Chapter 2
Preliminaries

In Section 2.1, we prepare some convention. In Section 2.2, we review basic results from the geometric invariant theory. In particular, we recall a sufficient condition for a quotient stack to be Deligne-Mumford and proper. We also recall Mumford-Hilbert criterion, and look at some easy examples. The results will be used in Chapter 4.

In Section 2.3, we review some basic facts on cotangent complexes. Then, we recall how to express cotangent complexes of quotient stacks in Subsection 2.3.2, which will be used in Chapter 5 frequently. We also study some more examples in Subsection 2.3.3, which will be used in Sections 6.3, 6.4 and 6.6.

In Section 2.4, we review obstruction theory in the sense of K. Behrend-B. Fantechi [6]. We explain a naive strategy to construct obstruction theories of moduli stacks in Subsection 2.4.2. We recall an obstruction theory of locally free subsheaves in Subsection 2.4.3. It gives obstruction theories of moduli spaces of torsion-free quotient sheaves over a smooth projective surface. The result will be used in Section 5.6. We also obtain the smoothness of moduli spaces of quotient torsion-sheaves over a smooth projective curve, although we will not use it later. In Subsection 2.4.4, we recall an obstruction theory of filtrations of a vector bundle on a smooth projective curve. It will be used to construct a relative obstruction theory for quasi-parabolic structures.

In Section 2.5, we recall some standard results for equivariant complexes on Deligne-Mumford stacks with GIT construction, which will be used in Section 5.9. In Section 2.6, we give some elementary remarks on extremal sets, which are used in Sections 4.3–4.4. In Section 2.7, we give remarks on the twist of line bundles.

2.1 Some Convention

2.1.1 Product and Projection

Let S be a scheme. Let Y be an algebraic stack over S. Let $g : T \longrightarrow U$ be a morphism of algebraic stacks over S. The naturally induced morphism $T \times_S Y \longrightarrow U \times_S Y$ is denoted by g_Y or simply by g.

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Let X and U be algebraic stacks over S. We use the symbol p_X to denote the projection forgetting the X-component:

$$p_X : U \times_S X \longrightarrow U, \quad p_X(u,x) = u$$

Similarly, p_U denotes the projection $U \times_S X \longrightarrow X$.

2.1.2 Vector Bundles

Let V be a vector bundle on an algebraic stack Y. The sheaf of local sections of V is also denoted by the same symbol V, if there are no risk of confusion. But, we use some particular notation in the following case: For vector bundles V_i $(i = 1,2)$, let $\mathcal{H}om(V_1, V_2)$ denote the sheaf of homomorphisms from V_1 to V_2. The corresponding vector bundle is denoted by $N(V_1, V_2)$.

Let F be a vector bundle on Y. The complement of the image of the 0-section in F is denoted by F^*, i.e., $F^* := F - Y$, and the dual bundle of F is denoted by $F^\vee$. The projectivization of F is denoted by $\mathbb{P}(F^\vee)$ or $\mathbb{P}_F$.

2.1.3 Coherent Sheaves on a Product

Let X be a flat scheme over S, and let U be an algebraic stack over S. A coherent sheaf E over $U \times_S X$ is called a U-coherent sheaf, if it is flat over U. A U-coherent sheaf E is called a U-torsion free sheaf, if $E_{|\{u\}\times_S X}$ is torsion-free for each $u \in U$. We will often omit to denote "U-", if there are no risk of confusion.

When we are given a line bundle $\mathcal{O}_X(1)$ on X which is relatively ample over S, we use the symbol $E(m)$ to denote $E \otimes p_U^* \mathcal{O}_X(m)$ for any coherent sheaf E on $U \times_S X$.

2.1.4 Quotient Stacks

Let Z be an algebraic stack over S provided with an action of a group scheme G over S. Then, we use the symbols Z_G or Z/G to denote the quotient stack.

2.1.5 Signature in Complexes

We follow the signature convention in [68]. We recall some of them for later use in our situation. Let X be an algebraic stack over S. For two bounded $\mathcal{O}_X$-complexes $C^\bullet$ and $D^\bullet$, let $\mathcal{H}om(C^\bullet, D^\bullet)$ denote the complex whose i-th terms are

$$\bigoplus_{k-j=i} \mathcal{H}om\big(C^j, D^k\big),$$

and whose differentials are given as follows:

$$\mathcal{H}om\big(C^j, D^k\big) \longrightarrow \mathcal{H}om\big(C^j, D^{k+1}\big) \oplus \mathcal{H}om\big(C^{j-1}, D^k\big)$$

$$a \longmapsto \big(d_D \circ a,\ (-1)^{k-j+1} a \circ d_C\big)$$

Let us look at some examples. For a complex $C^\bullet$, we denote the dual complex $\mathcal{H}om(C^\bullet, \mathcal{O}_X)$ by $C^{\bullet\vee}$. The differentials are as follows:

$$\mathcal{H}om(C^n, \mathcal{O}_X) \longrightarrow \mathcal{H}om(C^{n-1}, \mathcal{O}_X), \quad a \longmapsto (-1)^{n+1} \cdot a \circ d_X$$

For two term complexes $C^\bullet = (C^{-1} \to C^0)$ and $D^\bullet = (D^{-1} \to D^0)$, the differentials of the complex $\mathcal{H}om\big(C^\bullet, D^\bullet\big)$ are given as follows:

$$\mathcal{H}om\big(C^0, D^{-1}\big) \longrightarrow \mathcal{H}om\big(C^0, D^0\big) \oplus \mathcal{H}om\big(C^{-1}, D^{-1}\big)$$

$$a \longmapsto \big(d_D \circ a,\ a \circ d_C\big)$$

$$\mathcal{H}om\big(C^0, D^0\big) \oplus \mathcal{H}om\big(C^{-1}, D^{-1}\big) \longrightarrow \mathcal{H}om\big(C^{-1}, D^0\big)$$

$$(b_1, b_2) \longmapsto -b_1 \circ d_C + d_D \circ b_2$$

We will often use the dual $\mathcal{H}om(C^\bullet, D^\bullet)^\vee$ whose differentials are given as follows:

$$\mathcal{H}om\big(D^0, C^{-1}\big) \longrightarrow \mathcal{H}om(D^0, C^0) \oplus \mathcal{H}om(D^{-1}, C^{-1})$$

$$a \longmapsto (-d_C \circ a, a \circ d_D)$$

$$\mathcal{H}om(D^0, C^0) \oplus \mathcal{H}om(D^{-1}, C^{-1}) \longrightarrow \mathcal{H}om\big(D^{-1}, C^0\big)$$

$$(b_1, b_2) \longmapsto -b_1 \circ d_D - d_C \circ b_2$$

2.1.6 Filtrations and Complexes on a Curve

Let $\mathcal{D}$ be a smooth projective curve over S. Let E_a $(a = 1, 2)$ be coherent $\mathcal{O}_{\mathcal{D}}$-modules which are flat over S. Assume that we are given a decreasing filtration $F(E_a) = (F_i(E_a) \,|\, i = 1, \ldots, l)$ of E_a such that $\mathrm{Cok}_i(E_a) := E_a/F_{i+1}(E_a)$ are flat over S.

Let $V_{a,\bullet} = (V_{a,-1} \to V_{a,0})$ be locally free resolutions of E_a $(a = 1, 2)$. We set

$$V_a^{(1)} := V_{a,0}, \quad V_a^{(l+1)} = V_{a,-1},$$

$$V_a^{(i)} := \mathrm{Ker}\big(V_{a,0} \longrightarrow \mathrm{Cok}_i(E_a)\big), \quad (i = 2, \ldots, l).$$

Let $f_i : V_D^{(i+1)} \longrightarrow V_D^{(i)}$, $t_i : V_D^{(i)} \longrightarrow V_D^{(1)}$ and $s_i : V_D^{(l+1)} \longrightarrow V_D^{(i)}$ denote the inclusions. Let us consider the complex $C_1(V_1^*, V_2^*)$ given as follows:

$$\mathcal{H}om\bigl(V_1^{(1)}, V_2^{(l+1)}\bigr) \xrightarrow{d^{-1}} \bigoplus_{i=1}^{l+1} \mathcal{H}om\bigl(V_1^{(i)}, V_2^{(i)}\bigr) \xrightarrow{d^0} \bigoplus_{i=1}^{l} \mathcal{H}om\bigl(V_1^{(i+1)}, V_2^{(i)}\bigr)$$

Here, the first term stands in the degree -1. The differentials d^i are given as follows:

$$d^{-1}(a) = \bigl(s_i \circ a \circ t_i \,\big|\, i = 1, \dots, l+1\bigr) \tag{2.1}$$

$$d^0(b_1, \dots, b_l) = \bigl(-f_1 \circ b_1 + b_2 \circ f_1, -f_2 \circ b_2 + b_3 \circ f_2, \dots, -f_l \circ b_l + b_{l+1} \circ f_l\bigr) \tag{2.2}$$

We have the naturally defined morphism:

$$\varphi = (\varphi_i) : C_1(V_1^*, V_2^*) \longrightarrow \mathcal{H}om\bigl(V_{1,\bullet}, V_{2,\bullet}\bigr) \tag{2.3}$$

More precisely, φ_0 is the projection induced by the identifications $V_0 = V^{(1)}$ and $V_{-1} = V^{(l+1)}$, φ_1 is given by $\varphi_1(a_i) = \sum s_{i+1} \circ a_i \cdot t_i$, and φ_2 is the identity. We can directly check that φ is the morphism of complexes. We put

$$C_2(V_1^*, V_2^*) := \mathrm{Cone}(\varphi)[-1].$$

The following lemma is easy to check.

Lemma 2.1.1 *The complexes $C_i(V_1^*, V_2^*)$ and the morphism $\varphi : C_1(V_1^*, V_2^*) \longrightarrow \mathcal{H}om(V_{1\bullet}, V_{2\bullet})$ depend only on (E_1, F) and (E_2, F) in the derived category $D(\mathcal{D})$.* $\square$

Notation 2.1.2 *We denote $C_i(V_1^*, V_2^*)$ by $\mathcal{RH}om_i'(E_{1*}, E_{2*})$.* $\square$

If E_a and $E_a/F_j(E_a)$ are locally free sheaves for $a = 1, 2$ and $j = 1, \dots, l$, then we have vanishings

$$\mathcal{H}^i\bigl(\mathcal{H}om_1'(E_1, E_2)\bigr) = 0 \quad (i \neq 0),$$

and $\mathcal{H}^0\bigl(\mathcal{H}om_1'(E_1, E_2)\bigr)$ is isomorphic to the sheaf of homomorphisms of E_1 to E_2 which preserve the filtrations.

2.1.7 Virtual Vector Bundle

Let G be a group scheme over S. Let Y be an algebraic stack over S provided with a (possibly trivial) G-action. Let $K_G(Y)$ denote the K-group of G-equivariant perfect complexes. Elements of $K_G(Y)$ are called virtual G-equivariant vector bundles in this monograph. We often omit to distinguish G.

2.1.8 Compatible Diagrams

Let $A_{i,j}$ $(i = 1, 2)$ $(j = 1, 2, 3, 4)$ be objects in some category. Assume that we are given morphisms $\varphi_j : A_{1,j} \longrightarrow A_{2,j}$. We also assume that we are given commutative diagrams $(CD)_i$:

$$\begin{tikzcd} A_{i,1} \arrow[r, "a_i"] \arrow[d, "b_i"'] & A_{i,2} \arrow[d, "c_i"'] \\ A_{i,3} \arrow[r, "d_i"] & A_{i,4} \end{tikzcd}$$

We say that $(CD)_1$ and $(CD)_2$ are compatible with respect to the morphisms φ_j $(j = 1, 2, 3, 4)$, if every face of the naturally obtained cube is commutative. It is equivalent to the commutativity of the following diagrams:

$$\begin{tikzcd} A_{1,1} \arrow[r] \arrow[d] & A_{1,2} \arrow[d] & A_{1,1} \arrow[r] \arrow[d] & A_{1,3} \arrow[d] \\ A_{2,1} \arrow[r] & A_{2,2} & A_{2,1} \arrow[r] & A_{2,3} \\ A_{1,2} \arrow[r] \arrow[d] & A_{1,4} \arrow[d] & A_{1,3} \arrow[r] \arrow[d] & A_{1,4} \arrow[d] \\ A_{2,2} \arrow[r] & A_{2,4} & A_{2,3} \arrow[r] & A_{2,4} \end{tikzcd}$$

2.2 Geometric Invariant Theory

2.2.1 GIT Quotient and Algebraic Stacks

Let k be an algebraically closed field with characteristic 0. Let G be a linear reductive group over k. Let Y be a projective variety over k, provided with a G-action ρ. Let L be an ample line bundle on Y with a G-action which is a lift of ρ. The lift is also denoted by ρ.

We recall some basic definitions. A point $y \in Y$ is semistable with respect to L, if there exists a G-invariant section s of $L^{\otimes n}$ for some $n > 0$ such that $s(y) \neq 0$. A point $y \in Y$ is defined to be stable with respect to L, if there exists a G-invariant section s of $L^{\otimes n}$ for some $n > 0$ such that (i) $s(y) \neq 0$, (ii) any G-orbits contained in $Y - s^{-1}(0)$ are closed. Let $Y^s(L)$ (resp. $Y^{ss}(L)$) denote the set of the stable (resp. semistable) points with respect to L. The fundamental theorem of D. Mumford is the following.

Proposition 2.2.1 ([96]) *There exists a uniform categorical quotient $\pi : Y \longrightarrow Y^{ss}//G$. Moreover, the following holds:*

- *The map π is affine and universally submersive.*
- *$Y^{ss}//G$ is a projective variety.*
- *There exists the open subset $Y^s//G$ of $Y^{ss}//G$, such that (i) $\pi^{-1}(Y^s//G) = Y^s$, (ii) $\pi : Y^s \longrightarrow Y^s//G$ is a universal geometric quotient of Y^s.*

Proof See Proposition 1.9, Theorem 1.10 and Page 40 in [96]. □

We combine it with some results of A. Vistoli in [129]. Let Y^{sf} denote the set of the stable points of Y whose stabilizers are finite. In this situation, we obtain the quotient stack Y^{sf}/G, which is Deligne-Mumford. See Sections 2 and 7 of [129] for more details on such quotient stacks. We recall one of his results.

Proposition 2.2.2 ([129]) *The naturally induced morphism $Y^{sf}/G \longrightarrow Y^{sf}//G$ is proper.*

Proof The map $Y^{sf} \longrightarrow Y^{sf}//G$ is a universal geometric quotient. In particular, it is universally submersive, and the geometric fibers are precisely the orbits of geometric points of X. Therefore, $Y^{sf}//G$ is a quotient of Y^{sf} by G in the sense of Vistoli. (See Page 630 of [129].) Applying Proposition 2.11 of [129], we can conclude that the map $Y^{sf}/G \longrightarrow Y^{sf}//G$ is proper. □

Corollary 2.2.3 *Let Z be a variety over k with a G-action. Let $\Phi : Z \longrightarrow Y$ be a G-equivariant immersion with the following property:*

- *The stabilizer groups of any points of Z are finite.*
- *The image $\Phi(Z)$ is contained in $Y^s(L)$.*
- *$\Phi : Z \longrightarrow Y^{ss}(L)$ is proper.*

Then, Z/G is Deligne-Mumford and proper.

Proof We can regard Z/G as a substack of Y^{sf}/G. We can also regard $Z//G$ as a closed subscheme of $Y^{sf}//G \subset Y^{ss}//G$. Since $Y^{ss}//G$ is projective, $Z//G$ is also projective. According to the previous lemma, the morphism $Z/G \longrightarrow Z//G$ is proper. Therefore, Z/G is proper. □

2.2.2 Mumford-Hilbert Criterion and Some Elementary Examples

Let Y, L and G be as above. Let $\lambda : G_m \longrightarrow G$ be a one-parameter subgroup. We put $P(\lambda) := \lim_{t\to 0} \lambda(t)\cdot P$. Then, λ acts on the fiber $L_{|P(\lambda)}$. The weight is denoted by $\mu_\lambda(P, L)$.

Proposition 2.2.4 (Mumford-Hilbert criterion, [96]) *The point P is semistable (resp. stable) with respect to L, if and only if $\mu_\lambda(P, L) \geq 0$ (resp. $\mu_\lambda(P, L) > 0$) for any one-parameter subgroup λ.* □

Remark 2.2.5 *We use the convention to identify a vector bundle and the sheaf of its sections. Hence, the above definition of μ_λ is the same as that given in* [96]. □

For later use, we recall some elementary examples. Let V be a vector space over an algebraically closed field k of characteristic 0 with a base $u_1, \ldots, u_N$. Take $w_1, \ldots, w_N \in \mathbb{Z}$ such that $\sum w_i = 0$ and $w_i \leq w_{i+1}$. Let λ be the one-parameter subgroup of $\mathrm{SL}(V)$ given by $\lambda(t) \cdot u_i = t^{w_i} \cdot u_i$. Let $V^{(i)}$ denote the subspace generated by $u_1, \ldots, u_i$. Let $V = \bigoplus V_w$ denote the weight decomposition of λ, i.e., λ preserves the decomposition, and the action on V_w is the multiplication of t^w. We put $\mathcal{G}_j := \bigoplus_{w \leq j} V_w$.

