Chapter XXVI. Pseudo-Differential Operators of Principal Type

Summary

In Section 10.4 we saw that the strength of a differential operator with constant coefficients in $\mathbb{R}^n$ is determined by the principal part p if and only if $p=0$ implies $dp \neq 0$ in $\mathbb{R}^n \setminus 0$. Such operators were said to be of principal type. The purpose of this chapter is to study general operators $P \in \Psi^m_{\text{phg}}(X)$ on a manifold X assuming that the condition $dp \neq 0$ when $p=0$ is valid in a suitably strengthened form which makes the properties of P independent of lower order terms.

At first we assume that the principal symbol p is real valued. In the constant coefficient case we know then from Section 8.3 that singularities of solutions of the equation $Pu=f$ travel along bicharacteristic curves, that is, integral curves of the Hamilton field H_p of p with $p=0$, unless they are disturbed by singularities of f. In Theorem 23.2.9 and remarks at the beginning of Section 24.2 the result was extended to second order differential operators, and in Section 26.1 it is proved for pseudo-differential operators. After P is reduced to first order by multiplication with an elliptic operator of order $1-m$ we use the homogeneous Darboux theorem in Chapter XXI to reduce p locally to a coordinate ξ_1 by a homogeneous canonical transformation χ. The calculus of Fourier integral operators in Chapter XXV then shows that conjugation of P by a suitable Fourier integral operator associated with χ reduces p microlocally to the operator D_1 for which the propagation of singularities is quite obvious. Thus we obtain the desired extension of the theorem on propagation of singularities; it is non-trivial provided that H_p is non-radial at the characteristic points which is also required for the application of the homogeneous Darboux theorem. Existence theorems for the adjoint operator on a compact subset K of X follow when K is non-trapping for bicharacteristics of p, that is, no bicharacteristic remains forever over K. When this is true for every compact subset K of X we say that P is of principal type in X; locally this just means that dp is not proportional to the canonical one form $\langle \xi, dx \rangle$ at the characteristic points. Under appropriate conditions on convexity of X with respect to the bicharacteristic flow, related to those in Section 10.8, we can also construct global two sided parametrices of P.

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The situation is much more complicated when p is complex valued. This complexity is already seen in the geometry of the characteristic set $p^{-1}(0)$ which first of all may not be a manifold, and secondly may be complicated from the symplectic point of view since the rank of the symplectic form restricted to the characteristic set is variable. Two simple extreme cases are studied first. In Section 26.2 we assume that $p^{-1}(0)$ is an involutive manifold of codimension 2, thus $\{\operatorname{Re} p, \operatorname{Im} p\}=0$ when $p=0$. As in the real case we can then reduce P microlocally to the Cauchy-Riemann operator D_1+iD_2. This commutes with operators with symbol analytic in x_1+ix_2 which leads to a proof that if $Pu\in C^\infty$ then the regularity function

$$s_u^*(x,\xi)=\sup\{s; u\in H_{(s)} \text{ at } (x,\xi)\}$$

is superharmonic in the leaves of the foliation of the involutive manifold $p^{-1}(0)$. (These have a natural analytic structure defined by the complex tangent vector field H_p; the solutions of $H_p w=0$ are the analytic functions.)

In Section 26.3 we study the opposite extreme case where $\{\operatorname{Re} p, \operatorname{Im} p\} \neq 0$ which implies that $p^{-1}(0)$ is a symplectic manifold of codimension 2. A famous example is the Lewy operator

$$P=D_1+iD_2+i(x_1+ix_2)D_3$$

in $\mathbb{R}^3$. It appears as the tangential Cauchy-Riemann operator on the boundary of the strictly pseudo-convex domain

$$\Omega=\{(z_1,z_2)\in\mathbb{C}^2;\ |z_1|^2+2\operatorname{Im} z_2<0\}.$$

In fact, $\partial/\partial\bar z_1+a\partial/\partial\bar z_2$ is tangential to $\partial\Omega$ if and only if on $\partial\Omega$

$$0=(\partial/\partial\bar z_1+a\partial/\partial\bar z_2)(z_1\bar z_1-iz_2+i\bar z_2)=z_1+ai.$$

Writing $z_1=x_1+ix_2$, $z_2=x_3+ix_4$ and taking x_1,x_2,x_3 as parameters in $\partial\Omega$ we obtain the Lewy operator multiplied by $\frac{1}{2}i$. The fact that Ω is strictly pseudo-convex implies that for any point in $\partial\Omega$ one can find U analytic in $\bar\Omega$ except at the given point. Indeed, if $a\in\mathbb{C}$ then

$$\operatorname{Re}(z_1\bar a+z_2/i-|a|^2/2)\leqq\operatorname{Re}(z_1\bar a-|z_1|^2/2-|a|^2/2)\leqq 0, \qquad z\in\bar\Omega,$$

with strict inequality except when $z_1=a$ and $\operatorname{Im} z_2=-|a|^2/2$. Hence, if $b\in\mathbb{R}$

$$U(z)=1/(z_1\bar a+z_2/i-|a|^2/2+ib)$$

is analytic in $\bar\Omega$ except at $z_1=a$, $z_2=b-i|a|^2/2$. The boundary value u satisfies the equation $Pu=0$ and has a singularity which does not propagate. If $(x,\xi)\in WF(u)$ it is clear that $p(x,\xi)=\xi_1+i\xi_2+i(x_1+ix_2)\xi_3=0$, that is, $\xi_1=x_2\xi_3$ and $\xi_2=-x_1\xi_3$, and since u is a boundary value of a function analytic in z_2 in a lower half plane we must have $\xi_3<0$. Noting that

$$\{\operatorname{Re} p, \operatorname{Im} p\}=\{\xi_1-x_2\xi_3, \xi_2+x_1\xi_3\}=2\xi_3<0$$

we are led to the result proved in Section 26.3 that for every pseudodifferential operator P and characteristic point (x,ξ) with $\{\operatorname{Re} p, \operatorname{Im} p\}(x,\xi)<0$ one can find u with $Pu \in C^\infty$ and $WF(u)$ equal to the ray through (x, ξ). An essentially dual fact, first observed by Hans Lewy for the Lewy operator, is that the equation $Pu = f$ cannot be solved for most f if there is a characteristic point with $\{\operatorname{Re} p, \operatorname{Im} p\} > 0$; in fact, it is then usually impossible to solve the equation microlocally at (x, ξ). In the proofs of these facts we shall use Fourier integral operators to reduce to the model operator $D_1 + ix_1 D_2$, sometimes called the Mizohata operator, which is somewhat simpler than the Lewy operator. The existence and regularity of solutions of the equation $(D_1 + ix_1 D_2)u = f$ can be studied quite explicitly. At the same time we discuss the equation $(D_1 + ix_1^k D_2)u = f$ for every positive integer k. When k is even the properties are quite close to those of the Cauchy-Riemann operator ($k = 0$) and for all odd k we have properties similar to those of the Lewy operator.

The results of Section 26.3 suggest that solvability of the inhomogeneous equation $Pu = f$ requires that $\operatorname{Im} p$ has no sign change from $-$ to $+$ along bicharacteristics of $\operatorname{Re} p$. This condition was originally conjectured by Nirenberg and Treves and called condition (Ψ) by them. Section 26.4 is devoted to the proof of this conjecture by means of an idea of R. Moyers, after the functional analytic aspects of various notions of solvability have been discussed at some length and the condition (Ψ) has been given an appropriate global form invariant under multiplication by non-vanishing functions.

It is still unknown if condition (Ψ) is sufficient for solvability. From Section 26.5 on we therefore assume the stronger condition (P) which rules out all sign changes of $\operatorname{Im} p$ on bicharacteristics of $\operatorname{Re} p$. (For differential operators this is equivalent to (Ψ).) Condition (P) leads to considerably simplified properties of the characteristic set discussed in Section 26.5. The main point is that the flowout along $H_{\operatorname{Re} p}$ and $H_{\operatorname{Im} p}$ of the set of characteristic points with $d \operatorname{Re} p$ and $d \operatorname{Im} p$ linearly independent is an involutive manifold N_2^e of codimension 2. Thus N_2^e is foliated by two dimensional leaves where a degenerate Cauchy-Riemann equation is defined by the Hamilton field H_p. The propagation of singularities along bicharacteristics of $\operatorname{Re} p$ which leave the characteristic set at some time is discussed in Section 26.6 by means of energy integral estimates. Similar estimates are the basis of the study in Section 26.7 of degenerate Cauchy-Riemann equations

$$(D_1 + i a(x) D_2) u = f$$

with $a \geqq 0$, which implies condition (P). The results show in particular that there is an analytic structure in the leaves B of N_2^e, or rather in the sets $\tilde{B}$ obtained by collapsing to a point every embedded one dimensional bicharacteristic curve, that is, any curve where H_p is proportional to the tangent. In Section 26.9 we show that with this structure the superharmonicity of the regularity function s_u^* proved in Section 26.2 for the non-degenerate case remains valid in N_2^e. An essential ingredient in the proof is another version of the energy integral estimates, due to Nirenberg and Treves, which is given in Section 26.8. This estimate together with the advanced calculus of pseudo-differential operators in Section 18.5 leads also to the proof in Section 26.10 that when $Pu \in C^\infty$ then s_u^* is quasi-concave on any one dimensional bicharacteristic, that is, the minimum in any interval is taken at an end point.

All the results on singularities established in Sections 26.6–26.10 are combined in Section 26.11 to an existence theorem for a pseudo-differential operator P satisfying condition (P). It states that if no complete one or two dimensional bicharacteristic is trapped over the compact set K then the equation $Pu=f$ can be solved in a neighborhood of K for any f which is orthogonal to the finite dimensional space of solutions $v \in C_0^\infty(K)$ of the equation $P^*v=0$. When no bicharacteristic is trapped over a compact subset of X, we say that P is of principal type in X and have semi-global existence theorems for arbitrary lower order terms.

26.1. Operators with Real Principal Symbols

It was proved in Section 8.3 that the singularities of solutions of differential equations with constant coefficients and real principal part propagate along the bicharacteristics. We shall now show how the symplectic geometry and operator theory developed in Chapters XXI and XXV allow one to extend the result to variable coefficients. In doing so we shall start from scratch and do not rely on the results of Section 8.3.

Theorem 26.1.1. *Let X be a C^∞ manifold and let $P \in \Psi^m(X)$ be properly supported and have a principal symbol p which is real and homogeneous of degree m. If $u \in \mathscr{D}'(X)$ and $Pu=f$, it follows that $WF(u) \setminus WF(f)$ is contained in* $\operatorname{Char}(P)=p^{-1}(0)$ *and is invariant under the flow defined there by the Hamilton vector field H_p.*

By Theorem 18.1.28 we have

$$WF(u) \subset WF(f) \cup \operatorname{Char}(P)$$

so only the invariance under the Hamilton flow has to be proved. At a point where $H_p=0$ or H_p has the radial direction this invariance is also obvious, so in the proof we may assume that H_p and the radial direction are linearly independent. We shall prove the theorem by reducing it to the special case $P=D_1$ in $\mathbb{R}^n$ where it follows by explicit solution of the equation $Pu=f$. The study of this special case as well as the reduction will at the same time prepare for the construction of a parametrix later on in this section, so we shall also include some material which will be required then.

By E_1^+ and E_1^- we denote the forward and backward fundamental solutions of the operator D_1, the kernels of which are defined by

$$E_1^+ = iH(x_1-y_1)\otimes\delta(x'-y'), \qquad E_1^- = -iH(y_1-x_1)\otimes\delta(x'-y').$$

Here H is the Heaviside function, $H(t)=1$ for $t>0$ and $H(t)=0$ for $t<0$, and we have used the notation $x=(x_1, x')$ and $y=(y_1, y')$ for points in $\mathbb{R}^n$. Note that $E_1^+ - E_1^- = i\delta(x'-y')$ or, in Fourier integral form,

$$(E_1^+ - E_1^-)(x, y) = (2\pi)^{-(n-1)} \int e^{i\langle x'-y', \theta\rangle} i\, d\theta. \tag{26.1.1}$$

This is a conormal distribution with respect to $\{(x, y);\ x'=y'\}$, and the order is $-\frac{1}{2}$ since there are $n-1$ phase variables and $(2n-2(n-1))/4=\frac{1}{2}$. Thus we have $E_1^+ - E_1^- \in I^{-\frac{1}{2}}(\mathbb{R}^n \times \mathbb{R}^n, C_1')$ where

$$C_1 = \{(x, \xi, y, \eta);\ x'=y', \xi'=\eta' \neq 0, \xi_1=\eta_1=0\} \tag{26.1.2}$$

is the corresponding canonical relation. It follows that $\chi E_1^\pm$ belongs to $I^{-\frac{1}{2}}(\mathbb{R}^n \times \mathbb{R}^n, C_1')$ if $\chi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ vanishes in a neighborhood of the diagonal, for if $x \neq y$ then either E_1^+ or E_1^- vanishes in a neighborhood of (x, y). In particular, we conclude that $WF'(E_1^\pm)$ is contained in C_1 except over the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$. Since $(D_{x_j}+D_{y_j})E_1^\pm = 0$ for $j=1, \dots, n$ we have $\xi=\eta$ in $WF'(E_1^\pm)$ (see also (8.2.15)), and

$$WF'(E_1^\pm) \supset WF'(D_{x_1} E_1^\pm) = WF'(\delta(x-y)).$$

The right hand side is the diagonal in $(T^*(\mathbb{R}^n)\setminus 0) \times (T^*(\mathbb{R}^n)\setminus 0)$ (Theorem 8.1.5) so we have proved

Proposition 26.1.2. *Let E_1^+ and E_1^- be the forward and the backward fundamental solutions of $D_1 = -i\partial/\partial x_1$ in $\mathbb{R}^n$. Then we have*

(i) *$WF'(E_1^\pm)$ is the union of the diagonal in $(T^*(\mathbb{R}^n)\setminus 0) \times (T^*(\mathbb{R}^n)\setminus 0)$ and the part of the canonical relation C_1 defined by* (26.1.2) *where $x_1 \gtrless y_1$.*

(ii) *$E_1^+ - E_1^- \in I^{-\frac{1}{2}}(\mathbb{R}^n \times \mathbb{R}^n, C_1')$, and $\chi E_1^\pm \in I^{-\frac{1}{2}}(\mathbb{R}^n \times \mathbb{R}^n, C_1')$ if χ is in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and vanishes near the diagonal.*

The statement (i) is a more elementary analogue of Theorem 8.3.7 for the operator D_1. It is all that is needed to prove Theorem 26.1.1 for $P=D_1$ by repeating the proof of Theorem 8.3.3; this is left for the reader to do.

In the general proof of Theorem 26.1.1 we may assume that P is a first order operator, for if Q is an elliptic pseudo-differential operator with positive principal part, homogeneous of degree $1-m$, then $Pu=f$ implies $(QP)u=Qf$ where QP has the same characteristics and bicharacteristics as P, and $WF(Qf)=WF(f)$. As already pointed out it is also sufficient to consider characteristics (x_0, ξ_0) of P where H_p does not have the radial direction. This makes Theorem 21.3.1 applicable so we can find a homogeneous canonical transformation χ from an open conic neighborhood of $(0, \varepsilon_n) \in T^*(\mathbb{R}^n)\setminus 0$ to an open conic neighborhood of (x_0, ξ_0) such that $\chi^* p = \xi_1$. This geometrical construction can be lifted to the operator level:

Proposition 26.1.3. *Let $P \in \Psi^1(X)$ have real and homogeneous principal part p, let $p(x_0, \xi_0)=0$ and assume that the Hamilton field H_p at (x_0, ξ_0) and the radial direction are linearly independent. Let χ be any homogeneous canonical* *transformation from an open conic neighborhood of* $(0, \varepsilon_n) \in T^*(\mathbb{R}^n) \setminus 0$ *to a conic neighborhood of* (x_0, ξ_0) *such that* $\chi^* p = \xi_1$. *For any* $\mu \in \mathbb{R}$ *one can then find properly supported Fourier integral operators* $A \in I^\mu(X \times \mathbb{R}^n, \Gamma')$ *and* $B \in I^{-\mu}(\mathbb{R}^n \times X, (\Gamma^{-1})')$, *where* Γ *is the graph of* χ, *such that*

(i) $WF'(A)$ *and* $WF'(B)$ *are in small conic neighborhoods of* $(x_0, \xi_0, 0, \varepsilon_n)$ *and* $(0, \varepsilon_n, x_0, \xi_0)$ *respectively.*

(ii) $(x_0, \xi_0, x_0, \xi_0) \notin WF'(AB-I)$; $(0, \varepsilon_n, 0, \varepsilon_n) \notin WF'(BA-I)$.

(iii) $(x_0, \xi_0, x_0, \xi_0) \notin WF'(AD_1 B - P)$; $(0, \varepsilon_n, 0, \varepsilon_n) \notin WF'(BPA - D_1)$.

Thus D_1 *and* P *are microlocally conjugate to each other.*

Proof. Choose any $A_1 \in I^\mu(X \times \mathbb{R}^n, \Gamma')$ such that $WF'(A_1)$ is close to $(x_0, \xi_0, 0, \varepsilon_n)$ and A_1 is non-characteristic there. As observed after Definition 25.3.4 we can then choose $B_1 \in I^{-\mu}(\mathbb{R}^n \times X, (\Gamma^{-1})')$ so that (ii) is fulfilled. Then it follows from Theorem 25.3.5 that

$$(0, \varepsilon_n) \notin WF(B_1 P A_1 - D_1 - Q)$$

for some $Q \in \Psi^0(\mathbb{R}^n)$. We shall prove in a moment that there exist elliptic pseudo-differential operators $A_2, B_2 \in \Psi^0(\mathbb{R}^n)$ such that

$$B_2 A_2 - I \in \Psi^{-\infty}, \quad B_2(D_1 + Q) A_2 - D_1 \in \Psi^{-\infty}. \tag{26.1.3}$$

Admitting this for a moment we set $A = A_1 A_2$ and $B = B_2 B_1$. Then

$$(0, \varepsilon_n) \notin WF(B_2(B_1 A_1 - I) A_2) = WF(BA - I),$$
$$(0, \varepsilon_n) \notin WF(B_2(B_1 P A_1 - D_1 - Q) A_2) = WF(BPA - D_1)$$

which proves the second half of (ii) and (iii). The first half follows at once if we multiply left and right by A and by B.

To solve (26.1.3) we observe that by Theorem 18.1.24 one can for every elliptic A_2 of order 0 find $B_2 \in \Psi^0$ with $B_2 A_2 - I \in \Psi^{-\infty}$ and $A_2 B_2 - I \in \Psi^{-\infty}$, so (26.1.3) is equivalent to the condition $(D_1 + Q) A_2 - A_2 D_1 \in \Psi^{-\infty}$ for some elliptic A_2, that is,

$$[D_1, A_2] + Q A_2 \in \Psi^{-\infty}. \tag{26.1.3$'$}$$

If q^0 is a principal symbol of Q and a^0 is a principal symbol of A_2, then the principal symbol of (26.1.3)$'$ vanishes if

$$i^{-1}\{\xi_1, a^0\} + q^0 a^0 = 0,$$

that is, $\partial a^0 / \partial x_1 = -i q^0 a^0$. This equation is solved by

$$a^0(x, \xi) = \exp\left(-i \int_0^{x_1} q^0(t, x', \xi)\, dt\right)$$

which is an element of S^0 by Lemma 18.1.10. Choosing A^0 with principal symbol a^0 we can now successively choose $A^j \in \Psi^{-j}(\mathbb{R}^n)$ so that for every j

$$[D_1, A^0 + \ldots + A^j] + Q(A^0 + \ldots + A^j) = R_j \in \Psi^{-j-1}.$$

In fact, this only requires that a principal symbol a^j of A^j satisfies the equation

$$i^{-1}\partial a^j/\partial x_1 + q^0 a^j = -r^0_{j-1}$$

where r^0_{j-1} is a principal symbol of R_{j-1}. The solution

$$a^j(x,\xi) = a^0(x,\xi)\int_0^{x_1} -i r^0_{j-1}(t,x',\xi)/a^0(x',t,\xi)\,dt$$

is in S^{-j} since r^0_{j-1}/a^0 is. If the symbol of A_2 is chosen as the asymptotic sum of the symbols of $A^0, A^1, \dots$ we have satisfied (26.1.3)'.

Proof of Theorem 26.1.1. First recall that we have reduced the proof to the case $m=1$ and that the theorem has been proved for the operator D_1. So suppose that $m=1$ and let $(x_0,\xi_0)\in WF(u)\setminus WF(f)$, hence $p(x_0,\xi_0)=0$. As already pointed out we may assume that $H_p(x_0,\xi_0)$ and the radial direction are linearly independent. We then choose A, B according to Proposition 26.1.3 and set $v=Bu\in\mathscr{D}'(\mathbb{R}^n)$. Since

$$D_1 v = (D_1 - BPA)Bu + BP(AB-I)u + Bf$$

it follows from (ii) and (iii) in Proposition 26.1.3 that $(0,\varepsilon_n)\notin WF(D_1 v)$. On the other hand, $(0,\varepsilon_n)\in WF(v)$ since $(x_0,\xi_0)\in WF(u)$ and

$$u=(I-AB)u+Av, \qquad (x_0,\xi_0)\notin WF((I-AB)u).$$

Thus $(x_1,0,\varepsilon_n)\in WF(v)$ for small $|x_1|$, and since $WF(v)\subset\chi^{-1}WF(u)$ it follows that $WF(u)$ contains the image of this curve under χ. Now the definition of the Hamilton field is symplectically invariant so this means that $WF(u)$ contains a neighborhood of (x_0,ξ_0) on the bicharacteristic curve through (x_0,ξ_0) which completes the proof.

Theorem 26.1.1 can be given a more precise form if we take into account the $H_{(s)}$ classes of u and f. First recall that $f\in H^{\mathrm{loc}}_{(s)}$ at (x_0,ξ_0) means that $Af\in L^2_{\mathrm{loc}}$ for some $A\in\Psi^s$ which is non-characteristic at (x_0,ξ_0). If $f=Pu$ this means that $u\in H^{\mathrm{loc}}_{(s+m)}$ at (x_0,ξ_0) if $(x_0,\xi_0)\notin\operatorname{Char}(P)$. The $H_{(s)}$ regularity in the characteristic set propagates along the bicharacteristics:

Theorem 26.1.4. *Let P satisfy the hypotheses in Theorem* 26.1.1, *let I be an interval on a bicharacteristic curve where $f=Pu$ is in $H^{\mathrm{loc}}_{(s)}$. If $u\in H^{\mathrm{loc}}_{(s+m-1)}$ at some point on I it follows that this is true on all of I.*

Proof. The $H_{(s)}$ continuity properties of pseudo-differential and Fourier integral operators allow us to reduce the proof to the case $m=1$ and then, using Proposition 26.1.3 as before with $\mu=-s$, to the case $P=D_1$, $s=0$ and $(x_0,\xi_0)=(0,\varepsilon_n)$. Since $E_1^{\pm}$ maps L^2_{comp} to L^2_{loc} the proof works as before in this situation if the wave front set of a distribution is replaced throughout by the set of points in $T^*(\mathbb{R}^n)\setminus 0$ where it is not in L^2.

We shall now prove that bicharacteristics do carry the singularities of some solutions provided that they do not close on the cosphere bundle.

Theorem 26.1.5. *Assume that $P\in\Psi^m(X)$ is properly supported and has a real principal part p which is homogeneous of degree m. Let I be a compact interval on a bicharacteristic of p which has an injective projection to the cosphere bundle of X, let Γ be the cone generated by I in $T^*(X)\setminus 0$ and let Γ' be the cone generated by the end points of I. For any $s\in\mathbb{R}$ one can then find $u\in\mathscr{D}'(X)$ so that $u\in H_{(t)}^{\mathrm{loc}}(X)$ for every $t<s$ and*

$$WF(Pu)=\Gamma', \quad WF(u)=\Gamma, \quad u\notin H_{(s)}^{\mathrm{loc}} \text{ at } (x,\xi) \text{ if } (x,\xi)\in I.$$

If $X=\mathbb{R}^n$, $P=D_1$, $I=\{(x_1,0,\varepsilon_n),\ x_1\in\mathbb{R}\}$ then we can simply take

$$u(x)=(x_2^2+\ldots+x_{n-1}^2+0-ix_n)^{-n/4}.$$

Since the measure of $\{x'\in\mathbb{R}^{n-1},\ |x_2^2+\ldots+x_{n-1}^2-ix_n|\leqq t\}$ is $Ct^{(n-2)/2+1}=Ct^{n/2}$ for reasons of homogeneity, and $\int_0^1 t^{-a}d(t^{n/2})<\infty$ if and only if $a<n/2$, we have $u\in L^p_{\mathrm{loc}}$ if and only if $p<2$. Hence it follows from Theorem 7.1.13 that $\widehat{(\phi u)}\in L^q$ for every $q>2$ if $\phi\in C_0^\infty$, so $u\in H_{(t)}^{\mathrm{loc}}$ if $t<0$. It is clear that $D_1u=0$, and $WF(u)\subset\{(x,t\varepsilon_n),\ x'=0,\ t>0\}$ by Theorem 8.1.6. Since the projection sing supp u of $WF(u)$ in $\mathbb{R}^n$ is equal to the x_1 axis this inclusion is an equality and u is not in L^2 at any point on I.

If as in Theorem 26.1.5 we have a finite interval $I=\{(x_1,0,\varepsilon_n);\ a\leqq x_1\leqq b\}$ we shall cut off the function u at a and b with some care so that the wave front set does not grow. To do so we choose functions $\psi_j\in C_0^\infty((a,b)\times\mathbb{R}^{n-1})$ with $\sum_{-\infty}^{\infty}\psi_j=1$ in a neighborhood of $(a,b)\times\{0\}$ and supp $\psi_j\to\{a\}$ resp. $\{b\}$ as $j\to-\infty$ resp. $+\infty$. We can choose a regularization v_j of $u_j=\psi_j u$ such that if $U_j=u_j-v_j$ then supp $U_j\to\{a\}$ or $\{b\}$ as $j\to+\infty$ or $-\infty$, $\|U_j\|_{(-1/|j|)}\leqq 2^{-|j|}$ and

$$|\hat U_j(\xi)|\leqq(1+|\xi|)^{-|j|} \quad \text{when } |\xi_1|+\ldots+|\xi_{n-1}|\geqq\xi_n/|j|.$$

In fact, $\hat U_j(\xi)=\hat u_j(\xi)(1-\hat\chi(\delta_j\xi))$ where $\chi\in C_0^\infty$, $\hat\chi(0)=1$. We have

$$\int|\hat u_j(\xi)|^2(1+|\xi|)^t d\xi<\infty, \qquad t<0,$$

and $|\hat u_j(\xi)|(1+|\xi|)^{|j|}\to 0$ at ∞ outside any conic neighborhood of ε_n, so we just have to take δ_j small enough. Now we obtain $U=\sum U_j\in H_{(t)}$ for every $t<0$, $WF(U)\subset I$ and $U-u\in C^\infty$ at $(x_1,0)$ if $a<x_1<b$, so U is not in $H_{(0)}$ at any point on I. Since D_1U is only singular at $(a,0)$ and $(b,0)$ and since $WF(D_1U)\subset WF(U)=I$, it follows that U has the properties required in Theorem 26.1.5.

To prove Theorem 26.1.5 in general we need a global version of Theorem 21.3.1 and of Proposition 26.1.3 allowing us to conjugate P to D_1.

Proposition 26.1.6. *Let X be a C^∞ manifold and p a real valued C^∞ function in $T^*(X)\setminus 0$ which is homogeneous of degree 1. Let I be a compact interval on $\mathbb{R}$ and γ: $I\to T^*(X)\setminus 0$ a bicharacteristic, thus*

$$p\circ\gamma=0, \qquad \gamma'=H_p\circ\gamma.$$

We assume that the composition of γ and the projection π: $T^*(X)\setminus 0\to S^*(X)$ on the cosphere bundle is injective. Then one can find a conic neighborhood V of $\{(x_1, 0, \varepsilon_n);\ x_1\in I\}$ and a C^∞ homogeneous canonical transformation χ from V to an open conic neighborhood $\chi(V)\subset T^*(X)\setminus 0$ of $\gamma(I)$ such that $\chi(x_1, 0, \varepsilon_n)=\gamma(x_1)$ and $\chi^* p=\xi_1$.

Proof. Assume to simplify notation that $0\in I$. We can use Theorem 21.3.1 to find a homogeneous canonical transformation χ from a convex conic neighborhood V_0 of $(0, \varepsilon_n)$ to a conic neighborhood of $\gamma(0)$ such that $\chi^* p=\xi_1$ and $\chi(0, \varepsilon_n)=\gamma(0)$. Then χ_* maps the Hamilton field $\partial/\partial x_1$ of ξ_1 to H_p, so

$$\partial\chi(x, \xi)/\partial x_1=H_p(\chi(x, \xi)). \tag{26.1.4}$$

When $x'=0$ and $\xi=\varepsilon_n$ we also have the solution $\gamma(x_1)$, $x_1\in I$, with the same initial value when $x_1=0$. Hence we can uniquely extend χ to a conic neighborhood V of $I\times\{(0, \varepsilon_n)\}$, which is convex in the x_1 direction, so that (26.1.4) remains valid. The projected curves $x_1\mapsto\pi\chi(x, \xi)$ are the integral curves of the vector field induced by H_p on $S^*(X)$. (Functions on $S^*(X)$ can be identified with homogeneous functions f of degree 0 on $T^*(X)\setminus 0$, and $H_p f=\{p, f\}$ is then also homogeneous of degree 0.) Since χ is also homogeneous it follows from the hypothesis on $\pi\circ\gamma$ that χ is a diffeomorphism if V is small enough. If we write

$$\chi^{-1}=(X_1, \dots, X_n, \Xi_1, \dots, \Xi_n),$$

then the fact that χ_*^{-1} maps H_p to $\partial/\partial x_1$ means that

$$H_p X_1=1, \qquad H_p X_j=0 \quad \text{if } j>1, \qquad H_p\Xi_k=0 \quad \text{for all } k.$$

Hence the Poisson brackets $\{X_i, X_j\}$, $\{X_i, \Xi_k\}$, $\{\Xi_k, \Xi_l\}$ are constant along the orbits of the Hamilton field H_p, by the Jacobi identity. They vanish at some point since we started from a canonical transformation, so they vanish identically, which proves that also the extended map χ is canonical.

The following extension of Proposition 26.1.3 follows with the same proof:

Proposition 26.1.3′. *Let $P\in\Psi^1(X)$ have real and homogeneous principal part p, and let γ: $I\to T^*(X)\setminus 0$ be a bicharacteristic with the properties assumed in Proposition 26.1.6. If Γ is the graph of a canonical transformation χ from a* *conic neighborhood of $J = I \times (0, \varepsilon_n)$ to a conic neighborhood of $\gamma(I)$, satisfying the conclusion in Proposition 26.1.6, then one can for any $\mu \in \mathbb{R}$ find properly supported Fourier integral operators $A \in I^{\mu}(X \times \mathbb{R}^n, \Gamma')$ and $B \in I^{-\mu}(\mathbb{R}^n \times X, (\Gamma^{-1})')$ such that*

(i) *$WF'(A)$ and $WF'(B)$ lie in small conic neighborhoods of the graph of χ restricted to J and its inverse respectively,*

(ii) $\gamma(I) \cap WF(AB - I) = \emptyset$, $J \cap WF(BA - I) = \emptyset$.

(iii) $\gamma(I) \cap WF(AD_1B - P) = \emptyset$, $J \cap WF(BPA - D_1) = \emptyset$.

Proof of Theorem 26.1.5. We may again assume in the proof that $m = 1$. Changing notation in Theorem 26.1.5 so that I is replaced by $\gamma(I)$, $I \subset \mathbb{R}$, we have precisely the situation in Proposition 26.1.3'. Choose A and B according to Proposition 26.1.3' with $\mu = -s$. We have already constructed a distribution U in $\mathbb{R}^n$ with $WF(U)$ generated by J, $WF(D_1 U)$ generated by the end points of J, $U \in H^{\mathrm{loc}}_{(t)}$ for every $t < 0$ and $U \notin H^{\mathrm{loc}}_{(0)}$ at (x, ξ) for every $(x, \xi) \in J$. If we set $u = AU$ then $U \equiv Bu \mod C^{\infty}$, $Pu = PAU \equiv ABPAU \equiv AD_1 U \mod C^{\infty}$, so u has the required properties.

We shall now discuss existence theorems for the equation $Pu = f$ which follow from Theorems 26.1.4 and 26.1.5 applied to the adjoint P^* combined with abstract functional analysis. At first we shall only consider solvability on compact sets. All operators will tacitly be assumed to act on half densities so that the adjoints are well defined and of the same kind.

Theorem 26.1.7. *Assume that $P \in \Psi^m(X)$ is properly supported and has a real principal part p which is homogeneous of degree m. Let K be a compact subset of X such that no complete bicharacteristic curve is contained in K. Then it follows that*

$$N(K) = \{v \in \mathscr{E}'(K),\ P^* v = 0\}$$

is a finite dimensional subspace of $C_0^{\infty}(K)$ orthogonal to $P\mathscr{D}'(X)$. If $f \in H^{\mathrm{loc}}_{(s)}(X)$ for some $s \in \mathbb{R}$ (resp. $f \in C^{\infty}(X)$) and if f is orthogonal to $N(K)$, then one can find $u \in H^{\mathrm{loc}}_{(s+m-1)}(X)$ (resp. $u \in C^{\infty}(X)$) so that $Pu = f$ in a neighborhood of K.

Proof. The principal part of P^* is also p. Hence $N(K) \subset C^{\infty}$ by Theorem 26.1.1, for if $v \in N(K)$ and $(x, \xi) \in WF(v)$ then the bicharacteristic starting at (x, ξ) would have to remain over K. By the closed graph theorem the L^2 topology in $N(K)$ is equivalent to the C^{∞} topology, which shows that the unit ball in the L^2 topology is compact. Thus $\dim N(K) < \infty$.

The hypotheses of the theorem are also fulfilled if K is replaced by a sufficiently small compact neighborhood K'. To prove this we may assume that $m = 1$ and can then consider the bicharacteristics as curves in the cosphere bundle. Since this is compact over K', we would obtain a bicharacteristic staying over K for all values of the parameter if there is one over K'

for every compact neighborhood K' of K. This proves the assertion. Since $\dim N(K')$ decreases with K' and is finite, it is also clear that $N(K')=N(K)$ if K' is sufficiently close to K.

Let $\| \ \|_{(t)}$ denote a norm which defines the $H_{(t)}$ topology for distributions with support in an arbitrary fixed compact subset of X. Since $v\in\mathscr{E}'(K)$, $P^*v\in H_{(t)}$ implies $v\in H_{(t+m-1)}$ by Theorem 26.1.4, it follows from the closed graph theorem that

$$\|v\|_{(t+m-1)}\leqq C(\|P^*v\|_{(t)}+\|v\|_{(t+m-2)}),\qquad v\in C_0^\infty(K). \tag{26.1.5}$$

Let V be a supplementary space of $N(K)$ in $H_{(t+m-1)}\cap\mathscr{E}'(K)$. Then there is another constant C_1 such that

$$\|v\|_{(t+m-1)}\leqq C_1\|P^*v\|_{(t)},\qquad v\in V\cap C_0^\infty(K). \tag{26.1.6}$$

In fact, if this were false we could select a sequence $v_j\in V$ with

$$\|v_j\|_{(t+m-1)}=1,\qquad \|P^*v_j\|_{(t)}\to 0.$$

A weakly convergent subsequence must converge strongly in $H_{(t+m-2)}$ to a limit $v\in V$ with $P^*v=0$ and $1\leqq C\|v\|_{(t+m-2)}$, by (26.1.5). Hence v is a non-zero element of $N(K)$ belonging to V, which is a contradiction.

If $f\in H_{(s)}^{\text{loc}}(X)$ is orthogonal to $N(K)$ we set $t=1-m-s$ and have by (26.1.6) for some C

$$|(f,v)|\leqq C\|P^*v\|_{(t)},\qquad v\in C_0^\infty(K), \tag{26.1.7}$$

for this is true if $v\in V\cap C_0^\infty(K)$ and neither side changes if an element of $N(K)$ is added to v. By the Hahn-Banach theorem it follows that the anti-linear form $P^*v\mapsto(f,v)$, $v\in C_0^\infty(K)$, can be extended to an anti-linear form on $H_{(t)}^{\text{comp}}$ which is continuous for $\| \ \|_{(t)}$. Thus there is a distribution $u\in H_{(-t)}^{\text{loc}}=H_{(s+m-1)}^{\text{loc}}$ such that

$$(f,v)=(u,P^*v),\qquad v\in C_0^\infty(K),$$

which implies that $Pu=f$ in the interior of K. If we apply this conclusion to a suitable neighborhood K' of K, we obtain $Pu=f$ in a neighborhood of K.

To prove the C^∞ case of the theorem we denote by $C^\infty(K)$ the quotient of $C^\infty(X)$ by the subspace of functions vanishing of infinite order on K. The dual space of this Fréchet space is $\mathscr{E}'(K)$ (Theorem 2.3.3). To show that the range of the map $C^\infty(X)\to C^\infty(K)$ defined by P is the orthogonal space of $N(K)$ we have to show that $P^*\mathscr{E}'(K)$ is weakly closed in $\mathscr{E}'(X)$, or equivalently that the intersection of $P^*\mathscr{E}'(K)$ and the unit ball in $H_{(t)}\cap\mathscr{E}'(K_1)$ is weakly closed for every real t and compact $K_1\subset X$. (See Lemmas 16.5.8 and 16.5.9.) Now $v\in\mathscr{E}'(K)$, $P^*v\in H_{(t)}$ implies $v\in H_{(t+m-1)}$ by Theorem 26.1.4, and by (26.1.6) we have $v=v_1+v_2$ where $v_1\in N(K)$ and $\|v_2\|_{(t+m-1)}\leqq C$. Since the set of such $v_2\in\mathscr{E}'(K)$ is weakly compact and $P^*v=P^*v_2$, the assertion is proved.

Remark 1. When K consists of a point x_0 we conclude that for every $f \in C^\infty$ one can choose $u \in C^\infty$ so that $Pu = f$ in a neighborhood of x_0, provided that H_p does not have the radial direction at any characteristic point (x_0, ξ).

Remark 2. The condition on the bicharacteristics made in Theorem 26.1.7 is merely sufficient and in no way necessary for the conclusion to be valid. For example, if P is a differential operator with constant coefficients our assumption means that P is of principal type (Definition 10.4.11) but the conclusion is always valid in the C^∞ case and holds in the $H_{(s)}$ spaces also for example if P is the heat operator, which has multiple characteristics. Even when the characteristics are simple the condition is not necessary in the variable coefficient case. For example, the conclusions of Theorem 26.1.7 are valid for

$$P = x_2 \partial/\partial x_1 - x_1 \partial/\partial x_2 + c$$

in $X = \{(x_1, x_2);\ 1 < x_1^2 + x_2^2 < 2\}$ if c is a real constant $\neq 0$, although (the normals of) the circles $x_1^2 + x_2^2 = r^2$ are bicharacteristics. Thus the lower order terms may in general be essential. However, they are irrelevant when the hypotheses of Theorem 26.1.7 are fulfilled, and just as in Definition 10.4.11 we introduce a terminology which refers to this fact:

Definition 26.1.8. Let $P \in \Psi^m(X)$ be a properly supported pseudo-differential operator. We shall say that P is of real principal type in X if P has a real homogeneous principal part p of order m and no complete bicharacteristic strip of P stays over a compact set in X.

We shall now discuss global solvability of the equation $Pu = f$ modulo C^∞. The results should be compared with Sections 10.6 and 10.7 in the constant coefficient case.

Theorem 26.1.9. *Let P be of real principal type in the manifold X. Then the following conditions are equivalent:*

(a) *P defines a surjective map from $\mathscr{D}'(X)$ to $\mathscr{D}'(X)/C^\infty(X)$.*

(b) *For every compact set $K \subset X$ there is another compact set $K' \subset X$ such that*

$$u \in \mathscr{E}'(X), \quad \operatorname{sing\,supp} P^* u \subset K \Rightarrow \operatorname{sing\,supp} u \subset K'.$$

(c) *For every compact set $K \subset X$ there is another compact set $K' \subset X$ such that every bicharacteristic interval with respect to P having endpoints over K must lie entirely over K'.*

Proof. (b) $\Rightarrow$ (c) with the same K' by Theorem 26.1.5. Using Theorem 26.1.1 we shall also prove that (c) $\Rightarrow$ (b). In doing so we may assume that P is of order 1 since we can multiply P by an elliptic operator of order $1-m$ without affecting these conditions. When the degree is 1 the bicharacteristic strips can be considered as integral curves of a vector field on the cosphere bundle which is an advantage since the fibers are then compact.

Assuming that (c) is valid, that $u \in \mathscr{E}'(X)$, sing supp $P^*u \subset K$, $(x, \xi) \in WF(u)$, we shall show that there is a contradiction if $x \notin K'$. By Theorem 26.1.1 the bicharacteristic through (x, ξ) stays in $WF(u)$ until it reaches a point lying over K. In view of (c) and the assumption that $x \notin K'$ at least one half ray γ of the bicharacteristic starting at (x, ξ) contains no point above K, so $\gamma \subset WF(u)$. Choose (x_0, ξ_0) so that its projection in the cosphere bundle is a limit point of γ at infinity, which is possible since γ lies over the compact set supp u. Then the entire bicharacteristic strip with initial data (x_0, ξ_0) must stay over supp u, which contradicts the hypothesis that P is of principal type.

Since P is of principal type we know that $u \in C^\infty$ if $u \in \mathscr{E}'$ and $Pu \in C^\infty$. Combined with the purely functional analytic arguments in the proof of Theorem 10.7.8 this gives that (b) $\Rightarrow$ (a).

It remains to show that (a) $\Rightarrow$ (c). Assume that (c) is not valid. For some compact set $K \subset X$ we can then find a sequence of compact intervals $I_1, I_2, \dots$ on bicharacteristic strips with end points lying over K and points $(x_j, \xi_j) \in I_j$ with $x_j \to \infty$ in X, that is, only finitely many contained in any compact subset. We may assume that the intervals I_j are disjoint even when considered in the cosphere bundle. Let (y_j, η_j) be one end point of I_j and let Γ_j be the cone $\subset T^*(X) \setminus 0$ generated by the bicharacteristic between (y_j, η_j) and (x_j, ξ_j) while Γ_j' consists of the rays through these points. Now use Theorem 26.1.5 to determine $u_j \in \mathscr{E}'(X)$ such that

$$WF(u_j) = \Gamma_j, \quad WF(Pu_j) = \Gamma_j', \quad u_j \notin H_{(-j)} \text{ at any point in } \Gamma_j.$$

We can write $Pu_j = f_j + g_j$ where $WF(f_j)$ and $WF(g_j)$ are the rays through (x_j, ξ_j) and (y_j, η_j) respectively. In doing so we can take the support of f_j so close to x_j that the supports of the distributions f_j are locally finite. We can then form

$$f = \sum f_j.$$

Now we shall prove that $Pu - f$ is not in C^∞ for any $u \in \mathscr{D}'(X)$, which means that (a) is not valid. To do so we choose s so large negative that $u \in H_{(s)}^{\mathrm{loc}}$ in a neighborhood of K. When $-j \leqq s$ it follows that $u - u_j$ is not in $H_{(s)}^{\mathrm{loc}}$ at any point on I_j close to (y_j, η_j) whereas $u - u_j \in H_{(s)}^{\mathrm{loc}}$ at the other end point of I_j. By Theorem 26.1.4 this shows since $m = 1$ that $P(u - u_j)$ is not in $H_{(s)}$ at every point in the interior of I_j. However,

$$P(u - u_j) = Pu - f + \sum_{k \neq j} f_k - g_j$$

and the interior of I_j does not meet the wave front set of the sum nor that of g_j. Hence $Pu - f$ is not in $H_{(s)}$ at every point on I_j, which completes the proof.

When convexity conditions similar to those of Section 10.6 are fulfilled one can improve Theorem 26.1.9 to existence of genuine solutions. However, this does not differ very much from the discussion in Section 10.6 so we leave for the reader to contemplate such results or consult the references at the end of the chapter. Instead we shall study global parametrices for operators satisfying the condition in Theorem 26.1.9, for which it is convenient to introduce a name:

Definition 26.1.10. If P is of real principal type in X we shall say that X is pseudo-convex with respect to P when condition (c) in Theorem 26.1.9 is fulfilled.

To clarify the geometric properties of the Hamilton field on the characteristic set we need two lemmas on vector fields satisfying conditions like (c) in Theorem 26.1.9.

Lemma 26.1.11. *Let M be a C^∞ manifold and v a C^∞ vector field on M. Then the following conditions are equivalent:*

(a) *No complete integral curve of v is relatively compact, and for every compact set K in M there is another K' containing every compact interval on an integral curve of v with end points in K.*

(b) *v has no periodic integral curves, and the relation R consisting of all $(y_1, y_2) \in M \times M$ with y_1 and y_2 on the same integral curve of v is a closed C^∞ submanifold of $M \times M$.*

(c) *There exists a manifold M_0, an open neighborhood M_1 of $M_0 \times 0$ in $M_0 \times \mathbb{R}$ which is convex in the $\mathbb{R}$ direction, and a diffeomorphism $M \to M_1$ which carries v into the vector field $\partial/\partial t$ if points in $M_0 \times \mathbb{R}$ are denoted by (y_0, t).*

Proof. Let us first show that the first part of (a) implies

(a′) No integral curve of v defined for all positive or all negative values of the parameter is relatively compact.

In fact, suppose that $\mathbb{R}_+ \ni t \mapsto y(t)$ is an integral curve of v with compact closure K. Then we can find a sequence $t_j \to +\infty$ such that $x = \lim y(t_j)$ exists. Since $t \mapsto y(t_j + t)$ is an integral curve for $t \in (-t_j, \infty)$ it follows that K contains a complete relatively compact integral curve starting at x, which contradicts the first part of (a). (This argument was already used to prove that (c) $\Rightarrow$ (b) in Theorem 26.1.9.)

Next we prove that (a) $\Rightarrow$ (b). Denote the v flow by ϕ so that $t \mapsto \phi(y, t)$ is the solution of the equation $dx/dt = v(x)$ with $x(0) = y$, defined on a maximal open interval $\subset \mathbb{R}$. If D_ϕ is the domain of ϕ, then

$$R = \{(\phi(y, t), y);\ (y, t) \in D_\phi\}.$$

The map $(y, t) \mapsto (\phi(y, t), y)$ is injective since there are no closed integral curves, and it is clear that the differential is injective. To prove that R is a closed C^∞ submanifold it suffices therefore to show that the map is proper. Let $(y_j, t_j) \in D_\phi$ and assume that $y_j \to y$, $\phi(y_j, t_j) \to x$ as $j \to \infty$. We have to show that (y_j, t_j) has a limit point in D_ϕ. In doing so we may assume that $t_j \to T \in [-\infty, \infty]$. By the second part of condition (a) there is a compact set

K' such that $\phi(y_j, t) \in K'$ when $t \in [0, t_j]$. If $T = \pm\infty$ it follows that $\phi(y, s) \in K'$ for $s \geqq 0$ or for $s \leqq 0$. But this contradicts (a') so T is finite and $(y_j, t_j) \to (y, T) \in D_\phi$.

(b) $\Rightarrow$ (c). It follows from (b) that the quotient space $M_0 = M/R$ is a Hausdorff space, and identifying a neighborhood of the equivalence class of y with a manifold transversal to v at y we obtain a structure of C^∞ manifold in M_0. The map $M \to M_0$ has a C^∞ cross section $M_0 \to M$. This is obvious locally and using a partition of unity in M_0 we can piece local sections together to a global one, for only an affine structure is required to form averages. We can now take

$$M_1 = \{(y, t);\ y \in M_0,\ (y, t) \in D_\phi\}$$

and the map $M_1 \to M$ given by ϕ. Since the implication (c) $\Rightarrow$ (a) is trivial, this completes the proof.

In our applications of Lemma 26.1.11 we shall have a conic manifold M and a vector field v commuting with multiplication by positive scalars as is the case for the Hamilton field of a function which is homogeneous of degree 1. Thus vu is homogeneous of degree m if u is. In particular, if M_s is the quotient of M by multiplication with $\mathbb{R}_+$, then v induces a vector field v_s on M_s, as already observed in the proof of Proposition 26.1.6.

