(\omega) = \mathcal{F}\{\phi_{xy}(t)\} \overset{(1.4.12a)}{=} \mathcal{F}\{x(t) * y(-t)\}$$
$$= X(\omega)Y^*(\omega) = G^*(\omega)\,|X(\omega)|^2 \tag{P2.13.3}$$

(c) Prove that

$$\Phi_{yy}(\omega) = \mathcal{F}\{\phi_{yy}(t)\} \overset{(1.4.12a)}{=} \mathcal{F}\{y(t) * y^*(-t)\} = Y(\omega)Y^*(\omega)$$
$$= |G(\omega)|^2|X(\omega)|^2 \tag{P2.13.4}$$

2.14 Power Theorem - Generalization of Parseval's Relation
Prove the power theorem that, with $X(\omega) = \mathcal{F}\{x(t)\}$ and $Y(\omega) = \mathcal{F}\{y(t)\}$, we have

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)Y^*(\omega)\,d\omega \tag{P2.14.1}$$

2.15 Eigenfunction
If the output of a system to an input is just a (possibly complex-valued) constant times the input, then the input signal is called an *eigenfunction* of the system. Show that the output of an LTI (linear time-invariant) system with the (causal) impulse response $g(t)$ and system (transfer) function $G(s) = \mathcal{L}\{g(t)\}$ to a complex exponential input $e^{s_0 t}$ is,

$$G\{e^{s_0 t}\} = g(t) * e^{s_0 t} = G(s_0)e^{s_0 t} \text{ with } G(s_0) = G(s)|_{s=s_0} \tag{P2.15.1}$$

where $G(s)$ is assumed not to have any pole at $s = s_0$ so that the amplitude factor $G(s_0)$, called the *eigenvalue*, is finite. This implies that a complex exponential function is an eigenfunction of LTI systems.

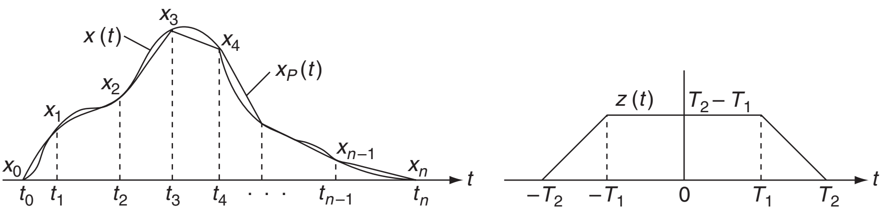
2.16 Fourier Transform of a Polygonal Signal (Problem 4.20 of [O-1])

(a) Show that the Fourier trnasform of $y(t) = (at + b)(u_s(t - t_1) - u_s(t - t_2))$ with $t_1 < t_2$ is

$$Y(\omega) = \frac{1}{\omega^2}(a + jb\omega)(e^{-j\omega t_2} - e^{-j\omega t_1}) + j\frac{a}{\omega}(t_1 e^{-j\omega t_2} - t_2 e^{-j\omega t_1}) \tag{P2.16.1}$$

(b) We can describe a triangular pulse having three vertices $(t_{i-1}, 0)$, (t_i, x_i), and $(t_{i+1}, 0)$ as

$$\lambda_i(t) = \begin{cases} x_i + \frac{x_i}{t_i - t_{i-1}}(t - t_i) = -\frac{x_i t_{i-1}}{t_i - t_{i-1}} + \frac{x_i}{t_i - t_{i-1}} t & \text{for } t_{i-1} \le t < t_i \\ x_i + \frac{x_i}{t_i - t_{i+1}}(t - t_i) = -\frac{x_i t_{i+1}}{t_i - t_{i+1}} + \frac{x_i}{t_i - t_{i+1}} t & \text{for } t_i \le t < t_{i+1} \end{cases} \tag{P2.16.2}$$



(a) A signal and its polygonal approximation (b) A trapezoidal signal

Fig. P2.16

Use Eq. (P2.16.1) to show that the Fourier transform of this pulse is

$$\Lambda_i(\omega) = \frac{1}{\omega^2}\left\{\frac{x_i}{t_i - t_{i-1}}(e^{-j\omega t_i} - e^{-j\omega t_{i-1}}) + \frac{x_i}{t_{i+1} - t_i}(e^{-j\omega t_i} - e^{-j\omega t_{i+1}})\right\} \tag{P2.16.3}$$