We denote a point of $\mathbb{P}(V^\vee)$ by $[v]$ by using a representative $v \in V - \{0\}$. Let us consider the right $\mathrm{SL}(V)$-action on $\mathbb{P}(V^\vee)$ given by $g \cdot [v] := [g^{-1}(v)]$, which can be lifted to the action on $\mathcal{O}_{\mathbb{P}(V^\vee)}(1)$.

Lemma 2.2.6 ([96]) $\mu_\lambda\big([v], \mathcal{O}_{\mathbb{P}(V^\vee)}(1)\big) = \min\big\{i \,\big|\, v_i \in \mathcal{G}_i\big\}$. *In other words,*

$$\mu_\lambda\big([v], \mathcal{O}(1)\big) = \sum_i w_i \cdot \big(\dim V^{(i)} \cap \langle v\rangle - \dim V^{(i-1)} \cap \langle v\rangle\big)$$
$$= \sum_j j \cdot \big(\dim \mathcal{G}_j \cap \langle v\rangle - \dim \mathcal{G}_{j-1} \cap \langle v\rangle\big). \quad (2.4)$$

Here $\langle v\rangle$ denotes the subspace generated by v.

Proof According to the weight decomposition $V = \bigoplus V_i$, we have the decomposition $v = \sum v_i$. In $\mathbb{P}(V^\vee)$, we have

$$\lambda(t)[v] = \big[\lambda(t)^{-1} v\big] = \Big[\sum t^{-i} \cdot v_i\Big].$$

We put $i_0 := \max\big\{i \,|\, v_i \neq 0\big\} = \min\big\{i \,\big|\, v \in \mathcal{G}_i\big\}$. It is easy to see

$$\lim_{t \to 0} \lambda(t)[v] = [v_{i_0}].$$

The weight of λ on $\mathcal{O}_{\mathbb{P}(V^\vee)}(1)_{|[v_{i_0}]}$ is i_0. Thus, the first claim is obtained. The second claim follows from the first one. □

Let $G_l(V)$ denote the Grassmann variety of l-dimensional subspaces of V:

$$G_l(V) := \big\{\iota : W \subset V \,\big|\, \dim W = l\big\}.$$

We have the Plücker embedding $G_l(V) \longrightarrow \mathbb{P}\big(\bigwedge^l V^\vee\big)$ given by $W \longmapsto \bigwedge^l W \subset \bigwedge^l V$. It induces a polarization $\mathcal{O}_{G_l(V)}(1)$ of $G_l(V)$. The group $SL(V)$ has the right action on $G_l(V)$ given by $\iota \longmapsto g^{-1} \circ \iota$, which can be lifted to that on $\mathcal{O}_{G_l(V)}(1)$.

Lemma 2.2.7 *For any point W of $G_l(V)$, we have the following equality:*

$$\mu_\lambda\big(W, \mathcal{O}_{G_l(V)}(1)\big) = \sum_{i=1}^N \big(\operatorname{rank} W \cap V^{(i)} - \operatorname{rank} W \cap V^{(i-1)}\big) \cdot w_i$$
$$= \sum_{j \in \mathbb{Z}} j \cdot \dim \frac{W \cap \mathcal{G}_j}{W \cap \mathcal{G}_{j-1}}. \quad (2.5)$$

Proof For any $J = (j_1 < j_2 < \cdots < j_l)$, we put $u_J := u_{j_1} \wedge \cdots \wedge u_{j_l}$ and $w_J := \sum_{i=1}^{l} w_{j_i}$. Collection of such u_J gives a base of $\bigwedge^l V$. Let $\widetilde{\lambda}$ denote the one-parameter subgroup of $\mathrm{SL}\bigl(\bigwedge^l V\bigr)$ induced by λ. We have $\widetilde{\lambda}(t)(u_J) = t^{w_J} \cdot u_J$.

Let us take a base $v_1, \ldots, v_l$ of W of the form $v_h = u_{i_h} + \sum_{j<i_h} a_{h,j} \cdot u_j$. Then, $z := v_1 \wedge \cdots \wedge v_l$ is expressed as the sum $\sum a_J \cdot u_J$, where $a_J = 1$ if $J = I = (i_1 < \cdots < i_l)$ and $a_J = 0$ if $w_J > w_I$. We have $\mu_\lambda\bigl(W, \mathcal{O}_{G_l(V)}(1)\bigr) = \mu_{\widetilde{\lambda}}\bigl(z, \mathcal{O}_{\mathbb{P}(\bigwedge^l V^\vee)}(1)\bigr) = w_I$ according to Lemma 2.2.6. Then, it is easy to derive the claim of the lemma. □

We also have the Grassmann variety G'_l of l-dimensional quotients:

$$G'_l(V) := \bigl\{ q : V \longrightarrow Q \bigm| \dim Q = l \bigr\}$$

We have the Plucker embedding $G'_l(V) \longrightarrow \mathbb{P}\bigl(\bigwedge^l V\bigr)$ given by the correspondence $q \longmapsto \bigl(\bigwedge^l q : \bigwedge^l V \longrightarrow \bigwedge^l Q\bigr)$. It induces a polarization $\mathcal{O}_{G'_l(V)}(1)$.

Lemma 2.2.8 ([96], [87]) *Let $q : V \longrightarrow Q$ be a point of $G'_l(V)$. We put $W := \mathrm{Ker}(q)$. Then, we have the following equality:*

$$\mu_\lambda\bigl(q, \mathcal{O}_{G'_l(V)}(1)\bigr) = \sum_{i=1}^{N} w_i \cdot \bigl(\dim V^{(i)} \cap W - \dim V^{(i-1)} \cap W - 1\bigr)$$

$$= \sum_{j=1}^{N} j \cdot \left(\dim \frac{W \cap \mathcal{G}_j}{W \cap \mathcal{G}_{j-1}} - \dim \frac{\mathcal{G}_j}{\mathcal{G}_{j-1}} \right). \quad (2.6)$$

Proof We put $W^{(i)} := \mathcal{G}_i \cap W \big/ \mathcal{G}_{i-1} \cap W$. By using the natural isomorphism $\mathcal{G}_i / \mathcal{G}_{i-1} \simeq V_i$, we regard $W^{(i)}$ as the subspaces of V_i. It is easy to see that the limit $\lim_{t \to 0} \lambda(t) \cdot q$ is given by the quotient $\widehat{q} : V \longrightarrow \bigoplus V_i / W^{(i)}$. The weight of λ on $\mathcal{O}_{G'_l(V)}(1)_{|\widehat{q}}$ is $-i \cdot \dim\bigl(V_i / W^{(i)}\bigr)$. Then, it is easy to deduce the claim. □

Remark 2.2.9 *We have the obvious isomorphism $G_l(V) \simeq G'_{N-l}(V)$. However, it does not preserve the semistability conditions on the varieties induced by the Plücker embeddings.* □

2.3 Cotangent Complex

2.3.1 Basic Facts

Recall some fundamental property of cotangent complexes from [64], [79] and [111]. Let $\mathcal{X}$ and $\mathcal{Y}$ be Deligne-Mumford stacks with étale site. For any morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ of Delinge-Mumford stacks, the cotangent complex was introduced by L. Illusie [64] as a complex of $\mathcal{O}_{\mathcal{X}}$-modules. It is denoted by $L_{\mathcal{X}/\mathcal{Y}}$ or L_f. Recall that the cotangent complex controls deformations of f in the following sense

(Section 3 [64]). Let T be a scheme over $\mathcal{Y}$, and let $h : T \longrightarrow \mathcal{X}$ be a $\mathcal{Y}$-morphism. Let $\overline{T}$ be a $\mathcal{Y}$-scheme such that T is a closed $\mathcal{Y}$-subscheme of $\overline{T}$ and the corresponding ideal J is square-zero, i.e., $J^2 = \{f \cdot g \,\big|\, f, g \in J\} = 0$.

Proposition 2.3.1 (Illusie, [64]) *We have the obstruction class*

$$o(h) \in Ext^1(h^* L_{\mathcal{X}/\mathcal{Y}}, J)$$

with the following property:

- *The morphism h can be extended over $\overline{T}$, if and only if $o(h)$ vanishes.*

In the case $o(h) = 0$, the set of the extension classes is a torsor over the group $Ext^0\big(h^* L_{\mathcal{X}/\mathcal{Y}}, J\big)$. $\square$

Cotangent complexes have a nice functorial property. For example, we have the distinguished triangle for a morphism $\mathcal{Y} \longrightarrow \mathcal{Z}$,

$$f^* L_{\mathcal{Y}/\mathcal{Z}} \longrightarrow L_{\mathcal{X}/\mathcal{Z}} \longrightarrow L_{\mathcal{X}/\mathcal{Y}} \longrightarrow f^* L_{\mathcal{Y}/\mathcal{Z}}[1]$$

in the derived category $D(\mathcal{X})$.

As for general Artin stacks with lisse-étale site, cotangent complexes with some good functorial property have been introduced by G. Laumon, L. Moret-Bailly and M. Olsson (Section 17 of [79] and Section 8 of [111]). For any Artin stack $\mathcal{X}$, Olsson introduced the category $D'_{\mathrm{qcoh}}(\mathcal{X})$ of the projective systems

$$K = (\cdots \to K_{\geq -n-1} \to K_{\geq -n} \to \cdots \to K_{\geq 0})$$

in $D^+(\mathcal{X})$ such that $K_{\geq -n} \longrightarrow \tau_{\geq -n} K_{\geq -n}$ and $\tau_{\geq -n} K_{\geq -n-1} \longrightarrow \tau_{\geq -n} K_{\geq -n}$ are isomorphisms. Here $\tau_{\geq -n}$ denotes the canonical n-th truncation functor. See [111] for the functorial property of $D'_{\mathrm{qcoh}}(\mathcal{X})$. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of Artin stacks. Then, we can associate

$$L_{\mathcal{X}/\mathcal{Y}} = L_f = \big(\cdots \to L^{\geq -n-1}_{\mathcal{X}/\mathcal{Y}} \to L^{\geq -n}_{\mathcal{X}/\mathcal{Y}} \to \cdots \to L^{\geq 0}_{\mathcal{X}/\mathcal{Y}}\big) \in D'_{\mathrm{qcoh}}(\mathcal{X})$$

to each f with the following property (Theorem 8.1 [111]):

- If $\mathcal{X}$ and $\mathcal{Y}$ are algebraic spaces, $L^{\geq -n}_{\mathcal{X}/\mathcal{Y}}$ are isomorphic to $\tau_{\geq -n} L_{\mathcal{X}/\mathcal{Y}}$ in $D^+_{\mathrm{qcoh}}(\mathcal{X})$, where the latter $L_{\mathcal{X}/\mathcal{Y}}$ denotes the usual cotangent complex defined by Illusie.
- When we are given a 2-commutative diagram of Artin stacks

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle g}\downarrow & & \downarrow \\ \mathcal{Y}' & \xrightarrow{h} & \mathcal{Y}, \end{array}$$

we have the functorial morphism $Lf^*L_{\mathcal{X}/\mathcal{Y}} \longrightarrow L_{\mathcal{X}'/\mathcal{Y}'}$. If the diagram is 2-Cartesian, and if one of g or h is flat, then the morphism $Lf^*L_{\mathcal{X}/\mathcal{Y}} \longrightarrow L_{\mathcal{X}'/\mathcal{Y}'}$ is an isomorphism.
- Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of Artin stacks. Let $g : \mathcal{Y} \longrightarrow \mathcal{Z}$ be another morphism. Then, we have the distinguished triangle

$$Lf^*L_{\mathcal{Y}/\mathcal{Z}} \longrightarrow L_{\mathcal{X}/\mathcal{Z}} \longrightarrow L_{\mathcal{X}/\mathcal{Y}} \longrightarrow Lf^*L_{\mathcal{Y}/\mathcal{Z}}[1]$$

in $D'_{\mathrm{qcoh}}(\mathcal{X})$.

The following properties can be derived directly from the construction. (See Section 8 of [111] for the construction of $L_{\mathcal{X}/\mathcal{Y}}$.)

- Each $L^{\geq -n}_{\mathcal{X}/\mathcal{Y}}$ is an object in $D^{[-n,1]}_{\mathrm{qcoh}}(\mathcal{X})$.
- If f is smooth and representable, then $L_{\mathcal{X}/\mathcal{Y}}$ is quasi-isomorphic to its 0-th cohomology sheaf, which is isomorphic to the locally free sheaves of Kahler differentials $\Omega_{\mathcal{X}/\mathcal{Y}}$. In general, if f is smooth, any $L^{\geq -n}_{\mathcal{X}/\mathcal{Y}}$ is of perfect amplitude contained in $[0, 1]$. In particular, they are isomorphic to $L^{\geq 0}_{\mathcal{X}/\mathcal{Y}}$.

Remark 2.3.2 *M. Aoki generalized the deformation theory of Illusie, and showed a natural generalization of Proposition* 2.3.1 *for Artin stacks* [2]. □

2.3.2 Quotient Stacks

Let S be a variety. Let G be a group scheme smooth over S. Let Y be a smooth S-scheme with a G-action. The quotient stack is denoted by Y_G. Let Z be an Artin stack over S with a morphism $F : Z \longrightarrow Y_G$. We have the corresponding G-torsor $P(F)$ over Z and the G-equivariant map $\widetilde{F} : P(F) \longrightarrow Y$:

$$\begin{array}{ccc} P(F) & \xrightarrow{\widetilde{F}} & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{F} & Y_G \end{array}$$

Let us describe $F^*L_{Y_G/S}$ on Z. We have a map $\alpha : \widetilde{F}^*\Omega_{Y/S} \longrightarrow \Omega_{P(F)/Z}$ on $P(F)$, which is the composite of the differential $\widetilde{F}^*\Omega_{Y/S} \longrightarrow \Omega_{P(F)/S}$ and the natural projection $\Omega_{P(F)/S} \longrightarrow \Omega_{P(F)/Z}$.

Proposition 2.3.3 *$F^*L_{Y_G/S}$ is represented by the descent of* $\mathrm{Cone}(-\alpha)[-1]$ *with respect to the natural G-action.*

Proof We recall the construction of $L_{Y_G/S}$ in this case. We set

$$Y^{(m)} := \overbrace{Y \times_{Y_G} \cdots \times_{Y_G} Y}^{m+1}.$$

We have the natural morphisms $Y^{(m)} \longrightarrow Y_G \longrightarrow S$. We obtain complexes $C^{(m)} := \big(\Omega_{Y^{(m)}/S} \longrightarrow \Omega_{Y^{(m)}/Y_G}\big)$ on $Y^{(m)}$, where $\Omega_{Y^{(m)}/S}$ stands in the degree 0. We have the strictly simplicial structure given by the naturally defined quasi-isomorphisms $\pi_i^* C^{(m-1)} \longrightarrow C^{(m)}$ $(i = 0, 1, \ldots, m)$, where π_i denote the projections $Y^{(m)} \longrightarrow Y^{(m-1)}$ forgetting the i-th components. By definition, $L_{Y_G/S} \in D'_{\mathrm{qcoh}}(Y_G)$ is represented by $\big(C^{(m)} \,\big|\, m = 0, 1, \ldots\big)$.

We put $P(F)^{(m)} := \overbrace{P(F) \times_Z \cdots \times_Z P(F)}^{m+1}$. We have the naturally defined morphisms $F^{(m)} : P(F)^{(m)} \longrightarrow Y^{(m)}$. Then, $F^* L_{Y_G/S} \in D'_{\mathrm{qcoh}}(Z)$ is represented by $\big(F^{(m)*} C^{(m)} \,\big|\, m = 0, 1, \ldots\big)$. We have the following commutative diagram:

$$\begin{array}{ccc}
F^{(m)\,*}\Omega_{Y^{(m)}/S} & \longrightarrow & F^{(m)\,*}\Omega_{Y^{(m)}/Y_G} \\
{\scriptstyle =}\big\downarrow & & {\scriptstyle \simeq}\big\downarrow \\
F^{(m)\,*}\Omega_{Y^{(m)}/S} & \xrightarrow{\;b\;} & \Omega_{P(F)^{(m)}/Z}
\end{array}$$

Here, b is the composite of the differential $F^{(m)\,*}\Omega_{Y^{(m)}/S} \longrightarrow \Omega_{P(F)^{(m)}/S}$ and the natural projection $\Omega_{P(F)^{(m)}/S} \longrightarrow \Omega_{P(F)^{(m)}/Z}$.