Lemma 26.1.12. *Let M be a conic manifold and v a C^∞ vector field on M commuting with multiplication by positive scalars, such that the vector field v_s induced on M_s has the properties in Lemma 26.1.11. Then there exists a C^∞ manifold M'_0, an open neighborhood M' of $M'_0 \times 0$ in $M'_0 \times \mathbb{R}$ which is convex in the direction of $\mathbb{R}$, and a diffeomorphism $M \to M' \times \mathbb{R}_+$, commuting with multiplication by positive scalars (defined as identity in M' and standard multiplication in $\mathbb{R}_+$) such that v is mapped to the vector field $\partial/\partial t$ if (y_0, t, r) denotes the variables in $M'_0 \times \mathbb{R} \times \mathbb{R}_+$.*

Proof. First note that by a partition of unity we can construct a positive C^∞ function $r(y)$ on M which is homogeneous of degree 1. If π is the projection of M on M_s we obtain a diffeomorphism

$$M \ni y \mapsto (\pi(y),\ r(y)) \in M_s \times \mathbb{R}_+$$

commuting with multiplication by $\mathbb{R}_+$. From condition (c) in Lemma 26.1.11 applied to v_s we now obtain a diffeomorphism $M \to M' \times \mathbb{R}_+$ with M' as in that lemma, which transforms v to a vector field of the form

$$v_1 = \partial/\partial t + a(y_0, t)\, r\, \partial/\partial r$$

since it is equal to $\partial/\partial t$ for functions independent of r. Now solve the equation

$$\partial b(y_0, t)/\partial t + a(y_0, t) = 0$$

with initial condition $b=0$ when $t=0$ for example. Then $b\in C^\infty(M')$, and if $R=r\exp b(y_0,t)$ we have $v_1 R=0$. If we take R as a new radial variable instead of r, there will be no term $\partial/\partial R$ in the new expression of v so the lemma is proved.

Remark 1. Under the hypotheses in Lemma 26.1.12 the vector field $(v,0)$ on $M\times M$ defines a vector field $\tilde{v}$ on the relation manifold R (see Lemma 26.1.11(b)) which satisfies the same conditions. This is obvious when the vector field is put in the form given by Lemma 26.1.12.

Remark 2. Let v satisfy the conditions in Lemma 26.1.12 and let $c\in C^\infty(M)$ be homogeneous of degree 0. Then the equation $(v+c)u=f$ has a solution $u\in S^m(M)$ for every $f\in S^m(M)$. In fact, if $c=0$ we just have to integrate f with respect to t from $t=0$ with the coordinates given by Lemma 26.1.12. For a general c we first obtain in this way a homogeneous function C with $vC=c$, and multiplication by e^C reduces to the case $c=0$.

Let us now return to an operator $P\in\Psi^m(X)$ of real principal type, with principal symbol p, assuming that X is pseudo-convex with respect to P. Denote by N the set of zeros of p in $T^*(X)\setminus 0$. This is a conic manifold, and the Hamilton field H_p is tangential to N. The integral curves are the bicharacteristics of P, and we define the bicharacteristic relation C of P by

$$C=\{((x,\xi),(y,\eta))\in N\times N;\ (x,\xi) \text{ and } (y,\eta) \text{ lie on the same bicharacteristic}\}. \tag{26.1.8}$$

The construction is invariant under the action of canonical transformations on p since the definition of the Hamilton field is. Multiplication of p by a non-vanishing function will change the parameter on the bicharacteristics but not affect C. Note that the set C_1 defined by (26.1.2) is the bicharacteristic relation of D_1.

By the preceding remarks we may assume that P is of degree 1 when studying C. By hypothesis the vector field induced by H_p on N_s satisfies condition (a) in Lemma 26.1.11 so Lemma 26.1.12 is applicable. It follows at once that C is a closed conic submanifold of $N\times N$, and since the positive homogeneous function r is constant along the bicharacteristics it is clear that C is also closed in $T^*(X\times X)\setminus 0$. Since C_1 is a canonical relation, that is, the product symplectic form vanishes in C_1, it follows in view of Proposition 26.1.6 that C is a canonical relation. In fact, if $((x,\xi),(y,\eta))\in C$ we can by a canonical transformation reduce p to ξ_1 in a neighborhood of the bicharacteristic between (x,ξ) and (y,η). Thus we have proved:

Proposition 26.1.13. *Assume that P is of real principal type in X and that X is pseudo-convex with respect to P. Then the bicharacteristic relation C of P is a homogeneous canonical relation from $T^*(X)\setminus 0$ to $T^*(X)\setminus 0$ which is closed in $T^*(X\times X)\setminus 0$.*

If Δ_N is the diagonal in N, then $C \setminus \Delta_N$ is the disjoint union $C^+ \cup C^-$ of the forward (backward) bicharacteristic relations C^+ and C^- defined as the set of all $((x, \xi), (y, \eta)) \in N \times N$ such that (x, ξ) lies after (resp. before) (y, η) on a bicharacteristic. These are open subsets of C and inverse relations. The definition is invariant under multiplication of p by positive functions but C^+ and C^- are interchanged if we multiply by a negative function. The importance of these sets is suggested by Proposition 26.1.2 which we shall now extend as follows:

Theorem 26.1.14. *Let $P \in \Psi^m(X)$ be of real principal type in X and assume that X is pseudo-convex with respect to P. Then there exist parametrices E^+ and E^- of P with*

$$WF'(E^+) = \Delta^* \cup C^+, \quad WF'(E^-) = \Delta^* \cup C^- \tag{26.1.9}$$

where Δ^* is the diagonal in $(T^*(X) \setminus 0) \times (T^*(X) \setminus 0)$. Any left or right parametrix E with $WF'(E)$ contained in $\Delta^* \cup C^+$ resp. $\Delta^* \cup C^-$ must be equal to E^+ resp. E^- modulo C^∞. For every $s \in \mathbb{R}$ the parametrices E^+ and E^- define continuous maps from $H_{(s)}^{\mathrm{comp}}(X)$ to $H_{(s+m-1)}^{\mathrm{loc}}(X)$. Finally

$$E^+ - E^- \in I^{\frac{1}{2}-m}(X \times X, C'), \tag{26.1.10}$$

and $E^+ - E^-$ is non-characteristic at every point of C'.

Before the proof we recall that a continuous operator $E: C_0^\infty(X) \to \mathscr{D}'(X)$ is called a right parametrix if

$$PE = I + R$$

where I is the identity and R has a C^∞ kernel. If instead $EP = I + R'$ with $R' \in C^\infty$ then E is called a left parametrix. We shall say that E is a parametrix if E is both a right and a left parametrix. Note that the theorem is an extension of Theorem 8.3.7 also.

Proof of Theorem 26.1.14. We begin with a proof of the uniqueness. Assume for example that E_1 is a right and E_2 a left parametrix with $WF'(E_j) \subset \Delta^* \cup C^+$, which implies that they map C_0^∞ to C^∞. To prove that $E_1 - E_2 \in C^\infty$ we would like to argue that $E_2 P E_1$ is congruent both to E_1 and to $E_2 \bmod C^\infty$ (cf. the proof of Theorem 18.1.9), but this is in no way obvious since E_1 and E_2 are not properly supported. However, we do know that $E_2 B E_1$ is defined if B is a pseudo-differential operator with kernel of compact support in $X \times X$, for B maps $\mathscr{D}'(X)$ to $\mathscr{E}'(X)$ then. If $(x, \xi, y, \eta) \in WF'(E_2 B E_1)$ but (x, ξ) and (y, η) are both in the complement of $WF(B)$ it follows that $(x, \xi, z, \zeta) \in C^+$ and that $(z, \zeta, y, \eta) \in C^+$ for some $(z, \zeta) \in WF(B)$. This implies that (x, ξ), (y, η), (z, ζ) are on the same bicharacteristic strip, with (z, ζ) between the other points. Let K and K' be as in condition (c) in Theorem 26.1.9. If $WF(B)$ has no point over K' it follows that $WF'(E_2 B E_1)$ has no point in $K \times K$. Now choose $\phi \in C_0^\infty(X)$ equal to 1 near K' and form

$$E_2\phi PE_1 - E_2 P\phi E_1 = E_2(\phi P - P\phi)E_1.$$

The wave front set of the right-hand side contains no point over $K \times K$, so the same is true of $E_2\phi - \phi E_1$. Since K is arbitrary it follows that $E_2 - E_1 \in C^\infty$.

Since $PE = I + R$ is equivalent to $E^*P^* = I + R^*$ and P^* has the same principal symbol as P, the existence of left parametrices with the properties listed in the theorem follows from the existence of right parametrices for P^*. To prove the theorem it is therefore sufficient to construct a right parametrix with the required regularity properties. In doing so we may assume that the order of P is 1, for P can otherwise be replaced by the product with an elliptic operator Q of degree $1-m$ with positive homogeneous principal symbol; Q has a pseudo-differential parametrix by Theorem 18.1.24.

The first step in the construction is local in the cotangent bundle near the diagonal.

Lemma 26.1.15. *Let $P \in \Psi^1(X)$ satisfy the hypotheses of Theorem 26.1.14 and let $(x_0, \xi_0) \in T^*(X) \setminus 0$, $p(x_0, \xi_0) = 0$. Choose A and B according to Proposition 26.1.3 with $\mu = 0$ and set $F_1^\pm = \psi E_1^\pm$ where $\psi \in C^\infty(\mathbb{R}^{2n})$ is equal to 1 in a neighborhood of the diagonal. If ψ vanishes outside a sufficiently small neighborhood of the diagonal, $T \in \Psi^0(X)$ has its wave front set in a sufficiently small conic neighborhood of (x_0, ξ_0), and $F^\pm = AF_1^\pm BT$, then*

(i) $WF'(F^\pm) \subset \Delta^* \cup C^\pm$,

(ii) $PF^\pm = T + R^\pm$ *where* $R^\pm \in I^{-\frac12}(X \times X, C')$ *and* $WF'(R^\pm) \subset C^\pm$,

(iii) $F^+ - F^- \in I^{-\frac12}(X \times X, C')$.

Proof. Conditions (i) and (iii) follow immediately from the corresponding conditions in Proposition 26.1.2. To prove (ii) we form

$$PF^\pm = PAF_1^\pm BT = (PA - AD_1)F_1^\pm BT + AD_1F_1^\pm BT. \tag{26.1.11}$$

By (iii) in Proposition 26.1.3 we have

$$(x_0, \xi_0, 0, \varepsilon_n) \notin WF'(PA - AD_1) \subset \Gamma.$$

It follows that there is a conical neighborhood V of $(0, \varepsilon_n)$ such that $(PA - AD_1)v \in C^\infty$ if $WF(v) \subset V$. Since $WF'(F_1^\pm)$ can be made arbitrarily close to the diagonal in $(T^*(\mathbb{R}^n)\setminus 0) \times (T^*(\mathbb{R}^n)\setminus 0)$ by choosing the support of ψ close to the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$, we can choose ψ and a conic neighborhood W of $(0, \varepsilon_n)$ such that $WF(F_1^\pm v) \subset V$ if $WF(v) \subset W$. If $WF(T) \subset \chi(W)$ it follows that the first term in the right-hand side of (26.1.11) is in C^∞. To study the last term in (26.1.11) we note that $D_1 F_1^\pm = I + R_1^\pm$ where

$$R_1^\pm = (D_{x_1}\psi(x, y))E_1^\pm \in I_1^{-\frac12}(\mathbb{R}^n \times \mathbb{R}^n, C_1'), \qquad WF'(R_1^\pm) \subset C_1^\pm.$$

Since $ABT - T = (AB - I)T \in C^\infty$ if $WF(T)$ is sufficiently close to (x_0, ξ_0), it follows that $PF^\pm = T + R^\pm$ where $R^\pm - AR_1^\pm BT \in C^\infty$, which proves (ii).

End of Proof of Theorem 26.1.14. If $(x_0, \xi_0) \in T^*(X) \setminus 0$ and $p(x_0, \xi_0) \neq 0$ then Theorem 18.1.24′ gives a stronger result than Lemma 26.1.15: we can find a pseudo-differential operator F such that $PF = T + R$ where $R \in C^\infty$ and $WF(F) = WF(T)$. Now choose a locally finite covering $\{V_i\}$ of $T^*(X) \setminus 0$ by open cones V_i such that either Lemma 26.1.15 or the preceding observation is applicable when $WF(T) \subset V_i$. We can choose V_i so that the projections W_i in X are also locally finite and can then write $I = \sum T_i$ where $WF(T_i) \subset V_i$ and the support of the kernel of T_i belongs to $W_i \times W_i$. For every i we choose $F_i^\pm$ according to Lemma 26.1.15 or as indicated above, with $\operatorname{supp} F_i^\pm \subset W_i \times W_i$. Then the sum

$$F^\pm = \sum F_i^\pm$$

is defined, (26.1.9) and (26.1.10) are satisfied by these operators, and $F^\pm$ maps $H_{(s)}^{\mathrm{comp}}(X)$ continuously into $H_{(s)}^{\mathrm{loc}}(X)$ for every s. In fact, this is true for $F_1^\pm = \psi E_1^\pm$ if ψ is taken as a function of $x - y$, for the operator $F_1^\pm$ is then convolution by a measure of compact support. All other factors are $H_{(s)}$ continuous by Corollary 25.3.2.

So far we just have

$$PF^\pm = I + R^\pm \quad \text{where } R^\pm \in I_1^{-\frac{1}{2}}(X \times X, C'), \qquad WF'(R^\pm) \subset C^\pm.$$

However, by Lemma 26.1.16 below we can choose $G^\pm \in I^{-\frac{1}{2}}(X \times X, C')$ so that

$$PG^\pm - R^\pm \in C^\infty(X \times X), \qquad WF'(G^\pm) \subset C^\pm \circ WF'(R^\pm).$$

Since corank $\sigma_C = 2$ it follows from Theorem 25.3.8 that $G^\pm$ is continuous from $H_{(s)}^{\mathrm{comp}}(X)$ to $H_{(s)}^{\mathrm{loc}}(X)$ for every s, so $E^\pm = F^\pm - G^\pm$ is a right parametrix which has this continuity property. The construction shows that $F^+ - F^-$ and therefore $E^+ - E^-$ is non-characteristic at the diagonal of N (cf. (26.1.1)). Since $P(E^+ - E^-) \in C^\infty$ it follows from Theorem 25.2.4 that the principal symbol satisfies a first order homogeneous differential equation along the bicharacteristics starting there. Hence $E^+ - E^-$ is non-characteristic everywhere. (Using Proposition 26.1.3′ instead of Proposition 26.1.3 we could in fact have computed the principal symbol directly at any point in C.) This implies that $WF'(E^+ - E^-) = C$, and since $WF'(E^\pm) \subset \Delta^* \cup C^\pm$ we conclude that $WF'(E^\pm) \supset C^\pm$. Since

$$\Delta^* = WF'(I) = WF'(PE^\pm) \subset WF'(E^\pm)$$

the proof of Theorem 26.1.14 will be completed by the following

Lemma 26.1.16. *If $F \in I^s(X \times X, C')$ and $WF'(F) \subset C^\pm$, then one can find $A \in I^s(X \times X, C')$ with*

$$PA - F \in C^\infty, \qquad WF'(A) \subset C^\pm \circ WF'(F) \subset C^\pm.$$

Proof. If a_0 and f are the principal symbols of A_0 and of F, then it follows from Theorem 25.2.4 that $PA_0 - F \in I^{s-1}$ if

$$i^{-1} \mathscr{L}_{H_p} a_0 + c a_0 = f,$$

where $c\in S^0$. Let ω be a non-vanishing section of $M_C\otimes\Omega_C^{\frac12}$ which is homogeneous of degree $n/2$. (As a complex vector bundle $M_C\otimes\Omega_C^{\frac12}$ is trivial.) If we set $a_0=\omega u$ and $f=\omega g$, the equation is of the form

$$i^{-1}H_p u+c'u=g$$

where c' is homogeneous of degree 0 and u, g are scalar symbols of degree s. It follows from the remarks after Lemma 26.1.12 that this equation has a unique solution $u\in S^s$ vanishing on the diagonal in N, and the support is contained in $C^\pm\circ WF'(F)$. The same argument can be applied to PA_0-F. Hence we obtain a sequence $A_j\in I^{s-j}(X\times X, C')$ with

$$WF'(A_j)\subset C^\pm\circ WF'(F)$$

and

$$P(A_0+\ldots+A_j)-F\in I^{s-j-1}(X\times X, C').$$

If we choose A so that $A-A_0-\ldots-A_j\in I^{s-j-1}$ for every j, the lemma is proved.

Theorem 26.1.14 can be generalized when the characteristic set N is not connected. In fact, if $N=N_+\cup N_-$ with N_+ and N_- disjoint and open, then we can find E^+ and E^- as in Theorem 26.1.14 with (26.1.9) replaced by

$$WF'(E^\pm)=\Delta^*\cup(C^\pm\cap(N_+\times N_+))\cup(C^\mp\cap(N_-\times N_-)). \tag{26.1.9$'$}$$

The very slight modification of the proof is left as an exercise for the reader. Important examples of this situation are the advanced and retarded fundamental solutions of the wave operator.

The most noteworthy feature of Theorem 26.1.14 is that a two sided parametrix is obtained. In the following sections we shall prove far reaching extensions of Theorem 26.1.4 concerning the propagation of singularities, and this will lead to existence theorems similar to Theorem 26.1.7. However, we do not have any general methods for constructing two-sided parametrices.

26.2. The Complex Involutive Case

The study of pseudo-differential operators $P\in\Psi^m(X)$ with homogeneous principal symbol p is far more intricate when p is complex valued than in the real case discussed in Section 26.1. Already the geometry of the characteristic set $N=p^{-1}(0)$ may then be very complicated even if $dp\neq 0$. At first we shall therefore only consider the subset

$$N_2=\{(x,\xi)\in T^*(X)\setminus 0;\ p(x,\xi)=0,\ d\operatorname{Re}p(x,\xi)\ \text{and}\ d\operatorname{Im}p(x,\xi)\ \text{are linearly independent}\} \tag{26.2.1}$$

which is a conic manifold of codimension 2. Section 26.3 will be devoted to the open subset

(26.2.2) $N_{2s}=\{(x,\xi)\in T^*(X)\setminus 0;\ p(x,\xi)=0,\ \{\operatorname{Re} p, \operatorname{Im} p\}(x,\xi)\neq 0\}$

which is a symplectic manifold. The purpose of the present section is to study the interior N_{2i} of $N_2\setminus N_{2s}$ which is an involutive manifold. We recall from Section 21.2 that as involutive manifold N_{2i} is foliated by 2 dimensional leaves Γ. In analogy with the real case we shall call them bicharacteristics of P. The Hamilton vector field

$$H_p=H_{\operatorname{Re} p}+iH_{\operatorname{Im} p}$$

is tangential to any leaf Γ and has linearly independent real and imaginary parts so it defines an analytic structure in Γ where the analytic functions are the solutions of the equation $H_p u=0$. By Theorem 21.2.7 a leaf Γ is either conic or else the radial direction is never tangential to Γ. We shall postpone the discussion of the first case until Section 26.7 and only discuss here the open subset N_{2i}^0 of N_{2i} where

(26.2.3) $H_{\operatorname{Re} p}$, $H_{\operatorname{Im} p}$ and the radial direction are linearly independent.

Whereas Theorem 26.1.4 reflects the fact that the equation $H_p u=0$ in the real case has only constant solutions on a bicharacteristic, we shall now have to take into account that this equation has a large solution space in the two dimensional bicharacteristic Γ. To state an analogous result we recall from Section 18.1 that if $u\in\mathscr{D}'(X)$ then the regularity of u at (x,ξ) can be measured by the function

$$s_u^*(x,\xi)=\sup\{t;\ u\in H_{(t)} \text{ at } (x,\xi)\},\qquad (x,\xi)\in T^*(X)\setminus 0,$$

which is lower semi-continuous and positively homogeneous of degree 0. We have by (18.1.38)

$$s_{Au}^*\geqq s_u^*-\mu \tag{26.2.4}$$

if A is a pseudo-differential operator of order μ, and by (18.1.39) there is equality in (26.2.4) where A is non-characteristic. If more generally A is a Fourier integral operator of order μ belonging to a canonical transformation χ then (26.2.4) is just modified to

$$\chi^* s_{Au}^*\geqq s_u^*-\mu, \tag{26.2.4$'$}$$

with equality at the non-characteristic points.

The following is an analogue Theorem 26.1.4:

Theorem 26.2.1. *Let $u\in\mathscr{D}'(X)$, $Pu=f$, and let $\Gamma\subset T^*(X)\setminus 0$ be an open subset of a leaf in the foliation of N_{2i}^0. If s is a superharmonic function in Γ such that $s_f^*\geqq s$ then*

$$\min(s_u^*, s+m-1)$$

is superharmonic in Γ.

When $\Gamma \cap WF(f)=\emptyset$ we can take $s=+\infty$ and conclude that s_u^* is superharmonic. Since a superharmonic function in an open connected set is identically $+\infty$ if it is $+\infty$ in an open subset, we obtain by applying Theorem 26.2.1 to all leaves close to a given one:

Corollary 26.2.2. *If $u\in\mathscr{D}'(X)$ and $Pu=f$, then*

$$(N_{2i}^0 \cap WF(u)) \setminus WF(f)$$

is invariant under the bicharacteristic foliation in $N_{2i}^0 \setminus WF(f)$.

Proof of Theorem 26.2.1. Choose a homogeneous function a of degree $1-m$ with $a(x_0,\xi_0)\neq 0$ at a given point in N_{2i}^0 and a homogeneous canonical transformation χ as in Theorem 21.3.2 such that

$$\chi^*(ap)=\xi_1+i\xi_2$$

in a conic neighborhood of $(0,\varepsilon_n)$. If $Q\in\Psi^{1-m}$ has principal symbol a, we can now repeat the proof of Proposition 26.1.3 to construct Fourier integral operators A and B of order 0 satisfying the conditions (i), (ii) there as well as (iii) with D_1 replaced by D_1+iD_2 and P replaced by QP. The only change is that to construct A_2 we must solve a Cauchy-Riemann equation in each step, and this can be done by Cauchy's integral formula. If $v=Bu$ and $(D_1+iD_2)v=g$ we obtain using (26.2.4)′ as in the proof of Theorem 26.1.1 or (26.1.4) that

$$s_v^*=\chi^* s_u^*, \qquad s_g^*=\chi^* s_f^*+m-1$$

in a neighborhood of $(0,\varepsilon_n)$. This reduces the proof to the special case $P=D_1+iD_2$ and the leaf through $(0,\varepsilon_n)$. It will then be made in three steps.

a) If $u\in\mathscr{E}'(\mathbb{R}^n)$ and $(D_1+iD_2)u=f\in L^2$ then $u\in L^2$. This is a very special case of Theorem 10.3.2. A direct proof follows from the fact that $u=E*f$ where the fundamental solution $E=(2\pi)^{-1}(x_1+ix_2)^{-1}\delta(x_3,\dots,x_n)$ is a measure.

b) (Localization) Let $u\in\mathscr{E}'(\mathbb{R}^n)$, $(D_1+iD_2)u=f$, and assume that for some compact set $K\subset\mathbb{R}^2$ we have, 0 denoting the origin in $\mathbb{R}^{n-2}$,

$$u\in L^2 \quad \text{at} \quad \partial K\times\{0\}\times\varepsilon_n, \qquad f\in L^2 \quad \text{at} \quad K\times\{0\}\times\varepsilon_n.$$

Then $u\in L^2$ at $K\times\{0\}\times\varepsilon_n$. For the proof we set

$$v=\chi_1(x_1,x_2)\chi_2(x_3,\dots,x_n)\chi_3(D)u$$

where $\chi_1\in C_0^\infty(\mathbb{R}^2)$ is equal to 1 in K, $\chi_2\in C_0^\infty(\mathbb{R}^{n-2})$ and $\chi_2(0)=1$, $\chi_3\in S^0(\mathbb{R}^n)$ and $\chi_3(t\varepsilon_n)=1$ when $t>1$. Then $(D_1+iD_2)v=g$ where

$$g=\chi_1\chi_2\chi_3(D)f+(D_1\chi_1+iD_2\chi_1)\chi_2\chi_3(D)u.$$

If $\operatorname{supp}\chi_1$ is sufficiently close to K, $\operatorname{supp}\chi_2$ is sufficiently close to 0 and $\operatorname{supp}\chi_3$ is in a sufficiently small conic neighborhood of ε_n then $g\in L^2$ so $v\in L^2$, by a), which proves the assertion.

c) Let $u \in \mathscr{D}'(\mathbb{R}^n)$, $(D_1 + iD_2)u = f$ and assume that for a compact set $K \subset \mathbb{R}^2$ and an entire function ϕ in $z = x_1 + ix_2$ we have

$$\min(s_u^*, s) > \operatorname{Re}\phi \quad \text{at} \quad \partial K \times \{0\} \times \varepsilon_n \quad \text{and} \quad s_f^* \geqq s \quad \text{at} \quad K \times \{0\} \times \varepsilon_n,$$

where $s(x_1, x_2)$ is superharmonic in a neighborhood of K. Hence $s > \operatorname{Re}\phi$ in K, and by Proposition 16.1.4 the superharmonicity of $\min(s_u^*, s)$ will follow if we show that $s_u^* \geqq \operatorname{Re}\phi$ at $K \times \{0\} \times \varepsilon_n$. Choose $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighborhood of $K \times \{0\}$, and set $U = a(x, D)(\chi u)$ where

$$a(x, \xi) = \chi(x)(1 + |\xi|^2)^{\phi(z)/2}.$$

If $\operatorname{Re}\phi < \mu$ at x, then $a \in S^\mu$ in a neighborhood since differentiation with respect to z can only give factors $\log(1 + |\xi|^2)$. We have $(D_1 + iD_2)U = F$,

$$F = a(x, D)\chi f + [D_1 + iD_2, a(x, D)\chi]u.$$

The commutator is of order $-\infty$ in a neighborhood of $K \times \{0\}$ since $\chi = 1$ there and ϕ is analytic. Hence

$$U \in L^2 \quad \text{at} \quad \partial K \times \{0\} \times \varepsilon_n \quad \text{and} \quad F \in L^2 \quad \text{at} \quad K \times \{0\} \times \varepsilon_n$$

so b) gives that $U \in L^2$ at $K \times \{0\} \times \varepsilon_n$. Set

$$b(x, \xi) = \chi(x)(1 + |\xi|^2)^{-\phi(z)/2}.$$

Then

$$V = b(x, D)U = \chi^3 u + c(x, D)u$$

where $c \in S^{\varepsilon - 1}$ for any $\varepsilon > 0$ in the neighborhood of $K \times \{0\}$ where $\chi = 1$. Since $\chi^3 + c(x, D)$ is non-characteristic there we obtain $s_u^* = s_V^*$, hence

$$s_u^* \geqq \operatorname{Re}\phi - \delta \quad \text{in} \quad K \times \{0\} \times \varepsilon_n$$

for any $\delta > 0$. This completes the proof.

For the operator $D_1 + iD_2$ in $\mathbb{R}^n$, $n \geqq 3$, we shall now prove an analogue of Theorem 26.1.5 which proves that the superharmonicity in Theorem 26.2.1 is exactly the right condition. The result can immediately be carried over locally to the leaves of N_{2i}^0 for a general P, by the argument used to prove Theorem 26.1.5 with Proposition 26.1.3' replaced by the modification of Proposition 26.1.3 at the beginning of the proof of Theorem 26.2.1. A global form of the result can be proved by working more directly with the operator P, but for this we refer to the literature indicated at the end of the chapter.

Theorem 26.2.3. *Let Ω be an open connected subset of $\mathbb{R}^2$ with boundary $\partial\Omega$, and set $\Gamma = \bar{\Omega} \times \{0\} \times \mathbb{R}_+ \varepsilon_n$, $\Gamma' = \partial\Omega \times \{0\} \times \mathbb{R}_+ \varepsilon_n$ where 0 is the origin in $\mathbb{R}^{n-2}$. Let s be a lower semi-continuous function in $\mathbb{R}^2$ with values in $(-\infty, +\infty]$ which is $+\infty$ in $\complement\bar{\Omega}$, superharmonic and not identically $+\infty$ in*

Ω. Then one can find $u \in \mathscr{D}'(\mathbb{R}^n)$ with $WF(u)=\Gamma$, $WF((D_1+iD_2)u) \subset \Gamma'$ and

$$s_u^* = \pi^* s \quad \text{in } \Gamma \setminus \Gamma', \qquad s_u^* \geqq \pi^* s \quad \text{in } \Gamma'. \tag{26.2.5}$$

Here π is the projection $\Gamma \to \bar{\Omega}$.

The proof is similar to that of Theorem 8.3.8, although more technical, so the reader may wish to recall that proof first. Using a functional analytic argument we shall show that u can be found so that $s_u^* \geqq \pi^* s$ with equality in a countable subset E of Γ. This will give (26.2.5) if E is suitably chosen:

Lemma 26.2.4. *For every lower semi-continuous function s in an open set $\Omega \subset \mathbb{R}^N$ there is a countable subset E of Ω such that for every lower semi-continuous function s' in Ω with $s' \leqq s$ in E we have $s' \leqq s$ in Ω.*

Proof. Let V_j be an enumeration of the closed balls with rational center and radius which are contained in Ω. Choose $x_j \in V_j$ such that $s(x_j) = \min_{V_j} s$ which is possible since s is lower semi-continuous, and let $E = \{x_j\}$. If now s' is lower semi-continuous and $s'(x) > s(x)$ for some $x \in \Omega$, we can find V_j with $x \in V_j$ such that $s'(y) > s(x)$ for every $y \in V_j$. Hence $s'(x_j) > s(x) \geqq s(x_j)$. This proves the lemma.

Remark. The choice of E here can be quite unique. For example, if $N=1$ and $s(x)=0$ for irrational x, $s(p/q) = -1/|q|$ when p/q is a reduced fraction, then $s/2 \leqq s$ at all irrational points but not at the rational ones. It is easily seen that E must in fact contain all rational points in this case.

We shall also need an analogue of Theorem 15.1.1 for open subsets of $\mathbb{C}$. (In Section 15.1.1 we only considered the whole of $\mathbb{C}^n$ to avoid technical difficulties which occur otherwise when $n>1$.)

Lemma 26.2.5. *Let ω be an open set in $\mathbb{C}$ and $\phi \in C^2(\omega)$ a strictly subharmonic function, that is, $\Delta\phi > 0$. If $f \in L^2(\omega, e^{-\phi}(\Delta\phi)^{-1} d\lambda)$, where $d\lambda$ is the Lebesgue measure, then one can find $u \in L^2(\omega, e^{-\phi} d\lambda)$ with $\partial u/\partial \bar{z} = f$ and*

$$\int |u|^2 e^{-\phi} d\lambda \leqq 4 \int |f|^2 e^{-\phi} (\Delta\phi)^{-1} d\lambda. \tag{26.2.6}$$

Proof. As in the proof of Theorem 15.1.1 we set

$$(u,v)_\phi = \int_\omega u \bar{v} e^{-\phi} d\lambda; \qquad u, v \in L^2_\phi = L^2(\omega, e^{-\phi} d\lambda).$$

The equation $\partial u/\partial \bar{z} = f$ means that

$$(f,w)_\phi = -(u, \delta w)_\phi, \qquad w \in C_0^\infty(\omega),$$
$$\delta w = e^\phi \partial(e^{-\phi} w)/\partial z = \partial w/\partial z - w \partial\phi/\partial z.$$

Now

$$\|\delta w\|_\phi^2 = -(\partial/\partial \bar{z}\, \delta w, w)_\phi = (\partial^2 \phi/\partial z \partial \bar{z} w, w)_\phi + \|\partial w/\partial \bar{z}\|_\phi^2$$
$$\geqq 4^{-1}((\Delta\phi) w, w)_\phi, \qquad w \in C_0^\infty(\omega).$$

Hence

$$|(f,w)_\phi| \leqq M\,\|\delta w\|_\phi, \quad w \in C_0^\infty; \qquad M^2 = 4\int |f|^2 e^{-\phi}(\Delta\phi)^{-1}\,d\lambda,$$

so the lemma follows from the Hahn-Banach theorem if we extend the map $\delta w \mapsto (f,w)_\phi$ to an antilinear map on L^2_ϕ without increasing the norm.

Just as in Section 15.1 we can use Lemma 26.2.5 to construct analytic functions with appropriate bounds:

Lemma 26.2.6. *Let ϕ, ω be as in Lemma 26.2.5, and let $z_0 \in \omega_0 \Subset \omega$. If t is a large positive number we can then find an analytic function f_t in ω such that*

$$f_t(z_0) = t^{\phi(z_0)}, \quad |f_t(z)| \leqq 2t^{\phi(z)}, \qquad z \in \omega_0. \tag{26.2.7}$$

There are constants C_α such that for all non-negative integers α

$$|D_z^\alpha f_t(z)| \leqq C_\alpha (\log t)^\alpha\, t^{\phi(z)}, \qquad z \in \omega_0. \tag{26.2.8}$$

Proof. Taylor's formula shows that

$$\phi(z) = \operatorname{Re} g(z) + \partial^2\phi(z_0)/\partial z\,\partial\bar{z}\,|z-z_0|^2 + o(|z-z_0|^2)$$

where g is the analytic polynomial

$$g(z) = \phi(z_0) + 2(z-z_0)\,\partial\phi(z_0)/\partial z + (z-z_0)^2\,\partial^2\phi(z_0)/\partial z^2.$$

If b and δ are sufficiently small positive numbers it follows that

$$\phi(z) \geqq \operatorname{Re} g(z) + b\,|z-z_0|^2, \qquad |z-z_0| < \delta.$$

Now choose $\chi \in C_0^\infty(\{z;\,|z-z_0|<\delta\})$ with $\chi(z)=1$ when $|z-z_0|<\delta/2$, and set

$$f_t(z) = \chi(z)\,t^{g(z)} - (z-z_0)\,u(z).$$

f_t is analytic if

$$\partial u/\partial\bar{z} = t^{g(z)}(z-z_0)^{-1}\,\partial\chi/\partial\bar{z} = h_t. \tag{26.2.9}$$

With $\varepsilon = b\delta^2/4 > 0$ we have

$$\int |h_t|^2\, t^{-2\phi}\,d\lambda \leqq C t^{-2\varepsilon}.$$

Shrinking ω if necessary we may assume that $\Delta\phi$ is bounded from below in ω and conclude, using Lemma 26.2.5 with ϕ replaced by $2\phi\log t$, that (26.2.9) has a solution with

$$\int |u|^2\, t^{-2\phi}\,d\lambda \leqq t^{-2\varepsilon}.$$

An application of Lemma 15.1.8 with $r = 1/\log t$ now gives

$$|u(z)| \leqq C' \log t\; t^{-\varepsilon+\phi(z)}, \qquad z \in \omega',$$

where $\omega_0 \Subset \omega' \Subset \omega$. This implies (26.2.7) for large t and $z \in \omega'$. Cauchy's inequality in discs with radius $1/\log t$ and center in ω_0 then proves (26.2.8).

Proof of Theorem 26.2.3. Let F be the Fréchet space of all $u\in\mathscr{D}'(\mathbb{R}^n)$ with $WF(u)\subset\Gamma$ and $s_u^*\geqq\pi^*s$ in Γ; the topology is the weakest one making the maps

$$F\ni u\mapsto Bu\in L^2_{\text{loc}}$$

continuous for every properly supported $B\in\Psi^\mu$ with $\mu<\pi^*s$ in $WF(B)\cap\Gamma$. (It suffices to use countably many operators B, so the topology is metrizable, and it is a routine exercise to verify the completeness.) By Lemma 26.2.4 we can choose a countable subset E of $\Gamma\setminus\Gamma'$ such that $u\in F$ and $s_u^*\leqq\pi^*s$ in E implies $s_u^*=\pi^*s$ in $\Gamma\setminus\Gamma'$.

The subset F_0 of F where $WF((D_1+iD_2)u)\subset\Gamma'$ is also a Fréchet space with the weakest topology making the inclusion $F_0\to F$ and the map

$$F_0\ni u\mapsto(D_1+iD_2)u\in C^\infty(\Omega\times\mathbb{R}^{n-2})$$

continuous. We shall prove that if $\gamma\in E$ and $T\in\Psi^{s(\pi\gamma)}$ is properly supported, with homogeneous principal symbol which does not vanish at γ, then

$$\{u\in F_0,\ Tu\in L^2_{\text{loc}}\} \tag{26.2.10}$$

is of the first category. If we use this fact for a countable number of operators T with $WF(T)$ shrinking to γ it follows that

$$\{u\in F_0,\ s_u^*(\gamma)>s(\pi\gamma)\} \tag{26.2.11}$$

is of the first category. Hence $s_u^*\leqq\pi^*s$ in E for all $u\in F_0$ except a set of the first category, and this will prove Theorem 26.2.3.

Suppose now that (26.2.10) is not of the first category. Then it follows from the closed graph theorem that the map

$$F_0\ni u\mapsto Tu\in L^2_{\text{loc}}$$

is continuous. Let $x_0=(\pi\gamma,0)$ be the projection of γ in $\mathbb{R}^n$ and let K be a compact neighborhood of x_0. Then we have

$$\|Tu\|_{L^2(K)}\leqq\|\chi(D_1+iD_2)u\|_{(M)}+\sum\|B_ju\|_{L^2(K_j)} \tag{26.2.12}$$

where $\chi\in C_0^\infty(\Omega\times\mathbb{R}^{n-2})$, M is a large integer, $B_j\in\Psi^{\mu_j}$ is properly supported, $\mu_j<\pi^*s$ in $\Gamma_j=WF(B_j)\cap\Gamma$, and the sum is finite. Let K_Ω be the union of $\pi\gamma$ and the projection of $\operatorname{supp}\chi$ in Ω. We shall choose u carefully near K_Ω so that the first term on the right-hand side drops out.

Choose open sets ω and ω_0 in $\mathbb{R}^2$ with

$$K_\Omega\subset\omega_0\Subset\omega\Subset\Omega$$

and then choose $\phi\in C^2(\omega)$ with $\Delta\phi>0$ and

$$\phi>-s \text{ in } \omega,\quad \phi<-\mu_j \text{ in } \omega\cap\pi\Gamma_j. \tag{26.2.13}$$

As in the proof of Theorem 15.1.6, for example, we can achieve this by regularizing $-s$ and adding a small multiple of $x_1^2+x_2^2$, for $-s<-\mu_j$ in $\pi\Gamma_j$ and $-s$ is semi-continuous from above. Choose $\chi_0\in C_0^\infty(\omega_0)$ equal to 1 in a neighborhood of K_Ω, $\Phi \in C_0^\infty(\mathbb{R}^{n-2})$ with $\Phi(0)=1$, and set

(26.2.14) $$a_t(x_1,x_2,\xi'')=\chi_0(x_1,x_2)\,f_t(z)\,t^{(2-n)/2}\,\Phi((\xi''/t-\varepsilon_n'')\log t).$$

Here $\xi''=(\xi_3,\dots,\xi_n)$, $z=x_1+ix_2$ and f_t is given by Lemma 26.2.6 with $z_0=\pi\gamma$. Thus $|\xi''/t-\varepsilon_n''|<C/\log t$ in $\operatorname{supp} a_t$, and since differentiation with respect to ξ'' will give a factor $\log t/t$, we obtain in view of (26.2.8)

(26.2.15) $$|D_\xi^\alpha D_x^\beta a_t(x_1,x_2,\xi'')|\le C_{\alpha\beta}(\log(2+|\xi|))^{|\alpha+\beta|}(1+|\xi''|)^{\phi(z)+(2-n)/2-|\alpha|}.$$

If $t=2^\nu$ where ν is an integer $>N_0$, say, the supports are disjoint so

$$A_N=\sum_{N_0}^{N} a_{2^\nu}$$

also has the bound (26.2.15). Note that any conic neighborhood of $\operatorname{supp}\chi_0\times\varepsilon_n''$ contains the supports of all terms except a finite number. Thus A_N is uniformly of order $-\mu$ outside such a neighborhood, for any μ.

We shall prove that (26.2.12) is not valid for the corresponding conormal distributions (with respect to the x_1x_2 plane)

$$u_N(x)=\int A_N(x_1,x_2,\xi'')\,e^{i\langle x'',\xi''\rangle}\,d\xi''$$

when $N\to\infty$. First of all we have $\chi(D_1+iD_2)u_N=0$ since $\chi_0=1$ in supp χ. This means that the first term in the right hand side of (26.2.12) vanishes when $u=u_N$. By (26.2.13) we have $\phi+\mu_j<-\varepsilon_j<0$ in $\omega\cap\pi\Gamma_j$ so (26.2.15) implies that A_N is bounded in $S^{-\varepsilon_j-\mu_j+(2-n)/2}$, in a neighborhood of $\operatorname{supp}\chi_0\cap\pi\Gamma_j$. Using (18.2.16) we now conclude that

$$B_j u_N=\int B_{jN}(x_1,x_2,\xi'')\,e^{i\langle x'',\xi''\rangle}\,d\xi''$$

where B_{jN} is bounded in $S^{-\varepsilon_j+(2-n)/2}$ as $N\to\infty$. If Λ is the conormal bundle of the x_1x_2 plane this means that B_ju_N is bounded in $I^{-\varepsilon_j-n/4}(\mathbb{R}^n,\Lambda)$ (Proposition 25.1.5). Hence B_ju_N in bounded in ${}^\infty H_{(\varepsilon_j)}^{\mathrm{loc}}$ which proves that $\|B_ju_N\|_{L^2(K_j)}$ is bounded. From (26.2.12) it follows now that Tu_N is bounded in $L^2(K)$, so $Tu_\infty\in L^2$ in a neighborhood of x_0. Now Tu_∞ can also be calculated by (18.2.16). If $t(x,\xi)$ is the principal symbol of T, which is homogeneous of degree $s(\pi\gamma)$, then the symbol of Tu_∞ is

$$t(x_1,x_2,0,\xi'')\,A_\infty(x_1,x_2,\xi'')$$

plus lower order terms. At $(\pi\gamma,2^\nu\varepsilon_n'')$ the symbol is thus asymptotically equal to

$$t(\gamma)(2^\nu)^{s(\pi\gamma)+\phi(\pi\gamma)+(2-n)/2}.$$

However, since $Tu_\infty\in{}^\infty H_{(0)}$ in a neighborhood of x_0 it follows from Theorem 25.1.4 that the symbol must be in $S^{\varepsilon+(2-n)/2}$ for every $\varepsilon>0$ in a neighborhood of $\pi\gamma$. This contradicts that $\phi+s>0$ by (26.2.13). The proof is now complete.

26.3. The Symplectic Case

In this section we shall study the pseudo-differential operator P with principal symbol p in the symplectic characteristic manifold N_{2s} defined by (26.2.2). By Theorem 21.3.3 we can locally in N_{2s} reduce p to $\xi_1+ix_1\xi_n$ by multiplication with a non-vanishing factor and composition with a canonical transformation. In Theorem 21.3.5 we have also given an invariant description of a more general situation where we can reduce p to the form $\xi_1+ix_1^k\xi_n$ where k is a positive integer. As in Proposition 26.1.3 we can lift these transformations to the operator level; in doing so we only consider the polyhomogeneous case for the sake of simplicity.

Proposition 26.3.1. *Let $P\in\Psi^m_{\mathrm{phg}}(X)$ have principal symbol p with $p(x^0,\xi^0)=0$, and assume that there is a homogeneous function a of degree $1-m$ in a conic neighborhood of (x^0,ξ^0) with $a(x^0,\xi^0)\neq 0$, and a homogeneous canonical transformation χ from a conic neighborhood of $(0,\pm\varepsilon_n)\in T^*(\mathbb{R}^n)\setminus 0$ to a conic neighborhood of $(x^0,\xi^0)\in T^*(X)\setminus 0$ such that $\chi^*(ap)=\xi_1+ix_1^k\xi_n$. Then we can find properly supported Fourier integral operators $A\in I^{1-m}_{\mathrm{phg}}(X\times\mathbb{R}^n,\Gamma')$ and $B\in I^0_{\mathrm{phg}}(\mathbb{R}^n\times X,(\Gamma^{-1})')$, where Γ is the graph of χ, such that*

(i) *$WF'(A)$ and $WF'(B)$ are in small conic neighborhoods of $(x^0,\xi^0,0,\pm\varepsilon_n)$ and $(0,\pm\varepsilon_n,x^0,\xi^0)$ respectively.*

(ii) *$BA\in\Psi^{1-m}(\mathbb{R}^n)$ is non-characteristic at $(0,\pm\varepsilon_n)$*

(iii) *$(0,\pm\varepsilon_n)\notin WF(BPA-D_1-ix_1^kD_n)$.*

Proof. Choose any $A_1\in I^{1-m}_{\mathrm{phg}}(X\times\mathbb{R}^n,\Gamma')$ and $B_1\in I^0_{\mathrm{phg}}(\mathbb{R}^n\times X,(\Gamma^{-1})')$ such that the principal symbol of A_1B_1 is equal to a in a neighborhood of (x^0,ξ^0). Then the principal symbol of B_1PA_1 is equal to $\xi_1+ix_1^k\xi_n$ in a neighborhood of $(0,\pm\varepsilon_n)$. Replacing P by B_1PA_1 it is then as in the proof of Proposition 26.1.3 sufficient to prove the theorem when $X=\mathbb{R}^n$, $m=1$ and the principal symbol of P is equal to $\xi_1+ix_1^k\xi_n$. The full symbol is then $\xi_1+ix_1^k\xi_n+p_0(x,\xi)+p_{-1}(x,\xi)+\dots$. We want to find pseudo-differential operators A and C of order 0, non-characteristic at $(0,\pm\varepsilon_n)$ such that the symbol of

$$PA-C(D_1+ix_1^kD_n) \tag{26.3.1}$$

is of order $-\infty$ in a conic neighborhood of $(0,\pm\varepsilon_n)$. If B is defined so that the symbol of $BC-I$ is of order $-\infty$ in another such neighborhood, we shall then have all statements in the proposition.

Let the symbols of A and C be $a_0+a_{-1}+\dots$ and $c_0+c_{-1}+\dots$. The leading symbol of (26.3.1) vanishes if $a_0=c_0$. The next term vanishes if

$$-i\{\xi_1+ix_1^k\xi_n,a_0\}+p_0a_0+(\xi_1+ix_1^k\xi_n)(a_{-1}-c_{-1})=0, \tag{26.3.2}$$

that is,

$$-i(\partial/\partial x_1+ix_1^k\partial/\partial x_n-ikx_1^{k-1}\xi_n\partial/\partial\xi_1)a_0+p_0a_0 +(\xi_1+ix_1^k\xi_n)(a_{-1}-c_{-1})=0. \tag{26.3.2$'$}$$

It suffices to solve this equation when $\xi_n=1$ and extend the solution by homogeneity after cutting it off outside a neighborhood of the origin. To do so we choose a_0 so that

$$-i(\partial/\partial x_1+ix_1^k\partial/\partial x_n-ikx_1^{k-1}\xi_n\partial/\partial\xi_1)a_0+p_0a_0 \tag{26.3.3}$$

vanishes of infinite order when $x_1=0$, and $a_0=1$ there. This means that (26.3.3) and all the x_1 derivatives shall vanish when $x_1=0$, which successively determines $\partial^j a_0/\partial x_1^j$ when $x_1=0$ for every j. By Theorem 1.2.6 we can choose a_0 with these derivatives. The quotient r of (26.3.3) by $\xi_1+ix_1^k\xi_n$ is then a C^∞ function r, homogeneous of degree -1, and (26.3.2) is valid if $a_{-1}-c_{-1}=-r$. Using this equation to eliminate c_{-1} from the next equation, it becomes an inhomogeneous equation of the form (26.3.2)′ which can be solved in the same way. Repeating the argument we obtain a solution of (26.3.1), and this completes the proof.

From Proposition 26.3.1 it follows as in the proof of Theorems 26.1.4 and 26.2.1 that any microlocal statement on the singularities of the equation

$$(D_1+ix_1^kD_n)u=f \tag{26.3.4}$$

at $(0,\pm\varepsilon_n)$ can be carried over to the equation $Pu=f$ at (x^0,ξ^0). We shall therefore study the equation (26.3.4) carefully. For odd values of k it will turn out that its properties differ significantly from those of the constant coefficient operators which served as models in Sections 26.1 and 26.2. Fourier transform of (26.3.4) with respect to x_n leads to an ordinary differential equation which we shall examine first.

Lemma 26.3.2. *If $u\in C_0^\infty(\mathbb{R})$ and k is an integer $\geqq 0$ then*

$$\int(|u'|^2+(1+x^{2k})|u|^2)\,dx\leqq C_k\int|u'-x^ku|^2\,dx. \tag{26.3.5}$$

Proof. We may assume that u is real valued. With $f=u'-x^ku$ we have

$$f^2=u'^2+x^{2k}u^2-x^k(u^2)',$$

hence

$$\int f^2\,dx=\int(u'^2+(x^{2k}+kx^{k-1})u^2)\,dx.$$

When k is *odd* the terms in the right hand side are all positive. If we just integrate for $|x|>1$ we also obtain then

$$u(-1)^2+u(1)^2\leqq\int_{|x|>1}f^2\,dx.$$

An integration by parts gives

$$\begin{aligned}\int_{-1}^{1}u^2\,dx&=u(1)^2+u(-1)^2-2\int xuu'\,dx\\ &\leqq u(1)^2+u(-1)^2+\int_{-1}^{1}u^2\,dx/2+\int_{-1}^{1}2u'^2\,dx,\end{aligned}$$

hence

$$\int_{-1}^{1} u^2\,dx \leqq 2(u(1)^2+u(-1)^2)+4\int u'^2\,dx \leqq 6\int f^2\,dx,$$

so (26.3.5) is valid with $C_k=7$. When k is *even* we first observe that

$$\int_1^\infty f^2\,dx \geqq \int_1^\infty (u'^2+x^{2k}u^2)\,dx+u(1)^2.$$

Set $ue^{-x^{k+1}/(k+1)}=v$, $fe^{-x^{k+1}/(k+1)}=g$. Then $v'=g$, so

$$\begin{aligned}\int_{-2}^{1} v^2\,dx &= \int_{-2}^{1} v^2\,d(x+3)=4v(1)^2-v(-2)^2-2\int_{-2}^{1} vv'(x+3)\,dx\\ &\leqq 4v(1)^2-v(-2)^2+\int_{-2}^{1} v^2\,dx/2+32\int_{-2}^{1} v'^2\,dx,\end{aligned}$$

which gives

$$2v(-2)^2+\int_{-2}^{1} v^2\,dx \leqq 8v(1)^2+64\int_{-2}^{1} g^2\,dx.$$

Since $|x|^{k+1}/(k+1)$ is bounded in $(-2,1)$ we have now proved that

$$\int_{-2}^{\infty} u^2\,dx+u(-2)^2 \leqq C_k' \int_{-2}^{\infty} f^2\,dx.$$

If we note that

$$\int_{-\infty}^{-2} f^2\,dx = \int_{-\infty}^{-2} (u'^2+(x^{2k}+kx^{k-1})u^2)\,dx-2^k u(-2)^2$$

and that $x^{2k}+kx^{k-1} \geqq x^{2k}(1-k/2^{k+1}) \geqq 3x^{2k}/4$ if $x<-2$, the estimate (26.3.5) follows.