(c) Fig. P2.16(a) shows a finite-duration signal $x(t)$ and its piece-wise linear or polygonal approximation $x_P(t)$ where $x(t) = x_P(t) = 0\ \forall t \le t_0$ and $t \ge t_n$. Use Eq. (P2.16.3) to show that the Fourier transform of $x_P(t)$ is

$$X_P(\omega) = \mathcal{F}\{x_P(t)\} = \mathcal{F}\left\{\sum\nolimits_{i=0}^{n} \lambda_i(t)\right\} = \frac{1}{\omega^2}\sum\nolimits_{i=0}^{n} k_i e^{-j\omega t_i} \tag{P2.16.4}$$

where

$$k_i = \frac{(t_{i-1} - t_i)x_{i+1} + (t_i - t_{i+1})x_{i-1} + (t_{i+1} - t_{i-1})x_i}{(t_i - t_{i-1})(t_{i+1} - t_i)} \text{ and}$$
$$x(t_{-1}) = x(t_{n+1}) = 0$$

Complete the following program "CTFT_poly ()" so that it implements this method of computing the Fourier transform.

```
function [X,w]=CTFT_poly(t,x,w)
% Computes the CTFT of x(t) for frequency w by using Eq.(P2.16.4).
N =length(x); n=1:N;
if nargin<3, w = [-100:100]*pi/100+1e-6; end
t=[2*t(1)-t(2) t 2*t(N)-t(N-1)]; x=[0 x 0];
X= 0;
for i=2:N+1
   ki=((t(i-1)-t(i))*x(i+1)+(t(i)-t(i+1))*x(i-1)+(t(i+1)-t(i-1))*x(i));
   k(i)= ki/(t(i)-t(i-1))/(t(i+1)-t(i));
   X= X + k(i)*exp(-j*w*t(i));
end
X = X./(w.^2+eps);
```

(d) For the trapezoidal signal $z(t)$ shown in Figure P2.16(b), find its Fourier transform $Z(\omega)$ in two ways:

(i) Regarding $z(t)$ as the convolution of two unit-height rectangular pulses each of length $D_1 = T_2 - T_1$ and $D_2 = T_2 + T_1$, you can use the convolution property (2.5.11) together with Eq. (E2.3.2).
(ii) Use Eq. (P2.16.4), which is the result of part (c).
(iii) Use the MATLAB routine "CTFT_poly ()" to get $Z(\omega) = \mathcal{F}\{z(t)\}$ with $T_1 = 1$ and $T_2 = 2$ and plot it to compare the result with that obtained (i) or (ii).
<Hint> You can complete and run the following program.

```
%sig02p_16.m
% Numerical approximation of CTFT for a trapezoidal signal
clear, clf
T1=1; T2=2; D1= T2-T1; D2= T1+T2;
t= [-T2 -T1 T1 T2];
x= [0 T2-T1 T2-T1 0];
w= [-200:200]/100*pi+1e-6; % frequency range on which to plot X(w)
X= D1*D2*sinc(w*D1/2/pi).*sinc(w*D2/2/pi);
X_poly= ?????????(?,?,?); % (P2.16.4)
Discrepancy=norm(X_poly-X)
plot(w,abs(X),'b', w,abs(X_poly),'r')
```

(e) Show that, if we choose the points t_i's sufficiently close together so that the piecewise linear approximation, $x_P(t)$, of $x(t)$ is accurate enough to satisfy the bounded error condition

$$|x(t) - x_P(t)| \le \varepsilon, \tag{P2.16.5}$$

then the Fourier transform (Eq. (P2.16.4)) of $x_P(t)$ is close to $X(\omega) = \mathcal{F}\{x(t)\}$ enough to satisfy

$$\int_{-\infty}^{\infty} |X(\omega) - X_P(\omega)|^2 d\omega \le 2\pi(t_n - t_0)\varepsilon^2 \tag{P2.16.6}$$

(cf.) This implies that even though a signal $x(t)$ cannot be expressed as a linear combination, a multiplication, or convolution of basic functions, we can find its approximate Fourier transform from the collection of sample values of $x(t)$ and that we can improve the accuracy by making the piecewise approximation $x_P(t)$ closer to $x(t)$.

(cf.) This result suggests us that it is possible to evaluate numerically the Fourier transform for a finite-duration signal whose values are measured experimentally at many (discrete) time instants, even though the closed-form analytic expression of the signal is not available or too complicated to deal with.