Let $q_i : Y \times_S G^m \longrightarrow Y$ $(i = 0, 1, \ldots, m)$ be the morphism given by

$$q_i(y, g_1, \ldots, g_m) = y \cdot g_1 \cdot \cdots \cdot g_i.$$

They induce an isomorphism $Y \times_S G^m \longrightarrow Y^{(m)}$. Under the identification, q_i is the projection onto the i-th component. Similarly, we have the identification $P(F) \times_S G^m \simeq P(F)^{(m)}$, under which $F^{(m)}$ is given by

$$F^{(m)}(y, g_1, \ldots, g_m) = \big(\widetilde{F}(y), g_1, \ldots, g_m\big).$$

Let ρ_m denote the projection of $P(F) \times_S G^m$ onto G^m. We have the subcomplex $\big(\rho_m^* \Omega_{G^m} \xrightarrow{\mathrm{id}} \rho_m^* \Omega_{G^m}\big)$ of $F^* C^{(m)}$. It is compatible with the simplicial structure. The quotient complexes are denoted by $\widehat{C}^{(m)}$, and $\big(\widehat{C}^{(m)} \,\big|\, m = 0, 1, \ldots\big)$ also represents $F^* L_{Y_G/S}$ in $D'_{qcoh}(Z)$. Then, it follows that $F^* L_{Y_G/S}$ is given as the descent of $\widehat{C}^{(0)} = \big(\widetilde{F}^* \Omega_{Y/S} \xrightarrow{\alpha} \Omega_{P(F)/Z}\big)$ with respect to the natural G-action. □

Example 2.1. Let k be a field. Let $\mathrm{GL}(R)$ denote the R-th general linear group over k. Let $k_{\mathrm{GL}(R)}$ denote the quotient stack of $\mathrm{Spec}(k)$ with the trivial $\mathrm{GL}(R)$-action. Let E be a vector bundle on a k-variety X of rank R, and let $f : X \longrightarrow k_{\mathrm{GL}(R)}$ be the classifying map. Then, we have $f^* L_{k_{GL(R)}/k} \simeq \mathrm{End}(E)[-1]$. □

Let H denote the composite of F and the canonical map $Y_G \longrightarrow S_G$. Let $P(H)$ denote the G-torsor over Z corresponding to H. Since we have the natural isomorphism $P(H) \simeq P(F)$, we do not distinguish them. Let $\widetilde{H} : P(F) \longrightarrow S$ be the lift of H. Let π denote the projection $P(F) \longrightarrow Z$. We have the canonical isomorphism $\pi^* H^* L_{S_G/S}[1] \simeq \widetilde{H}^* \Omega_{S/S_G} \simeq \Omega_{P(F)/Z}$. We also have the canonical isomorphism $\pi^* F^* L_{Y_G/S_G} \simeq \widetilde{F}^* \Omega_{Y/S}$. We obtain the following corollary.

Corollary 2.3.4 *The morphism $F^*L_{Y_G/S_G} \longrightarrow H^*L_{S_G/S}[1]$ on Z is obtained as the descent of $\alpha : \widetilde{F}^*\Omega_{Y/S} \longrightarrow \Omega_{P(F)/Z}$.*

Proof We have the distinguished triangle

$$H^*L_{S_G/S} \longrightarrow F^*L_{Y_G/S} \longrightarrow F^*L_{Y_G/S_G} \longrightarrow H^*L_{S_G/S}[1].$$

By Proposition 2.3.3, we understand the morphism $H^*L_{S_G/S} \longrightarrow F^*L_{Y_G/S}$. Then, we understand the morphism $F^*L_{Y_G/S_G} \longrightarrow H^*L_{S_G/S}[1]$. □

Let us argue the naturality of the expression in Proposition 2.3.3. Let G_i $(i = 1, 2)$ be smooth S-group schemes with a homomorphism $a : G_1 \longrightarrow G_2$. Let Y_i $(i = 1, 2)$ be S-schemes provided with G-actions. For simplicity, we assume that Y_i are smooth. Let $g : Y_1 \longrightarrow Y_2$ be an equivariant morphism through the morphism a. Let $[g] : Y_{1\,G_1} \longrightarrow Y_{2\,G_2}$ denote the induced morphism. Let $h_1 : Z \longrightarrow Y_{1\,G_1}$ be a morphism. The composite $[g] \circ h_1$ is denoted by h_2. We would like to obtain an expression of the morphism $h_2^*L_{Y_{2,G_2}/S} \longrightarrow h_1^*L_{Y_{1,G_1}/S}$.

We have corresponding G_i-torsors P_i over Z with G-equivariant morphisms $\widetilde{h}_i : P \longrightarrow Y_i$. We can identify $P_2 = (P_1 \times_S G_2)/G_1$, where the G_1-action on $P_1 \times_S G_2$ is given by $g_1(y, g_2) = (yg_1^{-1}, g_1g_2)$. Let $\iota : P_1 \longrightarrow P_2$ denote the natural inclusion. We have the following commutative diagram:

$$\begin{array}{ccc} P_1 & \xrightarrow{\widetilde{h}_1} & Y_1 \\ {\scriptstyle\iota}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ P_2 & \xrightarrow{\widetilde{h}_2} & Y_2 \end{array}$$

It induces the following commutative diagram of G_1-equivariant sheaves on P_1:

$$\begin{array}{ccc} \iota^*\widetilde{h}_2^*\Omega_{Y_2/S} & \xrightarrow{\iota^*\alpha_2} & \iota^*\Omega_{P_2/Z} \\ \big\downarrow & & \big\downarrow \\ \widetilde{h}_1^*\Omega_{Y_1/S} & \xrightarrow{\alpha_1} & \Omega_{P_1/Z} \end{array}$$

Note that the descent of $\mathrm{Cone}(-\iota^*\alpha_2)$ with respect to the G_1-action is naturally isomorphic to the descent of $\mathrm{Cone}(-\alpha_2)$ with respect to the G_2-action.

Lemma 2.3.5 *The morphism $c : h_2^*L_{Y_{2\,G_2}/S} \longrightarrow h_1^*L_{Y_{1\,G_1}/S}$ is the descent of the induced morphism $\mathrm{Cone}(-\iota^*\alpha_2)[-1] \longrightarrow \mathrm{Cone}(-\alpha_1)[-1]$.*

Proof According to the functorial construction of Proposition 2.3.3, we have only to consider the case $Y_1 = Y_2 =: Y$. Let us consider the case $Y = S$. Since $h_i^*L_{S_{G_i}/S}$ are isomorphic to the H^1-cohomology sheaves $\mathcal{H}^1\big(h_i^*L_{S_{G_i}/S}\big)$, we have only to identify the pull back of $\mathcal{H}^1(c)$ via the pull back $P_1 \longrightarrow Z$, which can be done easily. Let us consider the general case. Let $k_i : Z \longrightarrow S_{G_i}$ denote the naturally defined morphism. We have the distinguished triangles $k_i^*L_{S_{G_i}/S} \longrightarrow$

$h_i^* L_{Y_{i,G_i}/S} \longrightarrow h_i^* L_{Y_{i,G_i}/S_{G_i}}$. By the above argument, we know the induced morphism $k_2^* L_{S_{G_2}/S} \longrightarrow k_1^* L_{S_{G_1}/S}$. The isomorphism $h_2^* L_{Y_{2,G_2}/S_{G_2}} \simeq h_1^* L_{Y_{1,G_1}/S_{G_1}}$ is easy to understand. Hence, we can identify $h_2^* L_{Y_{2,G_2}/S} \simeq h_1^* L_{Y_{1,G_1}/S}$. □

Remark 2.3.6 *Let G_1 be a smooth group scheme over S. Assume that Y is provided with a G_1-action, which commutes with the G-action. It induces a G_1-action on Y_G. Moreover, assume that Z is also provided with a G_1-action such that F is G_1-equivariant. Then, we have the naturally induced G_1-action on the complex* $\mathrm{Cone}(-\alpha)[-1]$, *which commutes with the G-action. It induces a G_1-action on the descent of* $\mathrm{Cone}(-\alpha)[-1]$ *on Z. In particular, we obtain a G_1-equivariant representative of $F^* L_{Y_G/S}$.* □

Let $\pi : Y \longrightarrow Y_G$ denote the canonical projection. By Proposition 2.3.3, $L_{Y_G/S}$ on Y_G is the descent of $\big(\Omega_{Y/S} \xrightarrow{\alpha} \Omega_{Y/Y_G}\big)$ given on Y with respect to the natural G-action, where $\Omega_{Y/S}$ stands in the degree 0.

Lemma 2.3.7 *Let $\mathfrak{g}$ denote the tangent space of G at the unit, or equivalently the vector space of the right invariant vector fields, and let $\mathfrak{g}^\vee$ denote the dual. Then,* $\Omega_{Y/Y_G} \simeq \mathfrak{g}^\vee \otimes \mathcal{O}_Y$.

Proof Let $p_1, p_2 : Y \times_S G \longrightarrow Y$ be given by the natural projection and the G-action. Let $r_1, r_2, r_3 : Y \times_S G^2 \longrightarrow Y \times_S G$ be given by $r_1(y, g, h) = (y, g)$, $r_2(y, g, h) = (y, gh)$ and $r_3(y, g, h) = (yg, h)$. We have the following commutative diagram:

$$\begin{array}{ccccc} Y \times_S G^2 & \xrightarrow{r_1, r_2} & Y \times_S G & \xrightarrow{p_1} & Y \\ {\scriptstyle r_3}\downarrow & & {\scriptstyle p_2}\downarrow & & {\scriptstyle \pi}\downarrow \\ Y \times_S G & \xrightarrow{p_1, p_2} & Y & \longrightarrow & Y_G \end{array}$$

Then, Ω_π is obtained as the descent of Ω_{p_2} by the identification $r_1^* \Omega_{p_2} \simeq r_2^* \Omega_{p_2}$. Hence, the claim of the lemma follows. □

Let $\Theta_{Y/S}$ denote the relative tangent bundle of Y/S. The G-action on Y induces the map $A : \mathfrak{g} \otimes \mathcal{O}_Y \longrightarrow \Theta_{Y/S}$. The dual of A is denoted by $A^\vee$.

Lemma 2.3.8 *The map $\alpha : \Omega_{Y/S} \longrightarrow \mathfrak{g}^\vee \otimes \mathcal{O}_Y$ is given by the dual of $-A$. Namely, we have* $\pi^* L_{Y_G/S} \simeq \mathrm{Cone}(A^\vee)[-1]$.

Proof Let $p_i : Y \times_{Y_G} Y \longrightarrow Y$ denote the projection onto the i-th component. We have the following factorization of $p_1^* \alpha$:

$$p_1^* \Omega_{Y/S} \longrightarrow \Omega_{Y \times_{Y_G} Y/S} \longrightarrow \Omega_{p_2} \simeq p_1^* \Omega_{Y/Y_G}$$

Each morphism is induced by the natural differential. Let us take the identification $Y \times_{Y_G} Y \simeq Y \times_S G$, for which p_1 and p_2 correspond to the natural projection onto Y and the G-action, respectively.

Let y be any closed point of Y, and let e be the unit of G. We have $p_1(y, e) = p_2(y, e) = y$. We denote the differential of p_i at (y, e) by $T_{(y,e)} p_i$. Let us consider the specialization of the dual of $p_1^*\alpha$ at (y,e). Then, it is the composite of the inclusion $\mathrm{Ker}(T_{(y,e)}p_2) \subset T_{(y,e)}(Y\times_S G)$ and the natural projection $T_{(y,e)}(Y\times_S G) \longrightarrow T_yY$. Since we have $\mathrm{Ker}\bigl(T_{(y,e)}p_2\bigr) \simeq \bigl\{(-Av,v) \bigm| v \in \mathfrak{g}\bigr\} \simeq \mathfrak{g}$, the map is $-A$. Since α can be recovered from $p_1^*\alpha$, the claim of the lemma is proved. □

Remark 2.3.9 *Since $F^*L_{Y_G/S}$ is obtained as the descent of $\widetilde{F}^*\,\mathrm{Cone}(A^\vee)[-1]$ for a morphism $F : Z \longrightarrow Y_G$, Lemma* 2.3.8 *is useful for calculation.* □

Example 2.2. Let k be a field. Let W_i $(i=-1,0)$ be R_i-dimensional vector spaces over k. Let $N(W_{-1},W_0)$ denote the vector space of linear maps from W_{-1} to W_0. We have the right $GL(W_{-1})\times \mathrm{GL}(W_0)$-action on $N(W_{-1},W_0)$ given by

$$(g_{-1},g_0)\cdot f = g_0^{-1}\circ f\circ g_{-1}.$$

Hence, we obtain the quotient stack $Y(W_\bullet) := N(W_{-1},W_0)_{\mathrm{GL}(W_{-1})\times\mathrm{GL}(W_0)}$.

Let X and U be algebraic stacks over k. Let V_i $(i=-1,0)$ be vector bundles on $U\times X$ whose ranks are R_i. Let $f : V_{-1} \longrightarrow V_0$ be a morphism of $\mathcal{O}_{U\times X}$-modules. Then, we obtain a morphism $\Phi_f : U\times X \longrightarrow Y(W_\bullet)$. We claim that $\Phi_f^*L_{Y(W_\bullet)/k}$ is represented by the following complex:

$$\mathcal{H}om\bigl(V_0,V_{-1}\bigr) \xrightarrow{\alpha} \mathcal{H}om\bigl(V_0,V_0\bigr)\oplus\mathcal{H}om(V_{-1},V_{-1}).$$

Here $\mathcal{H}om\bigl(V_0,V_{-1}\bigr)$ stands in degree 0, and the map α is given by

$$\alpha(a) = \bigl(f\circ a, -a\circ f\bigr).$$

We remark that it is isomorphic to $\mathcal{H}om\bigl(V_\bullet,V_\bullet\bigr)^\vee_{\leq 0}[-1]$. (See Subsection 2.1.5.)

To show the claim, we have only to be careful on signatures. We can argue it formally. Let f be an element of $N(W_{-1},W_0)$. The differential of the action of $\mathrm{GL}(W_{-1})\times\mathrm{GL}(W_0)$ gives the map:

$$\mathrm{End}(W_{-1})\oplus\mathrm{End}(W_0) \longrightarrow T_fN(W_{-1},W_0) = N(W_{-1},W_0), \tag{2.7}$$

$$(a_{-1},a_0) \longmapsto -a_0\circ f + f\circ a_{-1}$$

If we regard $W_{-1} \xrightarrow{f} W_0$ as a complex, (2.7) can be regarded as $\mathcal{H}om(W_\bullet,W_\bullet)_{\geq 0}$. Then, the cotangent complex is represented by $\bigl(\mathcal{H}om(W_\bullet,W_\bullet)_{\geq 0}\bigr)^\vee[-1]$, according to Lemma 2.3.8. □

Example 2.3. Let X be an algebraic stack. Let E_a $(a=-1,0)$ be vector bundles on X with a morphism $f : E_{-1} \longrightarrow E_0$. We regard E_{-1} as a group scheme over X, which acts on E_0 through f. The quotient stack is denoted by $\mathcal{Q}(E_0,E_{-1})$. For simplicity, we assume that f is an injection as a morphism of $\mathcal{O}_X$-modules.

Let E_0/E_{-1} denote the quotient $\mathcal{O}_X$-module. A section of g of E_0/E_{-1} corresponds to a morphism $\Phi(g) : X \longrightarrow \mathcal{Q}(E_0,E_{-1})$. The correspondence is given as follows: From a section g, we obtain an extension $0 \longrightarrow E_{-1} \longrightarrow \mathcal{G} \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0$.

We obtain a E_{-1}-torsor $\mathcal{P} = \pi^{-1}(1)$ with an equivariant morphism $\mathcal{P} \longrightarrow E_0$, i.e., a morphism $\Phi(g) : X \longrightarrow \mathcal{Q}(E_0, E_{-1})$. The pull back of the cotangent complex $\Phi(g)^* L_{\mathcal{Q}(E_0,E_{-1})/X}$ is denoted by $(E_{-1} \longrightarrow E_0)^{\vee}$. □

Let us consider the following diagram:

$$\begin{tikzcd} Y \arrow[r, "\widetilde{\psi}"] \arrow[d, "\pi"'] & S \arrow[d, "\pi_1"'] \\ Y_G \arrow[r, "\psi"] & S_G \end{tikzcd}$$

We have the natural isomorphisms:

$$\pi^* L_{Y_G/S_G} \simeq L_{Y/S}, \quad \pi^*\psi^* L_{S_G/S}[1] \simeq \widetilde{\psi}^* L_{S/S_G} \simeq L_{Y/Y_G}.$$

Lemma 2.3.10 *Under the isomorphisms above, $\pi^* L_{Y_G/S_G} \longrightarrow \pi^*\psi^* L_{S_G/S}[1]$ is the same as the natural morphism $L_{Y/S} \longrightarrow L_{Y/Y_G}$.*

Proof We have the natural isomorphisms:

$$\pi^* L_{Y_G/S} \simeq \mathrm{Cone}\big(L_{Y/S} \longrightarrow L_{Y/Y_G}\big)[-1], \quad \pi^*\psi^* L_{S_G/S} \simeq \widetilde{\psi}^* L_{S/S_G}[-1]$$

The natural morphism $\pi^*\psi^* L_{S_G/S} \longrightarrow \pi^* L_{Y_G/S}$ is induced by $\widetilde{\psi}^* L_{S/S_G} \longrightarrow L_{Y/Y_G}$. The distinguished triangle $\pi^*\psi^* L_{S_G/S} \longrightarrow \pi^* L_{Y_G/S} \longrightarrow \pi^* L_{Y_G/S_G} \longrightarrow \pi^*\psi^* L_{S_G/S}[1]$ is identified with the following:

$$\widetilde{\psi}^* L_{S/S_G}[-1] \longrightarrow \mathrm{Cone}\big(L_{Y/S} \to L_{Y/Y_G}\big)[-1] \longrightarrow L_{Y/S} \longrightarrow \widetilde{\psi}^* L_{S/S_G}$$

Then, the claim of the lemma follows. □

2.3.3 Some More Examples

The technical results in this subsection will be used in Sections 6.3–6.6. The author recommends the reader to skip here.