If we replace x by θx in (26.3.5) we obtain

$$\int(|u'|^2+|\theta^{k+1}x^k u|^2+|\theta u|^2)\,dx \leqq C_k\int|u'-\theta^{k+1}x^k u|^2\,dx, \quad u\in C_0^\infty(\mathbb{R}). \tag{26.3.5$'$}$$

Here θ^{k+1} can be an arbitrary real number if k is even but must be positive when k is odd. This is significant for there is no estimate of the form (26.3.5) with $u'-x^k u$ replaced by $u'+x^k u$ when k is odd. In fact, the equation $u'+x^k u=0$ then has the solution $u(x)=e^{-x^{k+1}}/(k+1)$ in $\mathscr{S}$, and cutting u off far away we find that no such estimate exists. This distinction between even and odd values of k will be crucial in what follows and was in fact already observed in a geometric context in Theorem 21.3.5. From (26.3.5)$'$ we obtain the following estimate

Proposition 26.3.3. *Let $a(\xi)\in C^\infty(\mathbb{R}^n)$ be homogeneous of degree $1/(k+1)$ and assume that $|\xi|<K\xi_n$ in* supp *a for some constant K. Then we have with L^2 norms*

$$\|a(D)u\| \leqq C\|(D_1+ix_1^k D_n)u\|, \quad u\in C_0^\infty(\mathbb{R}^n). \tag{26.3.6}$$

Proof. If $(D_1+ix_1^k D_n)u=f$ then

$$(D_1+ix_1^k\xi_n)U=F$$

where U and F are the Fourier transforms of u and f with respect to x_n. When $\xi_n>0$ it follows from (26.3.5)′ with $\theta=\xi_n^{1/(k+1)}$ that

$$\int|\xi_n^{1/(k+1)}U|^2\,dx_1\leqq C_k\int|F|^2\,dx_1.$$

If $\hat{u}$, $\hat{f}$ are the Fourier transforms in all the variables it follows that

$$\int_{\xi_n>0}|\xi_n^{1/(k+1)}\hat{u}(\xi)|^2\,d\xi\leqq C_k\int|\hat{f}(\xi)|^2\,d\xi.$$

Since $a(\xi)/\xi_n^{1/(k+1)}$ is bounded in supp a the estimate (26.3.6) is proved.

The estimate (26.3.6) leads directly to a result on hypoellipticity:

Proposition 26.3.4. *The operator $D_1+ix_1^k D_n$ is microhypoelliptic where $\xi_n>0$ (and also where $\xi_n<0$ if k is even). More precisely, if (26.3.4) is valid and $f\in H_{(s)}$ at (x^0,ξ^0), then $u\in H_{(s+1/(k+1))}$ at (x^0,ξ^0) if $\xi_n^0>0$ (or $\xi_n^0<0$ and k is even).*

Proof. Assume first that $u\in L^2_{\text{comp}}$, $f\in L^2_{\text{comp}}$. Choose $\chi\in C_0^\infty$ with $\chi\geqq 0$ and $\int\chi\,dx=1$, and form the regularizations

$$u_\varepsilon=u*\chi_\varepsilon=\hat{\chi}(\varepsilon D)u.$$

Then $\|u_\varepsilon\|\leqq\|u\|$ and

$$(D_1+ix_1^k D_n)u_\varepsilon=f*\chi_\varepsilon+i\,[x_1^k D_n,\hat{\chi}(\varepsilon D)]\,u$$

is also bounded in L^2 as $\varepsilon\to 0$ since the symbol of the commutator

$$-\sum_{0<j\leqq k}\varepsilon^{j-1}(D_1^j\hat{\chi})(\varepsilon\xi)\,\varepsilon\xi_n x_1^{k-j}\binom{k}{j}$$

is bounded in S^0_{loc} (Proposition 18.1.2). Hence (26.3.6) shows that $a(D)u_\varepsilon$ is bounded in L^2 as $\varepsilon\to 0$, so $a(D)u\in L^2$ if a satisfies the condition in Proposition 26.3.3. This proves that $u\in H_{(1/(k+1))}$ when $\xi_n>0$; replacing x by $-x$ we obtain the same result when $\xi_n<0$ if k is even.

To prove the general statement assume that we already know that $u\in H_{(t)}$ at (x^0,ξ^0) for a certain $t\leqq s$. If $q(x,D)$ is of order t, q has compact support in x, and $WF(q)$ is in a sufficiently small conic neighborhood of (x^0,ξ^0), then $v=q(x,D)u\in L^2_{\text{comp}}$ and

$$(D_1+ix_1^k D_n)v=q(x,D)f+[D_1+ix_1^k D_n,q(x,D)]\,u\in L^2$$

since the commutator is also of order t. Hence $v\in H_{(1/(k+1))}$ by the first part of the proof. If q is chosen non-characteristic at (x^0,ξ^0) it follows that $u\in H_{(t+1/(k+1))}$ at (x^0,ξ^0). By iterating the argument a finite number of times we obtain $u\in H_{(s+1/(k+1))}$ at (x^0,ξ^0), which completes the proof.

In view of Theorems 21.3.3 and 21.3.5 we obtain from Propositions 26.3.1 and 26.3.4

Theorem 26.3.5. *Let $P \in \Psi^m_{\mathrm{phg}}(X)$ have principal part p, and let (x^0, ξ^0) be a point where*

$$p(x^0, \xi^0) = 0, \qquad \{\operatorname{Re} p, \operatorname{Im} p\}(x^0, \xi^0) > 0.$$

If $Pu = f \in H_{(s)}$ at (x^0, ξ^0) it follows then that $u \in H_{(s+m-\frac{1}{2})}$ at (x^0, ξ^0). More generally, $Pu = f \in H_{(s)}$ at (x^0, ξ^0) implies $u \in H_{(s+m-k/(k+1))}$ at (x^0, ξ^0) if $p(x^0, \xi^0) = 0$, $H_{\operatorname{Re} p}(x^0, \xi^0) \neq 0$, and $\operatorname{Im} p$ has just a zero of order exactly k near (x^0, ξ^0) on each bicharacteristic of $\operatorname{Re} p$ starting near (x^0, ξ^0), with a change of sign from $-$ to $+$ or no sign change at all. In particular, P is then microhypoelliptic at (x^0, ξ^0).

At a non-characteristic point we have of course the "elliptic" result that $f \in H_{(s)}$ implies $u \in H_{(s+m)}$. Thus Theorem 26.3.5 gives a loss of $k/(k+1)$ derivatives compared to the elliptic case. One calls P subelliptic with a loss of $k/(k+1) < 1$ derivatives. A complete discussion of subellipticity will be given in Chapter XXVII. In particular we shall then see that the constant $k/(k+1)$ in Theorem 26.3.5 cannot be decreased, which is also easy to prove by tracing the proof of Proposition 26.3.4 backwards.

When the sign change from $+$ to $-$ ruled out in Theorem 26.3.5 occurs, there is no microhypoellipticity at (x^0, ξ^0). Moreover, non-propagating singularities may appear there.

Theorem 26.3.6. *Let $P \in \Psi^m_{\mathrm{phg}}(X)$ have principal part p, let*

$$p(x^0, \xi^0) = 0, \qquad H_{\operatorname{Re} p}(x^0, \xi^0) \neq 0,$$

and assume that $\operatorname{Im} p$ on every bicharacteristic of $\operatorname{Re} p$ starting near (x^0, ξ^0) has a zero near (x^0, ξ^0) of order exactly k where the sign of $\operatorname{Im} p$ changes from $+$ to $-$. For any $s \in \mathbb{R}$ one can then find $u \in \mathscr{D}'(X)$ with $Pu \in C^\infty(X)$, $WF(u)$ generated by (x^0, ξ^0), and $u \in H^{\mathrm{loc}}_{(t)}$ if and only if $t < s$.

Proof. By Proposition 26.3.1 it suffices to prove the theorem when $P = D_1 + i x_1^k D_n$, $(x^0, \xi^0) = (0, -\varepsilon_n)$, and k is odd. Choose $\psi \in C^\infty(\mathbb{R})$ equal to 1 on $(2, \infty)$ and 0 on $(-\infty, 1)$, and set for real a

$$u_a(x) = \int e^{-\theta(i x_n + x_1^{k+1}/(k+1) + |x''|^2)} \theta^a \psi(\theta)\, d\theta$$

where $x'' = (x_2, \dots, x_{n-1})$. By Theorem 8.1.9 we have

$$WF(u_a) \subset \{(0, -\theta \varepsilon_n), \theta > 0\}.$$

Partial integration shows that u_a and all its derivatives are rapidly decreasing when $x \to \infty$, so $u_a \in H_{(t)}$ at $(0, -\varepsilon_n)$ if and only if $u_a \in H_{(t)}(\mathbb{R}^n)$. Moreover, $\hat{u}_a$ is rapidly decreasing outside any conic neighborhood of $(0, -\varepsilon_n)$. Denote the Fourier transform of $\exp(-x_1^{k+1}/(k+1))$ by Φ, thus $\Phi \in \mathscr{S}$. If $\varepsilon = 1/(k+1)$

then the Fourier transform of u_a with respect to x_1, x'' becomes

$$\int e^{-i\theta x_n} \Phi(\xi_1/\theta^\varepsilon)\theta^{-\varepsilon} e^{-|\xi''|^2/4\theta}(\pi/\theta)^{(n-2)/2} \theta^a \psi(\theta)\, d\theta$$

which means that

$$\hat{u}(\xi_1, \ldots, \xi_{n-1}, -\xi_n) = 2\pi^{n/2} \Phi(\xi_1/\xi_n^\varepsilon) e^{-|\xi''|^2/4\xi_n} \xi_n^{a-\varepsilon-(n-2)/2} \psi(\xi_n), \qquad \xi_n > 0.$$

The product by $(1+|\xi|^2)^{t/2}$ is square integrable when $|\xi_1| < \xi_n$, $|\xi''| < \xi_n$ if and only if

$$2(a-\varepsilon-(n-2)/2)+\varepsilon+(n-2)/2+2t < -1.$$

If we choose a so that

$$2a-\varepsilon-(n-2)/2+2s = -1,$$

the theorem is proved.

In Section 26.4 we shall prove a general form of Theorem 26.3.6 where hypotheses are only made on a single bicharacteristic of $\operatorname{Re} p$. At the same time it will be proved that there is an intimate connection between the existence of non-propagating singularities as in Theorem 26.3.6 and non-existence theorems for the adjoint operator.

As in Section 26.1 we shall finally give parametrix constructions, particularly for the model equation (26.3.4). First we assume that k is *even*. It is then easy to construct a twosided fundamental solution for (26.3.4) reduces to the Cauchy-Riemann equation if $x_1^{k+1}/(k+1)$ is introduced as a new variable instead of x_1. To simplify notation we first assume that $n=2$ and set for $x, y \in \mathbb{R}^2$

$$E(x,y) = \frac{i}{2\pi}(x_1^{k+1}/(k+1)+ix_2-y_1^{k+1}/(k+1)-iy_2)^{-1}. \tag{26.3.7}$$

This is a continuous function of x (or y) with values in L^1_{loc}, and a slight modification of the proof of (3.1.12) gives

$$(D_{x_1}+ix_1^k D_{x_2})E(x,y) = (-D_{y_1}-iy_1^k D_{y_2})E(x,y) = \delta(x-y). \tag{26.3.8}$$

In fact, if $u \in C_0^\infty(\mathbb{R}^n)$ then

$$\int_{|x-y|>\varepsilon} E(x,y)(-D_1 u(y)-iy_1^k D_2 u(y))\, dy = -\int_{|x-y|=\varepsilon} E(x,y)\, u(y)(y_1^k\, dy_1+i\, dy_2)$$

with the contour integral taken in the positive sense. The argument variation of $y_1^{k+1}/(k+1)+iy_2-x_1^{k+1}/(k+1)-ix_2$ around the circle is 2π, which gives the second part of (26.3.8). The first part follows since $E(x,y) = -E(y,x)$. For the operator E with kernel $E(x,y)$ we obtain

$$(D_1+ix_1^k D_2)Eu = E(D_1+ix_1^k D_2)u = u, \qquad u \in C_0^\infty(\mathbb{R}^2). \tag{26.3.9}$$

It is obvious from (26.3.7) that sing supp E is in the diagonal of $\mathbb{R}^2 \times \mathbb{R}^2$. If $(x,\xi,y,\eta) \in WF'(E)$ and $(x,\xi) \neq (y,\eta)$ then it follows from (26.3.9) that $\xi_1 = \eta_1 = 0$, hence $\xi_2 = \eta_2 \neq 0$ since $(D_{x_2}+D_{y_2})E(x,y) = 0$, and therefore $\xi_2 = \eta_2$.

Thus $WF'(E)$ is equal to the diagonal in $(T^*(\mathbb{R}^2)\setminus 0)\times(T^*(\mathbb{R}^2)\setminus 0)$ after all. - In case $n>2$ we have the fundamental solution

$$E(x,y)=\frac{i}{2\pi}(x_1^{k+1}/(k+1)+ix_n-y_1^{k+1}/(k+1)-iy_n)^{-1}\otimes\delta(x''-y'') \tag{26.3.7$'$}$$

where $x''=(x_2,\dots,x_{n-1})$ and $y''=(y_2,\dots,y_{n-1})$. It is clear that

$$(D_1+ix_1^kD_n)Eu=E(D_1+ix_1^kD_n)u=u,\qquad u\in C_0^\infty(\mathbb{R}^n). \tag{26.3.9$'$}$$

By Theorem 8.2.9 we have

$$\begin{aligned} WF'(E)=\{&(x,\xi;y,\eta)\in(T^*(\mathbb{R}^n)\setminus 0)\times(T^*(\mathbb{R}^n)\setminus 0);\\ &(x,\xi)=(y,\eta)\text{ or }x''=y'',\ \xi=\eta,\ \xi_1=\xi_n=0\}.\end{aligned} \tag{26.3.10}$$

From Proposition 26.3.4 it follows that E maps $H_{(s)}^{\mathrm{comp}}$ into $H_{(s+1/(k+1))}^{\mathrm{loc}}$ microlocally where $\xi_n\neq 0$.

The preceding results are essentially familiar from the Cauchy-Riemann equation. However, we shall now see that the situation changes drastically when k is *odd*. At first we assume again that $n=2$. The kernel $E(x,y)$ defined by (26.3.7) now has a singularity both for $x=(y_1,y_2)$ and $x=(-y_1,y_2)$. Instead of (26.3.8) we obtain, say,

$$(-D_{y_1}-iD_{y_2})E(x,y)=\delta(x-(|y_1|,y_2))-\delta(x-(-|y_1|,y_2)).$$

The definition must therefore be changed.

Let us first try to solve the equation

$$(D_1+ix_1^kD_2)u=f\in C_0^\infty(\mathbb{R}^2)$$

by introducing the Fourier transforms U and F of u and f with respect to x_2. This gives the equation $(D_1+ix_1^k\xi_2)\,U(x_1,\xi_2)=F(x_1,\xi_2)$ or

$$\partial_1(U(x_1,\xi_2)\exp(-x_1^{k+1}\xi_2/(k+1))=iF(x_1,\xi_2)\exp(-x_1^{k+1}\xi_2/(k+1)).$$

Since the exponential tends to 0 when $x_1\to\infty$ if $\xi_2>0$, the equation cannot have a solution in $\mathscr{S}$ unless the integral of the right-hand side vanishes. For a general F we take the L^2 orthogonal projection on this subspace, so we form

$$F(x_1,\xi_2)-c(\xi_2)\exp(-x_1^{k+1}\xi_2/(k+1)) \tag{26.3.11}$$

where $c(\xi_2)$ is determined by

$$c(\xi_2)\,I(\xi_2)=\int_{-\infty}^{\infty}F(y_1,\xi_2)\exp(-y_1^{k+1}\xi_2/(k+1))\,dy_1,\qquad \xi_2>0.$$

Here

$$I(\xi_2)=\int_{-\infty}^{\infty}\exp(-2y_1^{k+1}\xi_2/(k+1))\,dy_1=\xi_2^{-1/(k+1)}I(1).$$

($I(1)$ can of course be expressed in terms of $\Gamma(1/(k+1))$.) Let Q_+f be the inverse Fourier transform of the term removed from F in (26.3.11), (26.3.12) $$Q_+ f(x) = (2\pi)^{-1} \iint_{\xi_2 > 0} e^{i\xi_2\phi(x,y)} f(y)\,dy\,d\xi_2/I(\xi_2), \quad f \in C_0^\infty(\mathbb{R}^2)$$

where $\phi(x,y) = x_2 - y_2 + i(x_1^{k+1} + y_1^{k+1})/(k+1)$. It is the orthogonal projection in L^2 on solutions of the homogeneous equation $D_1 u - i x_1^k D_2 u = 0$. An elementary computation gives that the kernel is

(26.3.12)′ $$Q_+(x,y) = (2\pi)^{-1} 2^{-k/(k+1)} (k+1)^{-1/(k+1)} (\phi(x,y)/i)^{-(k+2)/(k+1)}.$$

The inverse Fourier transform of the solution of the differential equation with F replaced by (26.3.11) for $\xi_2 \geqq 0$, and 0 for $\xi_2 < 0$, is

(26.3.13) $$E_+ f(x) = \frac{i}{2\pi} \int_0^\infty d\xi_2 \int e^{i\xi_2\psi(x,y)} (H(x_1 - y_1) - G(x_1, \xi_2)) f(y)\,dy$$

where H is the Heaviside function, $\psi(x,y) = x_2 - y_2 + i(y_1^{k+1} - x_1^{k+1})/(k+1)$, and

(26.3.14) $$G(x_1, \xi_2) = \int_{-\infty}^{x_1} e^{-2t^{k+1}\xi_2/(k+1)}\,dt/I(\xi_2) = G(x_1 \xi_2^{1/(k+1)}, 1).$$

In view of the elementary estimates valid for $\xi_2 > 0$,

(26.3.15) $$\begin{aligned} |G(x_1, \xi_2)| &< C e^{-2x_1^{k+1}\xi_2/(k+1)}, \quad x_1 < 0, \\ |1 - G(x_1, \xi_2)| &< C e^{-2x_1^{k+1}\xi_2/(k+1)}, \quad x_1 > 0 \end{aligned}$$

it follows by partial integration with respect to ξ_2 that the inner integral in (26.3.13) is rapidly decreasing when $\xi_2 \to \infty$. In fact, $\operatorname{Im}\psi \geqq 0$ unless $|x_1| > |y_1|$ and then we have $H(x_1 - y_1) = 0$ if $x_1 < 0$ and $H(x_1 - y_1) = 1$ if $x_1 > 0$. Thus (26.3.13) defines a continuous map from C_0^∞ to C. From the definitions above we obtain

(26.3.16) $$E_+(D_1 + i x_1^k D_2) f = H(D_2) f, \qquad f \in C_0^\infty,$$

(26.3.17) $$(D_1 + i x_1^k D_2) E_+ f = H(D_2) f - Q_+ f, \quad f \in C_0^\infty.$$

Passing to adjoints in (26.3.16), (26.3.17) we obtain

$$(D_1 - i x_1^k D_2) E_+^* f = H(D_2) f, \quad E_+^*(D_1 - i x_1^k D_2) f = H(D_2) f - Q_+^* f, \quad f \in C_0^\infty.$$

We change the sign for x_2 which changes the adjoints to

(26.3.18) $$Q_-(x,y) = (2\pi)^{-1} \int_{\xi_2 < 0} e^{i\xi_2\overline{\phi(x,y)}}\,d\xi_2/I(-\xi_2)$$

(26.3.19) $$E_-(x,y) = -\frac{i}{2\pi} \int_{\xi_2 < 0} e^{i\xi_2\psi(x,y)} (H(y_1 - x_1) - G(y_1, -\xi_2))\,d\xi_2.$$

If we set $E = E_+ + E_-$ and note that $H(D_2) + H(-D_2)$ is the identity, we have

(26.3.20) $$(D_1 + i x_1^k D_2) E f = f - Q_+ f, \quad f \in C_0^\infty,$$

(26.3.21) $$E(D_1 + i x_1^k D_2) f = f - Q_- f, \quad f \in C_0^\infty.$$

The wave front sets of the kernels of these operators are easily determined. First of all we have

(26.3.22) $WF'(Q_\pm)=\{(x,\xi,y,\eta);\ x_1=y_1=\xi_1=\eta_1=0, \quad x_2=y_2,\ \xi_2=\eta_2 \gtrless 0\}$,

for $WF'(Q_\pm)$ is contained in the right hand side by (26.3.12), (26.3.18) and Theorem 8.1.9, and $Q_\pm$ is singular at (x,y) if $x_1=y_1=0$, $x_2=y_2$ by (26.3.12)′ and its analogue for Q_-. For the kernels (26.3.20), (26.3.21) mean that

$$(D_{x_1}+ix_1^k D_{x_2})\,E(x,y)=\delta(x-y)-Q_+(x,y),$$
$$(-D_{y_1}-iy_1^k D_{y_2})E(x,y)=\delta(x-y)-Q_-(x,y),$$

and the translation invariance in x_2 gives in addition

$$(D_{x_2}+D_{y_2})\,E(x,y)=0.$$

The common characteristics of these operators are defined by $\xi_1=\eta_1=0$, $\xi_2=-\eta_2 \neq 0$, $x_1=y_1=0$, and at these points one of the operators is microhypoelliptic by Proposition 26.3.4. It follows that $WF'(E)$ is contained in the diagonal, and since $WF'(Q_+)\cup WF'(Q_-)$ is nowhere dense there we must have equality. We are now ready to prove

Proposition 26.3.7. *Let E, Q_+, Q_- be defined as above. Then* (26.3.20), (26.3.21) *are valid for $f\in\mathscr{E}'$, $WF'(E)$ is the diagonal in $(T^*(\mathbb{R}^2)\setminus 0)\times(T^*(\mathbb{R}^2)\setminus 0)$, and $WF'(Q_\pm)$ is the subset defined by* (26.3.22). *If $f\in H_{(s)}$ at (x^0,ξ^0) then $Q_\pm f\in H_{(s)}$ and $Ef\in H_{(s+1/(k+1))}$ at (x^0,ξ^0), thus $H_{(s)}^{\mathrm{comp}}$ is mapped into $H_{(s)}^{\mathrm{loc}}$ and $H_{(s+1/(k+1))}^{\mathrm{loc}}$ by these operators.*

Proof. Only the continuity statements remain to be proved. We know already that $Q_\pm$ as orthogonal projections in L^2 are bounded there. Let J be the positive canonical ideal defined by the phase function $\xi_2\phi(x,y)$, $\xi_2>0$. It is generated by the functions $\phi(x,y)$, $\xi_1-i\xi_2x_1^k$, $\eta_1-i\eta_2y_1^k$ and $\xi_2-\eta_2$. Then $Q_+\in I^{1/(k+1)-\frac{1}{2}}(\mathbb{R}^4,J')$ is non-characteristic in the real set $J'_{\mathbb{R}}$. It follows that every $Q\in I^{1/(k+1)-\frac{1}{2}}(\mathbb{R}^4,J')$ defines an operator which is continuous from L^2_{comp} to L^2_{loc}, for the corresponding operator can be written in the form $QA \bmod C^\infty$, where A is a pseudo-differential operator of order 0. (Note that this follows from Theorem 25.5.6 if $k=1$ but not for larger values of k.) The $H_{(s)}$ continuity of Q_+ now follows immediately (see for example the proof of Corollary 25.3.2). Hence Q_+^* and therefore Q_- is $H_{(s)}$ continuous. If $f\in H_{(s)}^{\mathrm{comp}}$ then

$$(D_1+ix_1^kD_2)Ef=f-Q_+f\in H_{(s)}^{\mathrm{loc}}$$

so $Ef\in H_{(s+1/(k+1))}$ at (x^0,ξ^0) by Proposition 26.3.4 unless this is a characteristic point with $\xi_2^0<0$. For E^* we have the same result except at the characteristic points with $\xi_2^0>0$, and this completes the proof.

The operators Q_+ and Q_- are hermitian symmetric and

(26.3.23) $\qquad (D_1-ix_1^kD_2)Q_+=0, \qquad (D_1+ix_1^kD_2)Q_-=0.$

They are the projection operators on the cokernel and on the kernel of $D_1+ix_1^kD_2$. We shall draw some important conclusions from this after introducing the extra parameters which occur in the n dimensional case. From now on we therefore redefine E, Q_+, Q_- by substituting x_n, y_n for x_2, y_2 and taking the tensor product with $\delta(x''-y'')$, $x''=(x_2,\dots,x_{n-1})$.

Proposition 26.3.7′. *For the distributions E, Q_+, $Q_-\in\mathscr{D}'(\mathbb{R}^{2n})$ and the corresponding operators we have*

$$(D_1+ix_1^kD_n)Ef=f-Q_+f,\quad f\in\mathscr{E}', \tag{26.3.20$'$}$$

$$E(D_1+ix_1^kD_n)f=f-Q_-f,\quad f\in\mathscr{E}', \tag{26.3.21$'$}$$

$$(D_1-ix_1^kD_n)Q_+f=0,\quad (D_1+ix_1^kD_n)Q_-f=0,\quad f\in\mathscr{E}', \tag{26.3.23$'$}$$

$$\begin{aligned} WF'(Q_\pm)=\{(x,\xi,y,\eta)&\in(T^*(\mathbb{R}^n)\smallsetminus 0)\times(T^*(\mathbb{R}^n)\smallsetminus 0);\\ &(x,\xi)=(y,\eta),\ x_1=\xi_1=0,\ \xi_n\gtrless 0\\ &\text{or } x''=y'',\ \xi=\eta,\ \xi_1=\xi_n=0\}, \end{aligned} \tag{26.3.22$'$}$$

$$\begin{aligned} WF'(E)=\{(x,\xi,y,\eta)&\in(T^*(\mathbb{R}^n)\smallsetminus 0)\times(T^*(\mathbb{R}^n)\smallsetminus 0);\\ &(x,\xi)=(y,\eta) \text{ or } x''=y'',\ \xi=\eta,\ \xi_1=\xi_n=0\}. \end{aligned} \tag{26.3.24}$$

If $f\in\mathscr{E}'(\mathbb{R}^n)$ and $f\in H_{(s)}$ at (x^0,ξ^0) and $\xi_n^0\neq 0$, then $Q_\pm f\in H_{(s)}$ and $Ef\in H_{(s+1/(k+1))}$ at (x^0,ξ^0).

Proof. (26.3.22)′ and (26.3.24) are immediate consequence of Proposition 26.3.7 and Theorem 8.2.9. They show that E and $Q_\pm$ are continuous from $\mathscr{E}'$ to $\mathscr{D}'$, so (26.3.20)′, (26.3.21)′ follow since they hold in C_0^∞ by (26.3.20), (26.3.21). It also follows that $(x^0,\xi^0)\notin WF(Ef)\cup WF(Q_\pm f)$ if $(x^0,\xi^0)\notin WF(f)$ and $\xi_n^0\neq 0$. When proving the last statement we may therefore assume that $f\in H_{(s)}$ and that $WF(f)$ is in a small conic neighborhood of (x^0,ξ^0), thus

$$\hat{a}(D_n)f\in L^2$$

if $a\in\mathscr{E}'$ and $\hat{a}\in S^s(\mathbb{R})$. But $\hat{a}(D_n)$ commutes with E and $Q_\pm$ so it follows from Proposition 26.3.7 that $\hat{a}(D_n)Q_\pm f\in L^2_{\text{loc}}$, $\hat{a}(D_n)Ef\in H^{\text{loc}}_{(1/(k+1))}$. The statement follows if we multiply by $b(D)\chi(x)$ where $\chi\in C_0^\infty$ and $b\in S^0$ is chosen so that $|\xi|/|\xi_n|$ is bounded in $\operatorname{supp} b$, for $b(D)\chi(x)\hat{a}(D_n)$ is then a pseudo-differential operator which can be chosen non-characteristic at (x^0,ξ^0).

Proposition 26.3.7′ immediately gives back Proposition 26.3.4. Indeed, if $u\in\mathscr{E}'$ and $(D_1+ix_1^kD_2)u=f$, then

$$u=Ef+Q_-u.$$

If $f\in H_{(s)}$ at (x^0,ξ^0) and $\xi_n^0>0$ it follows that $u\in H_{(s+1/(k+1))}$ at (x^0,ξ^0) since $(x^0,\xi^0)\notin WF(Q_-u)$. We can also obtain Theorem 26.3.6 for the model operator if we observe that $(D_1+ix_1^kD_2)u=0$ when $u=Q_-f$. We choose $f\in\mathscr{E}'$ with $WF(f)$ equal to a ray in Σ_- where

$$\Sigma_\pm=\{(x,\xi)\in T^*(\mathbb{R}^n);\ x_1=\xi_1=0,\ \xi_n\gtrless 0\}.$$

Then $WF(u)$ is in the same ray by (26.3.22)'. We can choose f so that u is not smooth and then give u the desired regularity by applying a suitable convolution operator in D_n. We can also determine completely when the equation (26.3.4) can be solved microlocally at (x^0, ξ^0), provided that $\xi_n^0 \neq 0$. Let $f \in \mathscr{E}'$ and assume that

$$(x^0, \xi^0) \notin WF((D_1 + i x_1^k D_n) u - f) \tag{26.3.25}$$

for some $u \in \mathscr{D}'$. We can of course take $u \in \mathscr{E}'$ then. Since $Q_+(D_1 + i x_1^k D_2) = 0$ by (26.3.23)', because Q_+ is hermitian symmetric, it follows that

$$(x^0, \xi^0) \notin WF(Q_+ f). \tag{26.3.26}$$

Conversely, if (26.3.26) is valid then (26.3.25) is satisfied by $u = Ef$ in view of (26.3.20)', so we obtain

Proposition 26.3.8. *If $f \in \mathscr{E}'$, and $\xi_n^0 \neq 0$, then one can find $u \in \mathscr{D}'$ satisfying (26.3.25) if and only if (26.3.26) is fulfilled.*

The condition (26.3.26) is of course automatically fulfilled if $(x^0, \xi^0) \notin \Sigma_+$. However, if $(x^0, \xi^0) \in \Sigma_+$ we can as indicated above for Q_- find f so that $WF(Q_+ f)$ is generated by (x^0, ξ^0) and $Q_+ f$ has a prescribed regularity. Using Theorem 26.3.1 we can immediately carry this result over to operators satisfying the condition there. When $k=1$ we obtain in particular

Theorem 26.3.9. *For every $(x^0, \xi^0) \in T^*(X) \setminus 0$ where $p=0$ and $\{\operatorname{Re} p, \operatorname{Im} p\} > 0$ and for any given s one can find $f \in H_{(s)}^{\mathrm{loc}}(X)$ with $WF(f)$ generated by (x^0, ξ^0) and $(x^0, \xi^0) \in WF(Pu - f)$ for every $u \in \mathscr{D}'(X)$.*

It is also easy to extend Proposition 26.3.7' microlocally to operators satisfying the conditions in Proposition 26.3.1. In case N_{2s}, defined by (26.2.2), is the full characteristic variety one can also give a global version of Proposition 26.3.7'. To do so one just combines the local constructions with a pseudo-differential partition of unity placed to the right (left) except near $\Sigma_-(\Sigma_+)$,

$$\Sigma_\pm = \{(x,\xi) \in T^*(X) \setminus 0;\ p(x,\xi) = 0,\ \{\operatorname{Re} p, \operatorname{Im} p\}(x,\xi) \gtrless 0\}. \tag{26.3.27}$$

These constructions fit together in the complement of $\Sigma_- \cup \Sigma_+$ since E is uniquely determined there mod C^∞. The details are left for the reader who might also consult the references at the end of the chapter where it is shown that $Q_\pm$ and E become unique mod C^∞ if $Q_\pm$ are required to be hermitian symmetric.

26.4. Solvability and Condition (Ψ)

Let P be a properly supported pseudo-differential operator in a C^∞ manifold X of dimension n, and let K be a compact subset of X. In this section we shall prove a necessary condition for the equation $Pu=f$ to be solvable at K in a very weak sense suggested by Theorem 26.1.7.

Definition 26.4.1. We shall say that P is solvable at K if for every f in a subspace of $C^\infty(X)$ of finite codimension there is a distribution u in X such that

$$Pu=f \tag{26.4.1}$$

in a neighborhood of K.

In the definition we have not assumed that the neighborhood where (26.4.1) is valid or the order of the distribution u can be chosen independently of f. However, using Baire's theorem we shall now show that this is always possible. At the same time we shall show that solvability is equivalent to a solvability condition mod C^∞.

Theorem 26.4.2. *The following conditions on the properly supported pseudo-differential operator P in X and the compact set $K\subset X$ are equivalent:*

(i) *P is solvable at K.*

(ii) *There is an integer N and an open neighborhood $Y\subset X$ of K such that for every $f\in H_{(N)}^{\mathrm{loc}}(X)$ there is a distribution $u\in H_{(-N)}^{\mathrm{loc}}(X)$ such that $Pu-f\in C^\infty(Y)$.*

(iii) *There is an integer N such that for every $f\in H_{(N)}^{\mathrm{loc}}(X)$ there is a distribution u in X such that $Pu-f\in C^\infty$ in a neighborhood of K.*

(iv) *There is an integer N such that for every $f\in H_{(N)}^{\mathrm{loc}}(X)$ we can find $u\in\mathscr{D}'(X)$ with $Pu-f\in H_{(N+1)}^{\mathrm{loc}}$ in some neighborhood of K.*

(v) *There is an integer N and an open neighborhood $Y\subset X$ of K such that for every f in a subspace $W\subset H_{(N)}^{\mathrm{loc}}(X)$ of finite codimension the equation (26.4.1) is valid in Y for some $u\in H_{(-N)}^{\mathrm{loc}}(X)$.*

Proof. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (i) are obvious. We shall now prove that (i) $\Rightarrow$ (ii). Let $\|\ \|_{(s)}$ denote a norm in $H_{(s)}^{\mathrm{comp}}(X)$ which defines the topology in $H_{(s)}^{c}(M)=H_{(s)}^{\mathrm{loc}}\cap\mathscr{E}'(M)$ for every compact set $M\subset X$. Choose a fundamental decreasing system of open neighborhoods of K,

$$K\Subset\ldots\Subset Y_2\Subset Y_1\Subset X.$$

Since P is properly supported we can find $Z\Subset X$ so that $Pu=0$ in Y_1 if $u=0$ in Z. Fix $\phi\in C_0^\infty(X)$ with $\phi=1$ in Z. Then we have $Pu=P(\phi u)$ in Y_1 if $u\in\mathscr{D}'(X)$, and $\operatorname{supp}\phi u\subset M=\operatorname{supp}\phi$.

Condition (i) means that we can choose $f_1,\ldots,f_r\in C^\infty(X)$ so that for any $f\in C^\infty(X)$ we have

$$Pu=f+\sum_1^r a_j f_j \quad \text{in } Y_N \tag{26.4.2}$$

for some positive integer N, $a_j\in\mathbb{C}$ and $u\in\mathscr{D}'(X)$. Since u can be replaced by ϕu we can always choose $u\in\mathscr{E}'(M)$, hence $u\in H_{(-N)}^{c}(M)$ for some N. The union of the sets

$$F_N = \{f \in C^\infty(X);\ (26.4.2) \text{ is valid with } u \in H^c_{(-N)}(M),\quad \|u\|_{(-N)} + \sum |a_j| \leqq N\}$$

is therefore equal to $C^\infty(X)$. The set F_N is convex, symmetric and closed since the set of permitted $(u, a_1, \dots, a_r)$ in the definition is convex, symmetric and weakly compact. Hence it follows from Baire's theorem that F_N has 0 as an interior point when N is large. Thus we can find $\chi \in C_0^\infty(X)$ and N' so that

$$f \in C^\infty(X), \quad \|\chi f\|_{(N')} \leqq 1 \Rightarrow f \in F_N.$$

Using the same compactness argument again we conclude that (26.4.2) has a solution $u \in H^c_{(-N)}(M)$, $a_1, \dots, a_r$ with $\|u\|_{(-N)} + \sum |a_j| \leqq N$ for every $f \in H^{\text{loc}}_{(N')}$ with $\|\chi f\|_{(N')} \leqq 1$. This gives (ii) with N replaced by max (N, N').

It remains to prove that (iv) $\Rightarrow$ (v). To do so we now denote by G_ν the set of all $f \in H^c_{(N)}(\overline{Y}_1) = H$ such that

$$Pu = f + g \quad \text{in } Y_\nu$$

for some $g \in H^c_{(N+1)}(\overline{Y}_1)$ and $u \in H^c_{(-\nu)}(M)$ with

$$\|u\|^2_{(-\nu)} + \|g\|^2_{(N+1)} \leqq \nu^2. \tag{26.4.3}$$

Baire's theorem gives as above that G_ν contains the unit ball in H for large ν. The minimum of the left-hand side of (26.4.3) is attained precisely when (u, g) is orthogonal to all (u', g') with $Pu' = g'$ in Y_ν, so g is then a linear function Tf of f. (All norms are taken Hilbertian.) The map T: $H \to H^c_{(N+1)}(\overline{Y}_1)$ has norm $\leqq \nu$. Thus T defines a compact operator in H, which implies that the range of $I + T$ has finite codimension. The equation $Pu = h$ in Y_ν has a solution $u \in H^c_{(-\nu)}(M)$ for every $h \in H$ in the range of $I + T$. This proves (v) with N replaced by $\max(N, \nu)$.

Remarks. a) If P satisfies (v) and Q is of order $-2N-1$ then it is clear that $P + Q$ satisfies (iv). Thus solvability at K is not destroyed by perturbations of P of sufficiently low order.

b) In view of (iii) it follows from Theorem 26.3.9 that P is not solvable at $\{x\}$ if x is in the projection in X of the set Σ_+ defined by (26.3.27).

c) In proving that (i) $\Rightarrow$ (ii) it would have been sufficient to know that $\bigcup F_N$ is of the second category. If P is not solvable at K it follows therefore that for every finite dimensional subspace W of $C^\infty(X)$ the set

$$\{f + g;\ f \in C^\infty(X),\ Pu = f \text{ in a neighborhood of } K \text{ for some } u \in \mathscr{D}'(X),\ g \in W\}$$

is of the first category in $C^\infty(X)$. For any sequence K_j, W_j with these properties we can thus find $f \in C^\infty(X)$ so that the equation $Pu = f$ cannot be solved modulo W_j in a neighborhood of K_j for any j. In particular, we can choose $f \in C^\infty(X)$ so that the equation $Pu = f$ cannot be solved in a neighborhood of any point in the projection of Σ_+ in X. An example is the Lewy operator $P = D_1 + iD_2 + i(x_1 + ix_2)D_3$ in $\mathbb{R}^3$. Since

$$\Sigma_+ = \{(x, \xi);\ \xi_1 = x_2\xi_3,\ \xi_2 = -x_1\xi_3,\ \xi_3 > 0\}$$

has surjective projection we can choose $f \in C^\infty(\mathbb{R}^3)$ so that the equation $Pu = f$ does not have a distribution solution in any open set.

The condition (iii) in Theorem 26.4.2 suggests a definition of solvability at a set in the cosphere bundle:

Definition 26.4.3. If K is a compactly based cone $\subset T^*(X)\setminus 0$ we shall say that P is solvable at K if there is an integer N such that for every $f \in H_{(N)}^{\mathrm{loc}}(X)$ we have $K \cap WF(Pu-f) = \emptyset$ for some $u \in \mathscr{D}'(X)$.

Solvability at a compact set $M \subset X$ is equivalent to solvability at $T^*(X)|_M \setminus 0$, by condition (iii) in Theorem 26.4.2. Note that solvability at $K \subset T^*(X)\setminus 0$ implies solvability at any smaller closed cone, and that solvability at K only depends on the symbol of P in a conic neighborhood of K. This makes it possible to prove necessary conditions for solvability by local arguments where the following proposition can be used:

Proposition 26.4.4. *Let $K \subset T^*(X)\setminus 0$ and $K' \subset T^*(Y)\setminus 0$ be compactly based cones and let χ be a homogeneous symplectomorphism from a conic neighborhood of K' to one of K such that $\chi(K') = K$. Let $A \in I^{m'}(X \times Y, \Gamma')$ and $B \in I^{m''}(Y \times X, (\Gamma^{-1})')$ where Γ is the graph of χ, and assume that A and B are properly supported and non-characteristic at the restriction of the graphs of χ and χ^{-1} to K' and to K respectively, while $WF'(A)$ and $WF'(B)$ are contained in small conic neighborhoods. Then the pseudo-differential operator P in X is solvable at K if and only if the pseudo-differential operator BPA in Y is solvable at K'.*

Proof. Choose $A_1 \in I^{-m''}(X \times Y, \Gamma')$ and $B_1 \in I^{-m'}(Y \times X, (\Gamma^{-1})')$ properly supported so that

$$K' \cap WF(BA_1 - I) = \emptyset, \qquad K \cap WF(A_1 B - I) = \emptyset,$$
$$K' \cap WF(B_1 A - I) = \emptyset, \qquad K \cap WF(AB_1 - I) = \emptyset.$$

Assume that P is solvable at K and choose N as in Definition 26.4.3. Let $g \in H_{(N-m'')}^{\mathrm{loc}}(Y)$ and set $f = A_1 g \in H_{(N)}^{\mathrm{loc}}(X)$. We can then find $u \in \mathscr{D}'(X)$ with $K \cap WF(Pu-f) = \emptyset$. Let $v = B_1 u \in \mathscr{D}'(\mathbb{R}^n)$. Then

$$WF(Av - u) = WF((AB_1 - I)u)$$

does not meet K, so $K \cap WF(PAv - f) = \emptyset$. Hence

$$K' \cap WF(BPAv - Bf) = \emptyset.$$

Since $K' \cap WF((BA_1 - I)g) = \emptyset$ it follows that

$$K' \cap WF(BPAv - g) = \emptyset.$$

Hence BPA is solvable at K'. Conversely, if BPA is solvable at K' it follows that $A_1 BPAB_1$ is solvable at K. Since $K \cap WF(A_1 BPAB_1 - P) = \emptyset$ this means that P is solvable at K, which completes the proof.

As a final analytic preparation for the proof of necessary conditions for solvability we shall show that solvability of P implies an a priori estimate for the adjoint operator P^*.

Lemma 26.4.5. *Let K be a compactly generated cone $\subset T^*(X)\setminus 0$ such that P is solvable at K, and choose $Y\Subset X$ so that $K\subset T^*(Y)$. If N is the integer in Definition 26.4.3 we can find an integer ν and a properly supported pseudo-differential operator A with $WF(A)\cap K=\emptyset$ such that*

$$\|v\|_{(-N)}\leqq C(\|P^*v\|_{(\nu)}+\|v\|_{(-N-n)}+\|Av\|_{(0)}),\qquad v\in C_0^\infty(Y). \tag{26.4.4}$$

Proof. Let $Y\Subset Z\Subset X$. We claim that for fixed f in the Hilbert space $H^c_{(N)}(\bar{Z})$ we have for some C, ν and A as in the lemma

$$|(f,v)|\leqq C(\|P^*v\|_{(\nu)}+\|v\|_{(-N-n)}+\|Av\|_{(0)}),\qquad v\in C_0^\infty(Y). \tag{26.4.5}$$

In fact, by hypothesis we can find u and g in $\mathscr{E}'(X)$ so that $f=Pu+g$ and $K\cap WF(g)=\emptyset$. Thus

$$(f,v)=(u,P^*v)+(g,v),\qquad v\in C_0^\infty(Y).$$

Choose properly supported pseudo-differential operators B_1 and B_2 of order 0 with $I=B_1+B_2$ and $WF(B_1)\cap WF(g)=\emptyset$, $WF(B_2)\cap K=\emptyset$ which is possible since $WF(g)\cap K=\emptyset$. Then $B_1g\in C^\infty$ so (B_1g,v) can be estimated by $C\|v\|_{(-N-n)}$. We have for some μ

$$|(B_2g,v)|\leqq \|B_2^*v\|_{(\mu)}\leqq C(\|BB_2^*v\|_{(0)}+\|v\|_{(-N-n)})$$

if B is elliptic of order μ and properly supported. This gives (26.4.5) with $A=BB_2^*$.

Let V be the space $C_0^\infty(Y)$ equipped with the topology defined by the semi-norms $\|v\|_{(-N-n)}$, $\|P^*v\|_{(\nu)}$, $\nu=1,2,\dots$, and $\|Av\|_{(0)}$ where A is a properly supported pseudo-differential operator with $K\cap WF(A)=\emptyset$. It suffices to use a countable sequence $A_1,A_2,\dots$ where A_ν is noncharacteristic of order ν in a set which increases to $T^*(X)\setminus 0\setminus K$ as $\nu\to\infty$. Thus V is a metrizable space. The sesquilinear form (f,v) in the product of the Hilbert space $H^c_{(N)}(\bar{Z})$ and the metrizable space V is obviously continuous in f for fixed v, and by (26.4.5) it is also continuous in v for fixed f. Hence it is continuous, which means that for some ν and C

$$|(f,v)|\leqq C\|f\|_{(N)}(\|P^*v\|_{(\nu)}+\|A_\nu v\|_{(0)}+\|v\|_{(-N-n)}),$$
$$f\in H^c_{(N)}(\bar{Z}),\qquad v\in C_0^\infty(Y).$$

This implies (26.4.4).

Proposition 26.3.8 suggests that an operator $P\in\Psi^m_{\mathrm{phg}}$ with principal symbol p is not solvable at a characteristic point where $\operatorname{Im}p$ changes sign from $-$ to $+$ on the oriented bicharacteristic of $\operatorname{Re}p$. However, from Proposition 26.4.4 we know that a necessary condition for solvability stated in terms of p should be invariant under multiplication by non-vanishing homogeneous functions, so we are led to the following somewhat more complicated looking condition:

Definition 26.4.6. The positively homogeneous function $p \in C^\infty(T^*(X) \setminus 0)$ is said to satisfy condition (Ψ) in the open set $Y \subset X$ if there is no positively homogeneous complex valued function q in $C^\infty(T^*(Y) \setminus 0)$ such that $\operatorname{Im} qp$ changes sign from $-$ to $+$ when one moves in the positive direction on a bicharacteristic of $\operatorname{Re} qp$ over Y on which $q \neq 0$. (Sometimes $\bar{p}$ is then said to satisfy $(\bar{\Psi})$.)

Recall that a bicharacteristic of r is an integral curve of the Hamilton field H_r where $r=0$. We shall say that a bicharacteristic of $\operatorname{Re} qp$ where $q \neq 0$ is a *semi-bicharacteristic* of p. The main purpose of this section is to prove the following theorem.

Theorem 26.4.7. *Suppose that there is a C^∞ positively homogeneous function q in $T^*(X) \setminus 0$ and a bicharacteristic interval $t \mapsto \gamma(t)$, $a \leqq t \leqq b$, for* $\operatorname{Re} qp$ *such that $q(\gamma(t)) \neq 0$, $a \leqq t \leqq b$, and*

$$\operatorname{Im} qp(\gamma(a)) < 0 < \operatorname{Im} qp(\gamma(b)).$$

Then P is not solvable at the cone generated by $\gamma([a,b])$.

Corollary 26.4.8. *If P is solvable at the compact set $K \subset X$ then K has an open neighborhood Y in X where p satisfies condition (Ψ).*

Proof. By condition (v) in Theorem 26.4.2 we can find a neighborhood Y of K such that P is solvable at any compactly generated cone $M \subset T^*(Y)$. Hence the statement follows from Theorem 26.4.7.

Without using Theorem 26.4.7 but only results already established we can prove that $\operatorname{Im} qp$ cannot change sign from $-$ to $+$ on a bicharacteristic of $\operatorname{Re} qp$ at a point $(x^0, \xi^0) \in T^*(Y) \setminus 0$ where $\operatorname{Im} qp$ vanishes of finite order. In fact, if Q is a pseudo-differential operator with principal symbol q we know from Proposition 26.4.4 that QP must be solvable in a neighborhood of (x^0, ξ^0). On every bicharacteristic of $\operatorname{Re} qp$ nearby there must be a zero (x^1, ξ^1) where the same sign change occurs, and we choose it so that the order of the zero is minimal. Then qp satisfies the hypothesis of Theorem 21.3.5 at (x^1, ξ^1) so using Proposition 26.3.1 we can transform QP microlocally at (x^1, ξ^1) to the operator $D_1 + i x_1^k D_n$ at $(0, \varepsilon_n)$, where it is not solvable by Proposition 26.3.8. In view of Proposition 26.4.4 this is a contradiction proving the weaker form of condition (Ψ).