2.17 Impulse Response and Frequency Response of Ideal Bandpass Filter
Consider a sinc function multiplied by a cosine function, which is depicted in Fig. P2.17:

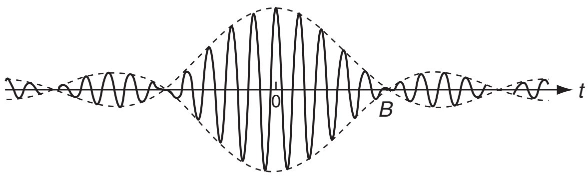


Fig. P2.17

$$g(t) = \frac{B}{\pi}\text{sinc}\left(\frac{B}{\pi}t\right)\cos\omega_p t \quad \text{with } \omega_p = 10B \tag{P2.17.1}$$

(a) Using Eqs. (E2.9.1&2) and the amplitude modulation (AM) property (E2.12.3), find the Fourier transform $G(\omega) = \mathcal{F}\{g(t)\}$ and sketch it.

(b) If Eq. (P2.17.1) describes the impulse response of a system, which type of filter is it?
(Lowpass, Bandpass, Highpass, Bandstop)

2.18 Design of Superheterodyne Receiver – Combination of Tunable/Selective Filters
We can build a tunable selective filter, called the *superheterodyne* or *supersonic heterodyne receiver* often abbreviated *superhet*), by combining a tunable, but poorly selective filter, a highly-selective, but untunable filter having

a fixed IF (intermediate frequency) passband, a mixer (multiplier), and a local oscillator as shown in Fig. P2.18.1.

Suppose the input signal $y(t)$ consists of many AM (amplitude modulated) signals which have been frequency-division-multiplexed (FDM) so that they each occupy different frequency bands. Let us consider one signal $y_1(t) = x_1(t)\cos\omega_c t$ with spectrum $Y_1(\omega)$ as depicted in Fig. P2.18.2(a). We want to use the system of Fig. P2.18.1 to demultiplex and demodulate for recovering the modulating signal $x_1(t)$, where the coarse tunable filter has the frequency response $H_c(\omega)$ shown in Fig. P2.18.2(b) and the fixed frequency selective filter is a bandpass filter (BPF) whose frequency response $H_f(\omega)$ is centered around fixed frequency ω_c as shown in Fig. P2.18.2(c).

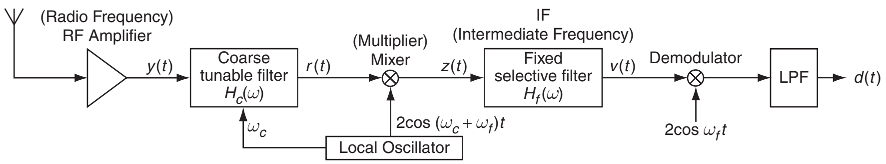


Fig. P2.18.1 Superheterodyne receiver – tunable and selective filter

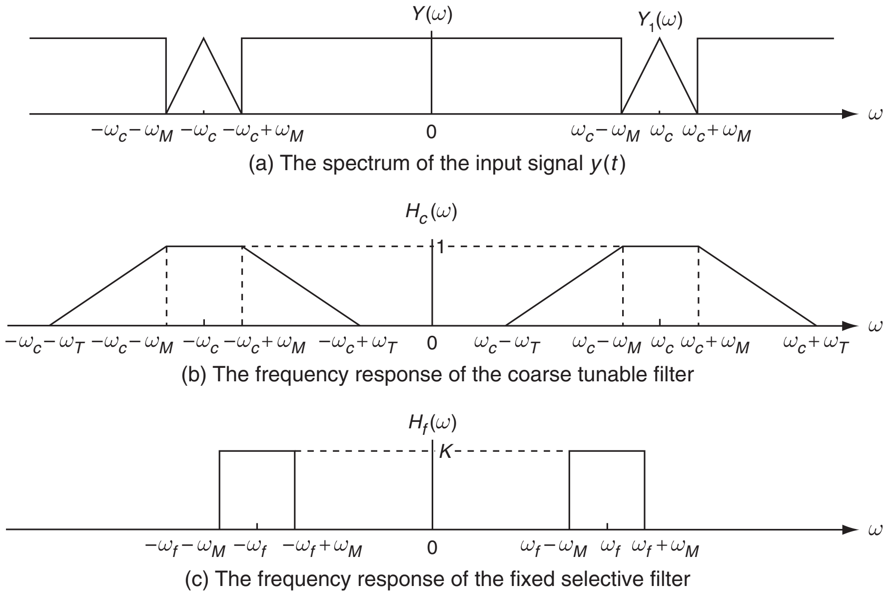


Fig. P2.18.2

(a) What constraint in terms of ω_c and ω_M must ω_T satisfy to guarantee that an undistorted spectrum of $x_1(t)$ is centered around ω_f so that the output of the fixed selective filter H_f can be $v(t) = x_1(t)\cos\omega_f t$?