Let X be a smooth connected projective surface, and let U_1 be a quasi-compact algebraic stack. Let U_0 be a substack of U_1. Let $\mathcal{F}$ be a U_1-coherent sheaf with a section φ over $U_1 \times X$. We assume the following:

(A): $p_{X\,*}\mathcal{F}$ is locally free, and U_0 is contained in the induced section $\overline{\varphi}$ of $p_{X\,*}\mathcal{F}$.

Assume we are given a data as follows:

- A commutative diagram on $U_1 \times X$:

$$\begin{array}{ccccc}
V_{0,-1} & \longrightarrow & V_{1,-1} & & \\
\big\downarrow & & \big\downarrow & & \\
V_{0,0} & \longrightarrow & V_{1,0} & & \\
\big\downarrow & & \big\downarrow & & \\
E_0 & \longrightarrow & E_1 & \longrightarrow & \mathcal{F}
\end{array}$$

Here, E_i are U_1-coherent sheaves, $V_{a,b}$ are locally free sheaves, and the sequences $0 \longrightarrow V_{a,-1} \longrightarrow V_{a,0} \longrightarrow E_a \longrightarrow 0$ and $0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \mathcal{F} \longrightarrow 0$ are exact.

- A section ϕ of E_1 such that the composite $\mathcal{O} \longrightarrow E_1 \longrightarrow \mathcal{F}$ is φ.

In this subsection, such a data is called a resolution of $(\mathcal{F}, \varphi)$. Note that the restriction of ϕ to $U_0 \times X$ induces a section ϕ_0 of E_0.

Note that there always exists such a resolution. For example, we have the following construction. Take a sufficiently large integer m_0, and we put

$$E_1' := p_X^*\big(p_{X*}\mathcal{F}(m_0)\big) \otimes p_{U_1}^*\mathcal{O}_X(-m_0), \quad E_1 := E_1' \oplus \mathcal{O}_{U_1 \times X}.$$

The natural morphism $E_1' \longrightarrow \mathcal{F}$ and φ induce a morphism $\pi_1 : E_1 \longrightarrow \mathcal{F}$. We set $E_0 := \operatorname{Ker} \pi_1$. We take a sufficiently large m_1, and we put

$$V_{a,0} := p_X^* p_{X*}\big(E_a(m_1)\big) \otimes p_{U_1}^*\mathcal{O}_X(-m_1), \quad V_{a,-1} := \operatorname{Ker}\Big(V_{a,0} \longrightarrow E_a\Big).$$

Then, we obtain a diagram with the desired property.

Let us return to the general situation. Let Z_a $(a = 0, 1)$ be the quotient stacks of $N(\mathcal{O}_{U_1 \times X}, V_{a,0})$ via the natural actions of $N(\mathcal{O}_{U_1 \times X}, V_{a,-1})$. We obtain the following commutative diagram:

$$\begin{array}{ccc}
U_1 \times X & \xrightarrow{\Phi(\phi)} & Z_1 \\
{\scriptstyle j_{1\,X}}\big\uparrow & & \big\uparrow \\
U_0 \times X & \xrightarrow{\Phi(\phi_0)} & Z_0
\end{array}$$

By using an argument in Subsection 2.3.2, $\Phi(\phi)^* L_{Z_0/Z_1}$ is represented by the following:

$$\mathfrak{k}(E_\bullet, V_{\bullet,\bullet}, \phi) := j_{1\,X}^* \operatorname{Cone}\Big(\mathcal{H}om(\mathcal{O}_{U_1 \times X}, V_{1\bullet})^\vee \longrightarrow \mathcal{H}om\big(\mathcal{O}_{U_1 \times X}, V_{0\bullet}\big)^\vee\Big)$$

Here, $j_{1X} : U_0 \times X \longrightarrow U_1 \times X$ denotes the inclusion. We have the induced morphism $\mathfrak{r}(E_\bullet, V_{\bullet,\bullet}, \phi) : \mathfrak{k}(E_\bullet, V_{\bullet,\bullet}, \phi) \longrightarrow L_{U_0 \times X/U_1 \times X}$. We set

$$\operatorname{Ob}(E_\bullet, V_{\bullet,\bullet}, \phi) := Rp_{X*}\Big(\mathfrak{k}(E_\bullet, V_{\bullet,\bullet}, \phi) \otimes \omega_X\Big).$$

Then, we have the induced morphism:

$$\mathrm{ob}(E_\bullet, V_{\bullet,\bullet}, \phi) : \mathrm{Ob}(E_\bullet, V_{\bullet,\bullet}, \phi) \longrightarrow L_{U_0/U_1}$$

Lemma 2.3.11 *$\mathfrak{k}(E_\bullet, V_{\bullet,\bullet}, \phi)$ and $\mathfrak{r}(E_\bullet, V_{\bullet,\bullet}, \phi)$ depend only on $(\mathcal{F}, \varphi)$ in the derived category $D(U_0 \times X)$. In particular, $\mathrm{Ob}(E_\bullet, V_{\bullet,\bullet}, \phi)$ and $\mathrm{ob}(E_\bullet, V_{\bullet,\bullet}, \phi)$ depend only on $(\mathcal{F}, \varphi)$ in the derived category $D(U_0)$. Hence, we denote them by $\mathfrak{k}(\mathcal{F}, \varphi)$, $\mathfrak{r}(\mathcal{F}, \varphi)$, $\mathrm{Ob}(\mathcal{F}, \varphi)$ and $\mathrm{ob}(\mathcal{F}, \varphi)$, respectively.*

Proof It is standard that $\mathfrak{k}(E_\bullet, V_{\bullet,\bullet}, \phi)$ and $\mathrm{Ob}(E_\bullet, V_{\bullet,\bullet}, \phi)$ are independent of the choice of resolutions. We would like to show the independence of $\mathfrak{r}(E_\bullet, V_{\bullet,\bullet}, \phi)$ and $\mathrm{ob}(E_\bullet, V_{\bullet,\bullet}, \phi)$. Assume we are given another $(E'_\bullet, V'_{\bullet,\bullet}, \phi')$. We set

$$E''_1 = E_1 \oplus E'_1, \quad E''_0 := \mathrm{Ker}\Big(E''_1 \longrightarrow \mathcal{F}\Big),$$

$$V''_{1,0} := V_{1,0} \oplus V'_{1,0}, \quad V''_{1,-1} := V_{1,-1} \oplus V'_{1,-1} := \mathrm{Ker}\big(V''_{1,0} \longrightarrow E''_1\big).$$

Let $\pi : V''_{1,0} \longrightarrow E''_1$ denote the natural morphism. We can take a locally free sheaf A with a surjection $A \longrightarrow \pi^{-1}(E''_0)$. We set

$$V''_{0,0} := V_{0,0} \oplus V'_{0,0} \oplus A, \quad V''_{0,-1} := \mathrm{Ker}\Big(V''_{0,0} \longrightarrow E''_0\Big).$$

Then, $(E''_\bullet, V''_{\bullet,\bullet}, \phi'')$ with a naturally defined diagram gives a resolution of $\mathcal{F}$. Moreover, we have the natural inclusions $E_a \subset E''_a$ and $V_{a,b} \subset V''_{a,b}$.

Let Z''_a $(a = 0, 1)$ be the quotient stacks of $N(\mathcal{O}_X, V''_{a,0})$ via the naturally induced actions of $N(\mathcal{O}_X, V''_{a,-1})$. We obtain the following diagram:

$$\begin{array}{ccccc} U_1 \times X & \xrightarrow{\Phi(\phi)} & Z_1 & \xrightarrow{c_1} & Z''_1 \\ \uparrow & & \uparrow & & \uparrow \\ U_0 \times X & \xrightarrow{\Phi(\phi_0)} & Z_0 & \xrightarrow{c_0} & Z''_0 \end{array} \tag{2.8}$$

The morphisms c_i are naturally induced ones. The composites $c_1 \circ \Phi(\phi)$ and $c_0 \circ \Phi(\phi_0)$ are the morphisms induced by ϕ'' and ϕ''_0, respectively. Hence, we obtain the following:

$$L_{U_0 \times X/U_1 \times X} \longleftarrow \mathfrak{k}(E_\bullet, V_{\bullet,\bullet}, \phi) \xleftarrow{\simeq} \mathfrak{k}(E''_\bullet, V''_{\bullet,\bullet}, \phi'')$$

Hence, we obtain the following factorization of $\mathrm{ob}(E''_\bullet, V''_{\bullet,\bullet}, \phi'')$:

$$L_{U_0/U_1} \longleftarrow \mathrm{Ob}(E_\bullet, V_{\bullet,\bullet}, \phi) \xleftarrow{\simeq} \mathrm{Ob}(E''_\bullet, V''_{\bullet,\bullet}, \phi'')$$

Similarly, we can compare $\mathrm{Ob}(E'_\bullet, V'_\bullet, \phi')$ and $\mathrm{Ob}(E''_\bullet, V''_{\bullet,\bullet}, \phi'')$. Thus, we are done. □

Lemma 2.3.12 *Let* $(\mathcal{F}_i, \varphi_i)$ $(i = 1, 2)$ *satisfy Condition (A). If we are given a morphism* $g : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ *such that* $\varphi_2 = g \circ \varphi_1$*, we have the factorization of* $\mathrm{ob}(\mathcal{F}_2, \varphi_2)$*:*

$$\mathrm{Ob}(\mathcal{F}_2, \varphi_2) \longrightarrow \mathrm{Ob}(\mathcal{F}_1, \varphi_1) \longrightarrow L_{U_0/U_1}$$

Proof We take sufficiently large integers m_j, and apply the above construction of resolutions to $\mathcal{F}_i$. Then, the claim is clear. $\Box$

Now, we assume $R^i p_{X\,*}\mathcal{F} = 0$ for $i > 0$. We put $\mathfrak{V} := p_{X\,*}\mathcal{F}$. A section $\overline{\varphi}$ of $\mathfrak{V}$ is induced by φ. We have the following commutative diagram:

$$\begin{array}{ccc} U_0 & \xrightarrow{j_2} & U_1 \\ {\scriptstyle j_1}\downarrow & & {\scriptstyle i}\downarrow \\ U_1 & \xrightarrow{\overline{\varphi}} & \mathfrak{V} \end{array} \tag{2.9}$$

Here i is the 0-section, and j_a are the natural inclusions.

Proposition 2.3.13 $\mathrm{Ob}(\mathcal{F}, \varphi)$ *is isomorphic to* $j_2^* L_{U_1/\mathfrak{V}} \simeq \mathfrak{V}^\vee[1]$*, and* $\mathrm{ob}(\mathcal{F}, \varphi)$ *is the same as the morphism* $\kappa : j_2^* L_{U_1/\mathfrak{V}} \longrightarrow L_{U_0/U_1}$ *induced from the diagram* (2.9).

Proof We have the naturally defined morphism $a_1 : p_X^*\mathfrak{V} \longrightarrow \mathcal{F}$, for which we have $\varphi = a_1 \circ p_X^*\overline{\varphi}$. By Lemma 2.3.12, we have the following factorization of $\mathrm{ob}(\mathcal{F}, \varphi)$:

$$\mathrm{Ob}(\mathcal{F}, \varphi) \xrightarrow{b_0} \mathrm{Ob}(p_X^*\mathfrak{V}, p_X^*\overline{\varphi}) \xrightarrow{\mathrm{ob}(p_X^*\mathfrak{V}, p_X^*\overline{\varphi})} L_{U_0/U_1}$$

Let us look at $\mathrm{ob}(p_X^*\mathfrak{V}, p_X^*\overline{\varphi})$ more closely. In the construction for $(p_X^*\mathfrak{V}, p_X^*\overline{\varphi})$, we may choose

$$E_1 = V_{1,0} = p_X^*\mathfrak{V}, \quad E_0 = V_{1,-1} = V_{0,0} = V_{0,-1} = 0.$$

Then, $Z_0 = U_1 \times X$ and $Z_1 = p_X^*\mathfrak{V}$. The diagram (2.8) is given as follows:

$$\begin{array}{ccccc} U_0 \times X & \xrightarrow{j_1} & Z_0 & = & U_1 \times X \\ {\scriptstyle j_2}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow \\ U_1 \times X & \xrightarrow{p_X^*\overline{\varphi}} & Z_1 & = & p_X^*\mathfrak{V} \end{array} \tag{2.10}$$

Here i denotes the 0-section. We have $\mathfrak{k} = j_1^* L_{Z_0/Z_1} \simeq p_X^*\mathfrak{V}^\vee_{|U_0 \times X}[1]$, and the morphism $\mathfrak{r} : \mathfrak{k} \longrightarrow L_{U_0 \times X/U_1 \times X}$ is the same as the pull back of κ. In particular, we have the following factorization of $\mathrm{ob}(p_X^*\mathfrak{V}, p_X^*\overline{\varphi})$:

$$\mathrm{Ob}(p_X^*\mathfrak{V}, p_X^*\overline{\varphi}) = \mathfrak{V}^\vee[1] \otimes Rp_{X\,*}\bigl(p_{U_0}^*\omega_X\bigr) \xrightarrow{b_1} \mathfrak{V}^\vee[1] \xrightarrow{\kappa} L_{U_0/U_1}$$

It is easy to see that the composite $b_1 \circ b_0$ is an isomorphism, under the assumption $R^i p_{X\,*}\mathcal{F} = 0$ $(i > 0)$. Thus, the proof of Proposition 2.3.13 is finished. $\square$

We have a similar result for a smooth projective curve. Since the argument is similar and simpler, we explain only the statement. Let D be a smooth projective curve. Let $\mathcal{F}$ be a U_1-coherent sheaf on $U_1 \times D$ with a section φ. Assume the following:

(A') $p_{D\,*}\mathcal{F}$ is locally free, and U_1 is contained in the 0-set of the induced section $\overline{\varphi}$ of $p_{D\,*}\mathcal{F}$.

Let $(E_0 \to E_1)$ be a locally free resolution of $\mathcal{F}$ on $U_1 \times D$ with a section of ϕ of E_1 such that the composite $\mathcal{O} \longrightarrow E_1 \longrightarrow \mathcal{F}$ is φ. It is called a resolution of $(\mathcal{F}, \varphi)$. A section ϕ_0 of $V_{0\,|\,U_0 \times D}$ is induced. We put $\mathfrak{k}(E_\bullet, \varphi) := \mathcal{H}om(\mathcal{O}_{U_0 \times D}, V_{\bullet|U_0 \times D})^\vee$.

Let us construct a morphism $\mathfrak{r}(E_\bullet, \varphi) : \mathfrak{k}(E_\bullet, \varphi) \longrightarrow L_{U_0 \times D/U_1 \times D}$. We put $Z_a := N(\mathcal{O}, E_a)$ for $a = 0, 1$. Then, we have the naturally defined morphism $Z_0 \longrightarrow Z_1$. The sections ϕ and ϕ_0 induce the following commutative diagram:

$$\begin{array}{ccc} U_0 \times D & \xrightarrow{j} & Z_0 \\ \Big\downarrow & & \Big\downarrow \\ U_1 \times D & \longrightarrow & Z_1 \end{array}$$

It induces a morphism $\mathfrak{r}(E_\bullet, \varphi) : \mathfrak{k}(E_\bullet, \varphi) \simeq j^* L_{Z_0/Z_1} \longrightarrow L_{U_0 \times D/U_1 \times D}$. It can be shown that $\mathfrak{r}(E_\bullet, \varphi)$ and $\mathfrak{k}(E_\bullet, \varphi)$ depend only on $(\mathcal{F}, \varphi)$, as in Lemma 2.3.11. Therefore, we use the symbols $\mathfrak{r}(\mathcal{F}, \varphi)$ and $\mathfrak{k}(\mathcal{F}, \varphi)$ to denote them. We set

$$\mathrm{Ob}(\mathcal{F}, \varphi) := Rp_{D\,*}\big(\mathfrak{k}(\mathcal{F}, \varphi) \otimes \omega_D\big).$$

We have the induced morphism $\mathrm{ob}(\mathcal{F}, \varphi) : \mathrm{Ob}(\mathcal{F}, \varphi) \longrightarrow L_{U_0/U_1}$. It is functorial as in Lemma 2.3.12.

Now, we assume $R^i p_{D\,*}\mathcal{F} = 0$ for $i > 0$. We put $\mathfrak{V} := p_{X\,*}\mathcal{F}$. We have the induced section $\overline{\varphi}$. We obtain the diagram (2.10). It induces a morphism $\kappa : \mathfrak{V}^\vee[1] \longrightarrow L_{U_0/U_1}$.

Proposition 2.3.14 *Under the assumption $R^i p_{D\,*}\mathcal{F} = 0$ for $i > 0$, we have the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{Ob}(\mathcal{F}, \varphi) & \xrightarrow{\mathrm{ob}(\mathcal{F}, \varphi)} & L_{U_0/U_1} \\ \simeq\Big\downarrow & & =\Big\downarrow \\ \mathfrak{V}^\vee[1] & \xrightarrow{\kappa} & L_{U_0/U_1} \end{array}$$

Proof It can be shown by an argument used in the proof of Proposition 2.3.13. $\square$

2.4 Obstruction Theory

2.4.1 Definition and Fundamental Theorems

In the study of Gromov-Witten theory, M. Kontsevich, J. Li-G. Tian, K. Behrend-B. Fantechi and K. Fukaya-K. Ono introduced the notion of virtual fundamental classes of moduli stacks with some good structure. (See [71], [82], [6] and [40]. See also the recent work of I. Ciocan-Fontanine and M. Kapranov [15].) In this paper, we follow the framework of Behrend-Fantechi. See [6] for more details and precise. The paper [55] of T. Graber-R. Pandharipande is also very useful, in which they studied localization of virtual fundamental classes.