Before proving Theorem 26.4.7 in complete generality we must study the geometrical situation in some detail; this will also lead to a simpler form of condition (Ψ). Suppose that the hypotheses of Theorem 26.4.7 are fulfilled, and choose a pseudo-differential operator Q with principal symbol q. Then the principal symbol of $P_1 = QP$ is $p_1 = qp$, so $\operatorname{Im} p_1$ changes sign from $-$ to $+$ along a bicharacteristic of $\operatorname{Re} p_1$. We then set $P_2 = Q_1 P_1$ where Q_1 is of degree $1 -$degree P_1 and has positive, homogeneous principal symbol. If p_2 is the principal symbol of P_2 then $\operatorname{Im} p_1$ and $\operatorname{Im} p_2$ have the same sign and $\operatorname{Re} p_2$ has the same bicharacteristics as $\operatorname{Re} p_1$ including the orientation. In view of Proposition 26.4.4 it is therefore sufficient to prove Theorem 26.4.7 in the case where $q = 1$ and p is of degree 1. The bicharacteristics of $\operatorname{Re} p$ can then be considered as curves on the cosphere bundle. If the curve where $\operatorname{Im} p$ changes sign is closed on $S^*(X)$ we can always pick an arc which is not closed where the change of sign still occurs, and this we assume done in what follows. We can then use Proposition 26.1.6 and Proposition 26.4.4 to reduce the proof further to the case $X = \mathbb{R}^n$, $\operatorname{Re} p = \xi_1$, and the bicharacteristic of $\operatorname{Re} p$ given by

$$a \leqq x_1 \leqq b, \quad x' = (x_2, \dots, x_n) = 0, \quad \xi = \varepsilon_n. \tag{26.4.6}$$

Global problems might occur in our constructions if $b - a$ is large so we shall examine how small the intervals can be where the crucial sign change occurs. To do so we set

$$L(x', \xi') = \inf\{t - s;\ a < s < t < b,\ \operatorname{Im} p(s, x', 0, \xi') < 0 < \operatorname{Im} p(t, x', 0, \xi')\}$$

when (x', ξ') is close to $(0, \varepsilon_n')$, and we denote by L_0 the lower limit of $L(x', \xi')$ as $(x', \xi') \to (0, \varepsilon_n')$. For small $\delta > 0$ we can choose an open neighborhood V_δ of $(0, \varepsilon_n')$ in $\mathbb{R}^{2n-2}$ with diameter $< \delta$ such that $L(x', \xi') > L_0 - \delta/2$ in V_δ. For some $(x_\delta', \xi_\delta') \in V_\delta$ and s_δ, t_δ with $a < s_\delta < t_\delta < b$ we have

$$t_\delta - s_\delta < L_0 + \delta/2, \qquad \operatorname{Im} p(s_\delta, x_\delta', 0, \xi_\delta') < 0 < \operatorname{Im} p(t_\delta, x_\delta', 0, \xi_\delta').$$

It follows that $\operatorname{Im} p(t, x', 0, \xi')$ and all derivatives with respect to x', ξ' must vanish at $(t, x_\delta', 0, \xi_\delta')$ if $s_\delta + \delta < t < t_\delta - \delta$, for otherwise we could choose $(x', \xi') \in V_\delta$ so close to (x_δ', ξ_δ') that

$$\operatorname{Im} p(t, x', 0, \xi') \neq 0, \qquad \operatorname{Im} p(s_\delta, x', 0, \xi') < 0 < \operatorname{Im} p(t_\delta, x', 0, \xi').$$

The required change of sign must then occur in one of the intervals (s_δ, t) and (t, t_δ) which is impossible since they are shorter than $L_0 - \delta/2$.

Choose a sequence $\delta_j \to 0$ such that $\lim s_{\delta_j} = a_0$ and $\lim t_{\delta_j} = b_0$ exist. Then $b_0 - a_0 = L_0$ and $\operatorname{Im} p^{(\alpha)}_{(\beta)}(t, 0, \varepsilon_n) = 0$ for all α, β with $\alpha_1 = 0$ if $a_0 < t < b_0$. If $a_0 < b_0$ it follows in particular that we have a one dimensional bicharacteristic in the following sense:

Definition 26.4.9. A one dimensional bicharacteristic of the pseudo-differential operator with homogeneous principal symbol p is a C^1 map $\gamma\colon I \to T^*(X) \smallsetminus 0$ where I is an interval on $\mathbb{R}$, such that

(i) $p(\gamma(t)) = 0$, $t \in I$,

(ii) $0 \neq \gamma'(t) = c(t) H_p(\gamma(t))$ if $t \in I$.

In order to achieve a simplification of p similar to that in Theorem 21.3.6 near a one dimensional bicharacteristic we shall now prove that the choice of the function q in Definition 26.4.6 is not very essential there.

Lemma 26.4.10. *Let $\gamma\colon I \to T^*(X)\setminus 0$ be the inclusion of a characteristic point for p or a compact one dimensional bicharacteristic interval and assume that for some $q \in C^\infty$ we have*

(i) $q \neq 0$ *and* $\operatorname{Re} H_{qp} \neq 0$ *on* $\gamma(I)$,

(ii) *there is a neighborhood U of $\gamma(I)$ where* $\operatorname{Im} qp$ *never changes sign from* $-$ *to* $+$ *along a bicharacteristic of* $\operatorname{Re} qp$.

Then (ii) *is valid for every q satisfying* (i).

Note that no homogeneity is assumed here so we could in fact have an arbitrary symplectic manifold. This will be allowed in the following more general statement of the result which is actually easier to prove.

Lemma 26.4.10′. *Let I be a point or a compact interval on $\mathbb{R}$, and let $\gamma\colon I \to M$ be an embedding of I in a symplectic manifold M as a one dimensional bicharacteristic of $p = p_1 + ip_2$, if I is not reduced to a point, and any characteristic point otherwise. Let*

$$f_j = \sum_1^2 a_{jk} p_k, \quad j = 1, 2,$$

where $\det(a_{jk}) > 0$ *on $\gamma(I)$. Assume that $H_{p_1} \neq 0$ and that $H_{f_1} \neq 0$ on $\gamma(I)$. If $\gamma(I)$ has a neighborhood U such that p_2 does not change sign from $-$ to $+$ along any bicharacteristic for p_1 in U, then U can be chosen so that f_2 has no such sign change along the bicharacteristics of f_1 in U.*

Proof. First note that if $p = 0$ at a point in U then

$$\{p_1, p_2\} = H_{p_1} p_2 \leqq 0.$$

Hence we have at the same point

$$\begin{aligned}\{f_1, f_2\} &= \{a_{11} p_1 + a_{12} p_2,\ a_{21} p_1 + a_{22} p_2\} \\ &= (a_{11} a_{22} - a_{12} a_{21}) \{p_1, p_2\} \leqq 0.\end{aligned}$$

The proof is now divided into two steps, the first of which is quite trivial.

(i) Assume first that $a_{12} = 0$. Since $a_{11} a_{22} > 0$ either a_{11} and a_{22} are both positive or both negative. Thus the bicharacteristics of $f_1 = a_{11} p_1$ are equal to those of p_1 with preserved and reversed orientation respectively, and $f_2 = a_{22} p_2$ when $p_1 = 0$ so f_2 has the same and opposite sign as p_2, respectively. The lemma is therefore true in this case.

(ii) Proposition 26.1.6 obviously has an analogue for a general symplectic manifold where we just drop everything referring to the multiplicative structure in $T^*(X)\setminus 0$. The proof is the same except that we start from Theorem 21.1.6 instead of Theorem 21.3.1. By a canonical change of variables we can therefore make $M=\mathbb{R}^{2n}$, $p_1=\xi_1$ and $\Gamma=\gamma(I)$ equal to an interval on the x_1 axis. Let T be a vector $\in\mathbb{R}^{2n}$ with

$$\langle T, dp_1\rangle=1, \quad \langle T, df_1\rangle\neq 0 \quad \text{on } \Gamma.$$

Since dp_1 and df_1 do not vanish on Γ, the existence of T is obvious if Γ consists of a single point. Otherwise dp_2 is proportional to dp_1 on Γ so df_1 is proportional to dp_1. We can take any T with ξ_1 coordinate equal to 1 then.

Set

$$q_2(x,\xi)=p_2((x,\xi)-\xi_1 T)$$

which means that $p_2=q_2$ when $\xi_1=0$ and that q_2 is constant in the direction T. Then there is a C^∞ function ϕ such that

$$q_2=\phi p_1+p_2$$

so it follows from step (i) that the hypotheses in the lemma are fulfilled for p_1+iq_2. We have

$$f_1=(a_{11}-a_{12}\phi)p_1+a_{12}q_2,$$

hence

$$0\neq\langle T, df_1\rangle=(a_{11}-a_{12}\phi) \quad \text{on } \Gamma.$$

In a neighborhood of Γ we can therefore divide f_1 by $a_{11}-a_{12}\phi$ and set

$$q_1=f_1/(a_{11}-a_{12}\phi)=p_1+a_{12}(a_{11}-a_{12}\phi)^{-1}q_2$$

which implies

$$f_j=\sum_1^2 b_{jk}q_k, \quad j=1,2,$$

where $b_{11}=a_{11}-a_{12}\phi$, $b_{12}=0$ and $\det b=\det a>0$. Thus it follows from step (i) that it is sufficient to prove that (q_1,q_2) satisfies the hypothesis made on (p_1,p_2) in the lemma. The difficulty here is that the surfaces $p_1=0$ and $q_1=0$ are not the same. We shall identify them by projecting in the direction T.

Let U be a neighborhood of Γ where q_2 does not change sign from $-$ to $+$ on the bicharacteristics of p_1. Since T is transversal to the surface $f_1=q_1=0$ we can choose U so small that

$$Y=\{(x,\xi)\in U;\, q_1(x,\xi)=0\}$$

is mapped diffeomorphically by the projection π in the direction T on

$$X=\{(x,\xi)\in U;\, \xi_1=0\}.$$

When $q_1=q_2=0$, thus $p_1=p_2=0$, we have $H_{q_1}q_2=H_{p_1}p_2\leqq 0$. At a point in Y where $q_2=0$ and dq_2 vanishes on the tangent space of Y, we have $dq_2=0$ since $\langle T, dq_2\rangle=0$. Hence $w=H_{q_1}=H_{p_1}$ there so $\pi_* w=H_{p_1}$. If we apply the following lemma to $f=q_2=\pi^* q_2$ and the vector fields $v=(\pi^{-1})_* H_{p_1}$ and $w=H_{q_1}$ in Y, it follows that q_2 cannot change sign from $-$ to $+$ along a bicharacteristic of q_1 in Y, which proves the lemma.

Lemma 26.4.11. *Let $f \in C^1(Y)$ where Y is a C^2 manifold and let v be a Lipschitz continuous vector field in Y such that for any integral curve $t \mapsto y(t)$ of v we have*

$$f(y(0))<0 \Rightarrow f(y(t)) \leqq 0 \quad \text{for } t>0. \tag{26.4.7}$$

Let w be another Lipschitz continuous vector field such that

$$\langle w, df \rangle \leqq 0 \quad \text{when } f=0 \tag{26.4.8}$$

$$w=v \quad \text{when } f=df=0. \tag{26.4.9}$$

Then (26.4.7) *remains valid if $y(t)$ is an integral curve of w.*

Note that (26.4.8) is empty when $f=df=0$ so it is natural that another condition must be imposed then.

Proof. Let F be the closure of the union of all forward orbits for v starting at a point with $f(y)<0$. By (26.4.7) we have $f \leqq 0$ in F, and F contains the closure of the set where $f<0$. Orbits of v which start in F must remain in F. If now $(y,\eta) \in N_e(F)$ (Definition 8.5.7) then y is in the boundary of F so $f(y)=0$. If $df(y) \neq 0$ then F is bounded by the surface $f=0$ in a neighborhood of y, so η must be a positive multiple of $df(y)$ and $\langle w(y), \eta \rangle \leqq 0$ by (26.4.8). If $df(y)=0$ we have $\langle w(y), \eta \rangle = \langle v(y), \eta \rangle$ by (26.4.9), and $\langle v(y), \eta \rangle \leqq 0$ by condition (ii) in Theorem 8.5.11. Hence w satisfies condition (ii) in Theorem 8.5.11 so condition (i) there is also fulfilled, which proves the lemma.

Before proceeding with the proof of Theorem 26.4.7 we digress to give two alternative forms of condition (Ψ).

Theorem 26.4.12. *Each of the following conditions is necessary and sufficient for the homogeneous C^∞ function p in $T^*(Y) \setminus 0$ to satisfy condition (Ψ):*

(Ψ_1) *There is no C^∞ complex valued function q in $T^*(Y) \setminus 0$ such that* $\operatorname{Im} qp$ *changes sign from $-$ to $+$ when one moves in the positive direction on a bicharacteristic of* $\operatorname{Re} qp$ *where $q \neq 0$.*

(Ψ_2) *If Γ is a characteristic point with $H_p \neq 0$ or a compact one dimensional bicharacteristic interval with injective regular projection in $S^*(Y)$ then there exists a C^∞ function q in a neighborhood Ω of Γ such that* $\operatorname{Re} H_{qp} \neq 0$ *in Ω and* $\operatorname{Im} qp$ *does not change sign from $-$ to $+$ when one moves in the positive direction on a bicharacteristic of* $\operatorname{Re} qp$ *in Ω.*

Proof. It is clear that $(\Psi_1) \Rightarrow (\Psi)$; the difference is just that q is not assumed homogeneous in (Ψ_1). To prove that $(\Psi) \Rightarrow (\Psi_2)$ we only have to show that $\operatorname{Re} H_{qp} \neq 0$ on Γ for some homogeneous q. This is clear if Γ is a point. Otherwise Γ has a parametrization $t \mapsto \Gamma(t)$ with $\Gamma'(t) = c(t) H_p(\Gamma(t))$. If the parameter is suitably normalized then $c(t)$ and $\Gamma(t)$ are C^∞ functions. If π: $T^*(Y) \setminus 0 \to S^*(Y)$ is the projection then $t \mapsto \pi \Gamma(t)$ is an embedding of an interval so we can find a C^∞ function q_s on $S^*(X)$ with $q_s(\pi\Gamma(t))=c(t)$. Thus $q=\pi^* q_s$ is homogeneous of degree 0 and $\operatorname{Re} H_{qp} \neq 0$ on Γ.

It remains to prove that $(\Psi_2) \Rightarrow (\Psi_1)$ or equivalently that (Ψ_2) is false if (Ψ_1) is. So let q be any function in $C^\infty(T^*(Y)\setminus 0)$ such that $\operatorname{Im} qp$ changes sign from $-$ to $+$ on a bicharacteristic γ of $\operatorname{Re} qp$ where $q \neq 0$. As above we can find a compact one dimensional bicharacteristic interval $\Gamma \subset \gamma$ or a point $\Gamma \in \gamma$ such that the sign change occurs on bicharacteristics of $\operatorname{Re} qp$ arbitrarily close to Γ. By Lemma 26.4.10 this remains true for any other choice of q with $H_{\operatorname{Re} qp} \neq 0$ on Γ, so (Ψ_2) will be proved false if we show that π is injective and has injective differential on Γ, when Γ is a one dimensional bicharacteristic interval. If H_p has the radial direction at some point on Γ then the whole orbit of $H_{\operatorname{Re} qp}$ starting at Γ, and in particular γ, would just be a ray where $p=0$ identically. This contradicts our assumptions so π restricted to Γ has injective differential. If $\pi\Gamma$ is a closed smooth curve then p would also vanish identically on γ which is again contradictory. Finally it cannot happen that $\pi \circ \Gamma$ returns to the same position with a change of orientation, for a one dimensional bicharacteristic is uniquely determined by its starting point and the choice of orientation there. If $\pi \circ \Gamma(t_1)=\pi \circ \Gamma(t_2)$, $t_1<t_2$, and the orientations are opposed, then we can for any $t_1'>t_1$ close to t_1 find t_2' with $t_1'<t_2'<t_2$ and $\pi\circ\Gamma(t_1')=\pi\circ\Gamma(t_2')$. The supremum t of such t_1' must be equal to the infimum of the corresponding t_2' which contradicts that $\pi\circ\Gamma$ has a nonzero tangent at $\pi\circ\Gamma(t)$. Thus (Ψ_2) is false and the theorem is proved.

The interest of condition (Ψ_2) is of course that it eliminates the need to consider arbitrary functions q. In case Γ is a point it suffices to check it for $q=1$ and for $q=i$.

To simplify the principal symbol near a one dimensional bicharacteristic we need a global version of Theorem 21.3.6.

Proposition 26.4.13. *Let p be a C^∞ homogeneous function on $T^*(X)\setminus 0$, let I be a compact interval on $\mathbb{R}$ not reduced to a point and $I \ni t \mapsto \gamma(t) \in T^*(X)\setminus 0$ a one dimensional bicharacteristic, $\gamma \in C^\infty$. Assume also that the composition of γ and the projection $T^*(X)\setminus 0 \to S^*(X)$ is injective, which means in particular that $H_p(\gamma(t))$ never has the radial direction. Then there is a homogeneous C^∞ canonical transformation χ from a conic neighborhood of $\{(x,\varepsilon_n), x_1\in I, x'=0\}$ in $T^*(\mathbb{R}^n)\setminus 0$ to a conic neighborhood of $\gamma(I)$ in $T^*(X)\setminus 0$ and a C^∞ homogeneous function a of degree $1-m$ with no zero on $\gamma(I)$ such that $\chi(x_1,0,\varepsilon_n)=\gamma(x_1)$, $x_1\in I$, and*

$$\chi^*(ap)=\xi_1+if(x,\xi') \tag{26.4.10}$$

where f is real valued, homogeneous of degree 1 and independent of ξ_1.

Proof. Essentially we just have to inspect the proof of Theorem 21.3.6 to see that it works globally. First choose as in the proof of Theorem 26.4.12 a C^∞

function q, homogeneous of degree $1-m$, such that $q(\gamma(t))=c(t)$ where c is the function in Definition 26.4.9. Then

$$\gamma'(t)=H_{\mathrm{Re}\,qp}(\gamma(t)), \quad d\,\mathrm{Im}\,qp=0 \quad \text{at } \gamma(t),\ t\in I.$$

From Proposition 26.1.6 it follows that we can find a canonical transformation χ satisfying the conditions in the theorem except that

$$\chi^*(qp)=\xi_1+ig(x,\xi)$$

where we only know that $dg=0$ on $I\times(0,\varepsilon_n)$. Using Malgrange's preparation theorem we can find h and r homogeneous of degree 0 and 1 respectively, and C^∞ in a neighborhood of $I\times(0,\varepsilon_n)$, so that

$$\xi_1=h(x,\xi)(\xi_1+ig(x,\xi))+r(x,\xi'). \tag{24.4.11}$$

In fact, it suffices to prove this when $\xi_n=1$ and then extend from there by homogeneity. As in the proof of Theorem 21.3.6 the preparation theorem gives a local solution at any point in $I\times(0,\varepsilon_n)$, and the local solutions can be pieced together by a partition of unity in x_1 to a solution in a neighborhood of $I\times(0,\varepsilon_n)$. Note that $h=1$ and $dr=0$ on $I\times(0,\varepsilon_n)$. Writing $r=r_1+ir_2$ we want to introduce

$$y_1=x_1, \quad \eta_1=\xi_1-r_1(x,\xi')$$

as new canonical variables. We choose

$$y_2=x_2,\ \eta_2=\xi_2,\ \dots,\ \eta_n=\xi_n \quad \text{when } x_1=0$$

and determine these canonical variables so that they are constant along the orbits of H_{η_1}. One of these contains $I\times(0,\varepsilon_n)$, so $y_2,\eta_2,\dots$ will be defined in a neighborhood. The commutation relations are fulfilled by the Jacobi identity since they hold when $x_1=0$. Hence we obtain a canonical transformation χ_1 keeping $I\times(0,\varepsilon_n)$ fixed, such that $h(x,\xi)(\xi_1+ig(x,\xi))$ composed with χ_1 is equal to $\eta_1+if(y,\eta)$ where $f(y,\eta)=-r_2(x,\xi')$. Now

$$\partial f/\partial\eta_1=\{f,y_1\}=-\{r_2,x_1\}=0$$

so $\chi\circ\chi_1$ and $q(\chi^{-1})^*h$ have the desired properties.

If we combine the discussion preceding Definition 26.4.9 with Proposition 24.4.13 or Theorem 21.3.6 we conclude in view of Lemma 26.4.10 that Theorem 26.4.7 follows if we prove

Theorem 26.4.7′. *Suppose that in a conic neighborhood of*

$$\Gamma=\{(x_1,0,0,\xi^0),\ a_0\leqq x_1\leqq b_0\}\subset T^*(\mathbb{R}^n)\setminus 0$$

the principal symbol of P has the form

$$p(x,\xi)=\xi_1+if(x,\xi')$$

where f is real valued and vanishes of infinite order on Γ if $b_0>a_0$. Assume also that in any neighborhood of Γ one can find an interval in the x_1

direction where f changes sign from $-$ to $+$ for increasing x_1. Then P is not solvable at Γ.

In the proof of Theorem 26.4.7′ we may also assume that the lower order terms $p_0, p_{-1}, \ldots$ in the symbol of P are independent of ξ_1 near Γ. In fact, Malgrange's preparation theorem implies that

$$p_0(x,\xi)=q(x,\xi)(\xi_1+if(x,\xi'))+r(x,\xi')$$

where q is homogeneous of degree -1 and r homogeneous of degree 0. (See the proof of Proposition 26.4.13.) The term of degree 0 in the symbol of $(I-q(x,D))P$ is equal to $r(x,\xi')$. Repetition of the argument allows us to make the lower order terms successively independent of ξ_1.

To prove Theorem 26.4.7′ we shall use Lemma 26.4.5 which shows that it suffices to construct approximate solutions of the equation $P^*v=0$ concentrated so near Γ that (26.4.4) cannot hold. Let us first show how this can be done in the simple case where $\Gamma=\{(0,\varepsilon_n)\}\in T^*(\mathbb{R}^n)$ and $P=D_1+ix_1D_n$. (In that case we know of course already from Proposition 26.3.8 that there is no solvability.) Set

$$v_\tau(x)=\phi(x)e^{i\tau w(x)} \tag{26.4.12}$$

where $\phi\in C_0^\infty(\mathbb{R}^n)$ is equal to 1 in a neighborhood of 0 and

$$w(x)=x_n+i(x_1^2+x_2^2+\ldots+x_{n-1}^2+(x_n+ix_1^2/2)^2)/2$$

satisfies the equation $P^*w=0$. If $\operatorname{supp}\phi$ is small enough then

$$\operatorname{Im} w(x)>|x|^2/4, \qquad x\in\operatorname{supp}\phi,$$

so $v_\tau\to 0$ in $C^\infty(\mathbb{R}^n\setminus 0)$ and $\tau^N P^*v_\tau=\tau^N(P^*\phi)e^{i\tau w}\to 0$ in $C_0^\infty(\mathbb{R}^n)$ for any N. We have $v_\tau(x)=e^{i\tau x_n}V_\tau(x\sqrt{\tau})$ where $V_\tau(x)\to V(x)=e^{-|x|^2/2}$ in $\mathscr{S}$ as $\tau\to+\infty$. Since $\hat v_\tau(\xi)=\tau^{-n/2}\hat V_\tau((\xi-\tau\varepsilon_n)/\sqrt{t})$ it is clear that $\hat v_\tau(\xi)(1+|\xi|)^N\to 0$ uniformly for any N outside any conic neighborhood of ε_n and on any compact set, so $Av_\tau\to 0$ uniformly for any properly supported pseudo-differential operator A such that $(0,\varepsilon_n)\notin WF(A)$. We also have

$$\|v_\tau\|_{(s)}^2\,\tau^{-2s+n/2}\to\|V\|_{L^2}^2 \qquad \text{as } \tau\to\infty,$$

and these statements together show that (26.4.4) cannot be valid.

Using Theorem 21.3.3 and Proposition 26.3.1 we can adapt the preceding construction to prove that (26.4.4) is not valid if there is a point $(x,\xi)\in T^*(Y)\setminus 0\setminus WF(A)$ where $p(x,\xi)=0$ and $\{\operatorname{Re}p,\operatorname{Im}p\}(x,\xi)>0$. When proving Theorem 26.4.7′ we may therefore assume that

$$f(x,\xi')=0\Rightarrow\partial f(x,\xi')/\partial x_1\leq 0 \tag{26.4.13}$$

in a neighborhood of Γ. This will be important for an application of Lemma 26.4.11 later on.

In the general proof of Theorem 26.4.7′ we shall take v_τ of the form

$$v_\tau(x) = e^{i\tau w(x)} \sum_0^M \phi_j(x)\tau^{-j} \tag{26.4.14}$$

where $\operatorname{Im} w \geqq 0$ with equality at some point and strict inequality outside a compact set, which makes v_τ very small and ϕ_j irrelevant there as $\tau \to \infty$. The principal symbol of P^* is $\xi_1 - if(x, \xi')$ near Γ. To make $P^* v_\tau$ small the first step is therefore to construct a phase function w satisfying the *eiconal equation*

$$\partial w/\partial x_1 - if(x, \partial w/\partial x') = 0 \tag{26.4.15}$$

approximately. When that has been done, which is the main problem, we shall choose appropriate *amplitude functions* $\phi_0, \phi_1, \dots$ successively. (Roughly speaking these steps correspond in the preceding proof of (26.4.13) to the application of Theorem 21.3.3 and of Proposition 26.3.1 respectively.)

To simplify notation we shall in what follows write t instead of x_1 and x instead of x', so (26.4.15) takes the form

$$\partial w/\partial t - if(t, x, \partial w/\partial x) = 0. \tag{26.4.15$'$}$$

To keep as much as possible of the qualitative properties of (26.4.12) we shall choose w so that $\operatorname{Im} w$ is strictly convex in x for fixed t and has its minimum on a smooth real curve $x = y(t)$. Thus we shall have

$$\operatorname{Im} \partial w(t, x)/\partial x = 0 \quad \text{when } x = y(t),$$

so we are led to looking for a solution of (26.4.15)′ which has the form

$$w(t, x) = w_0(t) + \langle x - y(t), \eta(t)\rangle + \sum_{2 \leqq |\alpha| \leqq M} w_\alpha(t)(x - y(t))^\alpha/|\alpha|!. \tag{26.4.16}$$

Here M is a large integer and it is convenient to use, during the present discussion only, the notation $\alpha = (\alpha_1, \dots, \alpha_s)$ for a sequence of $s = |\alpha|$ indices between 1 and the dimension $n-1$ of the x variable. w_α will be symmetric in these indices. If we make sure that the matrix $(\operatorname{Im} w_{jk})$ is positive definite then $\operatorname{Im} w$ will have a strict minimum when $x = y(t)$ as a function of the x variables, for $\eta(t)$ will be *real valued.*

On the curve $x = y(t)$ the equation (26.4.15)′ reduces to

$$w_0'(t) = \langle y'(t), \eta(t)\rangle + if(t, y(t), \eta(t)). \tag{0}$$

This is the only equation where w_0 occurs so it can be used to determine w_0 after y and η have been chosen. In particular

$$d \operatorname{Im} w_0(t)/dt = f(t, y(t), \eta(t)). \tag{0$'$}$$

If $f(t, y(t), \eta(t))$ has a sign change from $-$ to $+$ then $\operatorname{Im} w_0(t)$ will start decreasing and end increasing, so the minimum is attained at an interior point. We can normalize the minimum value to zero and have then for a suitable interval of t that $\operatorname{Im} w_0 > 0$ at the end points and $\operatorname{Im} w_0 = 0$ at some interior point. Thus $\operatorname{Im} w \geqq 0$ with equality attained but strict inequality valid outside a compact subinterval of the curve.

Our purpose is to make (26.4.15)′ valid apart from an error of order $M+1$ in $x-y(t)$. Actually $f(t, x, \xi)$ is not defined for complex ξ, but since

$$\partial w(t, x)/\partial x_j - \eta_j(t) = \sum w_{\alpha, j}(t)(x-y(t))^\alpha/|\alpha|!$$

this is given a meaning if $f(t, x, \partial w/\partial x)$ is replaced by the finite Taylor expansion

$$\sum_{|\beta| \leqq M} f^{(\beta)}(t, x, \eta(t))(\partial w(t, x)/\partial x - \eta(t))^\beta/|\beta|!.$$

Note that to compute the coefficient of $(x-y(t))^\alpha$ we just have to consider the terms with $|\beta| \leqq |\alpha|$. We have

$$\begin{aligned} \partial w/\partial t = w_0' - \langle y', \eta\rangle + \langle x-y, \eta'\rangle + \sum w_\alpha'(t)(x-y)^\alpha/|\alpha|! \\ -\sum_k \sum_{1 \leqq |\alpha| \leqq M-1} w_{\alpha, k}(t)(x-y)^\alpha \, dy_k/dt/|\alpha|!, \end{aligned}$$

so the first order terms in the equation (26.4.15)′ give

$$d\eta_j/dt - \sum_k w_{jk}(t)\, dy_k/dt = i(f_{(j)}(t, y, \eta) + \sum_k f^{(k)}(t, y, \eta)\, w_{jk}(t)). \tag{1}$$

Since y and η are real, this is a system of $2n$ equations

$$d\eta_j/dt - \sum_k \operatorname{Re} w_{jk}(t)\, dy_k/dt = -\sum_k \operatorname{Im} w_{jk}(t) f^{(k)}(t, y, \eta), \tag{1)$'$}$$

$$\sum_k \operatorname{Im} w_{jk}(t)\, dy_k/dt = -f_{(j)}(t, y, \eta) - \sum_k \operatorname{Re} w_{jk}(t) f^{(k)}(t, y, \eta). \tag{1)$''$}$$

When $\operatorname{Im} w_{jk}$ is positive definite we can solve these equations for dy/dt and $d\eta/dt$. At a point where $f = df = 0$ they just mean that $dy/dt = d\eta/dt = 0$.

When $2 \leqq |\alpha| \leqq M$ we obtain from (26.4.15)′ a differential equation

$$dw_\alpha/dt - \sum_k w_{\alpha, k}\, dy_k/dt = F_\alpha(t, y, \eta, \{w_\beta\}) \tag{α}$$

where F_α is a linear combination of the derivatives of f of order $\leqq |\alpha|$ multiplied by polynomials in w_β with $2 \leqq |\beta| \leqq |\alpha|+1$. (When $|\alpha| = M$ the sum on the left-hand side of (α) should of course be dropped and $|\beta| \leqq |\alpha|$.) Altogether (1)′, (1)″ and (α) form a quasilinear system of differential equations with as many equations as unknowns, so it is clear that we have local solutions with prescribed initial data. As seen above $F_\alpha(t, 0, \xi^0, .) = 0$ if $a_0 < t < b_0$, so when $a_0 < t < b_0$ we have the solution $y = 0$, $\eta = \xi^0 = (0, \dots, 0, 1)$, $w_\alpha = \text{constant}$. Hence we can find $c > 0$ such that the equations (1) and (α) with initial data

$$w_{jk} = i\delta_{jk}, \quad w_\alpha = 0 \quad \text{when } 2 < |\alpha| \leqq M, \ t = (a_0 + b_0)/2 \tag{26.4.17}$$

$$y = x, \quad \eta = \xi \quad \text{when } t = (a_0 + b_0)/2 \tag{26.4.18}$$

have a unique solution in $(a_0 - c, b_0 + c)$ for all x, ξ with $|x| + |\xi - \xi^0| < c$. Moreover,

(i) $(\operatorname{Im} w_{jk} - \delta_{jk}/2)$ is positive definite,

(ii) the map

$$(x, \xi, t) \mapsto (y, \eta, t); \quad |x| + |\xi - \xi^0| < c, \ a_0 - c < t < b_0 + c$$

is a diffeomorphism.

In the range X_c of the map (ii) we denote by v the image of the vector field $\partial/\partial t$ under the map. Thus v is the tangent vector field of the integral curves. Note that $v = \partial/\partial t$ when $f = df = 0$. Since we have assumed above that $f = 0$ implies $\partial f/\partial t \leqq 0$ in X_c (cf. (26.4.13)), if c is small enough, we can now apply Lemma 26.4.11 with the vector field v just defined and $w = \partial/\partial t$. The conclusion is that f must have a change of sign from $-$ to $+$ along an integral curve of v in X_c, for otherwise there would be no such sign change for increasing t and fixed (x, ξ), and that contradicts the hypothesis in Theorem 26.4.7′. Recalling the discussion of the equation (0) above we have therefore proved

Lemma 26.4.14. *Assume that the hypotheses of Theorem* 26.4.7′ *are fulfilled and that in a neighborhood of Γ we have $\partial f/\partial t \leqq 0$ when $f = 0$, the variables being denoted by (t, x) now. Given M one can then find*

(i) *a curve $t \mapsto (t, y(t), 0, \eta(t)) \in \mathbb{R}^{2n}$, $a' \leqq t \leqq b'$, as close to Γ as desired,*

(ii) *C^∞ functions $w_\alpha(t)$, $2 \leqq |\alpha| \leqq M$, with $(\operatorname{Im} w_{jk} - \delta_{jk}/2)$ positive definite when $a' \leqq t \leqq b'$*

(iii) *a function w_0 with $\operatorname{Im} w_0(t) \geqq 0$, $a' \leqq t \leqq b'$, $\operatorname{Im} w_0(a') > 0$,*

$$\operatorname{Im} w_0(b') > 0 \quad \textit{and} \quad \operatorname{Im} w_0(c') = 0 \quad \textit{for some } c' \in (a', b')$$

such that (26.4.16) *is a formal solution of* (26.4.15)′ *with an error which is $O(|x - y(t)|^{M+1})$.*

Before passing to the choice of the functions ϕ_j in (26.4.14) we shall make some general remarks which show what is required to disprove (26.4.4). In doing so we revert to the symmetric notation in (26.4.14) where x denotes all the variables in $\mathbb{R}^n$.

Lemma 26.4.15. *Let v_τ be defined by* (26.4.14) *where $w \in C^\infty(X)$, $\phi_j \in C_0^\infty(X)$, $\operatorname{Im} w \geqq 0$ in X and $d \operatorname{Re} w \neq 0$. Here X is an open set in $\mathbb{R}^n$. For any positive integer N we have then*

$$\|v_\tau\|_{(-N)} \leqq C\tau^{-N}, \quad \tau > 1. \tag{26.4.19}$$

If $\operatorname{Im} w(x_0) = 0$ *and* $\phi_0(x_0) \neq 0$ *for some* $x_0 \in X$ *then*

$$\|v_\tau\|_{(-N)} \geqq c\tau^{-n/2-N}, \quad \tau > 1, \tag{26.4.20}$$

for some $c > 0$. If $\tilde{\Gamma}$ is the cone generated by

$$\{(x, w'(x)),\ x \in \bigcup \operatorname{supp} \phi_j,\ \operatorname{Im} w(x) = 0\} \tag{26.4.21}$$

then $\tau^k v_\tau \to 0$ *in* $\mathscr{D}'_{\tilde{\Gamma}}$ *as* $\tau \to \infty$, *hence* $\tau^k A v_\tau \to 0$ *in* $C^\infty(\mathbb{R}^n)$, *if* A *is a pseudo-differential operator with* $WF(A) \cap \tilde{\Gamma} = \emptyset$, *and* k *is any real number.*

Proof. For every neighborhood U of the projection of (26.4.21) on the second component in $\mathbb{R}^n$ and every positive integer ν we have

$$|\hat{v}_\tau(\xi)| \leqq C_\nu(1+|\xi|+\tau)^{-\nu} \quad \text{if } \tau>1,\ \xi/\tau \notin U. \tag{26.4.22}$$

This follows from Theorem 7.7.1 since $x \mapsto (\tau w - \langle x, \xi \rangle)/(\tau+|\xi|)$ is in a compact set of functions with non-negative imaginary part and differential $\neq 0$ at the real points. If we choose U bounded with $0 \notin \bar{U}$ then

$$\int_{\tau U} |\hat{v}_\tau(\xi)|^2 (1+|\xi|^2)^{-N} d\xi = O(\tau^{-2N})$$

since v_τ is bounded in L^2. Together with (26.4.22) this gives (26.4.19). If $\chi \in C_0^\infty$ the estimate (26.4.22) is applicable to χv_τ as well. Hence

$$|\widehat{\chi v_\tau}(\xi)| \leqq C_\nu(1+|\xi|+|\tau|)^{-\nu}, \qquad \tau>1,\ \xi \in V,$$

if V is any closed cone with $\tilde{\Gamma} \cap (\operatorname{supp}\chi \times V) = \emptyset$; hence $\tau^k v_\tau \to 0$ in $\mathscr{D}'_{\tilde{\Gamma}}$ for every k. To prove (26.4.20) finally we assume that $x_0=0$ and observe that when $\psi \in C_0^\infty$ we have if $w(0)=0$

$$\begin{aligned} \tau^n \langle v_\tau, \psi(\tau\,.)\rangle &= \int e^{i\tau w(x/\tau)} \psi(x) \sum \phi_j(x/\tau)\tau^{-j} dx \\ &\to \int e^{i\langle x, w'(0)\rangle} \psi(x)\phi_0(0)\,dx, \end{aligned}$$

which is not equal to 0 for a suitable choice of ψ. Since

$$\|\psi(\tau\,.)\|_{(N)} = O(\tau^{N-n/2})$$

it follows that $c \leqq \tau^{N+n/2}\|v_\tau\|_{(-N)}$, which proves (26.4.20).

As already pointed out Theorem 26.4.7′ will be proved by showing that (26.4.4) cannot be valid. To do so we first use Lemma 26.4.14 to choose a function w in a neighborhood Y of $\{(x_1, 0);\ a_0 \leqq x_1 \leqq b_0\} \subset \mathbb{R}^n$ such that $\operatorname{Im} w > 0$ in Y except on a compact non-empty subset K of a curve $x' = y(x_1)$. In addition

$$\Gamma_0 = \{(x, w'(x)),\ x \in K\}$$

is in a small conic neighborhood of Γ which does not meet $WF(A)$ and where the symbol of P^* is of the form $\xi_1 + iF(x, \xi')$, $-f$ being the principal symbol of F. If we apply (26.4.4) to a function of the form (26.4.14) where $\phi_0 \neq 0$ at some point in K, it follows from Lemma 26.4.15 that the left-hand side has a lower bound $c\tau^{-n/2-N}$ and that there is a smaller bound for the last two terms in the right-hand side. If we can prove that

$$\|P^* v_\tau\|_{(\nu)} = O(\tau^{-N-(n+1)/2}) \tag{26.4.23}$$

it will follow that (26.4.4) is not valid.

If B is a pseudo-differential operator of order 0 with symbol 1 in a conic neighborhood of Γ_0 then $(I-B)P^*\tau^k v_\tau \to 0$ in C^∞ for any k. We can choose

B with $WF(B)$ so small that $\xi' \neq 0$ and the symbol of P^* is $\xi_1 + iF(x, \xi')$ in a conic neighborhood. Then it follows from Theorem 18.1.35 that the product $B(P^* - D_{x_1} - iF(x, D'))$ is a pseudo-differential operator with wave front set disjoint with Γ_0. Here $F(x, D')$ is a pseudo-differential operator in $n-1$ variables depending on x_1 as parameter. Hence (26.4.23) will follow if

$$\|(D_{x_1} + iF(x, D'))v_\tau\|_{(v)} = O(\tau^{-N-(n+1)/2}). \tag{26.4.24}$$

Since $F(x, D')$ is a pseudo-differential operator in the variables $(x_2, \dots, x_n)$ it is convenient to change notation again so that x_1 is denoted by t and the other variables are denoted by x. Thus (26.4.24) is written now

$$\|(D_t + iF(t, x, D))v_\tau\|_{(v)} = O(\tau^{-N-(n+1)/2}). \tag{26.4.24$'$}$$

In our construction we shall actually aim for the estimate

$$|(D_t + iF(t, x, D))v_\tau| \leq C\tau^{-N-(n+2)/2-v}, \tag{26.4.24$''$}$$

and we shall see afterwards that (26.4.24)$'$ is obtained at the same time.

To make the left-hand side of (26.4.24)$'$ small we need a formula for how $F(t, x, D)$ acts on functions of the form (26.4.14). This could be obtained from the work in Section 25.3, but we prefer a direct elementary proof. To simplify notation we suppress the parameter t in the proof.

Lemma 26.4.16. *Let $q(x, \xi) \in S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, let $\phi \in C_0^\infty(\mathbb{R}^{n-1})$, $w \in C^\infty(\mathbb{R}^{n-1})$, and assume that* $\operatorname{Im} w > 0$ *except at a point y where $w'(y) = \eta \in \mathbb{R}^{n-1} \setminus 0$ and* $\operatorname{Im} w''$ *is positive definite. Then*

$$|q(x, D)(\phi e^{i\tau w}) - \sum_{|\alpha| < k} q^{(\alpha)}(x, \tau\eta)(D - \tau\eta)^\alpha (\phi e^{i\tau w})/\alpha!| \leq C_k \tau^{\mu - k/2}; \tag{26.4.25}$$
$$\tau > 1, \; k = 1, 2, \dots.$$

Proof. Let us first observe that

$$|(D - \tau\eta)^\alpha \phi e^{i\tau w}| = |D^\alpha \phi e^{i\tau(w - \langle \cdot, \eta \rangle)}| \leq C\tau^{|\alpha|/2}. \tag{26.4.26}$$

In fact, if j of the $|\alpha|$ derivatives fall on the exponential they bring out a factor τ^j but also j factors $\partial w/\partial x_i - \eta_i$ vanishing at y. If $j > |\alpha|/2$ the remaining $|\alpha| - j$ derivatives can only reduce the order of the zero to $j - (|\alpha| - j) = 2j - |\alpha|$, so the term is bounded by a constant times

$$\tau^j |x - y|^{2j - |\alpha|} e^{-c\tau |x-y|^2} \leq C\tau^{|\alpha|/2}.$$

Note that (26.4.26) explains the power of τ in the right-hand side of (26.4.25).

To prove (26.4.25) we set $u_\tau(x) = \phi(x) e^{i\tau w(x)}$ and study

$$\hat{u}_\tau(\xi) = \int \phi(x) e^{i(\tau w(x) - \langle x, \xi \rangle)} dx.$$

As in the proof of (26.4.22) it is clear that for every v

$$|\hat{u}_\tau(\xi)| \leq C(|\xi| + \tau)^{-v}, \quad \tau > 1, \tag{26.4.27}$$

if $|\xi/\tau-\eta|\geqq|\eta|/2$, say. On the other hand, if $|\xi/\tau-\eta|<|\eta|/2$ we can also apply Theorem 7.7.1 with $f(x)=w(x)-\langle x, \xi/\tau\rangle$ noting that

$$|\eta-\xi/\tau|^2+|x-y|^2\leqq C(|\operatorname{grad} f|^2+\operatorname{Im} f)$$

since $f'(x)=w'(x)-\xi/\tau=\eta-\xi/\tau+O(|x-y|)$. Hence

$$|\hat{u}_\tau(\xi)|\leqq C_k\tau^{-k}|\eta-\xi/\tau|^{-2k}=C_k\tau^k|\tau\eta-\xi|^{-2k}. \tag{26.4.28}$$

In the integral

$$q(x, D)(\phi e^{i\tau w})=(2\pi)^{-n}\int e^{i\langle x,\xi\rangle} q(x,\xi)\,\hat{u}_\tau(\xi)\,d\xi$$

the contributions when $|\xi/\tau-\eta|>|\eta|/2$ are rapidly decreasing by (26.4.27). When $|\xi/\tau-\eta|<|\eta|/2$ we replace q by the Taylor expansion at $\tau\eta$ of order k. This gives the sum

$$\sum_{|\alpha|<k}(2\pi)^{-n}\int e^{i\langle x,\xi\rangle} q^{(\alpha)}(x,\tau\eta)(\xi-\tau\eta)^\alpha\,\hat{u}_\tau(\xi)\,d\xi/\alpha!.$$

Extending the integration to the whole space will only mean a change by a rapidly decreasing function, again by (26.4.27), and the sum is then equal to that in (26.4.25). The error term in Taylor's formula can be estimated by $C\tau^{\mu-k}|\tau\eta-\xi|^k$. If we use (26.4.28) with k replaced by $k/2$ it follows that the contribution from the error term to the integral when $|\xi/\tau-\eta|<|\eta|/2$ is $O(\tau^{\mu+n-k/2})$. This proves (26.4.25) apart from an extra factor τ^n in the right-hand side. If we apply this weaker result with k replaced by $k+2n$ and recall (26.4.26), we obtain (26.4.25).

If q is homogeneous of degree μ, then the sum in (26.4.25) consists apart from the factor $e^{i\tau w}$ of terms which are homogeneous in τ of degree μ, $\mu-1,\dots$. The terms of degree μ are those in

$$\phi\sum q^{(\alpha)}(x,\tau\eta)(\tau w'(x)-\tau\eta)^\alpha/\alpha!$$

which is the Taylor expansion at $\tau\eta$ of the possibly undefined quantity $q(x,\tau w')$, just as in the discussion of (26.4.15)$'$ above. The terms of degree $\mu-1$ where ϕ is differentiated are similarly

$$\sum_1^{n-1} q^{(k)}(x,\tau w'(x))\,D_k\phi$$

where $q^{(k)}$ should be replaced by the Taylor expansion at $\tau\eta$ representing the value at $\tau w'(x)$.

End of Proof of Theorem 26.4.7$'$. With v_τ defined by (26.4.14) we have now proved that

$$(D_t+iF(t,x,D))\,v_\tau=e^{i\tau w}\left(\sum_0^M\psi_j\tau^{-j}\right)+O(\tau^{(1-M)/2}),$$

where

$$\psi_j=D_t\phi_j+\sum_2^n c_k(t,x)\,D_k\phi_j+c_0\phi_j+R_j \tag{26.4.29}$$

with $R_0=0$ and R_j otherwise determined by $\phi_0, \dots, \phi_{j-1}$. Here c_k is a partial sum of the Taylor series at $\eta(t)$ for $-if^{(k)}(t,x,w'_x(t,x))$ but this is of no importance. Set

$$\phi_0(t,x)=\sum_{|\alpha|<M}\phi_{0\alpha}(t)(x-y(t))^\alpha$$

with $y(t)$ as in Lemma 26.4.14. Then

$$\psi_0=\left(D_t+\sum_2^n c_k(t,x)D_k+c_0\right)\phi_0=O((x-y(t))^M)$$

if $\phi_{0\alpha}$ satisfy a certain linear system of ordinary differential equations

$$D_t\phi_{0\alpha}+\sum_{|\beta|<M}a_{\alpha\beta}\phi_{0\beta}=0.$$

We can solve these equations so that $\phi_0(0)=1$ at a chosen point in K. In the same way we can successively choose ϕ_j so that

$$\psi_j(t,x)=O((x-y(t))^{M-2j}), \quad \text{when } j<M/2.$$

If M is chosen so that $(1-M)/2\leqq -N-(n+1)/2-\nu$, we obtain (26.4.24)″.

The asymptotic series in (26.4.25) remains valid if we differentiate with respect to x or a parameter t, though a factor τ may be lost in the estimate. In proving (26.4.24)″ we have also used that a function of the form $\chi(t,x)e^{i\tau w}$ can be estimated by $\tau^{-k/2}$ if χ vanishes of order k when $x=y(t)$. A differentiation can lead to a decrease of the order of the zero by one unit or to a factor τ when the exponential is differentiated, so the estimate may deteriorate by a factor τ. In any case it is clear that we obtain (26.4.24)′ when we compute the derivatives of order $\leqq\nu$, so (26.4.4) is not valid. This completes the proof of Theorem 26.4.7′.

26.5. Geometrical Aspects of Condition (P)

Unfortunately it is not yet known if the converse of Corollary 26.4.8 is valid. However, if P is a differential operator one can strengthen condition (Ψ) since the principal symbol $p(x,\xi)$ then has the symmetry property

$$p(x,-\xi)=(-1)^m p(x,\xi).$$

If $t\mapsto(x(t),\xi(t))$ is a bicharacteristic of $\operatorname{Re}p(x,\xi)$ it follows that $t\mapsto(x(t), -\xi(t))$ is also a bicharacteristic curve with the correct (reversed) orientation if m is odd (even). If the condition (Ψ) is fulfilled in X by p it follows that $\operatorname{Im}p(x,\xi)$ cannot take both positive and negative values on the bicharacteristic, that is, $\bar p$ also satisfies condition (Ψ).

Definition 26.5.1. A C^∞ homogeneous function p in $T^*(X)\setminus 0$ is said to satisfy condition (P) if p and $\bar p$ both satisfy condition (Ψ), that is, there is no

C^∞ complex valued function q in $T^*(X)\setminus 0$ such that $\operatorname{Im} qp$ takes both positive and negative values on a bicharacteristic of $\operatorname{Re} qp$ where $q \neq 0$.