(b) Determine and sketch the spectra $R(\omega)$, $Z(\omega)$, $V(\omega)$, and $D(\omega)$ of the output $r(t)$ of the coarse tunable filter H_c, the input $z(t)$ and output $v(t)$ of the fixed selective filter H_f, and the output $d(t)$ of the LPF.

(c) How could you determine the gain K of the fixed selective filter H_f required to have $v(t) = x_1(t)\cos\omega_f t$ at its output?

(d) Complete and run the following program "sig02p_18.m" to simulate the superheterodyne receiver twice, once with kT = 18 and once with kT = 19. Explain why the two simulation results are so different in terms of the error between $d(t)$ and $x_1(t)$. Note that the parameters kc, kf, kM, and kT correspond to ω_c, ω_f, ω_M, and ω_T, respectively.

```
%sig02p_18.m
% simulates a superheterodyne receiver combining a coarse tunable filter
%  and a fixed selective filter
% Copyleft: Won Y. Yang, wyyang53@hanmail.net, CAU for academic use only
clear, clf
N=128; kc=16; kf=12; kM=4; kT=12; Wcf=2*pi/N*(kc+kf); Wf=2*pi/N*kf;
%create a signal having the spectrum as Fig.P2.17.2(a)
n=[0:N-1]; kk=[-N/2:N/2]; % Time/Frequency ranges
y=make_signal(N,kc,kM);
Y=fftshift(fft(y)); Y_mag= abs([Y Y(1)]);
subplot(6,2,1), plot(n,y), title('y(t)')
subplot(6,2,2), plot(kk,Y_mag), title('Y(w)')
%tunable filter
r=??????????????(N,kc,kM,kT,?);
R=fftshift(fft(r)); R_mag= abs([R R(1)]);
subplot(6,2,3), plot(n,r), title('r(t) after tunable filter')
subplot(6,2,4), plot(kk,R_mag), title('R(w)')
%modulation
z=r.*(????????????);
Z=fftshift(fft(z)); Z_mag= abs([Z Z(1)]);
subplot(6,2,5), plot(n,z), title('z(t) after modulation')
subplot(6,2,6), plot(kk,Z_mag), title('Z(w)')
%selective filter
K=1; v=????????????????(N,kf,kM,K,?);
V=fftshift(fft(v)); V_mag= abs([V V(1)]);
subplot(6,2,7), plot(n,v), title('v(t) after selective filter')
subplot(6,2,8), plot(kk,V_mag), title('V(w)')
%demodulation
d0=v.*(???????????);
D0=fftshift(fft(d0)); D0_mag= abs([D0 D0(1)]);
subplot(6,2,9), plot(n,d0), title('d0(t) after demodulation')
subplot(6,2,10), plot(kk,D0_mag), title('D0(w)')
%tunable filter as LPF
d=tunable_filter(N,0,kM,kT,d0);
D=fftshift(fft(d)); D_mag= abs([D D(1)]);
x1=zeros(1,N);
for k=-kM:kM, x1(n+1)= x1(n+1)+(1-abs(k)/kM)*cos(2*pi*k*n/N); end
subplot(6,2,11), plot(n,d,'b', n,x1,'r')
title('detected/transmitted signal')
subplot(6,2,12), plot(kk,D_mag),
title('Spectrum D(w) of detected signal')
error_between_detected_transmitted_signals= norm(d-x1)
```

```
function x=make_signal(N,kc,kM)
% create a signal having the spectrum as Fig.P2.17.2(a)
n=1:N;
x=zeros(1,N);
for k=0:N/2-1
   tmp= 2*pi*k*(n-1)/N;
   if k<kc-kM, x(n)= x(n)+sin(tmp);
    elseif k>kc+kM, x(n)=x(n)+sin(tmp); % whatever, cos() or sin()
    else x(n)= x(n)+(1-abs(k-kc)/kM)*cos(tmp);
   end
end
```

```
function x_t=tunable_filter(N,kc,kM,kT,x)
% BPF with passband (kc-kM,kc+kM) and
%   stopband edge frequencies kc-kT and kc+kT
X=fft(x,N);
for k=1:N/2+1
   if k<=kc-kT|k>kc+kT+1, X(k)=0; X(N+2-k)=0;
    elseif k<=kc-kM+1
      X(k)=X(k)*(1-(kc-kM-k+1)/(kT-kM)); X(N+2-k)=conj(X(k));
    elseif k>kc+kM
      X(k)=X(k)*(1-(k-1-kc-kM)/(kT-kM)); X(N+2-k)=conj(X(k));
   end
end
x_t=real(ifft(X,N));
```

```
function x_t=selective_filter(N,kf,kM,K,x)
% passes only the freq (kf-kM,kf+kM) with the gain K
X=fft(x,N);
for k=1:N
   if (k>kf-kM&k<=kf+kM+1)|(k>N-kf-kM&k<=N-kf+kM+1), X(k)= K*X(k);
    else X(k)=0;
   end
end
x_t=real(ifft(X));
```