Definition 2.4.1 *Let S be an algebraic stack, and let $\mathcal{X}$ be an algebraic stack over S. Let $E^{\bullet}$ be an object in $D(\mathcal{X})$ such that $\mathcal{H}^i(E^{\bullet})$ are coherent $(i = -1, 0, 1)$. A homomorphism $\phi : E^{\bullet} \longrightarrow L_{\mathcal{X}/S}$ is called an obstruction theory for $\mathcal{X}/S$, if $\mathcal{H}^i(\phi)$ $(i \geq 0)$ are isomorphic and $\mathcal{H}^{-1}(\phi)$ is surjective. In that case, $E^{\bullet}$ is also called an obstruction theory for $\mathcal{X}/S$.* $\square$

Because $\mathcal{H}^i(L_{\mathcal{X}}) = 0$ for $i > 1$, the condition implies $\mathcal{H}^i(E^{\bullet}) = 0$ for $i > 1$. If $\mathcal{X}$ is Deligne-Mumford, we also have $\mathcal{H}^1(E^{\bullet}) = \mathcal{H}^1(L_{\mathcal{X}}) = 0$.

We will often use the following theorem of Behrend-Fantechi.

Proposition 2.4.2 (Theorem 4.5, [6]) *Let $\mathcal{X}$ be a Deligne-Mumford stack over S. Let $\phi : E^{\bullet} \longrightarrow L_{\mathcal{X}/S}$ be a morphism in $D(\mathcal{X})$. The following conditions are equivalent.*

- *ϕ is an obstruction theory.*
- *Let T and $\overline{T}$ be S-schemes such that T is a closed subscheme of $\overline{T}$ whose ideal sheaf J is square-zero. Let $g : T \longrightarrow \mathcal{X}$ be a morphism over S.*
  - *(A1) g can be extended to a morphism $\overline{g} : \overline{T} \longrightarrow \mathcal{X}$ over S, if and only if $\phi^*\big(o(g)\big) = 0$ in $\mathrm{Ext}^1\big(g^*E^{\bullet}, J\big)$, where $o(g)$ is the obstruction class of g. (See Proposition* 2.3.1.*)*
  - *(A2) If $\phi^*\big(o(g)\big) = 0$, the set of the extension classes of g is a torsor over the group $\mathrm{Ext}^0\big(g^*E^{\bullet}, J\big)$.* $\square$

We recall the notion of perfect obstruction theory in the sense of Behrend-Fantechi with a minor generalization.

Definition 2.4.3 *Let $\phi : E^{\bullet} \longrightarrow L_{\mathcal{X}/S}$ be an obstruction theory of an algebraic stack $\mathcal{X}$ over S. It is called perfect, if it is quasi-isomorphic to a complex of locally free sheaves $F^{-1} \to F^0 \to F^1$ in the derived category $D(\mathcal{X})$.* $\square$

In that case, the number $-\operatorname{rank} F^1 + \operatorname{rank} F^0 - \operatorname{rank} F^{-1}$ is well defined on each connected component of $\mathcal{X}$. It is called the expected dimension of $\mathcal{X}$ over S with respect to ϕ.

If $\mathcal{X}$ is Deligne-Mumford, we have $\mathcal{H}^1(E^\bullet) = \mathcal{H}^1(L_{\mathcal{X}}) = 0$ for any obstruction theory $E^\bullet$. Hence, a perfect obstruction theory is quasi-isomorphic to a two-term locally free complex $F^{-1} \to F^0$. The important and foundational theorem of Behrend and Fantechi is the following. (See also [82].)

Proposition 2.4.4 (Section 5 [6]) *Let $\mathcal{X}$ be a Deligne-Mumford stack over a smooth scheme S. Let $A_*(\mathcal{X})$ denote the Chow group of $\mathcal{X}$ with rational coefficient. A perfect obstruction theory $\phi : E^\bullet \longrightarrow L_{\mathcal{X}/S}$ induces an element $[\mathcal{X}, \phi] \in A_d(\mathcal{X})$ called virtual fundamental class, where d is the expected dimension with respect to ϕ.*

If $\mathcal{X}$ is smooth, $[\mathcal{X}, \phi]$ is the Euler class of the vector bundle $\mathcal{H}^1\big(E^{\bullet\vee}\big)$. □

We often use the notation $[\mathcal{X}]$ instead of $[\mathcal{X}, \phi]$, if there are no risk of confusion.

Remark 2.4.5 *In Definition 2.4.3, $E^\bullet$ is assumed to be quasi-isomorphic to a complex $F^\bullet$ of locally free sheaves. According to A. Kresch [75], the existence of such a global complex is not necessary for construction of virtual fundamental class. Namely, Proposition 2.4.4 holds, if $E^\bullet$ is locally quasi-isomorphic to a two-term locally free complex.* □

Let S, $\mathcal{X}$, $\phi : E^\bullet \longrightarrow L_{\mathcal{X}/S}$ be as in Proposition 2.4.4. Let S' be a smooth scheme, and let $\iota : S' \longrightarrow S$ be a morphism. We set $\mathcal{X}' := \mathcal{X} \times_S S'$ with the natural morphism $\iota : \mathcal{X}' \longrightarrow \mathcal{X}$.

Proposition 2.4.6 (Proposition 7.2, [6]) *The induced morphism $\iota^*\phi : \iota^* E^\bullet \longrightarrow L_{\mathcal{X}'/S'}$ is also a perfect obstruction theory.*

Let $[\mathcal{X}', \iota^*\phi]$ denote the associated virtual fundamental class. Assume that ι is either (i) a closed regular immersion, or (ii) flat. Then, we have the relation $\iota_1^*[\mathcal{X}, \phi] = [\mathcal{X}', \iota^*\phi]$. □

Let $\mathcal{X}_i$ $(i = 1, 2)$ be algebraic stacks over S provided with obstruction theories $\phi_i : E_i^\bullet \longrightarrow L_{\mathcal{X}_i/S}$. Assume we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{X}_1 & \xrightarrow{f} & \mathcal{X}_2 & & \\
{\scriptstyle g}\downarrow & & \downarrow & & \\
\mathcal{Y}_1 & \xrightarrow{h} & \mathcal{Y}_2 & \longrightarrow & S
\end{array}$$

Recall the following definition in [6].

Definition 2.4.7 *We say that ϕ_i are compatible over h, if we have the following morphism of distinguished triangles on $\mathcal{X}_1$:*

$$\begin{array}{ccccccc}
f^*E_2^\bullet & \longrightarrow & E_1^\bullet & \longrightarrow & g^*L_{\mathcal{Y}_1/\mathcal{Y}_2} & \longrightarrow & f^*E_2^\bullet[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
g^*L_{\mathcal{X}_2/S} & \longrightarrow & L_{\mathcal{X}_1/S} & \longrightarrow & L_{\mathcal{X}_1/\mathcal{X}_2} & \longrightarrow & g^*L_{\mathcal{X}_2/S}[1]
\end{array}$$

We will use the following theorem for comparison of virtual fundamental classes.

Proposition 2.4.8 (Proposition 7.5, [6]) *Assume $\mathcal{X}_i$ are Deligne-Mumford, and the obstruction theories ϕ_i are perfect. If ϕ_i are compatible over h, then we have the equality*

$$h^![\mathcal{X}_2, \phi_2] = [\mathcal{X}_1, \phi_1]$$

at least if h is smooth or $\mathcal{Y}_i$ are smooth over S. □

2.4.2 Easy Example

Let X be a smooth variety over k. We would like to construct an obstruction theory of the moduli spaces $\mathcal{M}$ of some objects on X. A naive strategy is summarized as follows (See [6], for example):

1. Take a classifying stack Y of such objects over X. It means that such objects over $U \times X$ bijectively correspond to morphisms $\Phi : U \times X \longrightarrow Y$ over X. For example, recall that vector bundles of rank R over $U \times X$ correspond to morphisms $U \times X \longrightarrow X_{\mathrm{GL}(R)}$ over X.
2. Any classifying maps $\Phi : U \times X \longrightarrow Y$ induce morphisms

$$\Phi^* L_{Y/X} \longrightarrow L_{U\times X/X}$$

on $U \times X$. Let ω_X denote the dualizing complex on X, i.e., it is the canonical line bundle shifted by the dimension of X. Then, we obtain the following morphisms on U:

$$\mathrm{Ob}_U := Rp_{X*}\big(\Phi^* L_{Y/X} \otimes \omega_X\big) \longrightarrow Rp_{X*}\big(p_X^* L_{U/k} \otimes \omega_X\big) \longrightarrow L_{U/k}.$$

In particular, a morphism $\mathrm{Ob}_{\mathcal{M}} \longrightarrow L_{\mathcal{M}}$ on $\mathcal{M}$ is induced by a universal object.
3. We hope that the morphism $\mathrm{Ob}_{\mathcal{M}} \longrightarrow L_{\mathcal{M}}$ is an obstruction theory, in some cases. Note that the property is local, once the morphism is given globally. Thus we have only to check the claim for sufficiently small étale open subsets of $\mathcal{M}$. Proposition 2.4.2 provides us with a useful tool to check it.

Remark 2.4.9 *In general, we need some modification in construction of* $\mathrm{Ob}_{\mathcal{M}}$ *to obtain a good obstruction theory.* □

Let us look at the easiest example. Let F and V be vector bundles on X. Let U be any scheme over k, and let $f : p_U^*(F) \longrightarrow p_U^*(V)$ be a morphism of $\mathcal{O}_{U\times X}$-modules over $U \times X$. It is easy to see that such a morphism f corresponds to a morphism $\Phi_f : U \times X \longrightarrow N(F, V)$ over X. We obtain a complex

$$\mathfrak{g}(f) := \Phi_f^* L_{N(F,V)/X}$$

and a morphism $\mathfrak{g}(f) \longrightarrow L_{U\times X/X}$ in the derived category $D(U \times X)$.

Lemma 2.4.10 $\mathfrak{g}(f)$ *is represented by* $p_U^* \mathcal{H}om(V, F)$.

Proof Let $\pi : N(F, V) \longrightarrow X$ denote the natural projection. Since the morphism $N(F, V) \longrightarrow X$ is smooth, the cotangent complex $L_{N(F,V)/X}$ is quasi-isomorphic to $\Omega_{N(F,V)/X} \simeq \pi^* \mathcal{H}om(V, F)$, and thus $\Phi_f^* L_{N(F,V)/X} \simeq p_U^* \mathcal{H}om(V, F)$. $\square$

We set $\mathrm{Ob}(f) := R p_{X*}\big(\mathfrak{g}(f) \otimes \omega_X\big)$. Then, we obtain morphisms

$$\mathrm{Ob}(f) \longrightarrow R p_{X*}\big(L_{U \times X/X} \otimes \omega_X\big) \longrightarrow L_U$$

in the derived category $D(U)$. The composite is denoted by $\mathrm{ob}(f)$.

Now, let $M(F, V)$ denote a moduli scheme of morphisms $F \longrightarrow V$, i.e., maps $U \longrightarrow M(F, V)$ bijectively correspond to $f : p_U^*(F) \longrightarrow p_U^*(V)$ on $U \times X$. It is easy to see that $M(F, V)$ is isomorphic to the vector space $H^0\big(X, \mathcal{H}om(F, V)\big)$. We have the universal morphism

$$f^u : p_{M(F,V)}^*(F) \longrightarrow p_{M(F,V)}^*(V)$$

over $M(F, V) \times X$. It induces a morphism $\mathrm{ob}(f^u) : \mathrm{Ob}(f^u) \longrightarrow L_{M(F,V)}$.

Lemma 2.4.11 *ob*(f^u) *gives an obstruction theory of* $M(F, V)$.

Proof It is almost obvious from the universal properties of $N(F, V)$ and $M(F, V)$. However, we give a rather detailed argument as an explanation. We have only to check the conditions (A1) and (A2) in Proposition 2.4.2.

Since the claim is local, we can check the claim for any sufficiently small open subset U of $M(F, V)$. Let T be an affine scheme over k. A morphism $g : T \longrightarrow U$ induces morphisms $g_X : T \times X \longrightarrow U \times X$ and

$$\widetilde{g}_X = \Phi_{f^u} \circ g_X : T \times X \longrightarrow N(F, V)$$

over X. Let $\overline{T}$ denote a scheme such that T is embedded in $\overline{T}$ whose ideal J is square-zero. Deformations of the morphisms g and $\widetilde{g}_X$ is controlled by the groups $\mathrm{Ext}^i\big(g^* L_{U/k}, J\big)$ and $\mathrm{Ext}^i\big(\widetilde{g}_X^* L_{N(F,V)/X}, J_X\big)$, respectively. We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ext}^i\big(g^* L_{U/k}, J\big) \arrow[r, "h"] \arrow[d] & \mathrm{Ext}^i\big(g^* \mathrm{Ob}(f^u), J\big) & \\
\mathrm{Ext}^i\big(g_X^* L_{U \times X/X}, J_X\big) \arrow[r] & \mathrm{Ext}^i\big(g_X^*(\mathfrak{g}), J_X\big) \arrow[u, "\simeq"] \arrow[r, "="] & \mathrm{Ext}^i\big(\widetilde{g}_X^* L_{N(F,V)/X}, J_X\big)
\end{tikzcd}$$

We have the obstruction classes

$$o(g) \in \mathrm{Ext}^1\big(g^* L_{U/k}, J\big), \quad o(\widetilde{g}_X) \in \mathrm{Ext}^1\big(g_X^* \mathfrak{g}, J_X\big)$$

of the morphisms g and $\widetilde{g}_X$ respectively. By the functoriality of the cotangent complex, the obstruction class $o(g)$ is mapped to the obstruction class $o(\widetilde{g}_X)$ in the above diagram.

If the image $h\big(o(g)\big)$ is 0, the class $o(\widetilde{g}_X)$ is 0. Hence, $\widetilde{g}_X$ can be extended to a morphism $\overline{T} \times X \longrightarrow N(F,V)$, which induces a morphism of $p_{\overline{T}}^*(F) \longrightarrow p_{\overline{T}}^*(V)$ on $\overline{T} \times X$. By the universal property of $M(F,V)$, we obtain a morphism

$$\overline{T} \longrightarrow M(F,V)$$

which is an extension of g. Therefore, the condition (A1) is satisfied.

Similarly, we can show that $\mathrm{Ext}^0\big(g^* L_{U/k}, J\big) \longrightarrow \mathrm{Ext}^0\big(\widetilde{g}_X^* L_{N(F,V)/X}, J\big)$ is an isomorphism by the universality of $M(F,V)$ and $N(F,V)$. Hence, the condition (A2) is also satisfied. Thus we are done. $\square$

2.4.3 Locally Free Subsheaves

Let X be a smooth projective variety over k with an ample line bundle $\mathcal{O}_X(1)$. Let V be a locally free sheaf on X. Let W denote an R-dimensional k-vector space. We denote $W \otimes \mathcal{O}_X$ by W_X. We have the natural right $\mathrm{GL}(W)$-action on $N(W_X, V)$. The quotient stack is denoted by $Y_{\mathrm{quo}}(W_\bullet)$.

We consider deformations of locally free subsheaves of V with rank R. Let U be any k-scheme. Any locally free subsheaf $f : F \longrightarrow p_U^* V$ on $U \times X$ with $\operatorname{rank} F = R$ induces a morphism $\Phi(F,f) : U \times X \longrightarrow Y_{\mathrm{quo}}(W_\bullet)$ over X. We put

$$\mathfrak{g}(F,f) := \Phi(F,f)^* L_{Y_{\mathrm{quo}}(W_\bullet)/X}$$

$$\mathrm{Ob}(F,f) := Rp_{X*}\big(\mathfrak{g}(F,f) \otimes \omega_X\big)$$

Then, we obtain morphisms $\mathfrak{g}(F,f) \longrightarrow L_{U\times X/X}$ on $U \times X$, and $\mathrm{ob}(F,f) : \mathrm{Ob}(F,f) \longrightarrow L_U$ on U. The following lemma can be shown by using the argument explained in Subsection 2.3.2.

Lemma 2.4.12 *$\mathfrak{g}(F,f)$ is represented by $\mathrm{Cone}(\alpha)[-1]$ of the morphism*

$$\alpha : \mathcal{H}om(p_U^* V, F) \longrightarrow \mathcal{H}om(F,F)$$

given by $\alpha(a) = a \circ f$. $\square$

Remark 2.4.13 *We put $V_{-1} := F$ and $V_0 := p_U^* V$, and we regard $V_\bullet = (V_{-1} \to V_0)$ as a complex, where V_0 stands in the degree 0. Then, $\mathrm{Cone}(\alpha)$ is naturally isomorphic to $\mathcal{H}om(V_{-1}[1], V_\bullet)^\vee[-1]$.* $\square$

Let H be a polynomial. We have a moduli scheme of quotients $(q : V \longrightarrow Q)$ of V such that the Hilbert polynomials of Q are H. Let $M(V,H)$ denote the open subscheme which consists of the points $(q : V \longrightarrow Q)$ such that $\mathrm{Ker}(q)$ are locally free. Then, we have the universal family $f^u : F^u \longrightarrow p_{M(V,H)}^*(V)$ defined over $M(V,H) \times X$. A morphism $\mathrm{ob}(F^u, f^u) : \mathrm{Ob}(F^u, f^u) \longrightarrow L_{M(V,H)}$ is induced.