Here we have chosen to use the equivalent form (Ψ_1) of (Ψ) in Theorem 26.4.12, which is applicable in any symplectic manifold. Of course we could have used (Ψ_2) or (Ψ) as well.

Thus condition (P) is necessary for solvability in the case of differential operators although not for general pseudo-differential operators. At the end of this chapter we shall prove that, conversely, an analogue of Theorem 26.1.7 is valid for every pseudo-differential operator P satisfying condition (P). The proof will be based on theorems concerning propagation of singularities which extend Theorems 26.1.4 and 26.2.1. These will be the main topic of the following sections. As a preparation we shall discuss in this section some geometrical properties of the characteristic set

$$N=\{(x,\xi)\in T^*(X)\setminus 0,\ p(x,\xi)=0\}$$

which follow from condition (P), *which we assume fulfilled throughout.*

As in Section 26.2 we set

$$N_2=\{(x,\xi)\in N;\ H_{\operatorname{Re} p} \text{ and } H_{\operatorname{Im} p} \text{ are linearly independent at } (x,\xi)\}.$$

This is a conic manifold of codimension 2. By condition (P) it is involutive, that is,

$$\{\operatorname{Re} p, \operatorname{Im} p\}(x,\xi)=0, \qquad (x,\xi)\in N_2,$$

for $\{\operatorname{Re} p, \operatorname{Im} p\}=H_{\operatorname{Re} p}\operatorname{Im} p$ must vanish at the zeros of $\operatorname{Im} p$ on a bicharacteristic of $\operatorname{Re} p$ since the sign would otherwise change. Thus $H_{\operatorname{Re} p}$ and $H_{\operatorname{Im} p}$ are tangents to N_2.

In Section 26.4 we introduced the term semi-bicharacteristic of p for a bicharacteristic of $\operatorname{Re} qp$ where $q \neq 0$. The advantage of this notion is that through every point in N with $dp \neq 0$ there is at least one semi-bicharacteristic curve. We shall now examine to what extent semi-bicharacteristics are one dimensional bicharacteristics in the sense of Definition 26.4.9. In doing so we shall use the following lemma, which is obtained by applying to $\operatorname{Re} qp$ the non-homogeneous version of Proposition 26.1.6. (This was also used in the proof of Lemma 26.4.10′.)

Lemma 26.5.2. *Let $I=[a,b]$ be a compact interval on $\mathbb{R}$ not reduced to a point, and let $I\ni t\mapsto\gamma(t)$ be a bicharacteristic arc for $\operatorname{Re} qp$ where $0\neq q\in C^\infty$. Then there is a symplectomorphism χ from a neighborhood V of $J=\{(x_1,0,\dots,0),\ x_1\in I\}\subset T^*(\mathbb{R}^n)$ to a neighborhood of $\gamma(I)\subset T^*(X)\setminus 0$ such that $\chi(x_1,0,\dots,0)=\gamma(x_1)$ and*

$$\chi^*(qp)(x,\xi)=\xi_1+if(x,\xi)$$

where f is real valued.

Assume now that $\gamma(a)\in N_2$. In a neighborhood of $a_0=(a,0,\dots,0)\in T^*(\mathbb{R}^n)$ the manifold $\chi^{-1}N_2$ is invariant under the vector field $\partial/\partial x_1$ so it is defined by $\xi_1=0$, $g(x',\xi')=f(a,x',0,\xi')=0$ where $x'=(x_2,\dots,x_n)$, $\xi'=(\xi_2,\dots,\xi_n)$. Since $\partial f/\partial x_1=\{\xi_1,f\}=0$ at a_0 we know that $dg\neq 0$ at 0. The parallels of the x_1 axis in the plane $\xi_1=0$ are semi-bicharacteristics so (P) gives in a neighborhood of J that $f(x,\xi)\geqq 0$ (resp. $f(x,\xi)\leqq 0$) when $\xi_1=0$ and $g(x',\xi')>0$ (resp. $g(x',\xi')<0$). Hence $f(x,\xi)=0$ when $\xi_1=0$ and $g(x',\xi')=0$, and for some $\varepsilon>0$

$$f(x,0,\xi')=g(x',\xi')h(x,\xi'), \quad \text{if } a-\varepsilon<x_1<b+\varepsilon,\ |x'|+|\xi'|<\varepsilon.$$

Here $h\geqq 0$, $h(a,x',\xi')>0$ if $|x'|+|\xi'|<\varepsilon$, $g(0)=0$ and $dg(x',\xi')\neq 0$ when $|x'|+|\xi'|<\varepsilon$. Thus $f=0$ on J and

$$df=\partial f/\partial\xi_1\, d\xi_1+h\,dg \quad \text{on } J,$$

so df is proportional to $d\xi_1$ except at points where $h\neq 0$, and they are in N_2. A semi-bicharacteristic starting in N_2 is therefore a one-dimensional bicharacteristic when it is not in N_2. (If an isolated point is not in N_2 the tangent is still proportional to H_p there.) Also note that if $f(x,0,\xi')=0$ and $g(x',\xi')\neq 0$ then $h(x,\xi')=0$ which implies $dh(x,\xi')=0$ since $h\geqq 0$, so $df=\partial f/\partial\xi_1\, d\xi_1$. If a semi-bicharacteristic starting in N_2 contains some point $\chi(x,\xi)$ with (x,ξ) in

$$V=\{(x,\xi);\ \xi_1=0,\ a-\varepsilon<x_1<b+\varepsilon,\ |x'|+|\xi'|<\varepsilon\}$$

and $g(x',\xi')\neq 0$ it must therefore run in the x_1 direction as a one dimensional bicharacteristic until it leaves V. Extending it if necessary so that it contains a point with $x_1=a$ we obtain a contradiction since $f(a,x',\xi')\neq 0$ when $g(x',\xi')\neq 0$. It follows that in $\chi(V)$ the union of all semi-bicharacteristics emanating from a point anywhere in N_2 is defined by $\xi_1=g(x',\xi')=0$, so it is an involutive manifold of codimension 2. Hence we have proved

Proposition 26.5.3. *The union N_2^e of all semi-bicharacteristics of p which meet N_2 is a locally closed conic involutive submanifold of $T^*(X)\setminus 0$ of codimension 2 on which $p=0$. Thus $H_{\operatorname{Re} p}$ and $H_{\operatorname{Im} p}$ are tangents to N_2^e, and N_2 is an open subset of N_2^e.*

From Theorem 21.2.7 and the remark following it we know that as involutive manifold N_2^e has a natural foliation with 2 dimensional leaves, having the symplectically orthogonal plane of the tangent plane of N_2^e as tangent planes. It is natural to extend the terminology used in Section 26.2 as follows:

Definition 26.5.4. The leaves of the natural foliation of N_2^e are called two dimensional bicharacteristics.

We recall from Theorem 21.2.7 that a leaf B is either conic or else the radial vector field is never tangent to it. In any case H_p is a complex tangent vector field of B. In Section 26.2 we used H_p to define a complex structure in the leaves of the foliation of N_2 such that the analytic functions are the solutions of the equation $H_p w = 0$. In a leaf B of the foliation of N_2^e this equation implies that w is constant on the one dimensional bicharacteristics which may be embedded in B. Let B_0 be the subset of B consisting of semi-bicharacteristics with both end points in N_2, and let $\tilde{B}_0$ be the reduced two dimensional bicharacteristic obtained by identifying points in B_0 which are connected by a one dimensional bicharacteristic. We shall prove in Section 26.7 that $\tilde{B}_0$ has a natural structure as Riemann surface such that the analytic functions lifted to B_0 are precisely the solutions of the equation $H_p w = 0$. In Section 26.9 we shall show that Theorem 26.2.1 can be extended to N_2^e with superharmonicity defined in terms of this analytic structure. In the proofs we shall use the following supplement to Proposition 26.4.13 to simplify the principal symbol.

Proposition 26.5.5. *Suppose with the notation in Proposition* 26.4.13 *that* $\gamma(I)$ *is a one dimensional bicharacteristic contained in* N_2^e *which cannot be extended at both end points as a one dimensional bicharacteristic. Then Proposition* 26.4.13 *is applicable, and in a neighborhood of* $I \times \{0\} \times \{\varepsilon_n\}$ *we have*

$$f(x, \xi') = g(x', \xi')\, h(x, \xi')$$

where $h \geqq 0$, $dg \neq 0$, *both factors are in* C^∞ *and* h *is homogeneous of degree* 0, g *homogeneous of degree* 1.

Proof. The tangent of $\gamma(I)$ cannot be radial and the projection of $\gamma(I)$ on $S^*(X)$ cannot be a closed curve since $\gamma(I)$ is maximal at one end. (In the proof that $(\Psi_2) \Rightarrow (\Psi_1)$ in Theorem 26.4.12 we saw that the projection cannot return to the same point with reversed orientation.) Hence Proposition 26.4.13 is applicable, and since $f(x_1, 0, \varepsilon_n) = 0$ for x_1 in a neighborhood of I and $df \neq 0$ at some point there, the factorization follows from the discussion of N_2^e following Lemma 26.5.2.

Remark. Proposition 26.5.5 and its proof remain valid if I is a point and $\gamma(I) \in N_2^e$ but does not belong to any one dimensional bicharacteristic. Proposition 26.4.13 is just replaced by Theorem 21.3.6.

Set $N_1 = N \setminus N_2^e$. We shall now discuss semi-bicharacteristics $I_0 \ni t \mapsto \gamma(t)$ such that $\gamma(t_0) \in N_1$ for some $t_0 \in I_0$. By the definition of N_2^e we know that $\gamma(I_0)$ cannot intersect N_2, so $H_{\mathrm{Re}\, p}$ and $H_{\mathrm{Im}\, p}$ are linearly dependent at every point in $\gamma(I_0) \cap N$. If $\gamma(I_0) \subset N$ it follows that γ is a one dimensional bicharacteristic. Let I be the largest subinterval of I_0 containing t_0 such that $\gamma(I) \subset N$. If I has an end point contained in the interior of I_0 it is clear that $\gamma(I)$ cannot be continued there as a one dimensional bicharacteristic, for it would have to coincide with $\gamma(I_0)$ then.

Proposition 26.5.6. *With the notation in Proposition* 26.4.13 *assume that* $\gamma(I) \cap N_2^e = \emptyset$ *and that* $\gamma(I)$ *cannot be extended in both directions as a one* *dimensional bicharacteristic. Then Proposition* 26.4.13 *is applicable and* $f \geqq 0$, *or* $f \leqq 0$, *in a neighborhood of* $I \times \{0\} \times \{\varepsilon_n\}$.

Proof. The maximality of $\gamma(I)$ shows as in the proof of Proposition 26.5.5 that Proposition 26.4.13 can be applied. If $f(x_1, 0, \varepsilon_n)$ vanishes for all x_1 in a neighborhood of I then $df = 0$ since $\gamma(I) \cap N_2^e = \emptyset$. Hence we would have a one dimensional bicharacteristic with $\gamma(I)$ in its interior. This is a contradiction proving that $f(x_1, 0, \varepsilon_n) \neq 0$ somewhere. By the continuity of f and condition (P) this sign is kept in the wide sense in a neighborhood of $I \times \{0\} \times \{\varepsilon_n\}$, since f does not depend on ξ_1.

We shall denote by N_{11} the set of all points in N_1 which lie on a semi-bicharacteristic with one non-characteristic end point. In Section 26.6 we shall show that Theorem 26.1.4 can be extended to control the singularities in N_{11} with the slight modification that regularity only propagates in one direction, determined by the sign of f in Proposition 26.5.6.

A semi-bicharacteristic starting at a point in $N_1 \setminus N_{11}$ is always a one dimensional bicharacteristic, for it never leaves the characteristic set by the definition of N_{11} and it cannot enter N_2 by the definition of N_2^e. Thus $N_1 \setminus N_{11}$ is the set through which one dimensional bicharacteristics can be prolonged indefinitely.

Let us now consider Proposition 26.4.13 when $\gamma(I) \subset N_1 \setminus N_{11}$. Condition (P) then states that the sign of f is independent of x_1, and $f = df = 0$ on $I \times \{0\} \times \{\varepsilon_n\}$. However, it is not always possible to factor f as in Proposition 26.5.5 into a product of a non-negative function and one which does not depend on x_1. (This can be done in the real analytic case.) When f just vanishes of second order, this is nearly possible though:

Proposition 26.5.7. *Suppose with the hypotheses and notation of Proposition* 26.4.13 *that for some* $t_0 \in I$ *and complex number* c *the quotient* dp/c *is real and the Hessian of* $\operatorname{Im} p/c$ *is not identically* 0 *at* $\gamma(t_0)$, *in the plane defined by* $dp = 0$. *Then we have near* $I \times \{0\} \times \{\varepsilon_n\}$

$$f(x, \xi') = g(x', \xi') h(x, \xi') + r(x, \xi')$$

where $g(x', \xi') = f(t_0, x', \xi')$, h *and* r *are in* C^∞, r *is homogeneous of degree* 1, $h \geqq 0$. *Moreover,* $r \equiv 0$ *unless the Hessian of* g *is positive (resp. negative) semi-definite at* 0, *and then* r *must still vanish when* $g < 0$ *(resp.* $g > 0$*).*

If f has constant sign near $I \times \{0\} \times \{\varepsilon_n\}$, we can of course take $r = f$ so the assertion is trivial then. Note that the Hessian of $\operatorname{Im} p/c$ is invariantly defined since $\operatorname{Im} p/c$ vanishes of the second order at $\gamma(t_0)$. If p is multiplied by a function $q_1 + iq_2$ which is real there, then $\operatorname{Im} p/c$ is replaced by $q_1 \operatorname{Im} p/c + q_2 \operatorname{Re} p/c$, and the Hessian of the second term is zero in the plane $dp = 0$. Thus the hypothesis is invariant under coordinate changes and multiplication by non-vanishing functions as well.

Proof of Proposition 26.5.7. It is enough to make the decomposition when $\xi_n=1$ and extend by homogeneity to $\xi_n>0$ afterwards. Let us write $t=x_1$ and $y=(x_2, \dots, x_n, \xi_2, \dots, \xi_{n-1})$. That $\gamma(I)$ is a one dimensional bicharacteristic means that $f=df=0$ on $I\times\{0\}$, and by hypothesis the Hessian of $g(y) = f(t_0, y, 1)$ with respect to y is not 0 when $y=0$, for the second derivatives containing some t or ξ_n must vanish. Assume for example that $\partial^2 g(0)/\partial y_1^2 \neq 0$. By Malgrange's preparation theorem (Theorem 7.5.5)

$$g(y)=k(y)(y_1^2+a_1(y')y_1+a_0(y'))$$

where $k(0)\neq 0$ and $y'=(y_2, \dots)$. We can take $y_1+a_1(y')/2$ as a new variable and divide by $k(y)$ so we may assume that with $a=a_0-a_1^2/4$

$$g(y)=y_1^2+a(y').$$

Now Malgrange's preparation theorem (Theorem 7.5.6) gives, near $I\times\{0\}$,

$$f(t,y,1)=h(t,y)g(y)+r(t,y); \qquad r(t,y)=b(t,y')y_1+c(t,y'),$$

where $h, b, c \in C^\infty$. When $a(y')<0$ we have two simple zeros $y_1=\pm(-a(y'))^{\frac{1}{2}}$ and obtain $b(t,y')=c(t,y')=0$ by condition (P), hence $r(t,y)=0$ when $g<0$. This completes the proof if the Hessian of g is semi-definite. Otherwise we can apply the same argument to prove that $f(t,y,1)/g(y)$ is also equal to a C^∞ function when $g(y)\geqq 0$. When $g(y)=0$ the two quotients must have the same Taylor expansion which proves that f is divisible by g.

An exact factorization is not always possible even if $n=2$. An example is given by

$$f(x_1,x_2,\xi_2)=(\xi_2-x_1\exp(-1/x_2))^2 \quad \text{if } x_2>0,$$
$$f(x_1,x_2,\xi_2)=\xi_2(\xi_2-\exp(1/x_2)) \quad \text{if } x_2<0; \quad f(x_1,0,\xi_2)=\xi_2^2.$$

The proof can be found in the references or may be supplied by the reader.

We shall denote by N_{12} the set of points in $N_1\setminus N_{11}$ such that there is some complex c for which dp/c is real and the Hessian of $\operatorname{Im} p/c$ is not identically zero in the plane $dp=0$, and we write N_{12}^i for the subset where the Hessian is not semi-definite. By N_{12}^e and N_{12}^{ie} we denote the union of the one dimensional bicharacteristics intersecting these sets. Proposition 26.5.7 gives a simple representation of the principal symbol, particularly in the set N_{12}^i. In the remaining part N_{13} of the characteristic set we can apply Proposition 26.4.13 and obtain a function f vanishing of third order on $I\times\{0\}\times\{\varepsilon_n\}$.

On all one dimensional bicharacteristics we shall prove a result on singularities which is considerably weaker than Theorem 26.1.4; roughly speaking it states when $Pu\in C^\infty$ that s_u^* is either monotonic or rises to a maximum value monotonically and then falls monotonically again. As a byproduct of the study of N_2^e we shall be able to prove more in N_{12}^{ie}; on one dimensional bicharacteristics there s_u^* is concave with respect to an affine structure which we shall now define. Under the hypotheses in Proposition 26.5.7 it will be defined by the differential $h(x_1, 0, \varepsilon_n)dx_1$ on $\gamma(I)$. We shall prove that this is invariantly defined apart from a constant factor, and this means that we have a unique affine structure if we identify points on a subinterval where $h=0$ identically.

Let us first note that if p is real and γ, γ' are two points on the same integral curve of H_p in the surface $p=0$, then the Hamilton flow gives a symplectic map between the tangent spaces of $p=0$ modulo H_p at γ and at γ'. (This is obvious if p is taken as a symplectic coordinate ξ_1 for example.) The observation remains true when p is complex valued provided that γ and γ' lie on a one dimensional bicharacteristic and we consider the complexified tangent planes, for analytically the same computations will be involved. When p has the special form in Proposition 26.5.7 and we consider two points on $\gamma(I)$, the map is obtained from the Hamilton equations for the Hamiltonian

$$\xi_1 + ih(x_1, 0, \varepsilon_n')Q(x', \xi')$$

where Q is the second order part of the Taylor expansion of g at $(0, \varepsilon_n)$. These equations

$$dx_1/dt=1, \quad dx'/dt=ih(x_1, 0, \varepsilon_n')\partial Q/\partial\xi', \quad d\xi'/dt=-ih(x_1, 0, \varepsilon_n')\partial Q/\partial x'$$

are easy to integrate but we only need the obvious fact that $Q(x', \xi')$ is constant along the orbits. This leads to the following general definition. Let $\gamma(t)$ be a one dimensional bicharacteristic of p,

$$H_p(\gamma(t))=c(t)\,d\gamma/dt,$$

and let Q_t be the Hessian of $\operatorname{Im} p/c(t)$ in the plane $dp=0$ modulo H_p, at $\gamma(t)$. Note that if p is replaced by qp where $q\neq 0$ then $c(t)$ must be replaced by $q(\gamma(t))c(t)$ and Q_t remains unchanged. When p has the special form in Proposition 26.5.7 we have just seen that the pullback of $Q_t=h(t, 0, \varepsilon_n')Q$ to the tangent space at t_0 is equal to

$$h(t, 0, \varepsilon_n')Q=(h(t, 0, \varepsilon_n')/h(0, 0, \varepsilon_n'))Q_0.$$

In general, if $\gamma(t_0)\in N_{12}$ and $Q_{t_0}\neq 0$ it follows that the pullback of Q_t to the tangent space at t_0 is of the form $h_0(t)Q_0$. The differential $h_0(t)dt$ is well defined on the bicharacteristic and changes only by a constant factor if the base point t_0 is changed. Now assume that we introduce another parameter s on the bicharacteristic. Then

$$H_p(\gamma(t))=c(t)\,ds/dt\,d\gamma/ds$$

so Q_t is multiplied by dt/ds which makes the differential invariant apart from the normalization which depends on the choice of base point. We have now proved

Proposition 26.5.8. *There is a natural affine structure in every bicharacteristic in N_{12}^e if one identifies points in a subinterval which does not meet N_{12}.*

Let us sum up the notation introduced in this section and which will be referred to in all the rest of the chapter:

N is the characteristic set consisting of all zeros of p;

N_2 is the subset where $H_{\operatorname{Re} p}$ and $H_{\operatorname{Im} p}$ are linearly independent;

N_2^e is the union of semi-bicharacteristics starting in N_2; it is an involutive manifold obtained by attaching one dimensional bicharacteristics to N_2;

N_{11} is the set of points in N which can be joined by a semi-bicharacteristic to a non-characteristic point;

N_{12} is the set of points in $N \setminus (N_2^e \cup N_{11})$ such that for some complex c with dp/c real the Hessian of $\operatorname{Im} p/c$ is not identically zero in the plane $dp = 0$;

N_{12}^i is the subset where the Hessian is not semi-definite;

N_{12}^e and N_{12}^{ie} are the unions of the one dimensional bicharacteristics intersecting these sets;

$N_{13} = N \setminus (N_2^e \cup N_{11} \cup N_{12}^e)$ is the rest of the characteristic set.

It should be kept in mind that this classification of the characteristic points has been made under the assumption that p satisfies (P).

26.6. The Singularities in N_{11}

We shall start with studying an operator for which the principal symbol has the special form which by Proposition 26.5.6 can be achieved by a homogeneous canonical transformation at any one dimensional bicharacteristic (or point) in N_{11}. Afterwards we shall put the result in a general invariant form.

The proof may seem a bit technical so it might be useful to see the simple idea first for the ordinary differential equation

$$du/dt + fu = g$$

in (a, b). If $f \geqq 0$ we obtain by multiplying with $\bar{u} e^{-2\lambda t}$, integrating and taking the real part

$$\begin{aligned}\operatorname{Re} \int_a^b g \bar{u} e^{-2\lambda t} dt &= \int_a^b e^{-2\lambda t} d|u|^2/2 + \int_a^b f |u|^2 e^{-2\lambda t} dt \\ &\geqq \lambda \int_a^b |u|^2 e^{-2\lambda t} dt - |u(a)|^2 e^{-2\lambda a}/2.\end{aligned}$$

An application of Cauchy-Schwarz' inequality now gives

$$(2\lambda - 1) \int_a^b |u|^2 e^{-2\lambda t} dt \leqq \int_a^b |g|^2 e^{-2\lambda t} dt + e^{-2\lambda a} |u(a)|^2.$$

When $\lambda > \frac{1}{2}$ we get control of the L^2 norm of u in terms of that of g and $u(a)$.

Proposition 26.6.1. *Let $P \in \Psi^1_{\mathrm{pgh}}(\mathbb{R}^n)$ be properly supported and have principal symbol p satisfying*

$$p(x,\xi) = \xi_1 + if(x,\xi), \quad f(x,\xi) \leqq 0, \tag{26.6.1}$$

in a conic neighborhood of $\Gamma = \{(x_1, 0, \varepsilon_n),\ a \leqq x_1 \leqq b\} \subset T^*(\mathbb{R}^n) \setminus 0$. If $u \in \mathscr{D}'(\mathbb{R}^n)$ and $Pu \in H_{(s)}$ at Γ, $u \in H_{(s)}$ at $(a, 0, \varepsilon_n)$ for some $s \in \mathbb{R}$, then $u \in H_{(s)}$ at Γ. Moreover, if $\Gamma \cap WF(Pu) = \emptyset$ and $(a, 0, \varepsilon_n) \notin WF(u)$, then $\Gamma \cap WF(u) = \emptyset$.

Proof. It suffices to prove the first statement for it implies the result on wave front sets when applied to all bicharacteristics of ξ_1 near Γ and all $s \in \mathbb{R}$. In the proof we may also make the additional hypothesis that $u \in H_{(s-\frac{1}{2})}$ at Γ. For suppose that the theorem is proved under that hypothesis. Since $u \in H_{(s-k/2)}$ at Γ for some positive integer k we can then conclude successively that $u \in H_{(s-(k-1)/2)}$ at $\Gamma, \dots, u \in H_{(s)}$ at Γ. We may even assume that

$$u \in H^{\mathrm{comp}}_{(s-\frac{1}{2})}, \text{ and } u \in H^{\mathrm{loc}}_{(s)} \text{ in a neighborhood of } \{x;\ x_1 = a\}. \tag{26.6.2}$$

In fact, we can choose $T \in \Psi^0$ with symbol 0 outside such a small conic neighborhood of Γ that Tu satisfies (26.6.2) but $WF(I-T) \cap \Gamma = \emptyset$, hence $PTu \in H_{(s)}$ at Γ since $PTu - Pu$ is in C^∞ at Γ. An application of Proposition 26.6.1 to Tu will then give $Tu \in H_{(s)}$ at Γ, hence $u \in H_{(s)}$ at Γ.

Now we come to the heart of the proof. Choose a compactly generated conic neighborhood V of Γ in $T^*(\mathbb{R}^n) \setminus 0$ such that (26.6.1) is valid in a neighborhood of V and $Pu \in H_{(s)}$ at V. Then choose $\chi \in C_0^\infty(\mathbb{R})$ and a real valued $C(x', \xi) \in S^s(\mathbb{R}^{n-1} \times \mathbb{R}^n)$ so that

(i) $\chi = 1$ on $[a, b]$, and C is non-characteristic at $(0, \varepsilon_n)$;

(ii) $\chi'(x_1) = \chi^-(x_1) - \chi^+(x_1)$ where $0 \leqq \chi^+$, $0 \leqq \chi^-$, $x_1 \leqq a$ in supp χ^-, $x_1 \geqq b$ in supp χ^+, and $\chi^-(x_1) u \in H_{(s)}$.

(iii) $\chi(x_1) C(x', \xi)$ vanishes outside V.

Set for $\lambda, \delta > 0$

$$Q_{\lambda,\delta}(x,\xi) = e^{-\lambda x_1} \chi(x_1) C(x',\xi)(1 + |\delta\xi|^2)^{-1}.$$

Then $Q_{\lambda,\delta} \in S^{s-2}$ and is uniformly bounded in S^s when $\delta \to 0$ for fixed λ. Writing $Q = Q_{\lambda,\delta}(x, D)$ for the sake of brevity we now form

$$(QPu, Qu) = (PQu, Qu) + ([Q, P]u, Qu)$$

which is legitimate since $Qu \in H_{(1)}$ and $QPu \in H_{(0)}$. Write $P = A + iB$ where A and B are self adjoint; thus the principal symbols are ξ_1 and f in a neighborhood of V. Taking the imaginary part we then obtain

$$\operatorname{Im}(QPu, Qu) = (BQu, Qu) + \operatorname{Re}([Q, B]u, Qu) + \operatorname{Im}([Q, A]u, Qu). \tag{26.6.3}$$

We shall discuss the terms in order from right to left.

The symbol of $[Q, A]$ is equal to the symbol $i\partial Q_{\lambda,\delta}/\partial x_1$ of $[Q, D_{x_1}]$ with an error which is bounded in S^{s-1} for fixed λ. Since

$$\partial Q_{\lambda,\delta}/\partial x_1 = -\lambda Q_{\lambda,\delta} + e^{-\lambda x_1}\chi'(x_1)\, C(x', \xi)(1+|\delta\xi|^2)^{-1}$$

and $\chi\chi' \leqq \chi\chi^-$, because $\chi \geqq 0$, we obtain

$$\operatorname{Im}([Q, A]u, Qu) \leqq -\lambda \|Qu\|^2 + \|Q^- u\|\,\|Qu\| + K_\lambda \|u\|_{(s-1)} \|Qu\| \tag{26.6.4}$$

where $Q^- = Q^-_{\lambda,\delta}(x, D)$ with $Q^-_{\lambda,\delta}$ defined as $Q_{\lambda,\delta}$ with χ replaced by χ^-.

The symbol of $Q^*[Q, B]$ is $-iQ_{\lambda,\delta}\{Q_{\lambda,\delta}, f\}$ with an error which is bounded in S^{2s-1}, so the symbol of the self adjoint part

$$\tfrac{1}{2}(Q^*[Q, B] + [Q, B]^* Q)$$

is bounded in S^{2s-1}, when $\delta \to 0$. Hence

$$|\operatorname{Re}([Q, B]u, Qu)| \leqq K_\lambda \|u\|^2_{(s-\frac{1}{2})}. \tag{26.6.5}$$

Choose $C_1 \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$ equal to 1 in $\{(x, \xi) \in V;\ |\xi| > 1\}$ and 0 outside such a small conic neighborhood of V that $0 \geqq F = C_1 f \in S^1$. By Theorem 18.1.14 we have

$$\operatorname{Re}(F(x, D)v, v) \leqq K\|v\|^2, \qquad v \in H_{(1)}. \tag{26.6.6}$$

If b_0 is the term of order 0 in the symbol of B then the symbol of $(B - F(x, D) - (C_1 b_0)(x, D))Q$ is bounded in S^{s-1} for fixed λ. Since $(C_1 b_0)(x, D)$ is bounded in L^2 we obtain with K' independent of λ

$$(BQu, Qu) \leqq K'\|Qu\|^2 + K_\lambda \|Qu\|\,\|u\|_{(s-1)}. \tag{26.6.7}$$

Summing up (26.6.3), (26.6.4), (26.6.5), (26.6.7) we have

$$(\lambda - K')\|Qu\|^2 \leqq \|Qu\|(\|Q^- u\| + \|QPu\| + K_\lambda \|u\|_{(s-1)}) + K_\lambda \|u\|^2_{(s-\frac{1}{2})}.$$

Using Cauchy-Schwarz' inequality we obtain

$$(\lambda - K' - 1)\|Qu\|^2 \leqq \|Q^- u\|^2 + \|QPu\|^2 + K'_\lambda \|u\|^2_{(s-\frac{1}{2})}. \tag{26.6.3$'$}$$

The symbol of

$$Q_{\lambda,\delta}(x, D) - \sum_{|\alpha| \leqq 1} \operatorname{Op} \partial_\xi^\alpha (1 + \delta|\xi|^2)^{-1} \operatorname{Op}(-D_x^\alpha) Q_{\lambda,0}(x, \xi)$$

is bounded in S^{s-2} when $\delta \to 0$, and $\operatorname{Op}(-D_x^\alpha Q_{\lambda,0})Pu \in L^2$ since $Pu \in H_{(s)}$ at V. Hence $\|Q_{\lambda,\delta}Pu\|$ is bounded when $\delta \to 0$. The same is true of $Q^-_{\lambda,\delta}u$ since $\chi^-(x_1)u \in H_{(s)}$. If we take $\lambda = K' + 2$ in (26.6.3)$'$ and let $\delta \to 0$, it follows now that $Q_{\lambda,0}(x, D)u \in L^2$. Hence $u \in H_{(s)}$ at Γ and the proposition is proved.

In the preceding proof the crucial point is the semi-boundedness (26.6.6). Now Theorem 18.6.8 shows that (26.6.6) is also valid if $0 \leqq F \in S^1_{1-\varepsilon,\varepsilon}$ and $0 < \varepsilon \leqq \frac{1}{4}$. This leads to an extension of Proposition 26.6.1 which will be important in Section 26.9.

Proposition 26.6.1′. *Let $P \in \Psi^1_{\text{phg}}(\mathbb{R}^n)$, $C_1 \in S^0_{1-\varepsilon,\varepsilon}(\mathbb{R}^n)$; assume that $C_1(x,\xi)=0$ for large $|x|$ and that the principal symbol p of P satisfies*

$$(26.6.1)' \qquad p(x,\xi)=\xi_1+if(x,\xi), \quad f(x,\xi)\leqq 0 \quad \textit{in } \operatorname{supp} C_1.$$

Let $\chi \in C_0^\infty(\mathbb{R})$ be equal to 1 in $[a,b]$ and $\chi'(t)=\chi^-(t)-\chi^+(t)$ where χ^- and χ^+ are non-negative with support to the left and right of a and b respectively. Finally let $C(x',\xi)\in S^s_{1-\varepsilon,\varepsilon}$ and assume that

$$(26.6.8) \qquad C_1(x,\xi)=1 \quad \textit{in a neighborhood of } \operatorname{supp}\chi(x_1)C(x',\xi).$$

If $0\leqq\varepsilon\leqq\frac14$ and

$$(26.6.9) \quad u\in H^{\text{comp}}_{(s+(3\varepsilon-1)/2)}, \quad \chi(x_1)C(x',D)Pu\in L^2, \quad \chi^-(x_1)C(x',D)u\in L^2$$

it follows then that

$$(26.6.10) \qquad \chi(x_1)C(x',D)u\in L^2.$$

Proof. We just have to inspect the proof of Proposition 26.6.1. The identity (26.6.3) is of course unchanged. In (26.6.4) we just have to replace $\|u\|_{(s-1)}$ by $\|u\|_{(s-1+\varepsilon)}$, and $s-1+\varepsilon<s+(3\varepsilon-1)/2$. The self-adjoint part of $Q^*[Q,B]$ is now bounded of order $2s-1+3\varepsilon$ so (26.6.5) remains valid with $\|u\|_{(s+(3\varepsilon-1)/2)}$ in the right-hand side. There is no change at all in (26.6.7), but the proof now relies on Theorem 18.6.8. The proof is then completed as before.

The important point in Proposition 26.6.1′ is that f does not have to be of constant sign in a conic neighborhood of $\operatorname{supp}\chi(x_1)C(x',\xi)$. In fact, we can choose $C_1\in S^0_{1-\varepsilon,\varepsilon}$ satisfying (26.6.8) and (26.6.1)′ if f is $\leqq 0$ at all points with a fixed distance to $\operatorname{supp}\chi(x_1)C(x',\xi)$ in the metric

$$|dx_1|^2+(1+|\xi'|^2)^\varepsilon|dx'|^2+(1+|\xi'|^2)^{\varepsilon-1}|d\xi|^2$$

(see Section 18.4). However, we shall not continue in this direction now but turn instead to more invariant formulations of Proposition 26.6.1.

Theorem 26.6.2. *Let $P\in\Psi^m_{\text{phg}}(X)$ be properly supported and have principal symbol p satisfying (P), and let*

$$[a,b]\ni t\mapsto\gamma(t), \quad a<b,$$

be a bicharacteristic of $\operatorname{Re} qp$ where $q\neq 0$ and $\operatorname{Im}(qp)(\gamma(a))<0$. If $u\in\mathscr{D}'(X)$ and $Pu\in H_{(s)}$ at $\gamma([a,b])$ it follows that $u\in H_{(s+m-1)}$ at $\gamma([a,b])$.

Proof. If $\gamma(t_0)\notin N$ we have $u\in H_{(s+m)}$ at $\gamma(t_0)$ so it suffices to prove that $u\in H_{(s+m-1)}$ at $\gamma(t_0)$ if $\gamma(t_0)\in N$. Let I be the maximal interval $\subset[a,b]$ containing t_0 such that $\gamma(I)\subset N$. Choose an interval $[a',b']$ with $I\subset[a',b']\subset[a,b]$ such that $\gamma(a')\notin N$ and $[a',b']$ is so close to I that Proposition 26.4.13 is applicable. Then we can find a C^∞ function $\tilde q$ which is homogeneous of degree $1-m$ and a homogeneous canonical transformation χ such that $\chi^*(\tilde q p)=\xi_1+if(x,\xi')$ in a neighborhood of $[a',b']\times\{0\}\times\{\varepsilon_n\}$.

We have $\chi(x_1, 0, \varepsilon_n) = \gamma(x_1)$ when $x_1 \in [a', b']$ and $\tilde{q} = q$ on this arc, for these properties are obvious after the first step of the proof of Proposition 26.4.13 and they are not affected by the second step. Now $\operatorname{Im}(qp)(\gamma(t)) \leqq 0$ when $t \in [a, b]$ by condition (P) since $\operatorname{Im}(qp)(\gamma(a)) < 0$, so we conclude that $f(x, \xi') \leqq 0$ in a neighborhood of $[a', b'] \times \{0\} \times \{\varepsilon_n\}$. If we now transform P as in Proposition 26.4.4 with Fourier integral operators A and B belonging to the graphs of χ and χ^{-1} respectively, and the principal symbol of AB is $\tilde{q}$ near $\gamma(I)$, the theorem follows from Proposition 26.6.1 since $u \in H_{(s+m)}$ at $\gamma(a')$.

We shall now study the propagation of singularities on one dimensional bicharacteristic arcs $\Gamma \subset N_{11}$. We can extend Γ as a one dimensional bicharacteristic so that it is maximal at one end point Γ_0 and choose $q \in C^\infty$ so that $q \neq 0$ and $H_{\operatorname{Re} qp} \neq 0$ on Γ. Then $\operatorname{Im} qp$ must be non-zero at some points arbitrarily close to Γ_0 on the bicharacteristic of $\operatorname{Re} qp$ extending Γ there, and by condition (P) it follows that either $\operatorname{Im} qp \geqq 0$ or else $\operatorname{Im} qp \leqq 0$ in a neighborhood of Γ in the surface $\operatorname{Re} qp = 0$. If $\tilde{q}$ is another function with the same properties as assumed for q, and $H_{\operatorname{Re} \tilde{q}p}$ has the same direction as $H_{\operatorname{Re} qp}$ on Γ, then we conclude that for each $t \in [0, 1]$ the imaginary part of $(tq + (1-t)\tilde{q})p$ has a fixed sign and is not identically 0 in a neighborhood of Γ in the surface $\operatorname{Re}(tq + (1-t)\tilde{q})p = 0$. Clearly the sign must then be independent of t. Put differently, if we choose q so that $\operatorname{Im} qp \leqq 0$ in a neighborhood of Γ when $\operatorname{Re} qp = 0$, then $H_{\operatorname{Re} qp}$ gives Γ an orientation which does not depend on the choice of q.

Definition 26.6.3. If p satisfies condition (P) and Γ is a one dimensional bicharacteristic arc $\subset N_{11}$ which is maximal at one end, then Γ is given the orientation of $H_{\operatorname{Re} qp}$ when q is chosen so that $q \neq 0$, $H_{\operatorname{Re} qp} \neq 0$ on Γ, and $\operatorname{Im} qp \leqq 0$ in a neighborhood of Γ (in the surface defined by $\operatorname{Re} qp = 0$).

Exactly as in the proof of Theorem 26.6.2 we now obtain from Proposition 26.6.1:

Theorem 26.6.4. *Let $P \in \Psi^m_{\mathrm{phg}}(X)$ be properly supported and have a principal symbol satisfying condition (P). If Γ is a compact one dimensional bicharacteristic interval $\subset N_{11}$ and $u \in \mathscr{D}'(X)$, $Pu \in H_{(s)}$ at Γ, $u \in H_{(s+m-1)}$ at the starting point on Γ, then $u \in H_{(s+m-1)}$ at Γ.*

The difference between this result and Theorem 26.1.4 is that regularity only propagates in the direction of the orientation, which of course agrees with the direction in which regularity enters from the non-characteristic set by Theorem 26.6.2. In general one cannot expect propagation in the opposite direction. We give an example.

Example 26.6.5. Let $P = D_1 D_n + iQ(D'')$ where Q is a real quadratic form in $D'' = (D_2, \dots, D_{n-1})$ which is not negative semi-definite. Then one can find a solution u of the equation $Pu=0$ such that

$$WF(u)=\{(x_1, 0, s\varepsilon_n), \quad s>0, \ x_1 \in \mathbb{R}\}$$

and $s_u^*(x_1, 0, \varepsilon_n)=h(x_1)$ where h is any given decreasing concave function on $\mathbb{R}$. If Q takes both positive and negative values then one can obtain an arbitrary concave function h. Since every (decreasing) concave function is the infimum of countably many (decreasing) linear functions the assertion follows from standard category arguments (cf. the proof of Theorem 26.2.3) if it is verified when h is linear.

Fourier transformation of the equation $Pu=0$ with respect to $x' =(x_2, \ldots, x_n)$ gives formally the equation

$$(\xi_n \partial/\partial x_1 - Q(\xi''))\,\hat{u}(x_1, \xi'')=0$$

with the solution $\hat{u}(x_1, \xi'')=c(\xi')\exp(x_1 Q(\xi'')/\xi_n)$. Choose $\theta \in \mathbb{R}^{n-2}$ with $Q(\theta)>0$, a function $\psi \in C_0^\infty(\mathbb{R}^{n-2})$ with $\psi(0)=1$ and a function $\phi \in C^\infty(\mathbb{R})$ with $\phi=0$ in $(-\infty, 1)$, $\phi=1$ in $(2, \infty)$, and set with some real number b

$$\hat{u}(x_1, \xi')=\xi_n^b \psi(\eta'')\,\phi(\xi_n)\exp(x_1 Q(\xi'')/\xi_n),$$

where

$$\eta''=\xi''((\log \xi_n)/\xi_n)^{\frac{1}{2}}-\theta \log \xi_n.$$

Note that

$$\begin{aligned} Q(\xi'')/\xi_n &= Q(\theta(\log \xi_n)^{\frac{1}{2}}+\eta''(\log \xi_n)^{-\frac{1}{2}}) \\ &= (\log \xi_n)\,Q(\theta)+L(\eta'')+Q(\eta'')/\log \xi_n \end{aligned}$$

where L is a linear function of η''. Since $|\eta''|<C$, $|\xi''-\theta(\xi_n \log \xi_n)^{\frac{1}{2}}| < C(\xi_n/\log \xi_n)^{\frac{1}{2}}$ in supp $\hat{u}$, and

$$\begin{aligned} &\partial\eta_j/\partial\xi_k=\delta_{jk}((\log \xi_n)/\xi_n)^{\frac{1}{2}} \quad \text{if } 1<j,k<n, \\ &\partial\eta_j/\partial\xi_n=\xi_j((\log \xi_n)/\xi_n)^{-\frac{1}{2}}\xi_n^{-2}(1-\log \xi_n)/2-\theta/\xi_n, \end{aligned}$$

it is easy to see that

$$|D_{\xi'}^\alpha \hat{u}(x_1, \xi')| \leqq C_\alpha(1+|\xi'|)^{b+x_1 Q(\theta)-\rho|\alpha|} \tag{26.6.11}$$

if $0<\rho<\frac{1}{2}$ is fixed. From (26.6.11) it follows that $\hat{u}$ is indeed the partial Fourier transform of a distribution u satisfying the equation $Pu=0$, and $x'^\alpha u(x)$ has N bounded derivatives where

$$b+x_1 Q(\theta)-\rho|\alpha|+N<-n.$$

Hence $x'=0$ in sing supp$u$. If $\chi \in C_0^\infty(\mathbb{R})$ then the Fourier transform of $\chi(x_1)u$ is

$$(\widehat{\chi u})(\xi)=\xi_n^b \psi(\eta'')\,\phi(\xi_n)\,\hat{\chi}(\xi_1+iQ(\xi'')/\xi_n).$$

We have $\xi_n>1$ and $|\xi''|=o(\xi_n)$ in the support. Since $Q(\xi'')/\xi_n=O(\log \xi_n)$ there we can also estimate $|\hat{\chi}(\xi_1+iQ(\xi'')/\xi_n)|$ by $C_N \xi_n^K(1+|\xi_1|)^{-N}$ for any N, with K independent of N. This proves that $\widehat{\chi u}$ is rapidly decreasing in any cone where $\xi_n<C|\xi_1|$, so $\widehat{\chi u}$ is rapidly decreasing outside any conic neigh-

borhood of $\varepsilon_n=(0, \dots, 0, 1)$. We have therefore proved that

$$WF(u)\subset\{(x_1, 0, s\varepsilon_n),\ s>0,\ x_1\in\mathbb{R}\}.$$

In view of the rapid decrease of u and all its derivatives as $x'\to\infty$ we can determine the microlocal $H_{(s)}$ class by just examining when $\chi(x_1)u\in H_{(s)}$, that is,

$$\int|\widehat{\chi u}(\xi)|^2(1+|\xi|^2)^s d\xi<\infty.$$

We have just seen that $\widehat{\chi u}$ is rapidly decreasing when $|\xi_1|>|\xi'|$, so this is equivalent to

$$\int|\widehat{\chi u}(\xi)|^2(1+|\xi'|^2)^s d\xi<\infty.$$

The integral with respect to ξ_1 can then be calculated by Parseval's formula, so an equivalent condition is that

$$\iint|\xi_n^b\psi(\eta'')\phi(\xi_n)\exp(x_1 Q(\xi'')/\xi_n)\chi(x_1)|^2(1+|\xi'|^2)^s dx_1\,d\xi'<\infty.$$

Here the exponential can be replaced by $\xi_n^{x_1 Q(\theta)}$ and $|\xi'|^2$ can be replaced by ξ_n^2. The integral with respect to ξ'' can then be worked out, so we obtain the simpler equivalent condition

$$\iint_{\xi_n>2}(\xi_n/\log\xi_n)^{(n-2)/2}\,\xi_n^{2(b+x_1Q(\theta)+s)}|\chi(x_1)|^2\,dx_1\,d\xi_n<\infty. \tag{26.6.12}$$

(26.6.12) implies that $n+4(b+x_1Q(\theta)+s)<0$ if $\chi(x_1)\neq 0$, and conversely (26.6.12) follows if this is true in $\operatorname{supp}\chi$. Hence

$$s_u^*(x_1, 0, 0, \varepsilon_n)=-x_1Q(\theta)-b-n/4$$

which is an arbitrary decreasing linear function. If we take θ with $Q(\theta)=0$ or $Q(\theta)<0$ we get instead a constant or an increasing linear function, which completes the verification.

That $s_u^*(x_1, 0, 0, \varepsilon_n)$ must be decreasing if $Q\geqq 0$ follows from Theorem 26.6.4. We shall see later (Theorem 26.9.6) that the concavity is also a necessary condition so the construction above is optimal.

26.7. Degenerate Cauchy-Riemann Operators

In Proposition 26.5.5 we have seen that the principal symbol of an operator satisfying condition (P) can be reduced to the form $\xi_1+ig(x', \xi')h(x, \xi')$ in a neighborhood of a one dimensional bicharacteristic (or a single point) embedded in a two dimensional bicharacteristic. Here $h\geqq 0$ and $dg\neq 0$. By a possibly non-homogeneous canonical transformation the function g can be taken as the ξ_2 variable. Then the two dimensional bicharacteristics are the leaves of the foliation of the plane $\xi_1=\xi_2=0$ by the planes parallel to the x_1x_2 plane, and the Hamilton field is

$$\partial/\partial x_1+ih(x, \xi')\partial/\partial x_2. \tag{26.7.1}$$

The purpose of this section is to discuss such first order differential operators. (An example is the model equation $D_1+ix_1^k D_2$ with even k studied in Section 26.3.) This will lead to a complex structure in reduced two dimensional bicharacteristics as indicated in Section 26.5, and this will be the basis of the extension of Theorem 26.2.1 in the next section. The study of singularities in N_{13} will require information on solutions of families of operators of the form (26.7.1) where there is only uniform control of the derivatives of h which do not depend on x_1. We shall therefore state the results in the generality which will be required then. The special role of the x_1 variable motivates a change of notation so that the x_1 variable in (26.7.1) becomes t and all other variables are called x.

Let $B^\infty(\mathbb{R}^{1+k})$ be the set of continuous function $u(t,x)$, where $t\in\mathbb{R}$ and $x=(x_1,\dots,x_k)\in\mathbb{R}^k$, which have continuous bounded derivatives of all orders with respect to x. It is a Fréchet space with the semi-norms

$$u\mapsto \sup|D_x^\alpha u(t,x)|.$$

We shall study first order differential operators of the form

$$P=D_t+ia(t,x)D_{x_1}+ib(t,x) \tag{26.7.2}$$

where $a,b\in B^\infty$. At first we take $k=1$ and prove a lemma closely related to Proposition 26.6.1′. The norms are L^2 norms unless otherwise indicated.

Lemma 26.7.1. *For every bounded subset B of $B^\infty(\mathbb{R}^2)$ there is a constant C such that if P is defined by* (26.7.2) *with $a,b\in B$ and $a\geqq 0$ then*

$$\|u\|+\|a^{\frac12}\Lambda^{\frac12}u\|\leqq C(\|Pu\|+\|\Lambda^{-1}u\|), \tag{26.7.3}$$

if $u\in\mathscr{S}$ and $u=0$ when $|t|>1$. Here $\Lambda^s u=(1+|D_x|^2)^{s/2}u$.