2.19 The Poisson Sum Formula (PSF) and Spectrum of Sampled Signal
Consider an LTI (linear time-invariant) system with impulse response $h(t)$ and frequency response $H(j\omega) = \mathcal{F}\{h(t)\}$. Note that the response (output) of the system to the impulse train can be expressed in terms of the impulse response as

$$\delta(t) \stackrel{G\{\}}{\rightarrow} h(t);\ \delta_T(t) = \sum\nolimits_{n=-\infty}^{\infty} \delta(t-nT) \stackrel{G\{\}}{\rightarrow} \sum\nolimits_{n=-\infty}^{\infty} h(t-nT) \quad \text{(P2.19.1)}$$

Likewise, the response (output) of the system to a single or multiple complex exponential signal can be written as

$$e^{jk\omega_0 t} \stackrel{G\{\}}{\rightarrow} H(jk\omega_0)e^{jk\omega_0 t};\ \frac{1}{T}\sum\nolimits_{k=-\infty}^{\infty} e^{jk\omega_0 t} \stackrel{G\{\}}{\rightarrow} \frac{1}{T}\sum\nolimits_{k=-\infty}^{\infty} H(jk\omega_0)e^{jk\omega_0 t} \quad \text{(P2.19.2)}$$

(a) Based on the above results (P2.19.1) & (P2.19.2) and Eq. (2.1.10), prove that, for any function $h(t)$ with its Fourier transform $H(j\omega) = \mathcal{F}\{h(t)\}$ well-defined, the following relation holds:

$$\sum_{n=-\infty}^{\infty} h(t-nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H(jk\omega_0)e^{jk\omega_0 t} \text{ with } \omega_0 = \frac{2\pi}{T} \tag{P2.19.3}$$

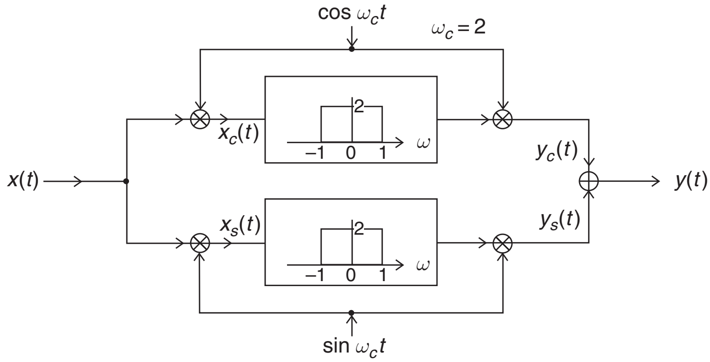
which is called the *Poisson sum formula*.

(b) Using the Poisson sum formula, prove the following relation pertaining to the spectrum of a sampled signal (see Eq. (E2.13.3)):

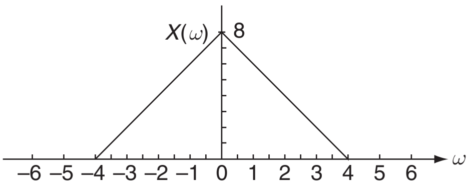
$$\sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega + k\omega_0)) \text{ with } X(j\omega) = \mathcal{F}\{x(t)\} \text{ and } \omega_0 = \frac{2\pi}{T} \tag{P2.19.4}$$

<Hint> Substitute $h(t) = x(t)e^{j\omega_1 t}$ and its Fourier transform $H(j\omega) = X(j(\omega + \omega_1))$ with $\omega = k\omega_0$ into Eq. (P2.19.3) and then substitute $t = 0$ and $\omega_1 = \omega$.

2.20 BPF (Bandpass Filter) Realization via Modulation-LPF-Demodulation
Consider the realization of BPF of Fig. P2.20(a), which consists of a modulator (multiplier), an LPF, and a demodulator (multiplier). Assuming that the spectrum of $x(t)$ is as depicted in Fig. P2.20(b), sketch the spectra of the signals $x_c(t)$, $x_s(t)$, $y_c(t)$, $y_s(t)$, and $y(t)$ to see to it that the composite system realizes a BPF and determine the passband of the realized BPF.