Proposition 2.4.14 $\mathrm{ob}(F^u, f^u)$ *is an obstruction theory of* $M(V, H)$.

Proof Let N be a sufficiently large number satisfying the condition O_N for the family F^u, i.e., we have $H^i\bigl(X, F^u(N)_{|\{q\}\times X}\bigr) = 0$ for any $q \in M(V, H)$ and $i > 0$, and $F^u_{|\{q\}\times X}(N)$ are globally generating for any $q \in M(V, H)$. We put $\widetilde{F}^u := p_X^* p_{X*}\bigl(F^u(N)\bigr) \otimes \mathcal{O}(-N)$. We have the natural surjection $g : \widetilde{F}^u \longrightarrow F^u$.

We put $\overline{F} = \mathcal{O}(-N)^{\oplus d}$, where $d = \operatorname{rank} \widetilde{F}^u$. We have the Grassmaniann bundle $\pi : Gr(\overline{F}, R) \longrightarrow X$ associated to the vector bundle $\overline{F}$, i.e., the fiber of π over a point $x \in X$ is the Grassmann variety of R-dimensional quotient spaces of the vector space $\overline{F}_{|x}$. We denote the universal quotient bundle over $Gr(\overline{F}, R)$ by Q. Then, we have the vector bundle $\widetilde{Y}_{\mathrm{quo}} := N(Q, \pi^* V)$ over $Gr(\overline{F}, R)$, which is a variety smooth over X. We have the natural morphism $\pi_1 : \widetilde{Y}_{\mathrm{quo}} \longrightarrow Y_{\mathrm{quo}}(W_\bullet)$.

We would like to check the conditions (A1) and (A2) in Proposition 2.4.2. Let U be any sufficiently small open set of $M(V, H)$, on which we can assume that there exists an isomorphism $\widetilde{F}^u \simeq p_U^* \overline{F}$. Thus, the surjection $\gamma : p_U^* \overline{F} \longrightarrow F^u$ is given on $U \times X$. From γ and f^u, we obtain a morphism

$$\Phi(\gamma, F^u, f^u) : U \times X \longrightarrow \widetilde{Y}_{\mathrm{quo}}$$

over X. By the argument in Subsection 2.3.2, we can show that the complex $\Phi(\gamma, F^u, f^u)^* L_{\widetilde{Y}_{\mathrm{quo}}/X}$ is represented by the cone $\mathrm{Cone}(\beta)[-1]$ for the morphism

$$\beta : \mathcal{H}om(p_U^* V, F^u) \oplus \mathcal{H}om(F^u, p_U^* \overline{F}) \longrightarrow \mathcal{H}om(F^u, F^u)$$

$$\beta(b_1, b_2) = b_1 \circ f^u - f^u \circ b_2.$$

We can also show that the natural morphisms

$$\mathrm{Cone}(\alpha)[-1] \longrightarrow \mathrm{Cone}(\beta)[-1] \longrightarrow L_{U\times X/X}$$

is the same as the factorization:

$$\Phi(F^u, f^u)^* L_{Y_{\mathrm{quo}}/X} \longrightarrow \Phi(\gamma, F^u, f^u)^* L_{\widetilde{Y}_{\mathrm{quo}}/X} \longrightarrow L_{U\times X/X}$$

associated to $U \times X \longrightarrow \widetilde{Y}_{\mathrm{quo}} \longrightarrow Y_{\mathrm{quo}}(W_\bullet)$. We set

$$\mathfrak{g}(\gamma, F^u, f^u) := \Phi(\gamma, F^u, f^u)^* L_{\widetilde{Y}_{\mathrm{quo}}/X}$$

$$\mathrm{Ob}(\gamma, F^u, f^u) := Rp_{X*}\bigl(\mathfrak{g}(\gamma, F^u, f^u) \otimes \omega_X\bigr).$$

Let T be an affine scheme. Let $g : T \longrightarrow U$ be a morphism, which induces $g_X : T \times X \longrightarrow U \times X$. We put $\widetilde{g}_X := \Phi(F^u, f^u) \circ g_X$ and $\widehat{g}_X := \Phi(\gamma, F^u, f^u) \circ g_X$. For any coherent sheaf J on T, we have the following commutative diagram:

$$
\begin{array}{ccc}
\operatorname{Ext}^i(g^*L_{U/k},J) & \longrightarrow & \operatorname{Ext}^i(g_X^*L_{U\times X/X},J_X) \\
{\scriptstyle h_1^i}\downarrow & & \downarrow \\
\operatorname{Ext}^i\big(g^*\operatorname{Ob}(\gamma,F^u,f^u),J\big) & \longrightarrow & \operatorname{Ext}^i\big(g_X^*\mathfrak{g}(\gamma,F^u,f^u),J_X\big) \\
{\scriptstyle h_2^i}\downarrow & & \downarrow \\
\operatorname{Ext}^i(g^*\operatorname{Ob}(F^u,f^u),J) & \longrightarrow & \operatorname{Ext}^i\big(g_X^*\mathfrak{g}(F^u,f^u),J_X\big)
\end{array}
\tag{2.11}
$$

Let $\overline{T}$ be an affine scheme into which T is embedded closedly such that the corresponding ideal J is square-zero. According to the deformation theory of Illusie, we have the obstruction classes of the morphisms g and $\widehat{g}_X$ in the groups $\operatorname{Ext}^1(g^*L_{U/k},J)$ and $\operatorname{Ext}^1\big(g_X^*\mathfrak{g}(\gamma,F^u,f^u),J\big)$ respectively. The classes are denoted by $o(g)$ and $o(\widehat{g}_X)$. By functoriality, $o(g)$ is mapped to $o(\widehat{g}_X)$ in the diagram (2.11). If $h_1^1(o(g))$ is 0, then the morphism $\widehat{g}_X$ can be extended.

Note that the cohomology sheaves $R^ip_{X*}\big(\mathcal{H}om(F^u,p_U^*\overline{F})\otimes\omega_X\big)$ vanish unless $i=0$, because of our choice of N. Thus, we have the isomorphisms

$$\operatorname{Ext}^i\big(g^*\operatorname{Ob}(\gamma,F^u,f^u),J\big)\simeq\operatorname{Ext}^i\big(g^*\operatorname{Ob}(F^u,f^u),J\big)$$

for any $i>0$ and for any coherent sheaf J on T. Hence, $h_2^1\circ h_1^1(o(g))=0$ implies $h_1^1\big(o(g)\big)=0$. Then, the morphism $\widehat{g}_X$ can be extended over $\overline{T}\times X$, and hence $\widetilde{g}_X$ can also be extended over $\overline{T}\times X$. Therefore, we obtain a locally free subsheaf $\widehat{\overline{F}}$ of $p_{\overline{T}}^*(V)$ on $\overline{T}\times X$, which is an extension of $g_X^*F^u$. By the universal property of $M(V,H)$, the morphism g can be extended over $\overline{T}$. Therefore, the condition (A1) is satisfied.

Let us check the condition (A2). We set

$$
\begin{aligned}
H_0 &:= \operatorname{Ext}^0\Big(p_{X*}\big(g_X^*\mathcal{H}om(F^u,p_U^*\overline{F})\otimes\omega_X\big),J\Big)\\
&\quad= H^0\Big(T,g^*\mathcal{E}nd\big(p_{X*}\big(F^u(m)\big)\big)\otimes J\Big)
\end{aligned}
$$

$$H_1:=\operatorname{Ext}^0\big(g^*\operatorname{Ob}(g,F^u,f^u),J\big)=\operatorname{Ext}^0\big(\widehat{g}_X^*L_{\widetilde{Y}_{\mathrm{quo}}/X},J_X\big)$$

$$H_2:=\operatorname{Ext}^0\big(g^*\operatorname{Ob}(F^u,f^u),J\big)$$

We obtain an exact sequence $0\longrightarrow H_0\longrightarrow H_1\longrightarrow H_2\longrightarrow 0$. According to the theory of Illusie, H_1 parameterizes the set of the extensions $\widehat{g}'_X:\overline{T}\times X\longrightarrow\widetilde{Y}_{\mathrm{quo}}$ of $\widehat{g}_X$. The natural action of H_0 on H_1 determines the equivalence relation on H_1, and it is easy to see that $\widehat{g}'_X\sim\widehat{g}''_X$ if and only if $\pi_1\circ\widehat{g}'_X=\pi_1\circ\widehat{g}''_X$, because H_0 parameterizes the deformations of the morphism $\overline{F}\longrightarrow F^u$. Thus the set of the extensions of the morphism $T\times X\longrightarrow Y_{\mathrm{quo}}(W_\bullet)$ over $\overline{T}\times X$ is a torsor over the group H_2.

By the universal property of $M(V,H)$ and $Y_{\mathrm{quo}}(W_\bullet)$, the set of the extensions of g over $\overline{T}$ is also a torsor over H_2. Namely the condition (A2) is satisfied. □

Usually, we consider deformations of quotients of V. Let H be a polynomial, and let $\mathrm{Quot}(V,H)$ denote a quot scheme which parameterizes the quotient sheaves of V whose Hilbert polynomials are H. We have the universal quotient $q^u : p^*_{\mathrm{Quot}(V,H)}(V) \longrightarrow Q^u$ on $\mathrm{Quot}(V,H) \times X$. We denote the kernel of q^u by F^u, and the inclusion $F^u \longrightarrow p^*_{\mathrm{Quot}(V,H)}(V)$ is denoted by f^u.

Let us consider the case $\dim X = 1$. Let H_V denote the Hilbert polynomial of V. Then, $\mathrm{Quot}(V,H)$ parameterizes the locally free subsheaves of V whose Hilbert polynomials are $H_V - H$. Therefore, we have obtained an obstruction theory $\mathrm{ob}(F^u, f^u) : \mathrm{Ob}(F^u, f^u) \longrightarrow L_{\mathrm{Quot}(V,H)}$.

Proposition 2.4.15 *In the case $\dim(X) = 1$, the obstruction theory $\mathrm{ob}(f^u)$ is perfect. The scheme $\mathrm{Quot}(V,H)$ is smooth, if H is a constant, i.e., H is a Hilbert polynomial of sheaves of finite length.*

Proof To show the perfectness of $\mathrm{Ob}(F^u, f^u)$, we have only to show that

$$Rp_{X\,*}\big(\mathfrak{g}(F^u, f^u)^\vee\big)$$

is perfect of amplitude contained in $[0,1]$. Let q be any point of $\mathrm{Quot}(V,H)$. We put $F := F^u_{|\{q\}\times X}$ and $Q := V/F$. The complex $\mathfrak{g}(F^u, f^u)^\vee_{|\{q\}\times X}$ is $\mathrm{Cone}(\gamma)[-1]$ for the natural morphism $\gamma : \mathcal{H}om(F,F) \longrightarrow \mathcal{H}om(F,V)$, which is quasi-isomorphic to $\mathcal{H}om(F,Q)$. Hence, $H^i\big(X, \mathfrak{g}(F^u, f^u)^\vee_{|\{q\}\times X}\big) = 0$ for any point $q \in \mathrm{Quot}(V,H)$ unless $i = 0, 1$. Then, the desired perfectness easily follows.

Let us show the second claim. If H is a constant, we have

$$H^1\big(X, \mathfrak{g}(F^u, f^u)^\vee_{|\{q\}\times X}\big) = 0$$

for any $q \in \mathrm{Quot}(V,H)$. Let T be any affine scheme over k, and let $g : T \longrightarrow \mathrm{Quot}(V,H)$ be a morphism. Then, $\mathrm{Ext}^1\big(g^*\,\mathrm{Ob}(F^u, f^u), J\big) = 0$ for any coherent $\mathcal{O}_T$-module J, and hence any obstruction classes vanish. Thus we obtain the smoothness. □

Remark 2.4.16 *Let us consider the case $\dim X = 2$. Let $Q^{tf}(V,H)$ denote the open subset of $Q(F,H)$ corresponding to torsion-free quotient sheaves. It gives an open subset of a moduli stack of locally free subsheaves of V. Then, $F^u_{|\{q\}\times X}$ are locally free for any $q \in Q^{tf}(V,H)$. Therefore, we obtain an obstruction theory $\mathrm{ob}(F^u, f^u) : \mathrm{Ob}(F^u, f^u) \longrightarrow L_{Q^{tf}(V,H)/k}$ from Proposition* 2.4.14. □

2.4.4 Filtrations of a Vector Bundle on a Curve

Let S be a scheme over k, and let $\mathcal{D}$ be a smooth projective curve over S provided with an ample line bundle $\mathcal{O}(1)$. The projection $\mathcal{D} \longrightarrow S$ is denoted by p. Let V and F be locally free sheaves on $\mathcal{D}$ provided with an injective morphism $f : F \longrightarrow V$. Assume that the quotient is S-flat.

Let H_i be polynomials. Let $g : T \longrightarrow S$ be an S-scheme. We have the induced curve $\mathcal{D}_T := \mathcal{D} \times_S T$ over T. The induced morphism $\mathcal{D}_T \longrightarrow \mathcal{D}$ is denoted by $\widetilde{g}$. We obtain a morphism $\widetilde{g}^* F \longrightarrow \widetilde{g}^* V$ on $\mathcal{D}_T$. Let $\mathcal{F}(T, g)$ denote the set of the filtrations V^* of $\widetilde{g}^* V$ on $\mathcal{D}_T$,

$$\widetilde{g}^* V = V^{(1)} \supset V^{(2)} \supset \cdots \supset V^{(l)} \supset V^{(l+1)} = \widetilde{g}^* F,$$

with the following property:

- The quotients $\mathrm{Cok}_i := V^{(1)}/V^{(i+1)}$ are T-flat.
- The Hilbert polynomials of $\mathrm{Cok}_{i|\mathcal{D}_t}$ are H_i for any $i = 1, \ldots, l$ and $t \in T$.

The functor $\mathcal{F}$ is representable by an S-scheme, which can be shown by the standard technique using quot schemes. Let $g_{M(H_*)} : M(H_*) \longrightarrow S$ denote a moduli S-scheme. We have the universal filtration on $M(H_*) \times_S \mathcal{D}$:

$$\widetilde{g}^*_{M(H_*)} V = \mathcal{V}^{(1)} \supset \mathcal{V}^{(2)} \supset \cdots \supset \mathcal{V}^{(l)} \supset \mathcal{V}^{(l+1)} = \widetilde{g}^*_{M(H_*)} F.$$

To construct an obstruction theory of $M(H_*)$, we introduce some stacks. Take vector spaces W_i $(i = 2, \ldots, l)$ over k such that $\operatorname{rank} W_i = \operatorname{rank} \mathcal{V}^{(i)} =: r_i$. We put $W^{(i)} := W_i \otimes \mathcal{O}_{\mathcal{D}}$ $(i = 2, \ldots, l)$. We set $W^{(1)} := V$ and $W^{(l+1)} := F$. We define $Y_0 := N(W^{(l+1)}, W^{(1)})$ and $R_1 := \prod_{i=1}^{l} N(W^{(i+1)}, W^{(i)})$. We put $G(W_*) := \prod_{i=2}^{l} \mathrm{GL}(W_i)$. We have the natural right $G(W_*)$-action on R_1. Let Y_1 denote the quotient stack of R_1 by the $G(W_*)$-action. By composition of the maps, we obtain a morphism $\phi : R_1 \longrightarrow Y_0$, which induces $Y_1 \longrightarrow Y_0$. We put $Y_2 := \mathcal{D}$. Then, the morphism $F \longrightarrow V$ induces a morphism $Y_2 \longrightarrow Y_0$. We put $Y := Y_1 \times_{Y_0} Y_2$.

Let V^* denote a filtered vector bundle on $\mathcal{D}_T$ as above. We obtain morphisms $\Phi_i(V^*) : \mathcal{D}_T \longrightarrow Y_i$. We have the naturally defined morphism:

$$G(V^*) : \Phi_0(V^*)^* L_{Y_0/\mathcal{D}} \longrightarrow \bigoplus_{i=1,2} \Phi_i(V^*)^* L_{Y_i/\mathcal{D}}$$

We use the notation in Subsection 2.1.6. We put $\mathfrak{g}(V^*) := C_2(V^*, V^*)^\vee[-1]$. By the argument in Subsection 2.3.2, the cone of $G(V^*)$ is represented by the complex $\mathfrak{g}(V^*)$. Thus, we have the naturally defined morphism $\mathfrak{g}(V^*) \longrightarrow L_{\mathcal{D}_T/\mathcal{D}}$. We put $\mathrm{Ob}(V^*) := Rp_*\big(\mathfrak{g}(V^*) \otimes \omega_{\mathcal{D}_T/T}\big)$. Then, we obtain a morphism

$$\mathrm{ob}(V^*) : \mathrm{Ob}(V^*) \longrightarrow L_{T/S}.$$

Applying the construction to the universal filtered bundle $\mathcal{V}^*$ on $M(H_*) \times_S \mathcal{D}$, we obtain

$$\mathrm{ob}(\mathcal{V}^*) : \mathrm{Ob}(\mathcal{V}^*) \longrightarrow L_{M(H_*)/S}.$$

Lemma 2.4.17 *The morphism* $\mathrm{ob}(\mathcal{V}^*)$ *gives an obstruction theory of* $M(H_*)$.