Proof. Choose $h\in C^\infty(\mathbb{R})$ decreasing, equal to 1 in $(-\infty,-2)$ and 0 in $(-1,\infty)$. We shall apply (26.6.3), that is,

$$\operatorname{Im}(QPu,Qu)=\operatorname{Re}((aD_x+b)Qu,Qu)+\operatorname{Im}([Q,P]u,Qu)$$

with $Q=e^{-\lambda t}h(D_x)$. The commutator is

$$[Q,P]=-i\lambda Q+ie^{-\lambda t}[h(D_x),aD_x+b].$$

Here aD_x+b can be regarded as a pseudo-differential operator in x with symbol bounded in $S^1(\mathbb{R}\times\mathbb{R})$, with t as parameter, and from the calculus it follows then that $[h(D_x),aD_x+b]$ has a symbol uniformly bounded in S^{-1} (in fact, bounded in S^{-N} for any N). Hence

$$\operatorname{Im}([Q,P]u,Qu)\leqq -\lambda\|Qu\|^2+K_\lambda\|\Lambda^{-1}u\|\,\|Qu\|.$$

We shall compare the term $\operatorname{Re}((aD_x+b)Qu,Qu)$ with the positive quantity

$$\|e^{-\lambda t}a^{\frac12}\Lambda^{\frac12}h(D_x)u\|^2=(e^{-\lambda t}\Lambda^{\frac12}a\Lambda^{\frac12}h(D_x)u,e^{-\lambda t}h(D_x)u).$$

The principal symbol of $\Lambda^{\frac12}a\Lambda^{\frac12}+aD_x+b$ is $a(|\xi|+\xi)=0$, $\xi<0$, so the symbol is the sum of one which is uniformly bounded in S^0 and one which is uniformly bounded in S^1 and supported where $\xi>0$, so the product by $h(D_x)$ is bounded in S^{-1}. Hence we obtain, with K independent of λ,

$$\operatorname{Re}((aD_x+b)Qu, Qu)+\|e^{-\lambda t}a^{\frac{1}{2}}\Lambda^{\frac{1}{2}}h(D_x)u\|^2 \\ \leqq K\|Qu\|^2+K_\lambda\|\Lambda^{-1}u\|\,\|Qu\|.$$

Summing up, we obtain

$$(\lambda-K)\|Qu\|^2+\|e^{-\lambda t}a^{\frac{1}{2}}\Lambda^{\frac{1}{2}}h(D_x)u\|^2 \\ \leqq\|Qu\|(\|QPu\|+K_\lambda\|\Lambda^{-1}u\|)+K_\lambda\|\Lambda^{-1}u\|^2.$$

We fix $\lambda=K+2$ now and obtain

$$\|Qu\|^2+\|e^{-\lambda t}a^{\frac{1}{2}}\Lambda^{\frac{1}{2}}h(D_x)u\|^2\leqq\|QPu\|^2+K'_\lambda\|\Lambda^{-1}u\|^2.$$

Now the same argument can be applied with $Q=e^{\lambda t}h(-D_x)$, which just amounts to changing the signs of both t and x. Since $h_0(\xi)=1-h(\xi)-h(-\xi)$ has compact support, we have

$$\|h_0(D)u\|\leqq C\|\Lambda^{-1}u\|, \qquad \|a^{\frac{1}{2}}\Lambda^{\frac{1}{2}}h_0(D)u\|\leqq C\|\Lambda^{-1}u\|.$$

The triangle inequality now gives (26.7.3).

When the support is small in the x direction we can eliminate the second term on the right-hand side of (26.7.3):

Lemma 26.7.2. *Under the hypotheses in Lemma* 26.7.1 *one can find positive constants c and C such that*

$$\|u\|\leqq C\|Pu\|, \quad \textit{if} \;\; u\in C_0^\infty(\{(t,x);\,|t|<1,\,|x|<c\}). \tag{26.7.3$'$}$$

Proof. If

$$\hat{u}(t,\xi)=\int e^{-ix\xi}u(t,x)\,dx$$

is the partial Fourier transform of u with respect to x, then

$$|\hat{u}(t,\xi)|^2\leqq c\int|u(t,x)|^2\,dx$$

by Cauchy-Schwarz' inequality. Hence

$$(2\pi)^{-1}\int(1+|\xi|^2)^{-1}|\hat{u}(t,\xi)|^2\,d\xi\leqq c/2\int|u(t,x)|^2\,dx,$$

so $\|\Lambda^{-1}u\|\leqq(c/2)^{\frac{1}{2}}\|u\|$. If we use this estimate in (26.7.3) and choose c so small that $C(c/2)^{\frac{1}{2}}<\frac{1}{2}$, we obtain (26.7.3)$'$ with twice the constant in (26.7.3).

Lemma 26.7.2 leads immediately to a local existence theorem for the adjoint operator. (Thanks to the lower order term in (26.7.2) the class of operators we consider does not change if we take adjoints and change the sign of x_1.) However, we want to find C^∞ solutions so we must also have estimates in $H_{(s)}$ norms for large negative s. We must then work directly with the parameters, so we allow an arbitrary dimension $1+k$ again now. The set of functions in $B^\infty(\mathbb{R}^2)$ obtained by fixing the variables $x_2,\dots,x_k$ in the functions belonging to a bounded subset of $B^\infty(\mathbb{R}^{1+k})$ is of course bounded, so (26.7.3)′ remains valid with norms in $L^2(\mathbb{R}^{k+1})$.

We shall actually work with the $H_{(s)}$ norm in the x variables only,

$$\|u\|'_{(s)}=((2\pi)^{-k}\int|\hat{u}(t,\xi)|^2(1+|\xi|^2)^s\,d\xi\,dt)^{\frac{1}{2}},\quad u\in\mathscr{S}, \tag{26.7.4}$$

where $\hat{u}(t,\xi)$ is the partial Fourier transform in the x variables. For technical reasons we must use equivalent norms defined as follows. Let $\varepsilon_1>\varepsilon_2>\dots$ be a decreasing positive sequence and set

$$E_N(\xi)=\prod_1^N(1+|\varepsilon_j\xi|^2)^{-1}.$$

Then

$$|||u|||_N=((2\pi)^{-k}\int|\hat{u}(t,\xi)|^2|E_N(\xi)|^2\,d\xi\,dt)^{\frac{1}{2}},\quad u\in\mathscr{S},$$

is equivalent to $\|u\|'_{(-2N)}$. We shall show that if ε_j are successively chosen small enough then (26.7.3)′ remains valid with a slight change of the constants for all the norms $|||\ |||_N$. The problem is of course that P does not commute with the operator $E_N(D_x)$. To be able to estimate commutators we must extract more information from the second term in the left-hand side of (26.7.3). In doing so we write

$$F_\varepsilon=(1+|\varepsilon D_x|^2)^{-1}.$$

With this notation we have

$$|||u|||_{N+1}=|||F_{\varepsilon_{N+1}}u|||_N=\|E_N(D_x)F_{\varepsilon_{N+1}}u\|.$$

Lemma 26.7.3. *If P is defined by* (26.7.2) *with a and b in a fixed bounded subset of $B^\infty(\mathbb{R}^{1+k})$, $a\geqq 0$, then*

$$\|[P,F_\varepsilon]v\|\leqq C\varepsilon^{\frac{1}{2}}(\|PF_\varepsilon v\|+\|F_\varepsilon v\|),\quad 0<\varepsilon<1, \tag{26.7.5}$$

if $v\in\mathscr{S}(\mathbb{R}^{1+k})$ and $v=0$ when $|t|>1$.

Proof. If a denotes multiplication by the function a then

$$[a,1+|\varepsilon D_x|^2]=\varepsilon^2\left(-\Delta_x a+2i\sum_1^k D_j\,\partial a/\partial x_j\right).$$

Multiplication left and right by F_ε gives (cf. the resolvent equation)

$$[a,F_\varepsilon]=\varepsilon^2F_\varepsilon\left((\Delta_x a)-2i\sum_1^k D_j\,\partial a/\partial x_j\right)F_\varepsilon. \tag{26.7.6}$$

Since $a\geqq 0$ and $a''_{x_jx_j}$ is uniformly bounded we have by Lemma 7.7.2

$$|\partial a/\partial x_j|\leqq Ca^{\frac{1}{2}}. \tag{26.7.7}$$

With the notation $\Lambda^{\frac{1}{2}}=(1+D_{x_1}^2)^{\frac{1}{4}}$ from Lemma 26.7.1 we now obtain

$$\begin{aligned}[aD_1,F_\varepsilon]=&-2i\varepsilon^2F_\varepsilon\sum D_jD_1\Lambda^{-\frac{1}{2}}\,\partial a/\partial x_j\Lambda^{\frac{1}{2}}F_\varepsilon+\varepsilon^2F_\varepsilon D_1(\Delta_x a)F_\varepsilon\\&-\varepsilon^2F_\varepsilon((D_1\Delta_x a)+2i\sum D_j[\partial a/\partial x_j,D_1\Lambda^{-\frac{1}{2}}]\Lambda^{\frac{1}{2}})F_\varepsilon.\end{aligned}$$

Here $\varepsilon^{\frac{1}{2}} F_\varepsilon D_j D_1 \Lambda^{-\frac{1}{2}}$ is bounded in L^2, and so are F_ε, $\varepsilon F_\varepsilon D_j$ and the commutator $[\partial a/\partial x_j, D_1 \Lambda^{-\frac{1}{2}}]\Lambda^{\frac{1}{2}}$. By (26.7.3) applied to $F_\varepsilon v$ and (26.7.7) this gives

$$\|[aD_1, F_\varepsilon]v\| \leqq C\varepsilon^{\frac{1}{2}}(\|PF_\varepsilon v\| + \|F_\varepsilon v\|),$$

and a similar estimate for $[b, F_\varepsilon]v$ follows from (26.7.6) with a replaced by b. The lemma is proved.

In the following lemma we collect information on some other commutators which will occur. We use the notation $E_N = E_N(D_x)$.

Lemma 26.7.4. *For fixed E_N and ϕ, ψ in a fixed bounded subset of B^∞, regarded as multiplication operators, we have for* $0<\varepsilon<1$, $u \in \mathscr{S}$

(i) $\|E_N F_\varepsilon \phi u\| \leqq C\|E_N F_\varepsilon u\|$,

(ii) $\|E_N[\phi, F_\varepsilon]u\| \leqq C\varepsilon\|E_N F_\varepsilon u\|$,

(iii) $\|E_N[D_j\phi, F_\varepsilon]u\| \leqq C\|E_N F_\varepsilon u\|$,

(iv) $\|E_N[D_j\phi, [\psi, F_\varepsilon]]u\| \leqq C\varepsilon\|E_N F_\varepsilon u\|$,

(v) $\|[D_j\phi, E_N]F_\varepsilon u\| \leqq C\|E_N F_\varepsilon u\|$,

(vi) $\|[F_\varepsilon, [D_j\phi, E_N]]u\| \leqq C\varepsilon\|E_N F_\varepsilon u\|$.

In (iii)–(vi) *j runs from* 1 *to k and D_j may be omitted.*

Proof. To prove (ii) we replace a by ϕ in (26.7.6) and multiply by E_N to the left and by $E_N^{-1}E_N$ to the right. The operators $E_N(\Delta_x\phi)E_N^{-1}$ and $E_N\partial\phi/\partial x_j E_N^{-1}$ have uniformly bounded symbols in S^0 so the L^2 norms are uniformly bounded. Since $\varepsilon F_\varepsilon D_j$ and F_ε have norm $\leqq 1$ we obtain (ii). If ϕ is multiplied by D_j to the left we just get another factor D_j to the left, and since the norm of $\varepsilon^2 F_\varepsilon D_j D_l$ is at most 1, this proves (iii). The estimate (i) follows from (ii), for

$$\|E_N\phi F_\varepsilon u\| = \|E_N\phi E_N^{-1}E_N F_\varepsilon u\| \leqq C\|E_N F_\varepsilon u\|.$$

To prove (iv) we observe that taking the commutator of $D_j\phi$ and (26.7.6) with $a=\psi$ gives three terms,

$$[D_j\phi, [\psi, F_\varepsilon]] = \varepsilon^4 F_\varepsilon A_1 F_\varepsilon B_1 F_\varepsilon + \varepsilon^2 F_\varepsilon B_2 F_\varepsilon + \varepsilon^4 F_\varepsilon B_1 F_\varepsilon A_2 F_\varepsilon$$

where A_j are second order differential operators and B_j are first order differential operators with coefficients in a bounded subset of B^∞. If we multiply by E_N to the left and insert factors $E_N^{-1}E_N$ at appropriate places, the estimate (iv) follows, for $\varepsilon^2 F_\varepsilon E_N A_j E_N^{-1}$ and $\varepsilon F_\varepsilon E_N B_j E_N^{-1}$ have uniformly bounded norms since differentiations can be moved through E_N to F_ε. (v) follows since

$$[D_j\phi, E_N]F_\varepsilon = [D_j\phi, E_N]E_N^{-1}E_N F_\varepsilon$$

and $[D_j\phi, E_N]E_N^{-1}$ is bounded. Now

$$[F_\varepsilon, [D_j\phi, E_N]]F_\varepsilon^{-1}E_N^{-1} = F_\varepsilon[[D_j\phi, E_N], F_\varepsilon^{-1}]E_N^{-1} = \varepsilon^2 F_\varepsilon(A_0 + \textstyle\sum D_i A_i)$$

where $A_0, \dots, A_k$ are uniformly bounded in L^2. The norm is therefore $O(\varepsilon)$, which completes the proof.

We can now give the main inductive step in the extension of (26.7.3)′ to the norms $|||\ |||_N$.

Lemma 26.7.5. *Assume that for a set of operators of the form* (26.7.2) *with coefficients in a bounded subset of* B^∞ *we have*

$$|||u|||_N \leqq C_N |||Pu|||_N \quad \text{if } u \in \mathscr{S}(\mathbb{R}^{k+1}) \text{ and } u=0 \text{ when } |t|>1 \text{ or } |x_1|>c_N. \tag{26.7.8}$$

If $C_{N+1} > C_N$ *and* $0 < c_{N+1} < c_N$ *it follows that* (26.7.8) *remains valid with* N *replaced by* $N+1$ *provided that* ε_{N+1} *is small enough.*

Proof. Choose $\psi \in C_0^\infty(-c_N, c_N)$ equal to 1 in $(-c_{N+1}, c_{N+1})$. To prove (26.7.8) with the next norm we shall apply (26.7.8) to $v = \psi F_\varepsilon u$ where $u=0$ when $|t|>1$ or $|x_1| > c_{N+1}$. Since $u = \psi u$ we have

$$F_\varepsilon u = F_\varepsilon \psi u = v - [\psi, F_\varepsilon] u,$$

so (ii) in Lemma 26.7.4 and (26.7.8) give

$$|||F_\varepsilon u|||_N \leqq |||v|||_N + C\varepsilon |||F_\varepsilon u|||_N \leqq C_N |||Pv|||_N + C\varepsilon |||F_\varepsilon u|||_N.$$

We have

$$Pv = P\psi F_\varepsilon u = PF_\varepsilon u + P[\psi, F_\varepsilon]u = F_\varepsilon P u + [P, F_\varepsilon]u + [\psi, F_\varepsilon]Pu + [P, [\psi, F_\varepsilon]]u.$$

By (ii) and (iv) in Lemma 26.7.4

$$|||[\psi, F_\varepsilon]Pu|||_N + |||[P, [\psi, F_\varepsilon]]u|||_N \leqq C\varepsilon(|||F_\varepsilon Pu|||_N + |||F_\varepsilon u|||_N).$$

Using the Jacobi identity and Lemma 26.7.3 we obtain

$$\begin{aligned} |||[P, F_\varepsilon]u|||_N &= \|E_N[P, F_\varepsilon]u\| \leqq \|[E_N, [P, F_\varepsilon]]u\| + \|[P, F_\varepsilon]E_N u\| \\ &\leqq \|[F_\varepsilon, [P, E_N]]u\| + \varepsilon^{\frac{1}{2}}(\|PF_\varepsilon E_N u\| + \|F_\varepsilon E_N u\|). \end{aligned}$$

The first term can be estimated by $C\varepsilon |||F_\varepsilon u|||_N$ by (vi) in Lemma 26.7.4, and

$$PF_\varepsilon E_N = E_N P F_\varepsilon + [P, E_N]F_\varepsilon = E_N F_\varepsilon P + E_N[P, F_\varepsilon] + [P, E_N]F_\varepsilon,$$

so by (iii) and (v) in Lemma 26.7.4 we have

$$\|PF_\varepsilon E_N u\| \leqq C(|||F_\varepsilon Pu|||_N + |||F_\varepsilon u|||_N).$$

Summing up, we have proved that

$$|||F_\varepsilon u|||_N \leqq C_N(1 + C\varepsilon^{\frac{1}{2}})|||F_\varepsilon Pu|||_N + C\varepsilon^{\frac{1}{2}}|||F_\varepsilon u|||_N.$$

Choosing ε so small that $C_N(1 + C\varepsilon^{\frac{1}{2}})/(1 - C\varepsilon^{\frac{1}{2}}) < C_{N+1}$ we obtain (26.7.8) with N replaced by $N+1$ if $\varepsilon_{N+1} < \varepsilon$. The proof is complete.

We can now prove an existence theorem by a standard duality argument.

Theorem 26.7.6. *Let M be a set of operators of the form (26.7.2) with coefficients in a bounded subset of $B^\infty(\mathbb{R}^{1+k})$ and $a \geqq 0$. Then there is a constant $c_1 > 0$ such that for every f in a bounded subset F of $B^\infty(\mathbb{R}^{1+k})$ one can find u with u and $\partial u/\partial t$ in another bounded subset U of $B^\infty(\mathbb{R}^{1+k})$, and satisfying the equation $Pu = f$ in $\{(t,x);\ |t|<1,\ |x_1|<c_1\}$.*

Proof. Application of Lemma 26.7.2 to the adjoint of P with x_2 replaced by $-x_2$ shows that for suitable positive constants C and c

$$\|v\| \leqq C\|P^* v\| \quad \text{if } v \in C_0^\infty(\{(t,x);\ |t|<1, |x_1|<c\}),\ P \in M.$$

Assume for the moment that all $f \in F$ have support in a set of fixed measure. Then there is a constant C' such that

$$\|f\| \leqq C', \quad f \in F.$$

Let $0 < c_0 < c$. We shall choose a sequence $\varepsilon_1 > \varepsilon_2 > \ldots > 0$ such that for $N = 0, 1, \ldots$

$$|||v|||_N \leqq C(2N+1)/(N+1)|||P^* v|||_N \tag{26.7.9}$$

$$\text{if } v \in C_0^\infty(\{(t,x);\ |t|<1,\ |x_1|<(Nc_0+c)/(N+1)\}),\ P \in M,$$

$$\|E_N^{-1} f\| \leqq C'(2N+1)/(N+1), \quad f \in F. \tag{26.7.10}$$

We know this already when $N=0$, and Lemma 26.7.5 states that if (26.7.9) is valid for one value of N then it is valid with N replaced by $N+1$ if ε_{N+1} is small enough. This is also true for (26.7.10), for

$$\|E_{N+1}^{-1} f\|^2 = \|E_N^{-1} f\|^2 + \varepsilon_{N+1}^2 \|E_N^{-1} \Delta_x f\|^2$$

where the last term tends to 0 uniformly for $f \in F$ when $\varepsilon_{N+1} \to 0$. If

$$v \in C_0^\infty(\Omega), \quad \Omega = \{(t,x); |t|<1, |x_1|<c_0\},$$

and if $f \in F$, we obtain

$$|(f,v)| = |(E_N^{-1} f, E_N v)| \leqq 4CC' |||P^* v|||_N \to 4CC' |||P^* v|||_\infty$$

as $N \to \infty$. Hence the Hahn-Banach theorem shows that we can find g with

$$(2\pi)^{-n} \int |g(t,\xi)|^2 \prod_1^\infty (1+|\varepsilon_j \xi|^2)\, d\xi \leqq (4CC')^2,$$

$$(v,f) = (2\pi)^{-n} \int \widehat{P^* v}(t,\xi)\, \overline{g(t,\xi)}\, d\xi, \quad v \in C_0^\infty(\Omega).$$

If we set

$$u(t,x) = (2\pi)^{-n} \int g(t,\xi)\, e^{i\langle x,\xi\rangle}\, d\xi$$

we obtain $Pu = f$ in Ω, and there is a fixed bound for $\|D_x^\alpha u\|$ for every α. Since $D_t u = f - iaD_{x_1} u - ibu$ in Ω we also have a uniform bound for $\|D_t D_x^\alpha u\|_{L^2(\Omega)}$. Hence $D_x^\alpha u$ is continuous and uniformly bounded in Ω for all α, and so is $D_t D_x^\alpha u$. After multiplying by a function $\psi \in C_0^\infty(-c_0, c_0)$ of x_1, which is 1 in a somewhat smaller interval $(-c_1, c_1)$ we can extend u to a function in $B^\infty(\mathbb{R}^{k+1})$ with fixed bounds for all x derivatives.

To remove the support condition on f we choose a function $\chi \in C_0^\infty(\mathbb{R}^{k-1})$ such that $\sum \chi(x'-g)^2 = 1$ in Ω if g runs through the lattice points in the variables $x' = (x_2, \ldots, x_k)$. After solving the equations

$$Pu_g = \chi(x'-g)f$$

as above we just have to set $u(t,x) = \sum \chi(x'-g)\, u_g(t,x)$ to get the desired solution of $Pu = f$ for we may assume that $|t| < 2$ and $|x_1| < 2\,C_0$ in $\operatorname{supp} f$.

Remark. By repeated differentiation of the equation $Pu = f$ it follows that $u \in C^\infty$ in any open set where a, b and f are in C^∞. Bounds for the derivatives of u follow from bounds for those of a, b and f.

The substitution $u = ve^w$ where $D_t w + iaD_{x_1} w + ib = 0$ can now be used to reduce the equation $Pu = f$ to the form $D_t v + iaD_{x_1} v = fe^{-w}$. In the following corollary we construct non-trivial solutions of the corresponding homogeneous equation.

Corollary 26.7.7. *Let $A \subset B^\infty(\mathbb{R}^{k+1})$ be a bounded set of non-negative functions. Then there exists a constant $c_1 > 0$ and another bounded set $U \subset B^\infty(\mathbb{R}^{k+1})$ such that for every $a \in A$ there is a solution of the equation*

$$D_t u + iaD_{x_1} u = 0 \quad \text{in } \Omega = \{(t,x);\ |t| < 1,\ |x_1| < c_1\}, \tag{26.7.11}$$

such that u and $\partial u/\partial t$ are in U, $\partial u/\partial x_1 = e^w$ in Ω, and $w, \partial w/\partial t \in U$.

Proof. If u satisfies (26.7.11) and $v = \partial u/\partial x_1$, then the equation

$$D_t v + iaD_{x_1} v + \partial a/\partial x_1\, v = 0 \tag{26.7.12}$$

follows by differentiation of (26.7.11). We solve this equation first by writing $v = e^w$, which gives the inhomogeneous equation

$$D_t w + iaD_{x_1} w + \partial a/\partial x_1 = 0. \tag{26.7.13}$$

By Theorem 26.7.6 there is a solution of this equation when $|t| < 1$, $|x_1| < c_1$ such that w and $D_t w$ are in a bounded subset of B^∞. Now

$$u(t,x) = \int_0^{x_1} v(t,s,x')\,ds - u_1(t,x'), \qquad x' = (x_2, \ldots, x_n),$$

satisfies (26.7.11) if

$$D_t u_1(t,x') = (av)(t,0,x').$$

We choose

$$u_1(t,x') = i\int_0^t (av)(s,0,x')\,ds.$$

It is then clear that u and $\partial u/\partial t$ are in a bounded subset of B after a cutoff for large $|t| + |x_1|$, and this proves the corollary.

In Section 26.10 we shall need solutions which are small at the boundary in the x direction:

Corollary 26.7.8. *Let $A \subset B^\infty(\mathbb{R}^{k+1})$ be a bounded set of non-negative functions. For every $\varepsilon>0$ one can then find arbitrarily small neighborhoods $V_0 \Subset V_1 \Subset V_2$ of the origin in $\mathbb{R}^k$ and $T>0$ such that the equation $D_t U + iaD_{x_1} U = 0$ for every $a \in A$ has a solution in $(-T,T)\times V_2$ with uniform bounds*

$$|D_x^\alpha U(t,x)| \leqq C_\alpha; \quad |t|<T,\ x\in V_2; \tag{26.7.14}$$

independent of a, and

$$\operatorname{Re} U(t,x) \geqq 0 \quad \text{if } |t|<T,\ x\in V_2, \tag{26.7.15}$$

$$\operatorname{Re} U(t,x) > 1 \quad \text{if } |t|<T,\ x\in V_2 \setminus V_1, \tag{26.7.16}$$

$$\operatorname{Re} U(t,x) < \varepsilon \quad \text{if } |t|<T,\ x\in V_0. \tag{26.7.17}$$

Proof. We start from the solution u in Corollary 26.7.7 where we may assume that $w(0)=0$, since this can be achieved by multiplication with a suitable constant. Decreasing c_1 if necessary we may then assume that $|w(0,x_1,0,\dots,0)|<\pi/4$ when $|x_1|\leqq c_1$. The curve

$$\Gamma = \{u(0,x_1,0,\dots,0);\ |x_1|\leqq c_1\}$$

then has slope $<\pi/4$ and the element of arc is $\geqq e^{-\pi/4}|dx_1|$. The distance between the end points $z_\pm = u(0,\pm c_1,0,\dots,0)$ is therefore at least $2^{\frac{1}{2}} c_1 e^{-\pi/4}$. The function

$$F_\delta(z) = 2\delta((\delta+z-z_-))^{-1} + (\delta+z_+-z)^{-1})$$

is analytic and $\operatorname{Re} F_\delta>0$ in the δ neighborhood of Γ, since $|\arg(z-z_-)|<\pi/4$ and $|\arg(z_+-z)|<\pi/4$ on Γ. We have $\operatorname{Re} F_\delta(z)<\varepsilon/2$ near $\{u(0,x_1,0,\dots,0);\ |x_1|\leqq c_1/2\}$ if δ is small, and $\operatorname{Re} F_\delta(z)>\frac{4}{3}$ when $|z-z_-|<\delta/2$ or $|z-z_+|<\delta/2$. Since u is uniformly Lipschitz continuous it follows that the function

$$U(t,z) = F_\delta(u(t,x)) + |x''|^2/\delta^4$$

has the desired properties for small δ if

$$V_0 = \{x;\ |x_1|<c_1/2, |x''|<\delta^2\varepsilon/2\},$$
$$V_1 = \{x;\ |x_1|<c_1-\delta^2, |x''|<\delta^2\},$$
$$V_2 = \{x;\ |x_1|<c, |x''|<2\delta^2\}.$$

In the following results we have no need to insist on uniformity with respect to the coefficients so we consider a single operator.

Corollary 26.7.9. *Let $0\leqq a\in B^\infty(\mathbb{R}^{1+k})$, and let u_0 be a solution of the equation*

$$D_t u_0 + ia(t,x_1,0) D_{x_1} u_0 = 0$$

when $|t|<1$ and $|x_1|<c_1$, with u_0 and $\partial u_0/\partial t \in B^\infty(\mathbb{R}^2)$. Then one can find $u\in B^\infty(\mathbb{R}^{k+1})$ with $\partial u/\partial t\in B^\infty(\mathbb{R}^{k+1})$ satisfying (26.7.11) *such that*

$$u(t,x_1,0) = u_0(t,x_1).$$

Proof. Since $f(t,x)=D_t u_0+ia(t,x)D_{x_1}u_0\in B^\infty$ vanishes in Ω when $x''=0$ we can find $f_j\in B^\infty(\mathbb{R}^{1+k})$ such that

$$f(t,x)=\sum_2^k x_j f_j(t,x)(1+|x''|^2)^{-\frac{1}{2}} \quad \text{in } \Omega;$$

we first define f_j in Ω and then extend to the whole space. By Theorem 26.7.6 we can find $v_j\in B^\infty$ with $\partial v_j/\partial t\in B^\infty$ and

$$D_t v_j+iaD_{x_1}v_j=f_j \quad \text{in } \Omega.$$

Now we just have to take

$$u(t,x)=u_0(t,x_1)-\sum_2^k x_j v_j(t,x)(1+|x''|^2)^{-\frac{1}{2}}.$$

We shall simplify the following discussion by dropping the parameters x'', so x will denote a real variable from now on. The characteristics of the operator

$$D_t+ia(t,x)D_x,$$

where we assume $0\leq a\in B^\infty(\mathbb{R}^2)$, are defined by $a(t,x)=0$ and $\tau=0$. Since $a=0$ implies $da=0$ the corresponding direction of the Hamilton field is that of the t axis, so the base projection of a one dimensional bicharacteristic is precisely an interval $I\times\{x_0\}$ in the direction of the t axis where a is identically 0. It is clear that every solution of the homogeneous equation

$$D_t u+ia(t,x)D_x u=0 \tag{26.7.11$'$}$$

is constant on $I\times\{x_0\}$. Differentiation with respect to x (cf. (26.7.12)) shows that also $\partial u/\partial x_1$ is constant in $I\times\{x_0\}$. For the solution given by Corollary 26.7.7 it follows then that w is also constant in $I\times\{x_0\}$. Now assume that $I\subset(-1,1)$ is a maximal compact interval such that a vanishes on $I\times\{x_0\}$. Choose a closed rectangle R with axes parallel to the coordinate axes so that $I\times\{x_0\}\subset R\subset\Omega$ and $a\neq 0$ on the sides parallel to the x axis. We choose R so small that w varies by less than $\pi/4$ in R. Then it is clear that intervals in R parallel to the t or x axis are mapped by u to C^1 curves with tangent direction differing by less than $\pi/4$ from $\operatorname{Im} w-\pi/2$ resp. $\operatorname{Im} w$, evaluated at a point in $I\times\{x_0\}$. If we join two points in R by a curve consisting of two line segments parallel to the coordinate axes, it follows that they have different images under u unless they lie on a line $x=$constant and a vanishes between them. In particular, the boundary of R is mapped to a Jordan curve Γ which has the value of u at $I\times\{x_0\}$ in its interior. If z is in the interior (exterior) of Γ then the winding number of $u(t,x)-z$ when (t,x) goes around Γ is 1 (resp. 0). Hence $u(R)$ contains the interior of Γ but no point in the exterior since regular values there would have to be taken an even number of times and we know that they can only be taken once. It follows that u is a homeomorphism from $\tilde{R}$ to $u(R)$ if $\tilde{R}$ is obtained from R by identifying points in R which lie on a line segment $x=$constant where a vanishes identically.

Now assume that v is any other solution of (26.7.11)′ in a neighborhood of R. Then $v=f(u)$ where f is a continuous function from $u(R)$ to $v(R)$. We claim that f is analytic in the interior of $u(R)$. This is obvious if $a\neq 0$ everywhere, for u is then a C^1 diffeomorphism which transforms (26.7.11)′ to the Cauchy-Riemann equation. In general we can obtain the same conclusion by using Corollary 26.7.9 to extend u to a solution $U(t,x,\varepsilon)$ of the equation

$$D_t U+i(a(t,x)+\varepsilon^2)D_x U=0$$

with $U(t,x,0)=u(t,x)$. We extend v similarly to a solution V and obtain

$$V(t,x,\varepsilon)=f_\varepsilon(U(t,x,\varepsilon)), \qquad \varepsilon\neq 0$$

where f_ε is a continuous map from $U(R,\varepsilon)$ to $V(R,\varepsilon)$ which is analytic in the interior of $U(R,\varepsilon)$. In particular, f_ε is uniformly bounded so there is a uniform limit f_0 of f_ε in the interior of $u(R)$. Thus f_0 is analytic and $f_0(u)=v$, so f_0 is the function we wanted to prove analytic. Thus we have proved:

Theorem 26.7.10. *Let $0\leqq a\in B^\infty(\mathbb{R}^2)$ and let u be the solution of the equation* (26.7.11)′ *given in Corollary* 26.7.7. *Set*

$$\Omega_0=\{(t,x)\in\Omega;\ a(t_1,x)\,a(t_2,x)\neq 0 \text{ for some } t_1,t_2 \text{ with } -1<t_1<t<t_2<1\},$$

and let $\tilde\Omega_0$ be the quotient of Ω_0 by the equivalence relation identifying (t,x) with (t',x) if $a(s,x)=0$ when $s\in[t,t']$. Then u defines a local homeomorphism $\tilde\Omega_0\to\mathbb{C}$ giving $\tilde\Omega_0$ an analytic structure such that the analytic functions in $\tilde\Omega_0$ lifted to Ω_0 are precisely the solutions of the equation (26.7.11)′.

Theorem 26.7.10 can be applied to the Hamilton field in a two dimensional bicharacteristic of an operator satisfying condition (P), for with the coordinates in Proposition 26.5.5 it is of the form

$$\partial/\partial x_1+ih\,H_g$$

where $h\geqq 0$. The two dimensional bicharacteristic is generated by the x_1 axis and a bicharacteristic of g in the x', ξ' variables. Hence we have a natural analytic structure in the reduced two dimensional bicharacteristics defined after Definition 26.5.4. The special case of a conic two dimensional bicharacteristic deserves a special discussion. Again with the coordinates in Proposition 26.5.5 the bicharacteristic through $I\times\{0\}\times\{\varepsilon_n\}$ is the product of the x_1 axis and the positive ξ_n axis, and the Hamilton field is

$$H_p=\partial/\partial x_1+ib(x_1)\xi_n\partial/\partial\xi_n$$

where $b(x_1)=-\partial f(x_1,0,\varepsilon_n')/\partial x_n$ is different from 0 at some points close to the end points of I. It follows that

$$u=\int b(x_1)\,dx_1+i\log\xi_n$$

is a solution of the equation $H_p u=0$ in B with $\operatorname{Re} u$ constant in the radial direction. Any other such solution is of the form $au+c$ for some real a. In fact, an analytic function f with $\operatorname{Re} f(z)$ depending only on $\operatorname{Re} z$ must be of the form $az+c$ with a real, for the harmonic function $\operatorname{Re} f(z)$ must be a linear function of $\operatorname{Re} z$. Thus we obtain:

Theorem 26.7.11. *Let p satisfy condition (P) and let B be a two dimensional bicharacteristic, $\tilde{B}_0$ the corresponding reduced bicharacteristic. Then $\tilde{B}_0$ has a natural analytic structure such that the (local) analytic functions on $\tilde{B}_0$ lifted to B are precisely the (local) solutions of the equation $H_p u=0$. If B is conic, then the set $\tilde{B}_0'$ obtained by identification of points on $\tilde{B}_0$ on the same ray has a natural affine structure such that the linear functions lifted to B are precisely the real parts of solutions of $H_p u=0$ which are constant in the radial direction.*

26.8. The Nirenberg-Treves Estimate

In this section we shall prove a general version of Lemma 26.7.1 where (26.7.2) is replaced by an ordinary differential equation

$$du/dt - A(t)Bu = f \tag{26.8.1}$$

in a Hilbert space H. We make the following assumptions:

(i) $A(t)$ is a bounded non-negative self adjoint operator which is uniformly continuous as a function of t.

(ii) B is bounded and self adjoint.

The boundedness condition on B could be dropped but it is convenient in the statement and proof of the following theorem, and it is quite harmless in our applications.

Theorem 26.8.1. *Assume that the conditions* (i) *and* (ii) *above are fulfilled and that when $|t|<T$*

$$10\,\|A(t)\|^{\frac14}\,\|[B,A(t)]\|^{\frac14}\,\|[B,[B,A(t)]]\|^{\frac12} \leqq M. \tag{26.8.2}$$

If u is a continuously differentiable function of t with values in H which satisfies (26.8.1) *and vanishes for $|t|>T$ and if $TM<\frac12$, then*

$$\int \|u(t)\|^2\,dt \leqq (4T/(1-2TM))^2 \int \|f(t)\|^2\,dt. \tag{26.8.3}$$

Proof. Let E_λ be the spectral projections of B and write $E_-=E_0$, $E_+=I-E_0$ for the projections corresponding to the half axes. They will replace the operators $h(\pm D_x)$ in the proof of Proposition 26.7.1. Set $u_\pm=E_\pm u$ and form

$$\begin{aligned}\operatorname{Re}(u_-,f_-)&=\operatorname{Re}(u_-,f)=\operatorname{Re}(u_-,\partial u/\partial t)-\operatorname{Re}(u_-,A(t)Bu)\\ &=\tfrac12 d\|u_-\|^2/dt-\operatorname{Re}(u_-,A(t)(B_+u_+-B_-u_-)).\end{aligned}$$

Here $B_+ = E_+ B$ and $B_- = -E_- B$ are positive operators and we have used that $\partial u_+/\partial t$ is orthogonal to u_-. Now

$$\begin{aligned}\operatorname{Re}(u_-, A(t)B_- u_-) &= \operatorname{Re}(u_-, [A(t), B_-^{\frac12}]B_-^{\frac12} u_-) + (B_-^{\frac12} u_-, A(t) B_-^{\frac12} u_-)\\ &\geqq (u_-, [[A(t), B_-^{\frac12}], B_-^{\frac12}]u_-)/2\end{aligned}$$

for the adjoint of $[A(t), B_-^{\frac12}]B_-^{\frac12}$ is $-B_-^{\frac12}[A(t), B_-^{\frac12}]$. Similarly

$$\begin{aligned}\operatorname{Re}(u_-, A(t)B_+ u_+) &= \operatorname{Re}(u_-, [A(t), B_+^{\frac12}]B_+^{\frac12} u_+)\\ &= \operatorname{Re}(u_-, [[A(t), B_+^{\frac12}], B_+^{\frac12}]u_+)\end{aligned}$$

since $B_+^{\frac12} u_- = 0$. In Lemma 26.8.2 we shall show that the norms of these commutators are $\leqq M/3$ by (26.8.2). If we now multiply by $T-t$ and integrate, we obtain

$$\tfrac12 \int \|u_-\|^2\,dt \leqq 2T \int \Big(\|u_-\|\,\|f_-\| + \frac{M}{3}(\|u_-\|^2/2 + \|u_-\|\,\|u_+\|)\Big)\,dt.$$

Taking scalar product with u_+ instead and multiplying by $-T-t$ we obtain

$$\tfrac12 \int \|u_+\|^2\,dt \leqq 2T \int \Big(\|u_+\|\,\|f_+\| + \frac{M}{3}(\|u_+\|^2/2 + \|u_-\|\,\|u_+\|)\Big)\,dt.$$

If we add and use Cauchy-Schwarz' inequality it follows that

$$\tfrac12 \int \|u\|^2\,dt \leqq 2T\Big(\int \|u\|^2\,dt\Big)^{\frac12}\Big(\int \|f\|^2\,dt\Big)^{\frac12} + TM\int \|u\|^2\,dt,$$

and this gives (26.8.3) when a factor $(\int \|u\|^2\,dt)^{\frac12}$ is cancelled.

Lemma 26.8.2. *Let A and B be bounded operators in a Hilbert space H, and assume that B is self-adjoint. If $B_\pm = (|B| \pm B)/2$ it follows that*

$$\|[B_\pm^{\frac12}, [B_\pm^{\frac12}, A]]\| \leqq \tfrac{10}{3} \|A\|^{\frac14}\, \|[B, A]\|^{\frac12}\, \|[B, [B, A]]\|^{\frac14}. \tag{26.8.4}$$

Proof. If $R(z) = (B-z)^{-1}$ is the resolvent of B and $\varepsilon > 0$, then

$$(1+\varepsilon|B|)^{-1} B_+^{\frac12} = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} z^{\frac12}(1+\varepsilon z)^{-1} R(z)\,dz. \tag{26.8.5}$$

Here $\sqrt{z}$ is analytic when $\operatorname{Re} z > 0$. In fact,

$$\begin{aligned}(2\pi i)^{-1} \int_{-i\infty}^{i\infty} z^{\frac12}(z-\lambda)^{-1}(1+\varepsilon z)^{-1}\,dz &= -\lambda^{\frac12}(1+\varepsilon\lambda)^{-1} \quad \text{if } \lambda > 0,\\ &= 0 \quad \text{if } \lambda \leqq 0,\end{aligned}$$

so (26.8.5) follows if we write $R(z) = \int (\lambda - z)^{-1}\,dE_\lambda$ by the spectral theorem. The integral in (26.8.5) is absolutely convergent since $\|R(z)\| \leqq 1/|\operatorname{Im} z|$. The resolvent equation

$$[A, R(z)] = R(z)\,[B, A]\,R(z)$$

follows by multiplying the equation $[B-z, A] = [B, A]$ left and right by $R(z)$, and it gives the estimate

$$\|[A, R(z)]\| \leqq \min\big(2\|A\|\,|\operatorname{Im} z|^{-1},\ \|[B, A]\|\,|\operatorname{Im} z|^{-2}\big).$$

Hence we have for every $T>0$

$$\begin{aligned}\|[A,(1+\varepsilon|B|)^{-1}B_+^{\frac{1}{2}}]\| &\leqq \frac{2}{\pi}\int_0^T t^{-\frac{1}{2}}\|A\|\,dt + 1/\pi\int_T^\infty t^{-\frac{3}{2}}\|[B,A]\|\,dt\\ &= 2/\pi(2T^{\frac{1}{2}}\|A\| + T^{-\frac{1}{2}}\|[B,A]\|).\end{aligned}$$

We minimize the right-hand side and conclude when $\varepsilon\to 0$ that

$$\|[A,B_+^{\frac{1}{2}}]\| \leqq 4\sqrt{2}/\pi\,\|A\|^{\frac{1}{2}}\|[B,A]\|^{\frac{1}{2}}. \tag{26.8.6}$$

To prove (26.8.4) we shall now estimate $[[A,B_+^{\frac{1}{2}}],B]$. We have

$$[[A,R(z)],B]=R(z)[[B,A],B]\,R(z),\ [[A,R(z)],B]=-[[B,A],R(z)]$$

since B and $R(z)$ commute, so $\|[[A,R(z)],B]\|$ can be estimated by

$$\min(2\|[B,A]\|\,|\mathrm{Im}\,z|^{-1},\ \|[[B,A],B]\|\,|\mathrm{Im}\,z|^{-2}).$$

Hence it follows from (26.8.5) as in the proof of (26.8.6) that

$$\|[[A,B_+^{\frac{1}{2}}],B]\| \leqq 4\sqrt{2}/\pi\,\|[B,A]\|^{\frac{1}{2}}\|[B,[B,A]]\|^{\frac{1}{2}}. \tag{26.8.7}$$

The preceding estimates are of course valid for $B_-^{\frac{1}{2}}$ also. If we now apply (26.8.6) with A replaced by $[A,B_\pm^{\frac{1}{2}}]$ we obtain by (26.8.7)

$$\begin{aligned}\|[[A,B_\pm^{\frac{1}{2}}],B_\pm^{\frac{1}{2}}]\|^2 &\leqq 32/\pi^2\,\|[A,B_\pm^{\frac{1}{2}}]\|\,4\sqrt{2}/\pi\,\|[B,A]\|^{\frac{1}{2}}\|[B,[B,A]]\|^{\frac{1}{2}}\\ &\leqq (32/\pi^2)^2\,\|A\|^{\frac{1}{2}}\|[B,A]\|\,\|[B,[B,A]]\|^{\frac{1}{2}}.\end{aligned}$$

Since $32/\pi^2=3.24\ldots<\frac{10}{3}$, the estimate (26.8.4) follows.

In our application of Theorem 26.8.1 A and B will be pseudo-differential operators with symbols bounded in S^0 and in S^1 respectively, which makes (26.8.2) valid for some M which we can estimate. It would have been possible to avoid the abstract operator theory in Theorem 26.8.1 by using Fourier integral operators corresponding to non-homogeneous canonical transformations to reduce to a situation where $B=D_{x_2}$ and pseudo-differential operators can be used as in the proof of Lemma 26.7.1. However, Theorem 26.8.1 may serve as a useful reminder that abstract operator theory may sometimes be more efficient than pseudo-differential operator theory; the operators $B_\pm^{\frac{1}{2}}$ are not pseudo-differential operators unless B is of a very special form.

In the proof of Theorem 26.8.1 we discarded a positive quantity corresponding to the one which gave the second term in the left-hand side of (26.7.3), which was essential for the commutator estimate in Lemma 26.7.3. We shall now prove an analogous estimate in the abstract context of Theorem 26.8.1 which will also allow us to control some commutators which occur in Sections 26.9 and 26.10.

Theorem 26.8.3. *If* $A(t)$ *and* B *satisfy* (i), (ii) *and* (26.8.2) *above, and if* $u \in C^1(\mathbb{R}, H)$ *vanishes for* $|t| > T$ *and satisfies* (26.8.1), *then*

$$\int (Bu, A(t)Bu)\,dt \leqq T\|B\| \int (18\|f\|^2 + 2M^2\|u\|^2)\,dt. \tag{26.8.8}$$

Proof. With the notation in the proof of Theorem 26.8.1 we have

$$\begin{aligned}\operatorname{Re}(B_-u_-, f) &= \operatorname{Re}(B_-u_-, du/dt - A(t)Bu)\\ &= \tfrac{1}{2}d\|B_-^{\frac{1}{2}}u_-\|^2/dt + (B_-u_-, A(t)B_-u_-) - \operatorname{Re}(B_-u_-, A(t)B_+u_+).\end{aligned}$$

Since $B_+^{\frac{1}{2}}B_- = 0$ the last term can be estimated by

$$|(B_-u_-, [[A(t), B_+^{\frac{1}{2}}], B_+^{\frac{1}{2}}]u_+)| \leqq \|B_-u_-\|\, M\|u_+\|/3$$

where we have used (26.8.4) and (26.8.2). If we integrate after multiplication by $2T - t$ we obtain

$$\begin{aligned}&\tfrac{1}{2}\int \|B_-^{\frac{1}{2}}u_-\|^2\,dt + T\int (B_-u_-, A(t)B_-u_-)\,dt\\ &\qquad \leqq T\int (3\|B_-u_-\|\,\|f_-\| + M\|B_-u_-\|\,\|u_+\|)\,dt.\end{aligned}$$

We have a similar estimate for $B_+^{\frac{1}{2}}u_+$, and since $Bu = B_+u_+ - B_-u_-$ we have

$$(Bu, A(t)Bu) \leqq 2(B_+u_+, A(t)B_+u_+) + 2(B_-u_-, A(t)B_-u_-),$$

for $A(t)$ is positive. Hence we obtain by adding

$$\int \||B|^{\frac{1}{2}}u\|^2\,dt + T\int (Bu, A(t)Bu)\,dt \leqq T\int (6\|Bu\|\,\|f\| + 2M\|Bu\|\,\|u\|)\,dt.$$

Here $\|Bu\| \leqq \|B\|^{\frac{1}{2}}\,\||B|^{\frac{1}{2}}u\|$ so the right-hand side is bounded by

$$\int \||B|^{\frac{1}{2}}u\|^2\,dt + T^2\|B\|\int (18\|f\|^2 + 2M^2\|u\|^2)\,dt.$$

Hence

$$T\int (Bu, A(t)Bu)\,dt \leqq T^2\|B\|\int (18\|f\|^2 + 2M^2\|u\|^2)\,dt,$$

which proves (26.8.8).

26.9. The Singularities in N_2^e and in N_{12}^e

We are now ready to extend Theorem 26.2.1 to a general two dimensional bicharacteristic B, that is, a leaf of the foliation of the involutive manifold N_2^e. More precisely we shall consider the corresponding reduced bicharacteristic $\tilde{B}_0$. (See Proposition 26.5.3, Definition 26.5.4 and the discussion after it, as well as Theorems 26.7.10 and 26:7.11.) If $u \in \mathscr{D}'(X)$ we define $\tilde{s}_u(\tilde{\gamma})$ for $\tilde{\gamma} \in \tilde{B}_0$ as $\inf s_u^*(x,\xi)$ for (x,ξ) in the inverse image γ of $\tilde{\gamma}$ in B, which is a compact maximal embedded one dimensional bicharacteristic. Since s_u^* is semi-continuous from below it is clear that $\tilde{s}_u(\tilde{\gamma})$ is the supremum of all $s \in \mathbb{R}$ such that $u \in H_{(s)}$ at every point in γ. The central result in this section is the following one.

Theorem 26.9.1. *Let $P \in \Psi^m_{\text{phg}}(X)$ be properly supported and satisfy condition (P). Let $u \in \mathscr{D}'(X)$, let B be a two dimensional bicharacteristic of P and $\tilde{s}$ a superharmonic function in an open subset ω of the corresponding reduced bicharacteristic $\tilde{B}_0$ such that $\tilde{s}_{Pu} \geqq \tilde{s}$ in ω. Then it follows that*

$$\min(\tilde{s}_u, \tilde{s}+m-1) \tag{26.9.1}$$

is superharmonic in ω. In the special case where B is conic, this means that (26.9.1) is a concave function on $\tilde{B}_0'$ if $\tilde{s}$ is concave; here $\tilde{B}_0'$ is obtained from $\tilde{B}_0$ by identifying points on the same ray.

Superhamonicity is a local property so it suffices to prove that Theorem 26.9.1 is valid for small neighborhoods ω of any point in $\tilde{B}_0$. Using Propositions 26.4.13 and 26.5.5 with the remark following the proof of the latter we can then transform P as in Proposition 26.4.4 (see also the proof of Theorem 26.6.2). Thus we may assume that $P \in \Psi^1_{\text{phg}}(\mathbb{R}^n)$ and that the principal symbol is of the form

$$p(x,\xi) = \xi_1 + ig(x',\xi')h(x,\xi'); \quad x' = (x_2,\dots,x_n),\ \xi' = (\xi_2,\dots,\xi_n) \tag{26.9.2}$$

in a conic neighborhood of $I' = I \times \{0\} \times \varepsilon_n \subset T^*(\mathbb{R}^n) \setminus 0$. Here $h \geqq 0$ is homogeneous of degree 0, g is homogeneous of degree 1, and $g(0,\varepsilon_n') = 0$, $dg(0,\varepsilon_n') \neq 0$. The interval $I \subset \mathbb{R}$ is compact and $h(x_1, 0, \varepsilon_n')$ vanishes in I but not in any strictly larger interval. The first step in the proof is to derive estimates from Theorems 26.8.1 and 26.8.3.