(a) A BPF realization via modulation–LPF–demodulation



(b) The spectrum of the input signal

Fig. P2.20

2.21 TDM (Time-Division Multiplexing)

As depicted in Fig. P2.21(a), *Time-Division multiplexing* (TDM) is to transmit two or more PAM (pulse amplitude modulation) signals not simultaneously as subchannels (separated in the frequency-domain), but physically by turns in one communication channel where each subchannel is assigned a timeslot of duration D every T s in such a way that the timeslots do not overlap. Sketch a TDM waveform of two signals $x_1(t)$ and $x_2(t)$ in Fig. P2.21(b).

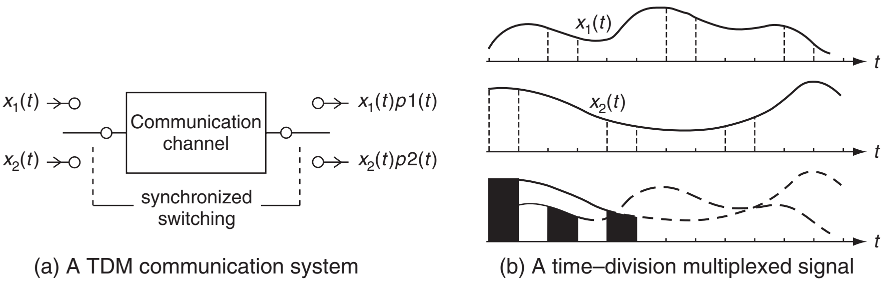


Fig. P2.21 TDM (Time-Division Multiplexing) communication

2.22 FDM (Frequency-Division Multiplexing)

As depicted in Fig. P2.22(a), *Frequency-Division multiplexing* (FDM) is to transmit two or more SAM (sinusoidal amplitude modulation) signals as *subcarrier*s (separated in the frequency-domain) in such a way that the frequency slots each carrying a signal (amplitude-modulated with different carrier frequency) do not overlap. Assuming that the channel is ideal so that $r(t) = s(t)$, sketch the spectrum of the FDM signal to be transmitted over the same communication channel where the spectra of two signals $x_1(t)$ and $x_2(t)$ to be frequency-multiplexed are shown in Fig. P2.22(b).

(cf.) Time/frequency-division multiplexing assigns different time/frequency intervals to each signal or subchannel.

2.23 Quadrature Multiplexing

Consider the quadrature multiplexing system depicted in Fig. P2.22, where the two signals are assumed to be bandlimited, that is,

$$X_1(\omega) = X_2(\omega) = 0 \text{ for } \omega > \omega_M \qquad \text{(P2.23.1)}$$

as illustrated in Fig. P2.23(b).

(a) Assuming that the channel is ideal so that $r(t) = s(t)$, express the spectra $S(\omega),\ V_1(\omega),\ V_2(\omega),\ Y_1(\omega),\ Y_2(\omega)$ of the signals $s(t),\ v_1(t),\ v_2(t),\ y_1(t)$, and $y_2(t)$ in terms of $X_1(\omega)$ and $X_2(\omega)$.

(b) Complete and run the following MATLAB program "sig02p_23.m" to see if $y_1(t) = x_1(t)$ and $y_2(t) = x_2(t)$.