Proof In the following, we will shrink S without mention. Take locally free sheaves $J^{(i)}$ $(i = 2, \ldots, l)$ on $\mathcal{D}$ such that (i) there exist surjections $J^{(i)} \longrightarrow \mathcal{V}^{(i)}$,

(ii) $R^1 p_* \mathcal{H}om(J^{(i)}, \mathcal{V}^{(i)}) = 0$. For any S-scheme $g : T \longrightarrow S$, let $\widetilde{\mathcal{F}}(T)$ denote the set of (V^*, φ_*) on $\mathcal{D}_T$ as follows:

- V^* denotes a filtration of $\widetilde{g}^* V$ as above.
- φ_* denotes a tuple of surjections of $\widetilde{g}^* J^{(i)}$ onto $V^{(i)}$.

The functor $\widetilde{\mathcal{F}}$ is representable by a scheme which is denoted by $\widetilde{M}(H_*)$. We have the locally free sheaves $N_i := \mathcal{H}om\big(\widetilde{g}^*_{M(H_*)} J^{(i)}, \mathcal{V}^{(i)}\big)$ on $M(H_*) \times_S \mathcal{D}$. Let $p : M(H_*) \times_S \mathcal{D} \longrightarrow M(H_*)$ denote the projection. Then, $\widetilde{M}(H_*)$ is isomorphic to an open subset of the vector bundle $\bigoplus p_* N_i$. On $\widetilde{M}(H_*) \times_S \mathcal{D}$, we have the universal filtration $\mathcal{V}^*$ with the tuple of surjective morphisms φ_*^u.

Let $Gr(J^{(i)}, r_i)$ be the Grassmannian bundles of r_i-dimensional quotient spaces associated to the vector bundles $J^{(i)}$. We have the universal quotient bundle Q_i. We put $Z := \prod_{i=2}^{l} Gr(J^{(i)}, r_i)$, where the fiber product is taken over $\mathcal{D}$. The pull back of Q_i via the projection $Z \longrightarrow Gr(J^{(i)}, r_i)$ are denoted by $\widetilde{W}^{(i)}$ ($i = 2, \ldots, l$). The pull back of V and F via the projection $Z \longrightarrow \mathcal{D}$ are denoted by $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(l+1)}$ respectively. Then, we put $\widetilde{Y}_0 := N\big(\widetilde{W}^{(l+1)}, \widetilde{W}^{(1)}\big)$, $\widetilde{Y}_1 := \prod_{i=1}^{l} N(\widetilde{W}^{(i+1)}, \widetilde{W}^{(i)})$ and $\widetilde{Y}_2 := Z$. We have the natural morphisms $\widetilde{Y}_i \longrightarrow \widetilde{Y}_0$ ($i = 1, 2$) as above. The fiber product $\widetilde{Y}_1 \times_{\widetilde{Y}_0} \widetilde{Y}_2$ is denoted by $\widetilde{Y}$. The inclusions $\widetilde{Y} \longrightarrow \widetilde{Y}_i$ are denoted by j_i. On $\widetilde{Y}$, we have the natural morphism $j_0^* L_{\widetilde{Y}_0/\mathcal{D}} \longrightarrow \bigoplus_{i=1,2} j_i^* L_{\widetilde{Y}_i/\mathcal{D}}$, whose cone is denoted by $\mathrm{Ob}(\widetilde{Y})$. Then, we have the naturally defined morphism $\mathrm{ob}(\widetilde{Y}) : \mathrm{Ob}(\widetilde{Y}) \longrightarrow L_{\widetilde{Y}/\mathcal{D}}$, and it gives an obstruction theory for $\widetilde{Y}$ over $\mathcal{D}$. (Basic example in [6]).

Let $g : T \longrightarrow S$ be an S-scheme. From (V^*, φ_*) on $\mathcal{D}_T$, we obtain morphisms $\Phi_i(V^*, \varphi_*) : \mathcal{D}_T \longrightarrow \widetilde{Y}_i$. Therefore, we obtain

$$\Phi(V^*, \varphi_*)^* \,\mathrm{Ob}(\widetilde{Y}) \longrightarrow L_{\mathcal{D}_T/T}.$$

We put $\widetilde{\mathrm{Ob}}(V^*, \varphi_*) := Rp_*\big(\mathrm{Ob}(\widetilde{Y}) \otimes \omega_{\mathcal{D}/S}\big)$, and then we obtain a morphism $\widetilde{\mathrm{ob}}(V^*, \varphi_*) : \widetilde{\mathrm{Ob}}(V^*, \varphi_*) \longrightarrow L_{T/S}$.

Let us describe the complex $\widetilde{\mathrm{Ob}}(V^*, \varphi_*)$. We have the morphisms

$$\mathcal{H}om\big(V^{(i)}, J^{(i)}\big) \longrightarrow \mathcal{H}om\big(V^{(i)}, V^{(i)}\big), \quad a_i \longmapsto \varphi_i \circ a_i.$$

It induces a morphism of the complexes

$$\alpha : \bigoplus_{i=2}^{l} \mathcal{H}om\big(V^{(i)}, J^{(i)}\big)[-1] \longrightarrow \mathfrak{g}(V^*)$$

We put $\widetilde{\mathfrak{g}}(V^*) := \mathrm{Cone}(\alpha)$. By using the argument in Subsection 2.3.2, we can show that $\widetilde{\mathfrak{g}}(V^*)$ represents $\Phi(V^*, \varphi_*)^* \,\mathrm{Ob}(\widetilde{Y})$.

Applying the above construction to $(\mathcal{V}^*, \varphi_*^u)$, we obtain a morphism

$$\widetilde{\mathrm{ob}}(\mathcal{V}^*, \varphi_*^u) : \widetilde{\mathrm{Ob}}(\mathcal{V}^*, \varphi_*^u) \longrightarrow L_{\widetilde{M}(H_*)/S}.$$

Lemma 2.4.18 *The morphism* $\widetilde{\mathrm{ob}}(\mathcal{V}^*, \varphi_*^u)$ *gives an obstruction theory of* $\widetilde{M}(H_*)$ *over* S.

Proof Let $h : T \longrightarrow \widetilde{M}(H_*)$ be a morphism, and let J be a coherent sheaf on T. The pull back of J via $\mathcal{D}_T \longrightarrow T$ is denoted by $J_{\mathcal{D}}$. We have the induced morphism $\widetilde{h} : \mathcal{D}_T \longrightarrow \widetilde{M}(H_*) \times_S \mathcal{D}$. We set $\widehat{h}_{\mathcal{D}} := \Phi(\mathcal{V}^*, \varphi_*^u) \circ h_{\mathcal{D}}$. We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ext}^1\big(h^* L_{\widetilde{M}(H_*)/S}, J\big) \arrow[r, "\psi"] \arrow[d] & \mathrm{Ext}^1\big(h^*\widetilde{\mathrm{Ob}}, J\big) \\
\mathrm{Ext}^1\big(h_{\mathcal{D}}^* L_{\widetilde{M}(H_*)\times_S\mathcal{D}/\mathcal{D}}, J_{\mathcal{D}}\big) \arrow[r] & \mathrm{Ext}^1\big(h_{\mathcal{D}}^*\widetilde{\mathfrak{g}}(\mathcal{V}^*), J_{\mathcal{D}}\big) \arrow[u, "\simeq"']
\end{tikzcd}$$

Let $\overline{T}$ be an S-scheme such that T is embedded as a closed subscheme and that the corresponding ideal J is square-zero. We have the obstruction classes $o(h)$ and $o(\widehat{h}_{\mathcal{D}})$ in $\mathrm{Ext}^1\big(h^* L_{\widetilde{M}(H_*)/S}, J\big)$ and $\mathrm{Ext}^1\big(\widehat{h}_{\mathcal{D}}^* L_{\widetilde{Y}/S}, J_X\big)$. It is easy to see that $\psi\big(o(h)\big) \in \mathrm{Ext}^1\big(h^*\widetilde{\mathrm{Ob}}, J\big)$ is the same as the image of $o(\widehat{h}_{\mathcal{D}})$ via the composite of the following morphisms:

$$\mathrm{Ext}^1\big(\widehat{h}_{\mathcal{D}}^* L_{\widetilde{Y}/S}, J_{\mathcal{D}}\big) \xrightarrow{b_1} \mathrm{Ext}^1\big(h_{\mathcal{D}}^*\widetilde{\mathfrak{g}}(\mathcal{V}^*), J_{\mathcal{D}}\big) \xrightarrow{b_2} \mathrm{Ext}^1\big(h^*\widetilde{\mathrm{Ob}}, J\big)$$

Hence, the vanishing of $\psi\big(o(h)\big)$ implies $b_1\big(o(\widehat{h}_{\mathcal{D}})\big) = 0$. Since $\widetilde{\mathrm{Ob}}$ gives an obstruction theory for $\widetilde{Y}$, it implies that $\widehat{h}$ can be extended to a morphism $\overline{T} \times_S \mathcal{D} \longrightarrow \widetilde{Y}$. Then, we obtain an extension of h to a morphism $\overline{T} \longrightarrow \widetilde{M}(H_*)$ by the universal property of $\widetilde{M}(H_*)$. Therefore, the condition (A1) of Proposition 2.4.2 is checked. The condition (A2) can also be checked easily, and the proof of Lemma 2.4.18 is finished. $\square$

Let π denote the projection $\widetilde{M}(H_*) \longrightarrow M(H_*)$, which is smooth. We have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{M}(H_*) \times_S \mathcal{D} \arrow[r, "\Phi_i(\mathcal{V}^*, \varphi_*^u)"] \arrow[d, "\pi_{\mathcal{D}}"'] & \widetilde{Y}_i \arrow[d] \\
M(H_*) \times_S \mathcal{D} \arrow[r, "\Phi_i(\mathcal{V}^*)"] & Y_i
\end{tikzcd}$$

We obtain the following morphism of the distinguished triangles on $\widetilde{M}(H_*) \times_S \mathcal{D}$:

$$\begin{array}{ccccc}
\pi_{\mathcal{D}}^*\mathfrak{g}(\mathcal{V}^*) & \longrightarrow & \Phi(\mathcal{V}^*, \varphi_*^u)^* \operatorname{Ob}(\widetilde{Y}) & \longrightarrow \\
\downarrow & & \downarrow & \\
\pi_{\mathcal{D}}^* L_{M(H_*)\times_S\mathcal{D}/\mathcal{D}} & \longrightarrow & L_{\widetilde{M}(H_*)\times_S\mathcal{D}/\mathcal{D}} & \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
\bigoplus_{i=2}^{l} \mathcal{H}om(\mathcal{V}^{(i)}, J^{(i)}) & \longrightarrow & \pi_{\mathcal{D}}^*\mathfrak{g}(\mathcal{V}^*)[1] \\
\downarrow & & \downarrow \\
L_{\widetilde{M}(H_*)\times_S\mathcal{D}/M(H_*)\times_S\mathcal{D}} & \longrightarrow & \pi_{\mathcal{D}}^* L_{M(H_*)\times_S\mathcal{D}/\mathcal{D}}[1]
\end{array} \tag{2.12}$$

Hence, we obtain the following morphism of the distinguished triangles:

$$\begin{array}{ccccc}
\pi^* \operatorname{Ob}(\mathcal{V}^*) & \longrightarrow & \widetilde{\operatorname{Ob}}(\mathcal{V}^*, \varphi^*) & \longrightarrow \\
\downarrow & & \downarrow & \\
\pi^* L_{M(H_*)} & \longrightarrow & L_{\widetilde{M}(H_*)/S} & \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
\bigoplus_i \big(p_*\mathcal{H}om(J^{(i)}, \mathcal{V}^{(i)})\big)^\vee & \longrightarrow & \pi^* \operatorname{Ob}(\mathcal{V}^*)[1] \\
{\scriptstyle\varphi}\downarrow & & \downarrow \\
L_{\widetilde{M}(H_*)/M(H_*)} & \longrightarrow & \pi^* L_{M(H_*)/S}[1]
\end{array} \tag{2.13}$$

It is easy to see that both $L_{\widetilde{M}(H_*/S)}$ and $\bigoplus_i \big(p_*\mathcal{H}om(J^{(i)}, \mathcal{V}^{(i)})\big)^\vee$ are isomorphic to their 0-th cohomology sheaves, and that φ is an isomorphism. Then, the claim of Lemma 2.4.17 immediately follows from Lemma 2.4.18. $\square$

2.5 Equivariant Complexes on Deligne-Mumford Stacks with GIT Construction

The results in this section will be used when we consider equivariant obstruction theory of master spaces in Section 5.8.

2.5.1 Locally Free Resolution

Let G_i $(i = 1, 2)$ be linear reductive groups over k. Let U be a quasi-projective variety over k provided with an action of $G_1 \times G_2$. We assume that there exists a

$G_1 \times G_2$-equivariant embedding into a projective space $\mathbb{P}^N$. The closure of U in $\mathbb{P}^N$ is denoted by $\overline{U}$. The $G_1 \times G_2$-equivariant polarization is denoted by $\mathcal{O}(1)$. We assume that U is contained in the open subset of the stable points of $\overline{U}$ with respect to the polarization $\mathcal{O}(1)$ and the G_2-action. We assume that $\mathcal{M} = U/G_2$ is a separated Deligne-Mumford stack. The projection $U \longrightarrow \mathcal{M}$ is denoted by π.

Lemma 2.5.1 *Let $\mathcal{F}$ be a G_1-equivariant coherent sheaf on $\mathcal{M}$. Then, there exists a G_1-equivariant locally free sheaf $\mathcal{V}$ on $\mathcal{M}$ with a G_1-equivariant surjection $\phi : \mathcal{V} \longrightarrow \mathcal{F}$.*

Proof There exists a coherent sheaf $\mathcal{G}$ on $\overline{U}$ such that $\mathcal{G}_{|U} = \pi^*\mathcal{F}$. There exists a large number N such that $\mathcal{G}(N)$ is globally generating. Then, $\pi^*\mathcal{F}(N)$ is also globally generating. We may take a $G_1 \times G_2$-equivariant subspace W of $H^0\big(U, \pi^*\mathcal{F}(N)\big)$ such that $W \otimes \mathcal{O}(-N) \longrightarrow \pi^*\mathcal{F}$ is surjective. We have only to take descent of $W \otimes \mathcal{O}(-N)$ and the morphism. □

Corollary 2.5.2 *Let $\mathcal{F}_\bullet$ be a bounded G_1-equivariant complex of coherent sheaves on $\mathcal{M}$. Assume that there exist integers M_1 and M_2 such that the following holds:*

- *For any point of $\mathcal{M}$, there exists a neighbourhood $\mathcal{U}$ such that $\mathcal{F}_{\bullet|\mathcal{U}}$ is isomorphic to a coherent locally free complex $\mathcal{G}^{\mathcal{U}}_\bullet$ in $D(\mathcal{U})$ where $\mathcal{G}^{\mathcal{U}}_i = 0$ unless $M_1 \leq i \leq M_2$.*

Then, there exists a global G_1-equivariant coherent locally free complex $\mathcal{G}_\bullet \simeq \mathcal{F}$ in $D(\mathcal{M})$, where $\mathcal{G}_i = 0$ unless $M_1 \leq i \leq M_2$. □

2.5.2 Equivariant Representative

We recall that the morphism of $\mathcal{M}$ to the coarse scheme is finite (Proposition 2.2.2).