Lemma 26.9.2. *Let $\psi \in C_0^\infty(\mathbb{R}^{2n-2})$ be equal to 1 in a neighborhood of 0, $0 \leqq \psi \leqq 1$ everywhere, and set*

$$\psi_{\delta,\lambda}(x',\xi') = \psi(x'/\delta, (\lambda\xi' - \varepsilon_n')/\delta); \quad \lambda, \delta > 0;$$

$I_\delta = \{t+t';\ t \in I, |t'| \leqq \delta\}$. Then there is a constant C such that

$$\|v\| \leqq C\|D_1 v + i(\psi^4_{\delta,\lambda} g h)(x,D')v\|, \tag{26.9.3}$$

$$\sum_{|\alpha+\beta|=1} \lambda^{|\beta|} \|(\psi^4_{\delta,\lambda} h^{(\alpha)}_{(\beta)} g)(x,D')v\| \leqq C\lambda^{\frac{1}{2}} \|D_1 v + i(\psi^4_{\delta,\lambda} g h)(x,D')v\|, \tag{26.9.4}$$

if $v \in C_0^\infty(I_\delta \times \mathbb{R}^{n-1})$, δ is sufficiently small and $0 < \lambda < \lambda_\delta$. The norms are L^2 norms.

Proof. After multiplying g and h by a cut off function which is equal to 1 in a conic neighborhood of $(0,\varepsilon_n')$ in $T^*(\mathbb{R}^{n-1})$ when $|\xi'| > \frac{1}{2}$ we may assume that $g \in S^1(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ and that $0 \leqq h \in S^0(\mathbb{R}^n \times \mathbb{R}^{n-1})$ are homogeneous of degree 1 and 0 for $|\xi'| > \frac{1}{2}$. We shall apply Theorems 26.8.1 and 26.8.3 to the self-adjoint operators in $L^2(\mathbb{R}^{n-1})$ defined by

$$A(x_1) = \psi_{\delta,\lambda}(x',D')^*(h(x,D') + h(x,D')^*)\psi_{\delta,\lambda}(x',D')/2 + C_0\lambda,$$
$$B = \psi_{\delta,\lambda}(x',D')^*(g(x',D') + g(x',D')^*)\psi_{\delta,\lambda}(x',D')/2,$$

where the constant C_0 will be determined so that $A(x_1)\geqq 0$. By Theorem 18.1.13 we have

$$(A(x_1)v,v)=C_0\lambda\|v\|^2+\operatorname{Re}(h(x,D')\psi_{\delta,\lambda}(x',D')v,\psi_{\delta,\lambda}(x',D')v)$$
$$\geqq C_0\lambda\|v\|^2-C\|(1+|D'|^2)^{-\frac14}\psi_{\delta,\lambda}(x',D')v\|^2.$$

If δ is small we have $\frac12<|\xi'\lambda|<2$ in the support of the symbol of $(1+|D'|^2)^{-\frac14}\psi_{\delta,\lambda}(x',D')$, the product of the symbol by $\lambda^{-\frac12}$ is uniformly bounded in S^0 for fixed δ, and the maximum has a bound independent of δ for small λ. Hence it follows from Theorem 18.1.15 that

$$A(x_1)\geqq C_0\lambda-\lambda(C_1+C_\delta\lambda^{\frac12})>0$$

for small λ if $C_1<C_0$. From Theorem 18.1.15 we also obtain

$$\|A(x_1)\|\leqq|\psi_{\delta,\lambda}^2 h|_\infty+C_\delta\lambda^{\frac12},$$
$$\|[B,A(x_1)]\|\leqq|\{\psi_{\delta,\lambda}^2 g,\psi_{\delta,\lambda}^2 h\}|_\infty+C_\delta\lambda^{\frac12}$$
$$\|[B,[B,A(x_1)]]\|\leqq|\{\psi_{\delta,\lambda}^2 g,\{\psi_{\delta,\lambda}^2 g,\psi_{\delta,\lambda}^2 h\}\}|_\infty+C_\delta\lambda^{\frac12},$$

for all symbols have support where $\frac12<|\xi'\lambda|<2$. The maximum norms on the right-hand side are independent of λ. Since g vanishes and h vanishes of second order on $I\times\{0\}\times\{\varepsilon_n'\}$ we have $D^\alpha(\psi_\delta,h/\delta^2,g/\delta)=O(\delta^{-|\alpha|})$ in $I_\delta\times\operatorname{supp}\psi_\delta$ so the maximum norms are $O(\delta^2)$, $O(\delta)$ and $O(1)$ respectively if $x_1\in I_\delta$. For $x_1\in I_\delta$ and $0<\lambda<\lambda_\delta$ we therefore have $\|B\|\leqq C/\lambda$ and

$$\|A(x_1)\|\leqq C\delta^2,\qquad \|[B,A(x_1)]\|\leqq C\delta,\qquad \|[B,[B,A(x_1)]]\|\leqq C.$$

When δ is small enough it follows from (26.8.3) and (26.8.8) that

$$\|v\|\leqq C\|D_1v+iABv\|,\quad v\in C_0^\infty(I_\delta\times\mathbb{R}^{n-1}),\tag{26.9.5}$$

$$\int(A(x_1)Bv,Bv)\,dx_1\leqq C/\lambda(\|D_1v+iABv\|^2+\|v\|^2),\quad v\in C_0^\infty(I_\delta\times\mathbb{R}^{n-1}).\tag{26.9.6}$$

The symbol of $A(x_1)B$ is

$$\psi_{\delta,\lambda}^4 hg+C_0\lambda\psi_{\delta,\lambda}^2 g+S_{\delta,\lambda}+R_{\delta,\lambda}$$

where $R_{\delta,\lambda}$ is bounded in S^{-1} for fixed δ and $S_{\delta,\lambda}$ is a finite linear combination of terms obtained from $\psi^4 gh$ by applying $\partial/\partial\xi_j$ and $\partial/\partial x_j$ to two factors, possibly the same. Thus $S_{\delta,\lambda}$ is bounded in S^0 for fixed δ and $|S_{\delta,\lambda}|_\infty\leqq C\delta$ with C independent of λ, by the arguments above. Hence Theorem 18.1.15 gives

$$\|A(x_1)B-(\psi_{\delta,\lambda}^4 gh)(x,D')\|\leqq C\delta,\qquad x_1\in I_\delta,$$

if λ is small enough. Hence (26.9.5) implies

$$\|v\|\leqq C\|D_1v+iABv\|\leqq C\|D_1v+i(\psi_{\delta,\lambda}^4 gh)(x,D')v\|+C'\delta\|v\|,$$
$$v\in C_0^\infty(I_\delta\times\mathbb{R}^{n-1}),$$

which gives (26.9.3) if δ is small enough and also shows that we may replace ABv by $(\psi_{\delta,\lambda}^4 gh)(x,D')v$ in the right-hand side of (26.9.6). To prove (26.9.4) we first observe that Lemma 7.7.2 implies

$$|h'_x|^2+(1+|\xi|^2)|h'_\xi|^2 \leqq Ch.$$

Hence it follows from Theorem 18.1.14 that

$$\sum \|h_{(j)}(x,D')v\|^2+\sum \|(1+|D'|^2)^{\frac{1}{2}} h^{(j)}(x,D')v\|^2 \leqq C\,\mathrm{Re}(h(x,D')v,v)+C_1\|v\|^2_{(-\frac{1}{2})}. \tag{26.9.7}$$

To combine (26.9.7) with (26.9.6) where the leading term in the symbol of BAB is $\psi^6_{\delta,\lambda}g^2h$ we replace v by $(\psi^3_{\delta,\lambda}g)(x',D')v$ in (26.9.7). The symbol of

$$(\psi^3_{\delta,\lambda}g)(x',D')^*\,h(x,D')(\psi^3_{\delta,\lambda}g)(x',D')-BA(x_1)B$$

is uniformly bounded in S^0 for fixed δ and as above it follows that the maximum of the symbol has a bound independent of δ when λ is small. Hence the norm has a bound independent of δ when $\lambda<\lambda_\delta$, again by Theorem 18.1.15. Combining (26.9.6) and (26.9.7) we now obtain

$$\lambda\sum \|h_{(j)}(x,D')(\psi^3_{\delta,\lambda}g)(x',D')v\|^2 \leqq C(\|D_1v+i(\psi^4_{\delta,\lambda}gh)(x,D')v\|^2+\|v\|^2), \qquad v\in C_0^\infty(I_\delta\times\mathbb{R}^{n-1}),$$

if δ is small and $\lambda<\lambda_\delta$. The symbol of

$$h_{(j)}(x,D')(\psi^3_{\delta,\lambda}g)(x',D')-(\psi^3_{\delta,\lambda}h_{(j)}g)(x',D')$$

is bounded in S^0 for fixed δ and the maximum has a bound independent of δ when $\lambda<\lambda_\delta$. Hence Theorem 18.1.13′ again gives a fixed bound for the norm, and (26.9.4) follows when we sum for $\alpha=0$, $|\beta|=1$. The estimate of the other terms follows in the same way by means of the second sum in (26.9.7), and this completes the proof, if we use (26.9.3) to estimate $\|v\|$.

In the proof of Theorem 26.9.1 we shall also need a description in terms of L^2 norms for the regularity function s^*_u defined in (18.1.41).

Lemma 26.9.3. *Let χ and ϕ be functions in $C_0^\infty(T^*(\mathbb{R}^n)\setminus 0)$ and set*

$$q_\lambda(x,\xi)=\chi(x,\lambda\xi)\lambda^{-\phi(x,\lambda\xi)}. \tag{26.9.8}$$

If $u\in\mathscr{E}'(\mathbb{R}^n)$ and $s^*_u>\mathrm{Re}\,\phi$ in $\operatorname{supp}\chi$, then $\|q_\lambda(x,D)u\|_{L^2}$ is bounded as $\lambda\to 0$. Conversely, if $\|q_\lambda(x,D)u\|_{L^2}$ is bounded as $\lambda\to 0$, then

$$s^*_u(x,\xi)\geqq \mathrm{Re}\,\phi(x,\xi) \quad \text{if } \chi(x,\xi)\neq 0.$$

Proof. First note that if $\mathrm{Re}\,\phi<\mu$ in $\operatorname{supp}\chi$, then q_λ is bounded in S^μ as $\lambda\to 0$, that is

$$|D^\alpha_\xi D^\beta_x q_\lambda(x,\xi)|\leqq C_{\alpha\beta}(1+|\xi|)^{\mu-|\alpha|}, \qquad \lambda<1. \tag{26.9.9}$$

In fact, $|\lambda\xi|$ lies between two fixed positive bounds when $(x,\lambda\xi)\in\operatorname{supp}\chi$, and we have

$$\partial q_\lambda/\partial x=((\partial\chi/\partial x-\chi\log\lambda\,\partial\phi/\partial x)\lambda^{-\phi})(x,\lambda\xi)$$

and a similar formula with another factor λ for $\partial q_\lambda/\partial\xi$. This proves (26.9.9), and it follows that $\|q_\lambda(x,D)u\|_{L^2}$ is bounded as $\lambda\to 0$ if $s_u^*>\mu>\operatorname{Re}\phi$ in $\operatorname{supp}\chi$. If we just have $s_u^*>\operatorname{Re}\phi$ in $\operatorname{supp}\chi$, we note that since s_u^* is semi-continuous from below we can write $\chi=\sum\chi_j$ where $\operatorname{Re}\phi<\mu_j<s_u^*$ in $\operatorname{supp}\chi_j$ for some μ_j. This proves the first part of the lemma.

It remains to prove the second part of the lemma. Let $\chi(y,\eta)\neq 0$ and choose for given $\mu<\operatorname{Re}\phi(y,\eta)$ a non-negative function $\chi_0\in C_0^\infty$ such that $\operatorname{Re}\phi>\mu$ and $\chi\neq 0$ in $\operatorname{supp}\chi_0$, $\chi_0(y,\eta)>0$. Let $u\in H_{(-M)}^{\mathrm{comp}}$. One can find $a_\lambda(x,\xi)$ bounded in S^0 and R_λ bounded in S^{-M} so that

$$\chi_0(x,\lambda\xi)\lambda^{-\mu}=a_\lambda(x,D)\,q_\lambda(x,D)+R_\lambda(x,D). \tag{26.9.10}$$

As a first approximation to a_λ we take

$$a_\lambda^0(x,\xi)=\chi_0(x,\lambda\xi)/\chi(x,\lambda\xi)\lambda^{-\mu+\phi(x,\lambda\xi)}$$

which gives an error of the desired form apart from a finite number of terms in the series

$$-\sum_{\alpha\neq 0}(iD_\xi)^\alpha a_\lambda^0(x,\xi)\,D_x^\alpha q_\lambda(x,\xi)/\alpha!.$$

These can again be handled in the same way, and after a finite number of iterations we obtain the desired function a_λ. It follows from (26.9.10) that

$$\|\chi_0(x,\lambda D)\lambda^{-\mu}u\|_{L^2}\leqq C,\qquad \lambda<1.$$

If we multiply by $\lambda^{\varepsilon-1}$ where $\varepsilon>0$ and integrate from 0 to 1, we obtain $\|r(x,D)u\|_{L^2}<\infty$ where

$$r(x,\xi)=\int_0^1\chi_0(x,\lambda\xi)\lambda^{\varepsilon-\mu-1}\,d\lambda.$$

When $|\xi|$ is large enough the upper bound in the integration may be replaced by ∞ which shows that $r(x,\xi)$ is homogeneous of degree $\mu-\varepsilon$ at ∞. It is clear that r is positive in the direction (y,η). Hence $s_u^*(y,\eta)\geqq\mu-\varepsilon$ which completes the proof.

Proof of Theorem 26.9.1. As already pointed out we may assume that $P\in\Psi_{\mathrm{phg}}^1(\mathbb{R}^n)$, that the principal symbol p satisfies (26.9.2) in a conic neighborhood of the inverse image $I'=I\times\{0\}\times\{\varepsilon_n\}\subset T^*(\mathbb{R}^n)\setminus 0$ of a point in $\tilde{B}_0$, and that B is the leaf containing $(0,\varepsilon_n)$ generated by the vector fields $\partial/\partial x_1$ and H_g. We may also assume that the term p_0 of order 0 in the symbol of p vanishes in I'. In fact, the equation $Pu=f$ implies $SPS^{-1}(Su)-Sf\in C^\infty$ if $S\in\Psi_{\mathrm{phg}}^0$ is elliptic with parametrix S^{-1}. We have $s_u^*=s_{Su}^*$, $s_f^*=s_{Sf}^*$, and the term of order 0 in the symbol of SPS^{-1} vanishes in I' if for the principal symbol S_0 of S we have

$$S_0D_1S_0^{-1}+p_0=0,\qquad \text{that is, } i\,\partial S_0/\partial x_1+S_0p_0=0 \text{ in } I'.$$

This ordinary differential equation is easy to solve, and S_0 can be extended to an elliptic symbol of order 0. Replacing P by SPS^{-1} we can thus assume that p_0 vanishes in I'. In view of Theorem 18.1.15 we can then choose $R \in S^0$ so that $p_0 - R$ is of order $-\infty$ in a conic neighborhood V of I' in $T^*(\mathbb{R}^n) \setminus 0$ and $C\|R(x,D)\| < \frac{1}{2}$ where C is the constant in Lemma 26.9.2.

We can find a neighborhood $V_0 \subset V$ of I' and functions $w, w_0 \in C^\infty(V_0)$ such that

$$(\partial/\partial x_1 + ihH_g)w = (\partial/\partial x_1 + ihH_g)w_0 = 0 \quad \text{and} \quad H_g w \neq 0 \quad \text{in } V_0, \tag{26.9.11}$$

$$C_1 d_B(x,\xi)^2 \leqq \operatorname{Re} w_0(x,\xi) \leqq C_2 d_B(x,\xi)^2, \quad (x,\xi) \in V_0, \tag{26.9.12}$$

where d_B is the distance to B. Note that (26.9.11) implies $H_p w = H_p w_0 = 0$ when $g=0$. When verifying this we may assume that $g = \xi_2$, for this can be achieved by a possibly non-homogeneous canonical transformation. Then we have the equations

$$(\partial/\partial x_1 + ih\,\partial/\partial x_2)w = (\partial/\partial x_1 + ih\,\partial/\partial x_2)w_0 = 0.$$

The existence of w is therefore an immediate consequence of Corollary 26.7.7. Recalling that w must be constant on I' we may assume that $w=0$ on I' and can then take

$$w_0(x,\xi) = (x_3^2 + \ldots + x_n^2 + \xi_1^2 + \ldots + \xi_{n-1}^2 + (\xi_n - 1)^2)\exp w(x,\xi).$$

Now we return to the original coordinates. Let $\tilde{K}$ be a compact subset of ω such that the inverse image K in B is contained in V_0, $x_1 \in I_{\delta/2}$ in K and the function $\psi_{\delta,1}$ in Lemma 26.9.2 is equal to 1 in a neighborhood of K for some δ such that (26.9.3), (26.9.4) are valid. We fix δ. Let H be a harmonic polynomial in $\mathbb{C}$ such that, with s denoting the lifting of $\tilde{s}$ to B,

$$H(w) < \min(s_u^*, s) \qquad \text{on the boundary of } K \text{ in } B_0.$$

The theorem will be proved if we show that the same inequality is then valid in K, for the restriction of w to B is the lifting to B of a local analytic coordinate in $\tilde{B}_0$. Since $H(w)$ is harmonic in $\tilde{B}_0$ and $\tilde{s}$ is superharmonic, we have $H(w) < s$ in K. Furthermore, the boundary ∂K of K in B is in the inverse image of the boundary of $\tilde{K}$ in $\tilde{B}_0$, for the inverse image of the interior is open. Hence we have if $f = Pu$

$$H(w) < s_u^* \quad \text{on } \partial K, \tag{26.9.13}$$

$$H(w) < s_f^* \quad \text{in } K. \tag{26.9.14}$$

We may also assume in the proof that $u \in \mathscr{E}'$ and that

$$H(w) - 1 < s_u^* \quad \text{in } K, \tag{26.9.15}$$

for if the assertion is proved under that additional hypothesis we can just start from the fact that $s_u^* > H(w) - k$ for some positive integer k and decrease k successively until $k=0$ and the theorem is proved.

Choose $\chi \in C_0^\infty(V_0)$ equal to 1 in a neighborhood of K but with $\operatorname{supp}\chi$ so close to K that $x_1 \in I_\delta$ and $\psi_{\delta,1}=1$ in $\operatorname{supp}\chi$ and (26.9.13)–(26.9.15) imply

$$s_u^* > H(w) \quad \text{in } B \cap \operatorname{supp} d\chi;$$
$$s_f^* > H(w) \quad \text{and} \quad s_u^* > H(w)-1 \quad \text{in } B \cap \operatorname{supp}\chi.$$

We can write $H = \operatorname{Re} F$ where F is an analytic polynomial in $\mathbb{C}$. In view of (26.9.12) we can choose a constant τ so large that if

$$\phi = F(w) - \tau w_0$$

we even have

$$s_u^* > \operatorname{Re}\phi \quad \text{in } \operatorname{supp} d\chi \tag{26.9.16}$$
$$s_f^* > \operatorname{Re}\phi \quad \text{in } \operatorname{supp}\chi \tag{26.9.17}$$
$$s_u^* > \operatorname{Re}\phi - 1 \quad \text{in } \operatorname{supp}\chi. \tag{26.9.18}$$

Note that $\operatorname{Re}\phi = H(w)$ in K and that $(\partial/\partial x_1 + ihH_g)\phi = 0$.

From (26.9.3) and the fact that $C\|R\| < \frac{1}{2}$ it follows that

$$\|v\| \leqq 2C\,\|D_1 v + i(\psi_{\delta,\lambda}^4 gh)(x,D')v + R(x,D)v\| \tag{26.9.3$'$}$$

if $v \in \mathscr{S}$ and $x_1 \in I_\delta$ in $\operatorname{supp} v$. We shall apply (26.9.3)$'$ to

$$v = q_\lambda(x,D)u, \qquad q_\lambda(x,\xi) = \chi(x,\lambda\xi)\lambda^{-\phi(x,\lambda\xi)}.$$

To estimate

$$M = \|(D_1 + i(\psi_{\delta,\lambda}^4 gh)(x,D') + R(x,D))\, q_\lambda(x,D)u\|$$

we want to commute $q_\lambda(x,D)$ through the operator in front. Choose μ so that $u \in H_{(-\mu)}$. Since

$$\partial q_\lambda(x,\xi)/\partial x = (\chi_x'(x,\lambda\xi) - \log\lambda\,\phi_x'(x,\lambda\xi))\lambda^{-\phi(x,\lambda\xi)}$$

and similarly for $\partial q_\lambda/\partial\xi$, the symbol of $[R(x,D),\ q_\lambda(x,D)]$ is, apart from an error which is bounded in $S^{-\mu}$, a finite sum of functions of the same form as q_λ but with ϕ replaced by $\phi - j$ where j is a positive integer, and multiplied by a power of $\log\lambda$. Hence it follows from (26.9.18) and Lemma 26.9.3 that

$$\|[R(x,D), q_\lambda(x,D)]u\| \leqq C_u, \qquad 0<\lambda<1.$$

The symbol of

$$[D_1 + i(\psi_{\delta,\lambda}^4 gh)(x,D'),\ q_\lambda(x,D)]$$

is similar apart from the first term in the symbol which is

$$-i(\partial/\partial x_1 + igH_h + ihH_g)q_\lambda.$$

(Here we have used that $\psi_{\delta,\lambda}=1$ in $\operatorname{supp} q_\lambda$.) In view of the differential equation $(\partial/\partial x_1 + ihH_g)\phi = 0$ we obtain

$$-i(\partial/\partial x_1 + igH_h + ihH_g)q_\lambda = \tilde{q}_\lambda(x,\xi) + i\log\lambda\, g\{h,\phi\}(x,\lambda\xi)q_\lambda. \tag{26.9.19}$$

Here

$$\tilde{q}_\lambda(x,\xi) = -i\tilde{\chi}(x,\lambda\xi)\lambda^{-\phi(x,\lambda\xi)}, \qquad \tilde{\chi} = H_p\chi,$$

is of the same form as q_λ but with $\operatorname{supp}\tilde{\chi}\subset\operatorname{supp}d\chi$. Hence it follows from (26.9.16) and Lemma 26.9.3 that $\|\tilde{q}_\lambda(x,D)u\|$ is bounded when $\lambda\to 0$. The main term in the symbol of

$$(26.9.20)\quad i\log\lambda\big(\sum\phi_{(j)}(x,\lambda D)(\psi^3_{\delta,\lambda}h^{(j)}g)(x,D')\,q_\lambda(x,D)\\ -\lambda\sum\phi^{(j)}(x,\lambda D)(\psi^3_{\delta,\lambda}h_{(j)}g)(x,D')\,q_\lambda(x,D)\big)$$

is equal to the last term in (26.9.19), and the others are of the same form but smaller by at least a factor $\lambda\log\lambda$. Estimating (26.9.20) by means of (26.9.4) we obtain in view of (26.9.16)

$$M\leqq\|q_\lambda(x,D)(D_1+i(\psi^4_{\delta,\lambda}gh)(x,D')+R(x,D))u\|+C\lambda^{\frac{1}{2}}|\log\lambda|\,M+C_u.$$

When λ is so small that $C\lambda^{\frac{1}{2}}|\log\lambda|<\frac{1}{2}$ we can cancel the middle term on the right-hand side against half the left hand side. The symbol of

$$q_\lambda(x,D)(D_1+i(\psi^4_{\delta,\lambda}gh)(x,D')+R(x,D)-P)$$

is bounded in $S^{-\infty}$ since $\psi_{\delta,\lambda}=1$ in $\operatorname{supp}q_\lambda$ and the complete symbol of P is $\xi_1+igh(x,\xi')+R(x,\xi)$ in V. Hence we obtain for small λ

$$M/2\leqq\|q_\lambda(x,D)Pu\|+C_u.$$

By (26.9.17) the right-hand side is bounded when $\lambda\to 0$, so using (26.9.3)$'$ with $v=q_\lambda(x,D)u$ we conclude that $\|q_\lambda(x,D)u\|$ is bounded as $\lambda\to 0$. By Lemma 26.9.3 it follows that $s^*_u\geqq\phi$ in K. In view of Theorem 26.7.11 the proof is now complete.

By the method of descent from an operator in a higher dimensional space we can derive from Theorem 26.9.1 a similar result on the singularities in N^{ie}_{12} (see the summary at the end of Section 26.5).

Theorem 26.9.4. *Let $P\in\Psi^m_{\mathrm{phg}}(X)$ be properly supported and satisfy condition (P), let I be a compact interval on a one dimensional bicharacteristic with end points in N^i_{12}, and let $\tilde{I}$ be the affine interval obtained by Proposition 26.5.8 when subintervals not meeting N^i_{12} are collapsed to points. If $u\in\mathscr{D}'(X)$, s is a concave function on $\tilde{I}$ and $\tilde{s}_{Pu}\geqq s$ on I, it follows that*

$$\min(\tilde{s}_u,s+m-1)$$

is a concave function on $\tilde{I}$.

Here the definition of $\tilde{s}_u$ and $\tilde{s}_{Pu}$ on $\tilde{I}$ is completely analogous to the definitions used in Theorem 26.9.1.

Proof. We may assume that $m=1$ and that the principal symbol p has the form ξ_1+igh in Propositions 26.4.13 and 26.5.7, with $h>0$ at the end points of I. Let $H(x,\xi)$ be a homogeneous function of degree 0 which is equal to $h(x,\xi')$ in a neighborhood of I. If we regard u and f as distributions U and

F in $\mathbb{R}^{n+1}$ independent of x_{n+1}, then

$$(P+iH(x,D)D_{n+1})U=F. \tag{26.9.21}$$

Strictly speaking the operator in (26.9.21) is not a pseudo-differential operator but it becomes one if we multiply by an operator with symbol

$$\chi(\xi_{n+1}/|\xi|) \quad \text{where } \chi \in C_0^\infty(\mathbb{R}), \quad \chi(0)=1 \quad \text{and} \quad |\xi|^2=\xi_1^2+\ldots+\xi_{n+1}^2.$$

Since $D_{n+1}U=0$ it is clear that $\xi_{n+1}=0$ in $WF(U)$. By Theorem 8.2.9

$$WF(U)=\{(x,x_{n+1},\xi,0);\ (x,\xi)\in WF(u)\},$$

and the same obvious proof gives the more precise statement

$$s_u^*(x,\xi)=s_U^*(x,x_{n+1},\xi,0).$$

Now the principal symbol of the operator in (26.9.21) is

$$\xi_1+ih(x,\xi')(\xi_{n+1}+g(x',\xi')).$$

Near $I\times(\mathbb{R}\times\{0\})$ it satisfies condition (P) and defines a two dimensional bicharacteristic which is the product of the x_1 axis, the x_{n+1} axis and $\varepsilon_n=(0,\ldots,0,1,0)$, for $H_g=0$ on I. The Hamilton field is

$$\partial/\partial x_1+ih(x_1,0,\varepsilon_n')\partial/\partial x_{n+1},$$

so introducing $\int h(x_1,0,\varepsilon_n')\,dx_1$ as a variable in $\tilde{I}$ we obtain the standard Cauchy-Riemann operator in $\tilde{I}\times\mathbb{R}$. A superharmonic function independent of x_{n+1} is then the same as a concave function on $\tilde{I}$, so Theorem 26.9.4 follows from Theorem 26.9.1.

The preceding argument does not fully use the hypothesis that the end points of I are in N_{12}^i; the conclusion is always valid when a factorization is available with $h\geqq 0$, $h>0$ at the end points of I, and g independent of x_1. In particular this is always true by Weierstrass' preparation theorem in the analytic case if the end points are just in N_{12}. However, it seems necessary to weaken the notion of concavity in general. We shall now discuss two such weaker concepts here, where the weakest will be useful in Section 26.10.

Definition 26.9.5. A function s defined on an interval $I\subset\mathbb{R}$ with values in $(-\infty,+\infty]$ will be called semi-concave if it is semi-continuous from below and for every compact interval $J\subset I$ and linear decreasing function L with $s\geqq L$ on ∂J we have $s\geqq L$ in J. We shall say that s is quasi-concave if this is true for all constants L.

Semi-concavity is well defined on a semi-bicharacteristic I with end points in $N_{12}\setminus N_{12}^i$. In fact, putting the principal symbol in the standard form $\xi_1+ig(x',\xi')h(x,\xi')+r(x,\xi')$ with $r(x,\xi')$ vanishing of infinite order on I, Hess $g\leqq 0$ and $h\geqq 0$, we obtain a natural orientation from the orientation of the x_1 axis; the form $h(x_1,0,\varepsilon_n')\,dx_1$ defines a natural oriented affine structure in $\tilde{I}$.

Semi-concavity is clearly invariant under linear increasing changes of variable whereas quasi-concavity is invariant under strictly monotonic changes of variable. Thus an affine structure and an orientation are required for the definition of semi-concavity. The meaning of the conditions is further clarified in the following

Proposition 26.9.6. *s is semi-concave in $I \subset \mathbb{R}$ if and only if either*

(i) *s is increasing and continuous to the left, or*

(ii) *s is decreasing and concave, or*

(iii) *there is a point $a \in I$ such that s satisfies* (i) *to the left of a and* (ii) *to the right of a, and $s(a) = s(a-0) \leqq s(a+0)$.*

s is quasi-concave if and only if either (i) *is valid or*

(ii)′ *s is decreasing and continuous to the right, or*

(iii)′ *there is a point $a \in I$ such that s satisfies* (i) *to the left of a and* (ii)′ *to the right of a, and $s(a) = \min(s(a+0), s(a-0))$.*

Proof. Assume first that s is quasi-concave. If s is monotonic we must of course have (i) or (ii)′. Otherwise s is not decreasing so we can choose $t_1 < t_2$ with $s(t_1) < s(t_2)$. Then $s(t) \leqq s(t_1)$ for $t < t_1$ since $s(t) > s(t_1) < s(t_2)$ would contradict the definition. Moreover, if $t' \leqq t \leqq t_1$ we have $s(t') \leqq s(t)$ for otherwise $s(t') > s(t) \leqq s(t_1) < s(t_2)$ which is also a contradiction. If a is the supremum of all $t_1 \in I$ with $s(t_1) < s(t_2)$ for some $t_2 > t_1$ in I, it follows that s is increasing to the left of a and decreasing to the right of a. The lower semi-continuity gives

$$s(a) \leqq \min(s(a+0), s(a-0)).$$

Inequality here would imply that $s(a) < s(t_1)$ and $s(a) < s(t_2)$ for suitable $t_1 < a$ and $t_2 > a$, which is also impossible. This proves that (iii)′ is valid. If s is even semi-concave it is obvious that s is concave to the right of a since the linear interpolation between two values there is decreasing. If $s(a-0) > s(a+0)$ we have for $a < t \in I$ and small $\varepsilon > 0$

$$s(a) \geqq (\varepsilon s(t) + (t-a)\, s(a-\varepsilon))/(t-a+\varepsilon) \to s(a-0), \qquad \varepsilon \to 0,$$

so $s(a) = s(a-0)$, which is also obvious from (iii)′ if $s(a-0) \leqq s(a+0)$. This proves the necessity of (i), (ii) or (iii) in the semi-concave case. Conversely, assume that s satisfies (iii) and let L be a linear decreasing function, J a compact interval $\subset I$ with $L \leqq s$ at ∂J. Then $s - L$ is increasing to the left of a so $L(t) \leqq s(t)$ if $a \geqq t \in J$. If $a \in J$ we obtain $s(a+0) \geqq L(a)$ by (iii). The concavity now gives $s \geqq L$ if $a < t \in I$ also. The sufficiency of the conditions in Proposition 26.9.6 is trivial in all other cases.

We can now give an exact analogue of Theorem 26.9.4:

Theorem 26.9.7. *Let $P \in \Psi^m_{\mathrm{phg}}(X)$ be properly supported and satisfy condition (P), let I be a compact interval on a one dimensional bicharacteristic with end points in $N_{12} \setminus N^i_{12}$, and let $\tilde{I}$ be the affine interval obtained by Proposition*

26.5.8 when subintervals not meeting N_{12} are identified to points, with the orientation just defined. If $u\in\mathscr{D}'(X)$, s is a semi-concave function on $\tilde{I}$ and $\tilde{s}_{Pu}\geqq s$ on $\tilde{I}$, it follows that

$$\min(\tilde{s}_u, s+m-1)$$

is a semi-concave function on $\tilde{I}$. Here $\tilde{s}_u$ and $\tilde{s}_{Pu}$ are defined as in Theorem 26.9.1.

Proof. We may assume that $P\in\Psi^1_{\mathrm{phg}}(\mathbb{R}^n)$, that $x'=0$, $\xi=\varepsilon_n$ on I and that

$$p(x,\xi)=\xi_1+if(x,\xi'),\qquad f(x,\xi')=g(x',\xi')h(x,\xi')+r(x,\xi') \tag{26.9.22}$$

in a neighborhood of I. Here Hess $g\leqq 0$ at $(0,\varepsilon_n')$, $h\geqq 0$, and $r=0$ when $g>0$, by Proposition 26.5.7. Since g is strictly concave in some variable the set where $g\leqq 0$ is the closure of the set where $g<0$ so $g(x',\xi')\leqq 0$ implies $f(x,\xi')\leqq 0$ for all x_1 by condition (P), for $g(x',\xi')=f(x,\xi')$ for a certain x_1. Let L be a linear decreasing function of $\int h(x_1,0,\varepsilon_n')\,dx_1$ and let J be a compact subinterval of I with end points in N_{12} such that

$$L<\min(s_u^*, s)\quad \text{at } \partial J. \tag{26.9.23}$$

(Here s denotes the function s lifted from $\tilde{I}$ to I.) The theorem will be proved if we show that the same estimate is valid in J. Since s is semi-concave we have $L<s$ in J, hence

$$L<s_{Pu}^*\quad \text{in } J. \tag{26.9.24}$$

If $g\leqq 0$ then $f\leqq 0$ in a neighborhood of I, so we can apply Proposition 26.6.1. Let γ_0 be the first point in J and γ an arbitrary point in J. Then $s_{Pu}^*\geqq L(\gamma)$ on $[\gamma_0,\gamma]\subset J$ and $s_u^*(\gamma_0)>L(\gamma_0)\geqq L(\gamma)$ so Proposition 26.6.1 gives $s_u^*(\gamma)>L(\gamma)$ which proves the statement.

We now allow g to change sign. To prove the theorem we shall then first use Proposition 26.6.1′ to get hold of u when $g<0$. We may then assume that we already know that

$$L-\tfrac{1}{8}<s_u^*\quad \text{in } J. \tag{26.9.25}$$

Let $0<\varepsilon\leqq\frac14$ and choose χ, $\chi_1\in S^0_{1-\varepsilon,\varepsilon}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ non-negative so that $\chi_1=1$ in supp χ, $\chi_1=0$ (resp. $\chi=1$) at all points in a conic neighborhood of $(0,\varepsilon_n')$ with distance <1 (resp. >2) to $\{(x',\xi');\ g(x',\xi')>0\}$ with respect to the metric

$$(1+|\xi'|^2)^\varepsilon(|dx'|^2+(1+|\xi'|^2)^{-1}|d\xi'|^2). \tag{26.9.26}$$

This is possible by Corollary 1.4.11 (or by using the closely related partitions of unity in Section 18.4). Choose ψ, $\psi_1\in S^0(\mathbb{R}^{n-1}\times\mathbb{R}^n)$ non-negative so that $\psi_1=1$ in supp ψ and $\psi=1$ at infinity in a conic neighborhood of $(0,\varepsilon_n)$. If we set

$$C^s(x',\xi)=(1+|\xi|^2)^{s/2}\psi(x',\xi)\chi(x',\xi'),$$
$$C_1(x',\xi)=\psi_1(x',\xi)\chi_1(x',\xi')$$

the hypotheses of Proposition 26.6.1′ are fulfilled in $[\gamma_0, \gamma]$ if $\operatorname{supp}\psi$ is contained in a sufficiently small conic neighborhood of $(0, \varepsilon_n)$ and $s = L(\gamma)$ for some $\gamma \in J$. (Note that $(3\varepsilon - 1)/2 \leqq -\frac{1}{8}$ so the first part of (26.6.9) follows from (26.9.25) in $[\gamma_0, \gamma]$.) Hence

$$C^s(x', D)u \in H_{(0)} \quad \text{at } \gamma \text{ if } \gamma \in J \quad \text{and} \quad s = L(\gamma).$$

Since $C^s(x', D) - (1 + |D|^2)^{s/2} C^0(x', D)$ is of order $s + \varepsilon - 1$ and $\varepsilon - 1 < -\frac{1}{8}$, it follows in view of (26.9.25) that $(1 + |D|^2)^{s/2} C^0(x', D)u \in H_{(0)}$ at γ, if $s = L(\gamma)$, that is,

$$C^0(x', D)u \in H_{(L)} \quad \text{at } J. \tag{26.9.27}$$

Now let $v = u - C^0(x', D)u$. From (26.9.23) and (26.9.27) it follows that

$$v \in H_{(L(\gamma))} \quad \text{if } \gamma \in \partial J.$$

We have

$$Pv = Pu - C^0(x', D) Pu - [P, C^0(x', D)]u.$$

The symbol of $[P, C^0(x', D)]$ is in $S^{2\varepsilon - 1}_{1-\varepsilon, \varepsilon}$ apart from the term

$$\{f, C^0\} = (\{f, C^0\}/C_1) C_1$$

which is in $S^\varepsilon_{1-\varepsilon}$. Thus we have

$$[P, C^0(x', D)] = Q_1(x, D)\, C_1(x, D) + Q_2(x, D)$$

where $Q_1 \in S^\varepsilon_{1-\varepsilon, \varepsilon}$ and $Q_2 \in S^{3\varepsilon - 1}_{1-\varepsilon, \varepsilon}$. The argument which led to (26.9.27) also gives

$$C_1(x', D)u \in H_{(L)} \quad \text{at } J,$$

so it follows that

$$Pv \in H_{(L-\varepsilon)} \quad \text{at } J.$$

Since $r = 0$ when $g > 0$ we can estimate $r(x, \xi')/|\xi'|$ by any power of the distance to this set in the norm (26.9.26) with $\varepsilon = 0$. In a conic neighborhood of I we therefore have $r(x, \xi') = O(|\xi|^{1 - N\varepsilon})$ for any N in supp $(1 - C^0)$, and all derivatives of r are also rapidly decreasing there. Hence the symbol of $r(x, D')(1 - C^0(x', D))$ is of order $-\infty$ in a conic neighborhood of J. If P_1 is obtained from P by changing the principal symbol to $\xi_1 + igh$ in a conic neighborhood of J, it follows that $P_1 v \in H_{(L-\varepsilon)}$ at J. The proof of Theorem 26.9.4 works for the operator P_1 since an exact factorization is available, so we may conclude that $v \in H_{(L-\varepsilon)}$ at J since this is true at ∂J. Hence

$$u \in H_{(L-\varepsilon)} \quad \text{at } J$$

for every $\varepsilon > 0$ and every L satisfying (26.9.23), which completes the proof.

26.10. The Singularities on One Dimensional Bicharacteristics

On a general one dimensional bicharacteristic we have no affine structure which permits us to define (semi-)concavity. However, quasi-concavity is meaningful (see Definition 26.9.5), and we shall prove

Theorem 26.10.1. *Let $P\in\Psi^m_{\mathrm{phg}}(X)$ be properly supported and satisfy condition (P), and let I be a one dimensional bicharacteristic interval. If $u\in\mathscr{D}'(X)$, s is a quasi-concave function on I and $s^*_{Pu}\geqq s$ on I, then*

$$\min(s^*_u, s+m-1)$$

is a quasi-concave function on I.

We have here chosen a statement which is analogous to Theorems 26.9.1, 26.9.4 and 26.9.7. However, it is useful to rephrase the result. Explicitly it means that if J is a compact interval $\subset I$ and $\min(s^*_u, s+m-1)\geqq t+m-1\in\mathbb{R}$ at ∂J, then this is also true in J. Since s is quasi-concave we have $s^*_{Pu}\geqq s\geqq t$ in J, so the assertion is that $s^*_u\geqq t+m-1$ in J. Theorem 26.10.1 is therefore equivalent to the following apparently weaker statement:

Theorem 26.10.1′. *Let $P\in\Psi^m_{\mathrm{phg}}(X)$ be properly supported and satisfy condition (P), and let I be a compact one dimensional bicharacteristic interval. If $u\in\mathscr{D}'(X)$, $s\in\mathbb{R}$ and $s^*_{Pu}\geqq s$ on I, $s^*_u\geqq s+m-1$ at ∂I, then $s^*_u\geqq s+m-1$ in I.*

In the proof we may by Proposition 26.4.13 assume that $P\in\Psi^1_{\mathrm{phg}}(\mathbb{R}^n)$, that $x'=(x_2,\dots,x_n)=0$ and $\xi=(0,\dots,0,1)=\varepsilon_n$ on I, and that

$$p(x,\xi)=\xi_1+if(x,\xi') \tag{26.10.1}$$

in a conic neighborhood of I, where f does not change sign for fixed (x',ξ'). Theorem 26.10.1′ is a consequence of Theorem 26.6.4 if $I\subset N_{11}$. If $I\subset N^e_{12}$ then Theorem 26.10.1′ follows from Theorems 26.9.4 and 26.9.7. In fact, if Γ is the one dimensional bicharacteristic containing I and $\varepsilon>0$ we can then choose $\phi\in S^0$ equal to 1 at infinity in a conic neighborhood of I and equal to 0 outside another neighborhood which is so small that

$$P\phi(x,D)u=\phi(x,D)Pu+[P,\phi(x,D)]u\in H_{(s-\varepsilon)}\quad \text{in } \Gamma,$$

and $\phi(x,D)u\in H_{(s-\varepsilon+m-1)}$ at every point in $\Gamma\setminus I$. If $\Gamma\setminus I$ contains points in N_{12} on both sides of I it follows from Theorem 26.9.4 or Theorem 26.9.7 that $\phi(x,D)u\in H_{(s-\varepsilon+m-1)}$ in I, hence that $u\in H_{(s-\varepsilon+m-1)}$ in I. Otherwise we can modify the definition of p outside $WF(\phi(x,D))$ so that such points occur, for example by changing the principal symbol to $p(y_1(x_1),x',\xi)$ where $y_1(x_1)=x_1$ in a neighborhood of I but $(y_1(x_1),0,\varepsilon_n)\in N_{12}$ for large $|x_1|$. This does not affect condition (P) or the condition that $P\phi(x,D)u\in H_{(s-\varepsilon)}$ on Γ.

If $I\subset N^e_2$ then f vanishes of third order on I since $f=gh$ and $g=h=0$ on I, $h\geqq 0$ in a neighborhood. In the proof of Theorem 26.10.1′ we may therefore always assume that

(26.10.2) $\qquad f$ vanishes of third order in I.

As in the proof of Theorem 26.4.7′ we may also assume that the term of order 0 in the symbol of P is of the form $r(x,\xi')$, and the argument at the beginning of the proof of Theorem 26.9.1 shows that by conjugation with an elliptic operator we can achieve that

(26.10.3) $$r=0 \quad \text{in } I.$$

As in Lemma 26.9.2 we shall cut f and r off outside a small neighborhood of $x'=0$, $\xi'=\varepsilon_n'/\lambda$ by introducing

(26.10.4)
$$\begin{aligned} f_{\delta,\lambda}(x,\xi') &= \psi(x'/\delta,(\lambda\xi'-\varepsilon_n')/\delta)\, f(x,\xi'),\\ r_{\delta,\lambda}(x,\xi') &= \psi(x'/\delta,(\lambda\xi'-\varepsilon_n')/\delta)\, r(x,\xi') \end{aligned}$$

where $\psi\in C_0^\infty$. Let $I=J\times\{0\}\times\{\varepsilon_n\}$ and $J_\delta=\{t+t';\ t\in J,\ |t'|\leqq\delta\}$. For $\lambda>0$ and small $\delta>0$ we have if $x_1\in J_\delta$

(26.10.5) $$|D_{\xi'}^\alpha D_{x'}^\beta f_{\delta,\lambda}(x,\xi')|\,\lambda/\delta^2+|D_{\xi'}^\alpha D_{x'}^\beta r_{\delta,\lambda}(x,\xi')|\leqq C_{\alpha\beta}\,\delta^{1-|\alpha|-|\beta|}\lambda^{|\alpha|}.$$

For reasons of homogeneity it suffices to prove (26.10.5) when $\lambda=1$ and then it follows from the fact that by Taylor's formula

$$D_{\xi'}^\alpha D_{x'}^\beta f(x,\xi')=O(\delta^{3-|\alpha|-|\beta|}),\qquad D_{\xi'}^\alpha D_{x'}^\beta r(x,\xi')=O(\delta^{1-|\alpha|-|\beta|})$$

when $|x'|^2+|\xi'-\varepsilon_n'|^2<C\delta^2$ and $x_1\in J_\delta$.

To simplify notation we shall write F and R instead of $f_{\delta,\lambda}$ and $r_{\delta,\lambda}$, taking $x'/\sqrt{\lambda}$, $\sqrt{\lambda}\xi'$ as variables instead of x', ξ'. Then the right-hand side of (26.10.5) becomes $C_{\alpha\beta}\delta\, a^{|\alpha|+|\beta|}$ where $a^2=\lambda/\delta^2$. (Note that the change of variables is the symplectic change of variables in $T^*(\mathbb{R}^{n-1})$ induced by the change of x' variables.) Dropping the primes and writing t instead of x_1 we have to study the operator

$$D_t+iF(t,x,D)+R(t,x,D)$$

where all x, ξ derivatives of F and R are continuous in t, x, ξ and

(i) $\qquad F$ is real valued and $F(t,x,\xi)\,F(s,x,\xi)\geqq 0$,

(ii) $$a^2|F_{x,\xi}^{(j)}(t,x,\xi)|+|R_{x,\xi}^{(j)}(t,x,\xi)|\leqq\delta\, C_j a^j,\qquad j=0,1,2,\dots.$$

Actually F and R are only defined in an interval, say $|t|\leqq T$, but since no differentiability with respect to t is assumed we can extend them to all $t\in\mathbb{R}$ so that they are independent of t when $t\geqq T$ or $t\leqq -T$. We can also introduce $t'=\delta t$ as a new variable instead of t. This gives the operator

$$\delta(D_{t'}+i\delta^{-1}F(t'/\delta,x,D)+\delta^{-1}R(t'/\delta,x,D)).$$

The interval $|t|<T$ has now become the interval $|t'|<\delta T$, and the operators involve functions satisfying (ii) with $\delta=1$. This is finally the situation in which we are going to work. It is no restriction to assume that $C_j=1$ for $j\leqq 2$. In the first lemma we write $X=(x,\xi)$.

Lemma 26.10.2. *Let $F(t,X)$ be a real valued function in $\mathbb{R}^{1+k}$ such that $F(t,X)F(s,X) \geqq 0$ for all t,s,X, all X derivatives are continuous and*

$$|F_X^{(j)}(t,X)| \leqq C_j a^{j-2}, \quad j=0,1,\ldots; \ (t,X) \in \mathbb{R}^{1+k}. \tag{26.10.6}$$

Let $1 \leqq \rho \leqq 1/a$ and set

$$\tilde{a}(X)^{-2} = \max(\rho^2, \sup_t |F(t,X)|, \sup_t |F_X'(t,X)|^2). \tag{26.10.7}$$

Then

$$a \leqq \tilde{a}(X) \leqq 1/\rho, \quad \tilde{a}(X+Y) \leqq 2\tilde{a}(X) \quad \text{if } |Y|\,\tilde{a}(X) < \tfrac{1}{2}, \tag{26.10.8}$$

$$\tilde{a}(X) \leqq \tilde{a}(X+Y)(1+|Y|\,\tilde{a}(X)), \tag{26.10.9}$$

$$|F_X^{(j)}(t,X)| \leqq C_j \tilde{a}(X)^{j-2}, \quad j=0,1,\ldots; \ (t,X) \in \mathbb{R}^{1+k}; \tag{26.10.10}$$

and for every X one of the following cases occurs:

I) $\tilde{a}(X) = 1/\rho$; *then* $\frac{1}{2} \leqq \rho\tilde{a}(X+Y) \leqq 1$ *and* $|F_Y^{(j)}(t,X+Y)| \leqq 4C_j\rho^{2-j}$, $j=0,1,\ldots$ *if* $\tilde{a}(X)|Y| \leqq \frac{1}{2}$.

II$_+$) $\tilde{a}(X)^{-2} = \sup_t F(t,X)$; *then* $F(s,X+Y) \geqq 0$ *if* $\tilde{a}(X)|Y| \leqq \frac{1}{2}$.

II$_-$) $\tilde{a}(X)^{-2} = \sup_t -F(t,X)$; *then* $F(s,X+Y) \leqq 0$ *if* $\tilde{a}(X)|Y| \leqq \frac{1}{2}$.