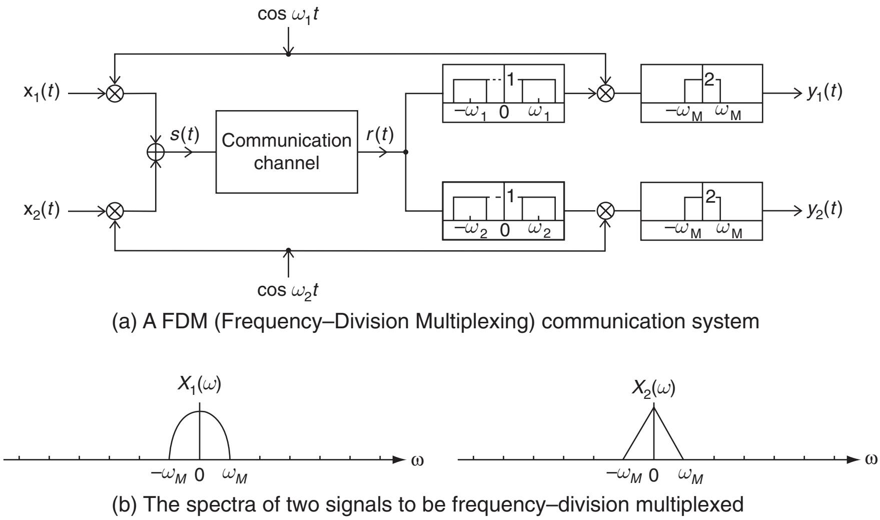


Fig. P2.22 FDM (Frequency-Division Multiplexing) communication

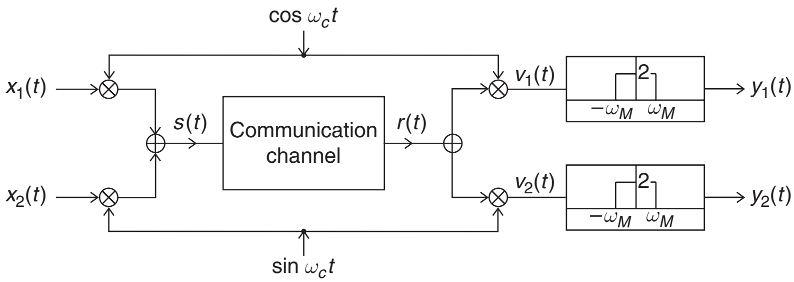


Fig. P2.23 Quadrature multiplexing

```
%sig02p_23.m
% Quadrature Multiplexing in Fig.P2.23
% Copyleft: Won Y. Yang, wyyang53@hanmail.net, CAU for academic use only
clear, clf
N=128; kc=16; kM=6; kx1=6; kx2=6; Wc=2*pi/N*kc;
n=[0:N-1]; kk=[-N/2:N/2];
x1= make_signal1(N,kx1); x2= make_signal2(N,kx2);
X1= fftshift(fft(x1)); X2= fftshift(fft(x2));
subplot(523), plot(kk,abs([X1 X1(1)])), title('X1(w)')
subplot(524), plot(kk,abs([X2 X2(1)])), title('X2(w)')
s= x1.*???(Wc*n) + x2.*sin(Wc*n);  S= fftshift(fft(s));
subplot(513), plot(kk,abs([S S(1)])),  title('S(w)')
v1= s.*cos(Wc*n); V1= fftshift(fft(v1));
v2= s.*???(Wc*n); V2= fftshift(fft(v2));
subplot(527), plot(kk,abs([V1 V1(1)])), title('V1(w)')
subplot(528), plot(kk,abs([V2 V2(1)])), title('V2(w)')
```

```
% selective filter(ideal LPF)
kf=0; K=2;
y1= selective_filter(N,kf,kM,K,v1); Y1= fftshift(fft(y1));
y2= ????????????????(N,kf,kM,K,v2); Y2= fftshift(fft(y2));
discrepancy1=norm(x1-y1), discrepancy2=norm(x2-y2)
subplot(529), plot(kk,abs([Y1 Y1(1)])), title('Y1(w)')
subplot(5,2,10), plot(kk,abs([Y2 Y2(1)])), title('Y2(w)')
```

```
function x=make_signal1(N,kx)
n=1:N; kk=1:kx-1; x= (1-kk/kx)*cos(2*pi*kk.'*n/N);
```

```
function x=make_signal2(N,kx)
n=1:N; kk=1:kx-1; x= kk/kx*cos(2*pi*kk.'*n/N);
```

2.24 Least Squares Error (LSE) and Fourier Series Representation

Consider a continuous-time function $x(t)$ on a time interval $[a, b]$ and a set of its sampled values, $\{x_1, x_2, \cdots, x_N\}$ (with $x_n = x(t_n)$). Let us find an approximate representation of $x(t)$ as a linear combination of some basis functions $\{\phi_k(t); k = 1 : K\}$:

$$\hat{x}(\mathbf{c}, t) = \sum_{k=1}^{K} c_k\, \phi_k(t) \text{ for } a \le t \le b \tag{P2.24.1}$$

where the coefficient vector $\mathbf{c} = [c_1\ c_2 \cdots c_K]^T$ is to be determined so that the following objective function can be minimized:

$$E^2(\mathbf{c}) = \int_a^b (x(t) - \hat{x}(\mathbf{c}, t))^2 dt \tag{P2.24.2}$$

$$E^2(\mathbf{c}) = \sum_{n=1}^{K} (x_n - \hat{x}(\mathbf{c}, t_n))^2 \tag{P2.24.3}$$