Lemma 2.5.3 *Let $C_{i\,\bullet}$ $(i = 1, 2)$ be G_1-equivariant bounded complexes of coherent sheaves on $\mathcal{M}$. We assume that $C_{1\,\bullet}$ is perfect. Let φ be an element of the G_1-invariant part of $\mathrm{Ext}^0(C_{1\,\bullet}, C_{2\,\bullet})$. Then, we can take a G_1-equivariant perfect complex $\widetilde{C}_{1\,\bullet}$ with a G_1-equivariant morphism $\psi : \widetilde{C}_{1\,\bullet} \longrightarrow C_2$, such that (i) $\widetilde{C}_{1\,\bullet}$ is G_1-equivariantly quasi-isomorphic to $C_{1\,\bullet}$, (ii) ψ represents φ.*

Proof We give only an indication. We may assume that $C_{2,i} = 0$ unless $|i| < N$. We take a sufficiently large number N_1, and we replace $C_{1\,\bullet}$ with a G_1-equivariant quasi-isomorphic complex $\widetilde{C}_{1\,\bullet}$ with the property $\mathrm{Ext}^k\big(\widetilde{C}_{1,i}, C_{2,j}\big) = 0$ for any $k > 0$ and $i > -N_1$, and for any j. Then, $\mathrm{Ext}^0(C_{1,\bullet}, C_{2,\bullet}) \simeq \mathrm{Ext}^0(\widetilde{C}_{1,\bullet}, C_{2,\bullet})$ is isomorphic to the first cohomology of the following:

$$\bigoplus_{-i+j=-1} \mathrm{Ext}^0(\widetilde{C}_{1,i}, C_{2,j}) \longrightarrow \bigoplus_{-i+j=0} \mathrm{Ext}^0(\widetilde{C}_{1,i}, C_{2,j}) \longrightarrow \bigoplus_{-i+j=1} \mathrm{Ext}^0(\widetilde{C}_{1,i}, C_{2,j})$$

Since G_1 is assumed to be reductive, the claim is clear. □

Let $B^{(i)}$ $(i=1,2)$ be G_1-equivariant bounded complexes on $\mathcal{M}$. We assume that $B^{(1)}$ is perfect. Let ϕ be an element of the G_1-invariant part of $\operatorname{Ext}^0\big(B^{(1)}, B^{(2)}\big)$. We take a G_1-equivariant perfect complex $\widetilde{B}^{(1)}$ with G_1-equivariant morphisms $a_i : \widetilde{B}^{(1)} \longrightarrow B^{(i)}$ such that (i) a_1 is a quasi-isomorphism, (ii) the diagram

$$B^{(1)} \xleftarrow{a_1} \widetilde{B}^{(1)} \xrightarrow{a_2} B^{(2)}$$

represents ϕ. We have the natural G_1-equivariant structure on the cone $\operatorname{Cone}(a_2)$. Assume we have another G_1-equivariant complex $\widehat{B}^{(1)}$ with G_1-equivariant morphisms $\widehat{a}_i : \widehat{B}^{(1)} \longrightarrow B^{(i)}$ such that the diagram

$$B^{(1)} \xleftarrow{\widehat{a}_1} \widehat{B}^{(1)} \xrightarrow{\widehat{a}_2} B^{(2)}$$

represents ϕ. Then, there exists a G_1-equivariant complex $\overline{B}^{(1)}$ with G_1-equivariant morphisms with morphisms $f : \overline{B}^{(1)} \longrightarrow \widetilde{B}^{(1)}$ and $g : \overline{B}^{(1)} \longrightarrow \widehat{B}^{(1)}$ such that the following diagrams are commutative up to homotopy for $i = 1, 2$:

$$\begin{tikzcd}
\overline{B}^{(1)} \arrow[r, "f"] \arrow[d, "g"'] & \widetilde{B}^{(1)} \arrow[d, "a_i"'] \\
\widehat{B}^{(1)} \arrow[r, "\widehat{a}_i"] & B^{(i)}
\end{tikzcd}$$

By an argument used in the proof of Lemma 2.5.3, we may assume that the homotopy is also G_1-equivariant. Then, we have the G_1-equivariant quasi-isomorphisms:

$$\operatorname{Cone}(\widehat{a}_2) \longleftarrow \operatorname{Cone}(\widehat{a}_2 \circ g) \simeq \operatorname{Cone}(a_2 \circ f) \longrightarrow \operatorname{Cone}(a_2)$$

In this sense, the G_1-equivariant complex $\operatorname{Cone}(a_2)$ is uniquely determined up to G_1-equivariant quasi-isomorphisms. We denote it by $\operatorname{Cone}(\varphi)$.

2.6 Elementary Remarks on Some Extremal Sets

The results in this section will be used when we study geometric invariant theory for enhanced master spaces in Sections 4.3–4.4.

2.6.1 Preparation for a Proof of Proposition 4.3.3

Let us consider a vector space $\mathcal{U} = \bigoplus_{i=1}^{N} \mathbb{Q} \cdot e_i$. We put

$$f_j := (j-N) \sum_{i \le j} e_i + j \cdot \sum_{i > j} e_i.$$

The following lemma is well known and easy to prove.

Lemma 2.6.1 *Take any element $\rho = \sum_{i=1}^{N} a_i \cdot e_i \in \mathcal{U}$ satisfying $\sum_{j=1}^{N} a_j = 0$ and $a_1 \leq a_2 \leq \cdots \leq a_N$. Then, there exist non-negative rational numbers b_j such that $\rho = \sum b_j \cdot f_j$.* $\square$

Let $r_1, \ldots, r_s$ be positive integers such that $\sum_{j=1}^{s} r_j = N$. We put $R_j := \sum_{i \leq j} r_i$. We set

$$v_j := \sum_{R_{j-1} < i \leq R_j} e_i \quad (j = 1, \ldots, s).$$

For an integer $j \in \{1, \ldots, s\}$, we put

$$y(j) := -(N - R_j) \sum_{h \leq j} v_h + R_j \sum_{h > j} v_h.$$

For a pair of integers (i_1, i_2) such that $1 \leq i_1 < i_2 \leq s$, we define

$$x(i_1, i_2) := -(N - R_{i_2}) \sum_{h \leq i_1} v_h + R_{i_1} \sum_{i_2 < h} v_h.$$

For an integer $i_0 \in \{1, \ldots, s\}$, we set

$$\mathcal{S}(i_0) := \left\{ (i_1, i_2) \in \mathbb{Z}^2 \,\middle|\, 1 \leq i_1 < i_0 < i_2 \leq s \right\}.$$

Lemma 2.6.2 *Let $v = \sum_{j=1}^{s} a_j \cdot v_j$ be an element of $\mathcal{U}$ satisfying the following:*

$$a_1 \leq a_2 \leq \cdots \leq a_s, \quad \sum_{j=1}^{s} r_j \cdot a_j = 0. \tag{2.14}$$

Take an integer i_0 such that $1 \leq i_0 \leq s$.

- *Assume $a_{i_0} > 0$. Then, there exist the non-negative rational numbers $b(i_1, i_2)$ for $(i_1, i_2) \in \mathcal{S}(i_0)$ and the non-negative rational numbers c_j $(1 \leq j < i_0)$ such that the following equality holds:*

$$v = \sum_{(i_1, i_2) \in \mathcal{S}(i_0)} b(i_1, i_2) \cdot x(i_1, i_2) + \sum_{j=1}^{i_0 - 1} c_j \cdot y(j). \tag{2.15}$$

- *Assume $a_{i_0} = 0$. Then, there exist the non-negative rational numbers $b(i_1, i_2)$ for $(i_1, i_2) \in \mathcal{S}(i_0)$ such that the following holds:*

$$v = \sum_{(i_1, i_2) \in \mathcal{S}(i_0)} b(i_1, i_2) \cdot x(i_1, i_2).$$

- *Assume $a_{i_0} < 0$. Then, there exist the non-negative rational numbers $b(i_1, i_2)$ for $(i_1, i_2) \in \mathcal{S}(i_0)$ and the non-negative rational numbers c_j $(i_0 < j \leq N)$ such that the following holds:*

$$v = \sum_{(i_1,i_2)\in\mathcal{S}(i_0)} b(i_1,i_2)\cdot x(i_1,i_2) + \sum_{j=i_0+1}^{N} c_j\cdot y(j).$$

Proof We use an induction on the number $d(v) := \#\{i \mid a_i \neq a_{i+1}\}$. In the case $d(v) = 0$, the claim is obvious. Let v be as in the lemma such that $d(v) = m+1$. Take the integers h_1 and h_2 satisfying the following:

$$a_1 = a_2 = \cdots = a_{h_1} < a_{h_1+1}, \quad a_s = a_{s-1} = \cdots = a_{h_2+1} > a_{h_2}.$$

We remark the following:

- In the case $a_{i_0} > 0$, we have $h_1 < i_0$.
- In the case $a_{i_0} = 0$, we have $h_1 < i_0 < h_2$.
- In the case $a_{i_0} < 0$, we have $i_0 < h_2$.

Let us argue the case $a_{i_0} > 0$. If we have $i_0 \leq h_2$, we put as follows:

$$v' := v - f\cdot x(h_1,h_2) = \sum a_i'\cdot v_i, \quad f := \min\left\{\frac{a_{h_1+1}-a_{h_1}}{N-R_{h_2}}, \frac{a_{h_2+1}-a_{h_2}}{R_{h_1}}\right\}$$

If we have $i_0 \geq h_2+1$, we put as follows:

$$v' := v - g\cdot y(h_2), \quad g := \frac{a_{i_0}-a_{h_2}}{2R_{h_2}}$$

It is easy to see that the numbers a_i' satisfy the condition (2.14), and that we have $d(v') \leq d(v) - 1$. By the hypothesis of the induction, we have the expression for v' as in (2.15) with the non-negative coefficients. Then, we obtain the desired expression for v.

The cases $a_{i_0} = 0$ or $a_{i_0} < 0$ can be argued similarly. □

2.6.2 Preparation for a Proof of Proposition 4.4.4

Let $N^{(\alpha)}$ $(\alpha = 1,2)$ be positive integers. Let us consider a vector space as follows:

$$\mathcal{U} = \mathcal{U}^{(1)} \oplus \mathcal{U}^{(2)}, \quad \mathcal{U}^{(\alpha)} = \bigoplus_{i=1}^{N^{(\alpha)}} \mathbb{Q}\cdot e_i^{(\alpha)}.$$

Let $r_1^{(\alpha)},\ldots,r_{s(\alpha)}^{(\alpha)}$ $(\alpha = 1,2)$ be positive integers such that $\sum_{j=1}^{s(\alpha)} r_j^{(\alpha)} = N^{(\alpha)}$. We put $R_j^{(\alpha)} = \sum_{h\leq j} r_h^{(\alpha)}$. We set $\Omega^{(\alpha)} = \sum_i e_i^{(\alpha)}$. We also put

$$v_j^{(\alpha)} := \sum_{R_{j-1}^{(\alpha)} < i \leq R_j^{(\alpha)}} e_i^{(\alpha)} \quad \bigl(j = 1,\ldots,s(\alpha)\bigr).$$

For each integer $j \in \{1, \dots, s(2)\}$, we set

$$y^{(2)}(j) := -(N - R_j^{(2)}) \cdot \sum_{h \le j} v_h^{(2)} + R_j^{(2)} \cdot \sum_{h > j} v_h^{(2)}.$$

For each integer $j \in \{1, \dots, s(1)\}$, we put

$$x_1(j) := -N^{(2)} \cdot \sum_{h \le j} v_h^{(1)} + R_j^{(1)} \cdot \Omega^{(2)},$$

$$x_2(j) := N^{(2)} \cdot \sum_{h \ge j} v_h^{(1)} + (R_{j-1}^{(1)} - N^{(1)}) \cdot \Omega^{(2)}.$$

Lemma 2.6.3 *Let $v = \sum_{\alpha=1,2} \sum_j a_j^{(\alpha)} \cdot v_j^{(\alpha)}$ be any element of $\mathcal{U}$ satisfying the following conditions:*

$$a_1^{(\alpha)} \le a_2^{(\alpha)} \le \dots \le a_{s(\alpha)}^{(\alpha)}, \quad \sum_{\alpha=1,2} \sum_j r_j^{(\alpha)} \cdot a_j^{(\alpha)} = 0. \tag{2.16}$$

Take an integer i_0 such that $1 \le i_0 \le s(1)$. Then, there exist non-negative rational numbers $c(j) \ge 0$ $(j = 1, \dots, s(2))$, $d_1(i) \ge 0$ $(i = 1, \dots, i_0)$, $d_2(i) \ge 0$ $(i = i_0 + 1, \dots, s(1))$ and a rational number A such that the following holds:

$$v = \sum_{j=1}^{s(2)} c(j) \cdot y^{(2)}(j) + \sum_{i < i_0} d_1(i) \cdot x_1(i) + \sum_{i > i_0} d_2(i) \cdot x_2(i) + A \cdot \big(N^{(2)} \Omega^{(1)} - N^{(1)} \cdot \Omega^{(2)}\big). \tag{2.17}$$

Proof By Lemma 2.6.1, we may assume $a_1^{(2)} = \dots = a_{s(2)}^{(2)}$ from the beginning. We use an induction on the number $d(v) = \#\big\{i \,|\, a_i^{(1)} \ne a_{i+1}^{(1)}\big\}$. If $d(v) = 0$, we have $v = A \cdot \big(N^{(2)} \cdot \Omega^{(1)} - N^{(1)} \cdot \Omega^{(1)}\big)$ for some A, and hence the claim is clear. Let v be an element as in the lemma such that $d(v) = m + 1 > 0$. Let us take the integer h_1 satisfying $a_1^{(1)} = a_{h_1}^{(1)} < a_{h_1+1}^{(1)}$. In the case $i_0 > h_1$, we put as follows:

$$v' := v - \frac{a_{h_1+1}^{(1)} - a_{h_1}^{(1)}}{N^{(2)}} x_1(h_1)$$

In the case $i_0 \le h_1$, we put as follows:

$$v' = v - \frac{a_{h_1+1}^{(1)} - a_{h_1}^{(1)}}{N^{(2)}} \cdot x_2(h_1).$$

Then, v' satisfies the conditions (2.16) and $d(v') < d(v)$. By the hypothesis of the induction, we have the expression for v' as in (2.17). Hence, we obtain the desired expression for v. $\square$

2.7 Twist of Line Bundles

This subsection is a preparation for Section 4.6.

2.7.1 Construction

Let Y be an algebraic stack over a field k. Let G_m denote the one dimensional algebraic torus $\mathrm{Spec}\, k[t, t^{-1}]$. Let I denote the trivial line bundle on Y. A point of I is denoted by (y, u) where $y \in Y$ and $u \in I_{|y}$. For each integer n, $\mathcal{T}(n)$ denote the line bundle I with the G_m-action by $t \cdot (y, u) := (y, t^n \cdot u)$.

Let L be any line bundle on Y. Let L^* denote the complement of the image of the 0-section, i.e., $L^* := L - Y$. Let $\pi : L^* \longrightarrow Y$ denote the naturally defined projection. A point of L^* is also denoted by (y, v), where $y \in Y$ and $v \in \pi^{-1}(y)$.

Let us fix an integer r. We consider the G_m-action on L^* given by $t \cdot (y, v) := (y, t^r v)$. We have the naturally defined G_m-action on $\pi^*\mathcal{T}(n)$. It induces a line bundle $\mathcal{I}_n$ on the algebraic stack L^*/G_m. Let $\varphi : L^*/G_m \longrightarrow Y$ denote the naturally induced morphism.

Lemma 2.7.1 *We have canonical isomorphisms $\mathcal{I}_n \otimes \mathcal{I}_m \simeq \mathcal{I}_{n+m}$ and $\mathcal{I}_{-n} \simeq \mathcal{I}_n^{-1}$ and $\mathcal{I}_0 \simeq \mathcal{O}_{L^*/G_m}$. We also have a canonical isomorphism $\mathcal{I}_{-r} \simeq \varphi^* L$.*

Proof The first claim is obvious. Let us show the second claim. Let us denote a point of $\pi^* L$ by (y, v, u'), where $y \in Y$, $v \in \pi^{-1}(y)$ and $u' \in L_{|y}$. The trivial G_m-action on L induces the G_m-action on $\pi^* L$ over L^*, which is given by $t \cdot (y, v, u') = \big(y, t^r \cdot v, u'\big)$.

On the other hand, let us denote a point of $\pi^*\mathcal{T}(-r)$ by (y, v, u) where $y \in Y$, $v \in \pi^{-1}(y)$ and $u \in \mathcal{T}(-r)_{|y}$. The action is denoted by $t \cdot (y, v, u) = \big(y, t^r v, t^{-r} u\big)$.

We have the naturally defined isomorphism $\pi^*\mathcal{T}(-r) \longrightarrow \pi^* L$ given by $(y, v, u) \longmapsto (y, v, u \cdot v)$, which is G_m-equivariant. Therefore, we obtain the isomorphism $\mathcal{I}_{-r} \simeq \varphi^* L$. $\square$

2.7.2 Weight of the Induced Action

We use the notation in the previous subsection. Let $G_m^{(i)}$ $(i = 1, 2)$ denote one-dimensional tori $\mathrm{Spec}\, k[t_i, t_i^{-1}]$. Let us consider the action of $G_m^{(1)} \times G_m^{(2)}$ on L given by $(t_1, t_2) \cdot (y, v) := \big(y, t_1 \cdot t_2 \cdot v\big)$.

Let $\mathcal{T}(n_1, n_2)$ denote the trivial line bundle I with the $G_m^{(1)} \times G_m^{(2)}$-action given by $(t_1, t_2) \cdot (y, u) = \big(y, t_1^{n_1} \cdot t_2^{n_2} \cdot u\big)$. Then, we have the $G_m^{(1)} \times G_m^{(2)}$-line bundle $\pi^*\mathcal{T}(n_1, n_2)$ on L^*. We obtain the line bundle $\mathcal{I}_{n_2}$ on $L^*/G_m^{(2)}$, and we have the induced $G_m^{(1)}$-action on $\mathcal{I}_{n_2}$.

Lemma 2.7.2 *The weight of the $G_m^{(1)}$-action on $\mathcal{I}_{n_2}$ is $n_1 - n_2$.*

Proof We put $\widetilde{G}_m^{(i)} := \operatorname{Spec} k[s_i]$. Let us consider the morphism $\widetilde{G}_m^{(1)} \times \widetilde{G}_m^{(2)} \longrightarrow G_m^{(1)} \times G_m^{(2)}$ given by $(s_1, s_2) \longmapsto \big(s_1, s_1^{-1} \cdot s_2\big)$. The induced $\widetilde{G}_m^{(1)} \times \widetilde{G}_m^{(2)}$-action on L^* and $\mathcal{T}(n_1, n_2)$ is given by $(s_1, s_2) \cdot (y, v) = \big(y, s_2^r \cdot v\big)$ and $(s_1, s_2) \cdot (y, u) = \big(y, s_1^{n_1 - n_2} \cdot s_2^{n_2} \cdot u\big)$. Therefore, the weight of the $G_m^{(1)}$-action on $\mathcal{I}_{n_2}$ is given by $n_1 - n_2$. $\square$

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