The following notations will be used:

$$\mathbf{x} = [x_1\ x_2\ \cdots\ x_N]^T \text{ with } x_n = x(t_n) \tag{P2.24.4a}$$

$$\mathbf{c} = [c_1\ c_2\ \cdots\ c_K]^T \tag{P2.24.4b}$$

$$\varphi(t) = [\phi_1(t)\ \phi_2(t)\ \cdots\ \phi_K(t)]^T \tag{P2.24.4c}$$

$$\varepsilon = [\varepsilon_1\ \varepsilon_2\ \cdots\ \varepsilon_N]^T \tag{P2.24.4d}$$

$$\text{with } \varepsilon_n = x_n - \hat{x}(\mathbf{c},\ t_n) = x_n - \sum_{k=1}^{K} c_k \phi_k(t_n) = x_n - \varphi^T(t_n)\mathbf{c}$$

$$\Phi = \begin{bmatrix} \varphi^T(t_1) \\ \varphi^T(t_2) \\ \bullet \\ \bullet \\ \varphi^T(t_N) \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{21} & \bullet & \phi_{K1} \\ \phi_{12} & \phi_{22} & \bullet & \phi_{K2} \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \phi_{1N} & \phi_{2N} & \bullet & \phi_{KN} \end{bmatrix} \text{ with } \phi_{kn} = \phi_k(t_n) \qquad \text{(P2.24.4e)}$$

Based on these notations, the above objective function (P2.24.3) can be written as

$$\begin{aligned} E^2(\mathbf{c}) &= \varepsilon^{*T}\varepsilon = [\mathbf{x}^* - \Phi^*\mathbf{c}^*]^T[\mathbf{x} - \Phi\mathbf{c}] \\ &= \mathbf{x}^{*T}\mathbf{x} - \mathbf{x}^{*T}\Phi\mathbf{c} - \mathbf{c}^{*T}\Phi^{*T}\mathbf{x} + \mathbf{c}^{*T}\Phi^{*T}\Phi\mathbf{c} \\ &= \mathbf{x}^{*T}\left[I - \Phi[\Phi^{*T}\Phi]^{-1}\Phi^{*T}\right]\mathbf{x} \\ &\quad + [\Phi^T\Phi^*\mathbf{c}^* - \Phi^T\mathbf{x}^*]^T[\Phi^{*T}\Phi]^{-1}[\Phi^{*T}\Phi\mathbf{c} - \Phi^{*T}\mathbf{x}] \qquad \text{(P2.24.5)} \end{aligned}$$

where only the second term of this equation depends on $\mathbf{c}$. Note that this objective function is minimized for

$$\Phi^{*T}\Phi\mathbf{c} - \Phi^{*T}\mathbf{x} = 0 \text{ (a normal equation); } \mathbf{c} = [\Phi^{*T}\Phi]^{-1}\Phi^{*T}\mathbf{x} \qquad \text{(P2.24.6)}$$

(a) Let the K functions $\{\phi_k(t),\ k = 1 : K\}$ constitute an orthogonal set for the N discrete times $\{t_n,\ n = 1 :\ N\}$ in the sense that

$$\sum_{n=1}^{N} \phi_{kn}^*\phi_{mn} = 0 \ \forall\ k \neq m \qquad \text{(P2.24.7)}$$

so that $[\Phi^{*T}\Phi]$ is diagonal. Show that the coefficients can be determined as

$$c_k = \frac{\sum_{n=1}^{N} x_n\phi_{kn}^*}{\sum_{n=1}^{N} \phi_{kn}^*\phi_{kn}}, \text{ for } k = 1, 2, \cdots, K \qquad \text{(P2.24.8)}$$

Likewise, if the K functions $\{\phi_k(t),\ k = 1 :\ K\}$ constitute an orthogonal set for the continuous time intrerval $[a, b]$ in the sense that

$$\int_a^b \phi_k^*(t)\phi_m(t)\,dt = 0 \ \forall\ k \neq m, \qquad \text{(P2.24.9)}$$

then the coefficients can be determined as

$$c_k = \frac{\int_a^b x(t)\phi_k^*(t)\,dt}{\int_a^b \phi_k^*(t)\phi_k(t)\,dt}, \text{ for } k = 1, 2, \cdots, K \qquad \text{(P2.24.10)}$$

(b) Using Eq. (P2.24.1), find an approximate representation of a real-valued function $x(t)$ for $-P/2 \leq t \leq P/2$ in terms of the following basis functions

$$\phi_k(t) = \frac{1}{P} e^{jk\omega_0 t} \text{ with } \omega_0 = \frac{2\pi}{P} \text{ and } k = -K : K \qquad \text{(P2.24.11)}$$

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