III) $\tilde{a}(X)^{-1} = \sup|F_X'(t,X)|$; *then* $F(t,Y) = G(Y)H(t,Y)$ *if* $\tilde{a}(X)|Y-X| < \frac{1}{2}$, *and then we have* $H \geqq 0$, $|G'(X)| = 1/\tilde{a}(X)$,

$$|G^{(j)}(Y)| \leqq C_j \tilde{a}(Y)^{j-2}, \quad |H_Y^{(j)}(t,Y)| \leqq C_j' \tilde{a}(Y)^j \tag{26.10.11}$$

where the constants C_j' only depend on $C_0, \ldots, C_{j+1}$.

Proof. That $a \leqq \tilde{a}(X) \leqq 1/\rho$ follows at once from the definition and the hypothesis. Since

$$|F(t,X+Y) - F(t,X)| \leqq |Y|\,|F_X'(t,X)| + |Y|^2/2 \leqq |Y|/\tilde{a}(X) + |Y|^2/2,$$
$$|F'(t,X+Y) - F'(t,X)| \leqq |Y|,$$

we obtain if $\tilde{a}(X)|Y| \leqq \frac{1}{2}$ that

$$|F(t,X+Y)| \geqq |F(t,X)| - 3\tilde{a}(X)^{-2}/4,$$
$$|F'(t,X+Y)| \geqq |F'(t,X)| - \tfrac{1}{2}\tilde{a}(X)^{-1}$$

which proves the second inequality in (26.10.8). Also (26.10.9) follows for

$$|F(t,X+Y)| \leqq \tilde{a}(X)^{-2}(1+|Y|\,\tilde{a}(X))^2,$$
$$|F'(t,X+Y)| \leqq \tilde{a}(X)^{-1}(1+|Y|\,\tilde{a}(X)).$$

The estimate (26.10.10) follows from (26.10.6) if $j \geqq 2$, since $a \leqq \tilde{a}$, and from the definition of $\tilde{a}$ if $j=0$ or $j=1$. In case I) we have $\frac{1}{2} \leqq \rho\tilde{a}(X+Y) \leqq 1$ when $\tilde{a}(X)|Y| \leqq 1$ in view of (26.10.9), hence $\tilde{a}(X+Y)^{j-2} \leqq 4\rho^{2-j}$, $j \geqq 0$. In case II$_+$) we have

$$F(t,X+Y) \geqq F(t,X) - 3\tilde{a}(X)^{-2}/4 > 0, \quad \tilde{a}(X)|Y| < \tfrac{1}{2},$$

provided that $F(t,X)>3\tilde{a}(X)^{-2}/4$. By hypothesis it follows that we have $F(s,X+Y)\geqq 0$ for all s then. Case II$_-$) is of course similar.

In case III) we choose t_j so that $|F'_X(t_j,X)|\to\tilde{a}(X)^{-1}$ and $F(t_j,Y)\to G(Y)$. Then the first estimate in (26.10.11) follows from (26.10.6), and $|G'(X)| =\tilde{a}(X)^{-1}$. Hence $G'(Y)\neq 0$ if $|X-Y|\,\tilde{a}(X)<1$. Since the zeros of G are simple then, they are limits of simple zeros of $F(t_j,Y)$. At such zeros we have $F(s,Y)=0$ for all s since Y is in the closure of the set where $F(t_j,.)>0$ as well as the set where $F(t_j,.)<0$. It follows that $F=0$ when $G=0$ and that $H(t,Y)=F(t,Y)/G(Y)$ is a non-negative C^∞ function when $\tilde{a}(X)\,|Y-X|<1$. To estimate H we assume for example that $\partial G(X)/\partial X_1>0$, $\partial G(X)/\partial X_j=0$, $j>1$. Then

$$\partial G(X+Y)/\partial Y_1>\tilde{a}(X)^{-1}-|Y|.$$

If $\tilde{a}(X)\,|Y|<\frac{1}{2}$ and $G(X+Y+se_1)\neq 0$ for $|s|<1/(2\tilde{a}(X))$, where $e_1=(1,0,\dots,0)$, then $G(X+Y)>\int_0^{1/(2\tilde{a}(X))} s\,ds=1/(8\tilde{a}(X)^2)$. Since $|F(t,X+Y)|\leqq\tilde{a}(X)^{-2}(1+\frac{1}{2}+\frac{1}{8})$ we obtain $|H(t,X+Y)|\leqq 13$ and of course similar estimates for the derivatives with respect to Y. On the other hand, if $G(X+Y+se_1)=0$ for some s with $\tilde{a}(X)\,|s|<\frac{1}{2}$ then $F(X+Y+se_1)/s$ and $G(X+Y+se_1)/s$ are equal to averages of $\partial_1 F$ and of $\partial_1 G$ on the intervals between $X+Y$ and $X+Y+se_1$ so the ratio is bounded by

$$\int_{\frac{1}{2}}^{1}(1+s)\,ds\Big/\int_{\frac{1}{2}}^{1}(1-s)\,ds=7.$$

Hence $|H(t,X+Y)|\leqq 7$, and we have similar estimates for the derivatives of H since this is true of the averages of $\partial_1 G$ and $\partial_1 F$. The proof is complete.

Note that the proof also shows in the third case that if $G(Y)$ has no zero with $|Y-X|\,\tilde{a}(X)<\frac{1}{2}$ then $|G(X)|\geqq\frac{3}{8}\tilde{a}(X)^{-2}$, so we could essentially have classified this situation as case II. There are similar borderline cases between the others, but this will not be important. What is crucial is that the lemma cleanly separates areas where F is bounded and therefore controlled, or F is of constant sign so that the methods of Section 26.6 are applicable, or finally where F can be factored so that we have essentially the situation studied in Section 26.9.

Since the proof of the following proposition will depend on the advanced calculus of pseudo-differential operators in Sections 18.5.6, we shall use the Weyl calculus already in the statement. Note that the hypotheses (26.10.2) and (26.10.3) remain valid for the Weyl symbols.

Proposition 26.10.3. *Let $P=D_t+iF^w(t,x,D)+R^w(t,x,D)$ where F is real valued, $F(t,x,\xi)\,F(s,x,\xi)\geqq 0$ for all $(s,t,x,\xi)\in\mathbb{R}^{2n}$ and*

$$a^2|F^{(j)}_{x,\xi}(t,x,\xi)|+|R^{(j)}_{x,\xi}(t,x,\xi)|\leqq\delta\,C_j a^j,\quad j=0,1,\dots. \tag{26.10.12}$$

If a and δT are smaller than positive constants depending only on $C_0, C_1, \dots$ then

(26.10.13) $\|u\| \leqq 16T\|Pu\|$, *if* $u \in \mathscr{S}(\mathbb{R}^n)$ *and* $u(t,x)=0$ *when* $|t|>T$.

Proof. As already observed we may assume in the proof that $\delta=1$ and that $C_j=1$ when $j \leqq 2$. At first we also assume that $R=0$. From Lemma 26.10.2 with ρ still to be chosen we obtain a metric

$$g=\tilde{a}(x,\xi)^2(|dx|^2+|d\xi|^2)$$

which is slowly varying and σ temperate (cf. (18.5.11)′) by (26.10.8) and (26.10.9), for $1+|Y|\,\tilde{a}(X) \leqq 1+|Y| \leqq 1+|Y|/\tilde{a}(X)$. Choose $X_\nu=(x_\nu,\xi_\nu)$ so that the balls

$$B_\nu(r)=\{X; \tilde{a}(X_\nu)|X-X_\nu|<r\}$$

cover $\mathbb{R}^{2(n-1)}$ when $r=\frac{1}{5}$ and there is a fixed bound for the number of balls $B_\nu(\frac{1}{2})$ with a non-empty intersection. Choose $\phi_\nu \in C_0^\infty(B_\nu(\frac{1}{4}))$ real valued so that $\sum \phi_\nu^2=1$ and $\{\phi_\nu\}$ is uniformly bounded as a symbol in $S(1,g)$ with values in l^2. (See Lemma 18.4.4.) We have

$$\sum \|\phi_\nu^w u\|^2=\sum(\phi_\nu^w\phi_\nu^w u,u)=(u,u)+(\phi^w u,u)$$

where ϕ is uniformly bounded in $S(\tilde{a}^2,g)$, hence in $S(\rho^{-2},g)$. It follows that $\|\phi^w\| \leqq C\rho^{-2}$. Fixing ρ now so that $\rho^2=\max(1,2C)$ we obtain

$$\|u\|^2 \leqq 2\sum \|\phi_\nu^w u\|^2 \leqq 3\|u\|^2. \tag{26.10.14}$$

Choose $\psi_\nu \in C_0^\infty(B_\nu(\frac{1}{2}))$ non-negative and equal to 1 in $B_\nu(\frac{1}{3})$ so that $\{\psi_\nu\}$ is uniformly bounded in $S(1,g)$, and set

$$F_\nu(t,x,\xi)=\psi_\nu(x,\xi)^2 F(t,x,\xi).$$

F_ν is uniformly bounded in $S(\tilde{a}(X_\nu)^{-2}, \tilde{a}(X_\nu)^2(|dx|^2+|d\xi|^2))$. If X is in case I) of Lemma 26.10.2 then $\tilde{a}(X_\nu)^{-2}=\rho^2$, so Theorem 26.8.1 applied with $A=B=0$ gives

$$\|u\| \leqq 4T\|D_t u\| \leqq 4T\|D_t u+iF_\nu^w(t,x,D)u\|+CT\rho^2\|u\|.$$

When $CT\rho^2<\frac{1}{2}$ it follows that

$$\|u\| \leqq 8T\|(D_t+iF_\nu^w(t,x,D))u\|. \tag{26.10.15}$$

In case II$_+$) we observe that by Theorem 18.6.7 (in fact, by Theorem 18.1.14) there is a constant c such that

$$F_\nu^w(t,x,D)+cI \geqq 0.$$

Applying Theorem 26.8.1 with $B=I$ and $A(t)=F_\nu^w(t,x,D)+cI$ we obtain

$$\|u\| \leqq 4T\|(D_t+iF_\nu^w+c)u\| \leqq 4T\|(D_t+iF_\nu^w)u\|+4Tc\|u\|,$$

so (26.10.15) is valid if $4Tc<\frac{1}{2}$. The same is true in case II$_-$; we just have to change the sign of t. In case III we set $G_\nu=\psi_\nu G$ and $H_\nu=\psi_\nu H$ where $F=GH$ is the local factorization in Lemma 26.10.2. We can apply Theorem 26.8.1

with $B=G_\nu^w(x,D)$ and $A(t)=H_\nu^w(t,x,D)+c\tilde{a}(X_\nu)^2$ with c chosen so large that $A(t)\geqq 0$. Then $B\in S(\tilde{a}(X_\nu)^{-2},\ \tilde{a}(X_\nu)^2(|dx|^2+|d\xi|^2))$ and $A(t)\in S(1,\ \tilde{a}(X_\nu)^2(|dx|^2+|d\xi|^2))$ uniformly, so $\|A\|$, $\|[B,A]\|$ and $\|[B,[B,A]]\|$ are uniformly bounded; later we shall need that $\tilde{a}(X_\nu)^2\|B\|$ is also uniformly bounded. Hence (26.8.3) gives if T is small enough

$$\|u\|\leqq 6T\|(D_t+iA(t)B)u\|\leqq 6T\|(D_t+iF_\nu^w(t,x,D))u\|+CT\|u\|$$

since the symbol of $A(t)B-F_\nu^w$ is uniformly bounded in $S(1,g)$. When $CT<\frac{1}{4}$ we obtain (26.10.15), which is now established in all cases.

Now we apply (26.10.15) to $\phi_\nu^w u$ and obtain using (26.10.14)

$$\|u\|^2\leqq 2\sum\|\phi_\nu^w u\|^2\leqq 2^7T^2\sum\|(D_t+iF_\nu^w(t,x,D))\phi_\nu^w u\|^2.$$

Regarding $\{F_\nu\}$ and $\{\phi_\nu\}$ as symbols with values in diagonal matrices in $\mathscr{L}(l^2,l^2)$ or in $l^2=\mathscr{L}(\mathbb{C},l^2)$, we obtain from the calculus that

$$(D_t+iF_\nu^w(t,x,D))\phi_\nu^w u=\phi_\nu^w(D_t+iF^w(t,x,D))u+K_\nu^w(t,x,D)$$

where $\{K_\nu\}$ is uniformly bounded in $S(1,g)$ (with values in l^2). Since

$$\sum\|\phi_\nu^w(x,D)(D_t+iF^w(t,x,D))u\|^2\leqq\tfrac{3}{2}\|(D_t+iF^w(t,x,D))u\|^2$$

by (26.10.14) and

$$\sum\|K_\nu^w(t,x,D)u\|^2\leqq C\|u\|^2,$$

we obtain for small T

$$\|u\|\leqq 14T\|(D_t+iF^w(t,x,D))u\|.$$

This proves (26.10.13) if $R=0$ with the constant 14 instead of 16. When T is so small that

$$14T\|R^w(t,x,D)u\|\leqq\|u\|/8,$$

the estimate (26.10.13) follows.

In case III) the estimate (26.10.15) is analogous to (26.9.3). We shall also need the analogue of (26.9.4) obtained when Theorem 26.8.3 is applied to the operators $A(t)=H_\nu^w(t,x,D)+c\tilde{a}(X_\nu)^2$ and $B=G_\nu^w(x,D)$ in the preceding proof. Using (26.10.15) we obtain for small T

$$\int(B^w u,H_\nu^w(t,x,D)B^w u)\,dt\leqq CT\tilde{a}(X_\nu)^{-2}\|P_\nu u\|^2;\qquad P_\nu=D_t+iF_\nu^w(t,x,D).$$

Since $H_\nu\geqq 0$ and the second derivatives of $\tilde{a}(X_\nu)^{-2}H_\nu$ have uniform bounds, it follows from Lemma 7.7.2 that

$$|\tilde{a}(X_\nu)^{-2}H_\nu'|^2\leqq C\tilde{a}(X_\nu)^{-2}H_\nu$$

if $H_\nu'=\partial H_\nu/\partial x_j$ or $\partial H_\nu/\partial\xi_j$ for some j. Hence

$$0\leqq C\tilde{a}(X_\nu)^2H_\nu-(H_\nu')^2\in S(\tilde{a}(X_\nu)^2,\tilde{a}(X_\nu)^2(|dx|^2+|d\xi|^2))$$

so we can find another constant C' such that the Weyl operator with symbol

$$C\tilde{a}(X_\nu)^2H_\nu-H_\nu'^2+C'\tilde{a}(X_\nu)^4$$

is non-negative. Since the symbol of $(H_\nu'^2)^w-((H_\nu')^w)^2$ belongs to $S(\tilde a(X_\nu)^4, \tilde a(X_\nu)^2(|dx|^2+|d\xi|^2))$ we obtain

$$\|H_\nu'^w v\|^2 \leqq C\tilde a(X_\nu)^2(H_\nu^w v, v)+C'\tilde a(X_\nu)^4(v,v).$$

Taking $v=G_\nu^w(x,D)u$ we obtain with another constant C

$$\|H_\nu'^w G_\nu^w(x,D)u\|^2 \leqq CT\|P_\nu u\|^2+C\|u\|^2.$$

The symbol of $H_\nu'^w G_\nu^w-(H_\nu' G_\nu)^w$ is bounded in $S(1,\tilde a(X_\nu)^2(|dx|^2+|d\xi|^2))$. In view of (26.10.15) it follows for small T that with still another C

$$\|(H_\nu' G_\nu)^w(t,x,D)u\|^2 \leqq CT\|P_\nu u\|^2 \tag{26.10.16}$$

if $u\in\mathscr{S}$, $u=0$ for $|t|>T$.

Proposition 26.10.3 is all one needs to prove local existence theorems. However, the proof of Theorem 26.10.1′ requires a localized form of (26.10.13) which we shall now prove.

Proposition 26.10.4. *Let the hypotheses of Proposition* 26.10.3 *be fulfilled, let* χ_0 *and* χ_1 *be uniformly bounded in* $S(1,a^2(|dx|^2+|d\xi|^2))$, *and assume that* $\chi_1=1$ *in* $\operatorname{supp}\chi_0$. *For every* $\varepsilon>0$ *we have then if* a *and* δT *are sufficiently small*

$$a^\varepsilon\|\chi_0^w(x,D)u\| \leqq CT(\|\chi_1^w(x,D)Pu\|+a^3\|Pu\|)+a^{\frac12}\|u\|, \tag{26.10.17}$$

for all $u\in\mathscr{S}(\mathbb{R}^n)$ *vanishing when* $|t|>T$.

Proof. We may assume in the proof that $|\chi_1|>\frac12$ at all points with distance $\leqq 1/a$ from supp χ_0, for this is true at distance $\leqq\gamma/a$ for some fixed $\gamma\in(0,1)$, and changing the constants in (26.10.12) we may replace a by a/γ in the hypothesis. As in the proof of Proposition 26.10.3 we may also assume that $\delta=1$ so that a and T are small. Now we apply Lemma 26.10.2 with $\rho=a^{-\frac12}$, define ϕ_ν, ψ_ν and F_ν as in the proof of Proposition 26.10.3 and set $R_\nu(t,x,\xi)=\psi_\nu(x,\xi)^2R(t,x,\xi)$, $P_\nu=D_t+iF_\nu^w(t,x,D)+R_\nu^w(t,x,D)$. If a is small enough we have (26.10.14), hence

$$\|\chi_0^w(x,D)u\|^2 \leqq 2\sum\|\phi_\nu^w(x,D)\chi_0^w(x,D)u\|^2. \tag{26.10.18}$$

Application of Proposition 26.10.3 to $\phi_\nu^w(x,D)\chi_0^w(x,D)u$ and P_ν gives

$$\|\phi_\nu^w(x,D)\chi_0^w(x,D)u\| \leqq 16T\|P_\nu\phi_\nu^w(x,D)\chi_0^w(x,D)u\|.$$

We shall estimate the right-hand side by writing

$$P_\nu\phi_\nu^w\chi_0^w=[P_\nu,\phi_\nu^w]\chi_0^w+\phi_\nu^w[P_\nu,\chi_0^w]+\phi_\nu^w(P_\nu-P)+\phi_\nu^w\chi_0^w P.$$

When computing $[P_\nu,\phi_\nu^w]=(iF_\nu^w+R_\nu^w)\phi_\nu^w-\phi_\nu^w(iF_\nu^w+R_\nu^w)$ we regard the right-hand factor as a symbol with values in $l^2=\mathscr{L}(\mathbb{C},l^2)$ and the left factor as a symbol with values in diagonal matrices in $\mathscr{L}(l^2,l^2)$. (All later calculations are made similarly.) With the notation $g=\tilde a^2(|dx|^2+|d\xi|^2)$ it follows that the symbol is $\{F_\nu-iR_\nu,\phi_\nu\}\in S(1,g)$ (with l^2 values) apart from an error in $S(\tilde a^2,g)$

$\subset S(a, g)$. Hence

$$\sum \|[P, \phi_\nu^w]\chi_0^w u\|^2 \leqq C\|\chi_0^w u\|^2$$

which is harmless to have multiplied by T^2 in the right hand side of our estimates since $\|\chi_0^w u\|^2$ will occur on the left. The symbol of $\phi_\nu^w[P_\nu, \chi_0^w]$ is $\phi_\nu\{F_\nu - iR_\nu, \chi_0\}$ with an error in $S(\tilde{a}^2, g)$, for $\{F_\nu - iR_\nu\} \in S(\tilde{a}^{-2}, g)$ and $\chi_0 \in S(1, g)$. Since $\chi_0 \in S(1, a^2(|dx|^2 + |d\xi|^2))$ the symbol is bounded in $S(a/\tilde{a}, g) \subset S(a^{\frac{1}{2}}, g)$ if we restrict ν to the set N_{I} of indices such that X_ν is in case I) of Lemma 26.10.2. Thus

$$\sum_{N_{\mathrm{I}}} \|\phi_\nu^w[P_\nu, \chi_0^w]u\|^2 \leqq Ca\|u\|^2.$$

The symbol of $\phi_\nu^w \chi_0^w(P_\nu - P)$ is bounded in $S(a, g)$ for all terms in the composition series vanish because $\psi_\nu = 1$ in $\operatorname{supp}\phi_\nu$. These terms have therefore an even better estimate. Finally the symbol of $\chi_0^w - \chi_0^w \chi_1^w$ belongs to $S(a^4, a^2(|dx|^2 + |d\xi|^2))$ so

$$\|\chi_0^w P u\|^2 \leqq C(\|\chi_1^w P u\|^2 + a^8\|Pu\|^2).$$

Using (26.10.14) again we therefore obtain

$$\sum_{N_{\mathrm{I}}} \|\phi_\nu^w(x, D)\chi_0^w(x, D)u\|^2 \leqq CT^2(\|\chi_1^w(x, D)Pu\|^2 + a^8\|Pu\|^2 + a\|u\|^2 + \|\chi_0^w(x, D)u\|^2). \tag{26.10.19}$$

In cases II) and III) the commutator of P_ν and χ_0^w is too large to make the preceding estimates useful. There is no point in keeping the factor χ_0^w then. Since $\phi_\nu^w \chi_0^w = \chi_0^w \phi_\nu^w + [\phi_\nu^w, \chi_0^w]$ and $[\phi_\nu^w, \chi_0^w]$ has a vector valued symbol in $S(a^{\frac{1}{2}}, g)$ (recall that $\tilde{a} \leqq a^{\frac{1}{2}}$), we have

$$\sum_{\nu \notin N_{\mathrm{I}}} \|\phi_\nu^w \chi_0^w u\|^2 \leqq C(\sum_{N_{\mathrm{II}} \cup N_{\mathrm{III}}} \|\phi_\nu^w u\|^2 + a^3\|u\|^2), \tag{26.10.20}$$

where N_{II} and N_{III} denote the sets of indices ν such that X_ν is in case $\mathrm{II}_\pm$ resp. III of Lemma 26.10.2 and $\phi_\nu \chi_0 \neq 0$. We recall that this implies $|\chi_1| > \frac{1}{2}$ in $B_\nu(\frac{1}{2})$. (See the beginning of the proof.)

If $\nu \in N_{\mathrm{II}}$ we go back to the proofs of Proposition 26.6.1 and of Theorem 26.8.1. Thus we start from the fact that for fixed t

$$\operatorname{Im}(\phi_\nu^w Pu, \phi_\nu^w u) = \operatorname{Im}(P\phi_\nu^w u, \phi_\nu^w u) + \operatorname{Im}([\phi_\nu^w, P]u, \phi_\nu^w u). \tag{26.10.21}$$

We have in case II_+

$$\begin{aligned} 2\operatorname{Im}(P\phi_\nu^w u, \phi_\nu^w u) &= -\frac{\partial}{\partial t}\|\phi_\nu^w u\|^2 + 2(F^w\phi_\nu^w u, \phi_\nu^w u) + 2\operatorname{Im}(R^w\phi_\nu^w u, \phi_\nu^w u) \\ &\geqq -\frac{\partial}{\partial t}\|\phi_\nu^w u\|^2 - C\|\phi_\nu^w u\|^2, \end{aligned}$$

since F^w is bounded from below and R^w is bounded. Furthermore

$$2\operatorname{Im}([\phi_\nu^w, P]u, \phi_\nu^w u) = ([\phi_\nu^w, [\phi_\nu^w, F^w]]u, u) + 2\operatorname{Im}(\phi_\nu^w[\phi_\nu^w, R^w]u, u),$$

and the symbols of

$$\sum_{N_{\mathrm{II}+}} [\phi_\nu^w, [\phi_\nu^w, F^w]], \qquad \sum_{N_{\mathrm{II}+}} \phi_\nu^w [\phi_\nu^w, R^w]$$

are bounded in $S(\tilde{a}^2, g) \subset S(a, g)$. If we multiply (26.10.21) by $T+t$, sum and integrate with respect to t also, we obtain with L^2 norms in (t, x)

$$\sum_{N_{\mathrm{II}}} \|\phi_\nu^w u\|^2 \leqq CT(\sum_{N_{\mathrm{II}}} \|\phi_\nu^w u\|^2 + a\|u\|^2 + \sum_{N_{\mathrm{II}}} \|\phi_\nu^w Pu\| \, \|\phi_\nu^w u\|).$$

(The case II_- is reduced to II_+ by changing the sign of t.) Since

$$CT\|\phi_\nu^w Pu\| \, \|\phi_\nu^w u\| \leqq (C^2 T^2 \|\phi_\nu^w Pu\|^2 + \|\phi_\nu^w u\|^2)/2$$

we obtain for small T and another C

$$\sum_{N_{\mathrm{II}}} \|\phi_\nu^w u\|^2 \leqq CT^2 \sum_{N_{\mathrm{II}}} \|\phi_\nu^w Pu\|^2 + CTa\|u\|^2. \tag{26.10.22}$$

If $\tilde{\phi}_\nu = \phi_\nu/\chi_1 + \{\phi_\nu, 1/\chi_1\}/2i$ then $\{\tilde{\phi}_\nu\}_{\nu \in N_{\mathrm{II}}}$ is bounded in $S(1, g)$ since $|\chi_1| > \frac{1}{2}$ in $\operatorname{supp} \phi_\nu$ when $\nu \in N_{\mathrm{II}}$, and the symbol of $\phi_\nu^w - \tilde{\phi}_\nu^w \chi_1^w$ is bounded in $S(a^3, g)$ since $\phi_\nu - \chi_1 \tilde{\phi}_\nu - \{\tilde{\phi}_\nu, \chi_1\}/2i = \{\{\phi_\nu, 1/\chi_1\}, \chi_1\}/4$. Hence

$$\sum_{N_{\mathrm{II}}} \|\phi_\nu^w Pu\|^2 \leqq C(\|\chi_1^w Pu\|^2 + a^6 \|Pu\|^2). \tag{26.10.23}$$

In case III) we shall use operators commuting approximately with P which are similar to the operators with symbol (26.9.8) used in the proof of Theorem 26.9.1. They are constructed in the following lemma, which will be proved after completion of the proof of Proposition 26.10.4.

Lemma 26.10.5. *One can find an integer J and for every $\nu \in N_{\mathrm{III}}$ and $j = 1, \dots, J$ functions $\phi_{\nu j}, \tilde{\phi}_{\nu j} \in C_0^\infty(B_\nu(\frac{1}{3}))$, $v_{\nu j} \in C_0^\infty(B_\nu(\frac{1}{2}))$ such that if $|t| < T$ and T is small*

(i) $\{\phi_{\nu j}\}_{\nu \in N_{\mathrm{III}}}, \{\tilde{\phi}_{\nu j}\}_{\nu \in N_{\mathrm{III}}}$, *and* $\{v_{\nu j}\}_{\nu \in N_{\mathrm{III}}}$ *are uniformly bounded in* $S(1, g)$,

(ii) $\phi_{\nu j} = 1$ *in* $\operatorname{supp} \tilde{\phi}_{\nu j}$, $\sum_j \tilde{\phi}_{\nu j} = 1$ *in* $B_\nu(\frac{1}{4})$,

(iii) $\partial v_{\nu j}/\partial t - iH_\nu\{v_{\nu j}, G_\nu\} = 0$ *in* $\operatorname{supp} \phi_{\nu j}$,

(iv) $v_{\nu j} > \varepsilon/3$ *in* $\operatorname{supp} \phi_{\nu j}$, $v_{\nu j} < 2\varepsilon/3$ *in* $\operatorname{supp} \tilde{\phi}_{\nu j}$, $v_{\nu j} > 1 + \varepsilon/3$ *in* $\operatorname{supp} d\phi_{\nu j}$.

End of proof of Proposition 26.10.4. With

$$m_{\nu j} = \phi_{\nu j} a^{v_{\nu j}}$$

we have by Proposition 26.10.3 applied to P_ν and $m_{\nu j}^w u$

$$\|m_{\nu j}^w u\| \leqq 16T\|P_\nu m_{\nu j}^w u\|. \tag{26.10.24}$$

Here we want to commute P_ν and $m_{\nu j}^w$. The first part of (iv) shows that $\{m_{\nu j}\}_{\nu \in N_{\mathrm{III}}}$ is bounded in $S(1, g)$, for the powers of $\log a$ which occur when $m_{\nu j}$ is differentiated can be estimated by $a^{-\varepsilon/3}$. The symbol of $[m_{\nu j}, P_\nu]$ is therefore in $S(a, g)$ (with values in l^2) apart from the term

$$-i\{m_{\nu j}, \tau + iF_\nu\} = -i\{\phi_{\nu j}, \tau + iF_\nu\} a^{v_{\nu j}} + (\log a) G_\nu \{v_{\nu j}, H_\nu\} m_{\nu j}. \tag{26.10.25}$$

Here we have used (iii) when calculating $\{v_{\nu j}, \tau + iF_\nu\}$. Since $\{\phi_{\nu j}, \tau + iF_\nu\}$ is in $S(1, g)$ with values in l^2, the last part of (iv) gives

$$\{\phi_{\nu j}, \tau + iF_\nu\} a^{v_{\nu j}} \in S(a, g)$$

with values in l^2, and with a uniform bound. The second term in (26.10.25) differs from the symbol of

$$\log a \Big(\sum_k ((\partial_{\xi_k} v_{\nu j})^w (G_\nu \partial_{x_k} H_\nu)^w - (\partial_{x_k} v_{\nu j})^w (G_\nu \partial_{\xi_k} H_\nu)^w)\Big) m_{\nu j}^w$$

by a symbol in $S(a, g)$ with values in l^2. Since $\partial_{\xi_k} v_{\nu j}$ and $\partial_{x_k} v_{\nu j}$ are bounded in $S(a^{\frac{1}{2}}, g)$, it follows from (26.10.16) that

$$\|P_\nu m_{\nu j}^w u\| \leqq \|m_{\nu j}^w P u\| + C a^{\frac{1}{2}} \log a \|P_\nu m_{\nu j}^w u\| + \|\rho_{\nu j}^w u\|$$

where $\{\rho_{\nu j}\}$ is bounded in $S(a, g)$. (Note that the composition series of $m_{\nu j}^w (P - P_\nu)$ has only zero terms.) Hence

$$\|P_\nu m_{\nu j}^w u\| \leqq 2\|m_{\nu j}^w P u\| + 2\|\rho_{\nu j}^w u\|$$

if a is so small that $2 C a^{\frac{1}{2}} \log a < 1$, so we obtain using (26.10.24)

$$\sum \|m_{\nu j}^w u\|^2 \leqq C T^2 \Big(\sum \|m_{\nu j}^w P u\|^2 + a^2 \|u\|^2\Big). \tag{26.10.26}$$

The proof of (26.10.23) also gives

$$\sum \|m_{\nu j}^w P u\|^2 \leqq C(\|\chi_1^w P u\|^2 + a^6 \|P u\|^2). \tag{26.10.27}$$

Since (ii) in Lemma 26.10.5 implies

$$\phi_\nu = \sum_j \phi_\nu \tilde{\phi}_{\nu j} = a^{-\varepsilon} \sum_j \psi_{\nu j} m_{\nu j}, \qquad \psi_{\nu j} = \phi_\nu \tilde{\phi}_{\nu j} a^{\varepsilon - v_{\nu j}},$$

and $\{\psi_{\nu j}\}$ is bounded in $S(1, g)$ by the second part of (iv), it follows that the symbol of $a^\varepsilon \phi_\nu^w - \sum_j \psi_{\nu j}^w m_{\nu j}^w$ is bounded in $S(a, g)$ with values in l^2, hence

$$\sum_{N_{\mathrm{III}}} \|a^\varepsilon \phi_\nu^w u\|^2 \leqq C\Big(\sum \|m_{\nu j}^w u\|^2 + a^2 \|u\|^2\Big).$$

If we combine this estimate with (26.10.26), (26.10.27), and recall the estimates (26.10.18), (26.10.19), (26.10.20), (26.10.22) and (26.10.23), we have proved that

$$a^{2\varepsilon} \|\chi_0^w(x, D) u\|^2 \leqq C T^2 (\|\chi_1^w(x, D) P u\|^2 + a^6 \|P u\|^2 + a^{2\varepsilon} \|\chi_0^w(x, D) u\|^2) + C a \|u\|^2.$$

When T is so small that $C T^2 < \frac{1}{2}$, the estimate (26.10.17) follows.

Proof of Lemma 26.10.5. The essential point is to use Corollary 26.7.8 to construct a solution of the equation

$$\partial v / \partial t - i H_\nu \{v, G_\nu\} = 0$$

for $|t| < T$, where T is small, and for (x, ξ) in a neighborhood of an arbitrary $Y \in B_\nu(\frac{1}{4})$ with diameter proportional to $1/a_\nu$ where $a_\nu = \tilde{a}(X_\nu)$. To do so we set $\kappa_\nu(y,\eta)=(x_\nu+y/a_\nu, \xi_\nu+\eta/a_\nu)$ and obtain the equation

$$\partial(\kappa_\nu^* v)/\partial t + i\kappa_\nu^* H_\nu\{a_\nu^2 \kappa_\nu^* G_\nu, \kappa_\nu^* v\}=0.$$

(Note that κ_ν is not symplectic but multiplies the symplectic form by a constant factor.) Here $\kappa_\nu^* H_\nu$ and $a_\nu^2\kappa_\nu^* G_\nu$ have uniformly bounded $y\eta$ derivatives. There is a fixed positive lower bound for $a_\nu^2|d\kappa_\nu^* G_\nu(0)|$, so Theorem 21.1.6 and its proof show that there is a canonical transformation $\tilde\kappa_\nu$ from a fixed neighborhood of 0 to a neighborhood of $\chi_\nu^{-1}(Y)$ with uniform bounds for all derivatives of $\tilde\kappa_\nu$ and $\tilde\kappa_\nu^{-1}$ such that

$$a_\nu^2 \tilde\kappa_\nu^* \kappa_\nu^* G_\nu(y,\eta)=\eta_1.$$

Thus we obtain the equation

$$\frac{\partial}{\partial t}((\kappa_\nu\tilde\kappa_\nu)^* v) + i(\kappa_\nu\tilde\kappa_\nu)^* H_\nu \partial((\kappa_\nu\kappa_\nu)^* v)/\partial y_1 = 0$$

which we solve using Corollary 26.7.8 with ε replaced by $\varepsilon/3$. Since $\kappa_\nu\tilde\kappa_\nu(0) = Y\in B_\nu(\frac14)$ the neighborhoods can be chosen so small that $\kappa_\nu\tilde\kappa_\nu V_2\subset B_\nu(\frac13)$. Choose $\Phi\in C_0^\infty(V_2)$ equal to 1 in V_1 and $\Psi\in C_0^\infty(V_2)$ equal to 1 in $\operatorname{supp}\Phi$. Then

$$v=(\tilde\kappa_\nu^{-1}\kappa_\nu^{-1})^*(\Psi(U+\varepsilon/3)), \qquad \phi=(\tilde\kappa_\nu^{-1}\kappa_\nu^{-1})^*\Phi$$

are in $C_0^\infty(B_\nu(\frac13))$ and satisfy for small T the conditions on $\phi_{\nu j}$, $v_{\nu j}$ in (i), (iii), (iv). In addition we have $\phi=1$ and $v<2\varepsilon/3$ in $\{X;\, a_\nu|X-Y|<c\}$ where c is a fixed constant. Now we can cover $B_\nu(\frac14)$ by a fixed number J of such neighborhoods so that there is a subordinate partition of unity $\tilde\phi_{\nu j}$ with uniform bounds in $S(1, a_\nu^2(|dx|^2+|d\xi|^2)$. The corresponding functions v and ϕ are denoted by $v_{\nu j}$ and $\phi_{\nu j}$. This completes the proof of the lemma.

Proof of Theorem 26.10.1'. We must show that if $u\in\mathscr{E}'(\mathbb{R}^n)$, $s_u^*\geqq s$ at ∂I and $s_{Pu}^*\geqq s$ in I then $s_u^*\geqq s$ in I. In doing so we may assume that $s_u^*\geqq s-\frac14$ in I, for if the theorem is known then we can start from the fact that $s_u^*\geqq s-k/4$ in I for some integer k and deduce that $s_u^*\geqq s-(k-1)/4, \dots, s_u^*\geqq s$ in I. The hypothesis $s_{Pu}^*\geqq s$ in I is then preserved if we change the terms of order $\leqq -1$ in the symbol of P, so we may assume that the Weyl symbol of P is equal to $\xi_1+if(x,\xi')+r(x,\xi')$ in a conic neighborhood V of I, when $|\xi|>1$. We can take V so small that $s_u^*>s-\frac14-\varepsilon$ and that $s_{Pu}^*>s-\varepsilon$ in V where ε is an arbitrary positive number kept fixed in the following discussion. The conditions (26.10.2), (26.10.3) are also assumed valid. Choose χ_0, χ_1, $\psi\in C_0^\infty(\mathbb{R}^{2n-2})$ so that

$$\chi_0(0)=1, \qquad \chi_1=1 \text{ in } \operatorname{supp}\chi_0, \qquad \psi=1 \text{ in } \operatorname{supp}\chi_1,$$

and define $f_{\delta,\lambda}$, $r_{\delta,\lambda}$ by (26.10.4) and similarly

$$\chi_{j,\delta,\lambda}(x',\xi')=\chi_j(x'/\delta, (\lambda\xi'-\varepsilon_n')/\delta).$$

After a symplectic dilation we can then, as observed after (26.10.5), apply Proposition 26.10.4 with $a^2=\lambda/\delta^2$ and obtain for sufficiently small δ and λ

(26.10.28) $$(\lambda/\delta^2)^{\varepsilon/2}\|\chi^w_{0,\delta,\lambda}(x',D')v\| \leqq C(\|\chi^w_{1,\delta,\lambda}(x',D')P_{\delta,\lambda}v\|+(\lambda/\delta^2)^{\frac12}\|P_{\delta,\lambda}v\|+(\lambda/\delta^2)^{\frac12}\|v\|),$$

if $v\in\mathscr{S}$ and $v=0$ when $x_1\notin J$. (Recall that $I=J\times\{0\}\times\{\varepsilon_n\}$.) Here

$$P_{\delta,\lambda}=D_1+if^w_{\delta,\lambda}(x,D')+r^w_{\delta,\lambda}(x,D').$$

Choose a compact interval I_0 in the interior of I such that $s^*_u>s-\varepsilon$ in $I\setminus I_0$ and then a function $\chi\in C_0^\infty(V)$ with $x_1\in J$ in $\operatorname{supp}\chi$ and $\chi=1$ in a neighborhood of I_0. We shall apply (26.10.28) to $v=q_\lambda(x,D)u$ where

$$q_\lambda(x,\xi)=\chi(x,\lambda\xi)\lambda^{\varepsilon-s}.$$

(As this point we prefer not to use the Weyl calculus to be sure that $x_1\in J$ in supp v.) By Lemma 26.9.3 there is a bound for $\|\lambda^{\frac12}q_\lambda(x,D)u\|$ as $\lambda\to0$. Since $\lambda D_1 q_\lambda$ is a sum of two operators of the same form with $\chi(x,\xi)$ replaced by $\xi_1\chi(x,\xi)$ or by $\lambda D_1\chi(x,\xi)$ and since $\lambda(f^w_{\delta,\lambda}(x,D')+r^w_{\delta,\lambda}(x,D'))$ is uniformly bounded, it follows that $\lambda^{\frac12}\|P_{\delta,\lambda}q_\lambda u\|\to0$ as $\lambda\to0$.

When computing the symbol of

$$\chi^w_{1,\delta,\lambda}P_{\delta,\lambda}q_\lambda(x,D)$$

we note that the symbol of $P_{\delta,\lambda}$ is equal to the symbol of P in the intersection of $\operatorname{supp}q_\lambda$ and $\operatorname{supp}\chi_{1,\delta,\lambda}$. By Theorem 18.5.4 the symbol of

$$\chi^w_{1,\delta,\lambda}(x,D)P_{\delta,\lambda}q_\lambda(x,D)-\chi^w_{1,\delta,\lambda}(x,D)q_\lambda(x,D)P$$

is therefore equal to $\lambda^{\varepsilon-s}\tilde\chi(x,\lambda\xi)+\rho_{\delta,\lambda}$ where

$$\tilde\chi(x,\xi)=\chi_{1,\delta,1}\{p,q_1\}/i$$

and $\rho_{\delta,\lambda}$ is uniformly bounded in S^{s-1} as $\lambda\to0$, for q_λ is uniformly bounded in $S^{s-\varepsilon}$. Hence $\|\rho_{\delta,\lambda}u\|$ is bounded as $\lambda\to0$. If δ is small we have $s^*_u>s-\varepsilon$ in $\operatorname{supp}\tilde\chi\subset\operatorname{supp}\chi_{1,\delta}\cap\operatorname{supp}d\chi$, for $s^*_u>s+1-\varepsilon$ when $\xi_1\neq0$ since P is non-characteristic then, and $s^*_u>s-\varepsilon$ in $\operatorname{supp}d\chi$ when $x'=0$, $\xi=\varepsilon_n$, by the choice of χ. Hence

$$\|\lambda^{\varepsilon-s}\tilde\chi(x,\lambda D)u\|$$

is bounded as $\lambda\to0$. This is also true for $\|q_\lambda(x,D)Pu\|$, so (26.10.28) shows that

$$\lambda^{\varepsilon/2}\|\chi^w_{0,\delta,\lambda}(x',D')q_\lambda(x,D)u\|$$

is bounded as $\lambda\to0$. The symbol of $\chi^w_{0,\delta,\lambda}q_\lambda(x,D)$ is $\chi_{0,\delta,\lambda}q_\lambda$ apart from a term which is bounded in S^{s-1}, so it follows that

$$\lambda^{\varepsilon/2}\|(\chi_{0,\delta,\lambda}q_\lambda)(x,D)u\|$$

is bounded as $\lambda\to0$. Hence $s^*_u\geqq s-3\varepsilon/2$ on I_0 by Lemma 26.9.3, and this proves the theorem since ε is an arbitrary positive number.

26.11. A Semi-Global Existence Theorem

For arbitrary operators satisfying condition (P) we have now proved substitutes for Theorem 26.1.4 which permit us to prove an analogue of Theorem 26.1.7 with essentially the same arguments. Before stating it we shall examine the geometrical conditions involved. The notation is that in the summary at the end of Section 26.5.

Theorem 26.11.1. *Let P be a pseudo-differential operator in $\Psi^m_{\mathrm{phg}}(X)$ satisfying condition (P), and let K be a compact subset of X. Then the following two conditions are equivalent:*

(i) *Every characteristic point over K lies on a compact semi-bicharacteristic interval with no characteristic endpoint over K.*

(ii) *No two dimensional bicharacteristic and no complete one dimensional bicharacteristic in $N \setminus (N_{11} \cup N_2^e)$ lies entirely over K.*

Proof. It is clear that (i) $\Rightarrow$ (ii). Assume now that (ii) is fulfilled. We can also assume that the order of P is 1. The Hamilton field H_p of the principal symbol p can then be regarded as a vector field v on the cosphere bundle $S^*(X)$. It follows from (ii) that v cannot vanish anywhere over K in the characteristic set for then there would exist a radial bicharacteristic curve which contradicts (ii). If $\gamma_0 \in N \setminus N_2^e$ then a semi-bicharacteristic through γ_0 is a one dimensional bicharacteristic until it leaves the characteristic set. If (i) is false for some $\gamma_0 \in N \setminus N_2^e$ we can therefore find a C^1 map $\mathbb{R}_+ \ni t \mapsto \gamma(t) \in S^*(X)|_K$ with

$$p(\gamma(t)) = 0, \quad \gamma'(t) = c(t)\, v(\gamma(t)), \quad |c(t)| = 1, \quad \gamma(0) = \pi \gamma_0$$

where π is the projection $T^*(X) \setminus 0 \to S^*(X)$. Now choose a sequence $t_j \to \infty$ such that $\gamma(t_j)$ converges. Then it follows that

$$\tilde{\gamma}(t) = \lim_{j \to \infty} \gamma(t + t_j)$$

exists and is a complete one dimensional bicharacteristic curve, which contradicts (ii). Assume now that $\gamma_0 \in N_2^e$ and let B be the two dimensional bicharacteristic containing γ_0. We may assume that B contains some point $\hat{\gamma} \in N_2$ over the complement of K, for Proposition 26.5.5 shows that without violating condition (P) one can modify the symbol at a point in $N_2^e \setminus N_2$ to make it lie in N_2. If $\gamma_0 \in B_0$ (see the discussion after Definition 26.5.4) then the assertion (i) follows since in the Riemann surface $\tilde{B}_0$ we can obviously choose a smooth curve through the class of γ_0 with endpoints near $\hat{\gamma}$. If $\gamma_0 \in B \setminus B_0$ one can still find a semi-bicharacteristic from $\hat{\gamma}$ to γ_0 by the definition of N_2^e, and (i) follows again unless it continues indefinitely in the opposite direction as a one dimensional bicharacteristic over K. But we saw in the first part of the proof that this would contradict (ii), so the proof is now complete.

Our microlocal regularity theorems have the following consequence:

Theorem 26.11.2. *Let P be a pseudo-differential operator in $\Psi^m_{\text{phg}}(X)$ where X is a manifold. Assume that P satisfies condition (P), and let K be a compact subset of X such that the equivalent conditions in Theorem 26.11.1 are fulfilled. If $u \in \mathscr{E}'(K)$ and $s^*_{Pu} \geqq s$ where s is a real number or $+\infty$, it follows then that $s^*_u \geqq s+m-1$.*

Proof. Assume that the assertion is false so that

$$s_0 = \inf s^*_u < s+m-1. \tag{26.11.1}$$

We shall prove that this leads to a contradiction. Since s^*_u is lower semi-continuous, there is some $\gamma \in T^*(X) \setminus 0$ over K such that $s^*_u(\gamma) = s_0$. We have $s^*_u \geqq s+m$ outside N so it is clear that $\gamma \in N$. Choose a semi-bicharacteristic interval Γ containing γ with no characteristic end point over K. Then $s^*_u \geqq s+m$ at the end points of Γ. If Γ is not contained in N it follows from Theorems 26.6.2 and 26.6.4 that $s^*_u \geqq s+m-1$ at Γ, which contradicts (26.11.1). Thus $\Gamma \subset N$. If Γ is a one dimensional bicharacteristic we also obtain a contradiction in view of Theorem 26.10.1′. The remaining possibility is that $\Gamma \subset N^e_2$. Without violating condition (P) we can then as in the proof of Theorem 26.11.1 change the principal symbol at the end points of Γ so that they are in N_2. This does not affect the condition $s^*_{Pu} \geqq s$ if the change is made in a small enough set, for $u \in \mathscr{E}'(K)$. Let B be the leaf of the foliation of N^e_2 containing γ. Then $\gamma \in B_0$ and the function

$$S = \min(\tilde{s}_u, s+m-1)$$

which is superharmonic by Theorem 26.9.1 is $\geqq s_0$ in $\tilde{B}_0$ with equality in the class of γ. Hence S is identically equal to s_0 which contradicts the fact that $S = s+m-1$ at the class of any end point of Γ. This completes the proof.

We can now prove a slightly weakened analogue of Theorem 26.1.7.

Theorem 26.11.3. *Assume that $P \in \Psi^m_{\text{phg}}(X)$ is properly supported and satisfies condition (P). Let K be a compact subset of X such that the equivalent conditions in Theorem 26.11.1 are fulfilled. Then it follows that*

$$N(K) = \{v \in \mathscr{E}'(K),\ P^* v = 0\}$$

is a finite dimensional subspace of $C^\infty_0(K)$ orthogonal to $P\mathscr{D}'(X)$. For every $f \in H^{\text{loc}}_{(s)}(X)$ with $(f, N(K)) = 0$ and every $t < s+m-1$ one can find $u \in H^{\text{loc}}_{(t)}(X)$ satisfying the equation $Pu = f$ in a neighborhood of K. (If $s = \infty$ one can take $t = \infty$.)

Proof. That $N(K)$ is a finite dimensional subspace of $C^\infty_0(K)$ follows from Theorem 26.11.2 exactly as in the proof of Theorem 26.1.7. By condition (i) in Theorem 26.11.1 we can choose a compact neighborhood K' of K for which the hypotheses are still fulfilled and $N(K') = N(K)$, so it suffices to prove that the equation can be satisfied in the interior of K. The proof then proceeds as that of Theorem 26.1.7 except that in (26.1.5) and (26.1.6) we must replace t by a larger number in $\|P^*v\|_{(t)}$, so we obtain (26.1.7) for any $t>1-m-s$. The existence of a solution then follows as before.

It is now natural to extend Definition 26.1.8 and end the chapter by defining the terminology used in the title:

Definition 26.11.4. Let $P\in\Psi^m_{\text{phg}}(X)$ be properly supported and satisfy condition (P) in X. We shall then say that P is of principal type in X if the conditions in Theorem 26.11.1 are satisfied for every K.

When P is of principal type we have proved in this chapter that the equation $Pu=f$ can be solved on an arbitrary compact set when f satisfies a finite number of compatibility conditions there.

Notes

For operators of real principal type a local existence theorem was proved in Hörmander [1]. The example $D_1+iD_2+i(x_1+ix_2)D_3$ due to Lewy [1] showed that the result was not true in general for complex coefficients. This led to the proof in Hörmander [11] of a necessary condition for solvability. Solvability was proved in Hörmander [10] under a stronger form of this condition, and the results were made semi-global in "Linear partial differential operators". (See also Calderón [2].) Mizohata [4] observed that the same methods are applicable in some other cases such as the "Mizohata operators" $D_1+ix_1^kD_2$. The importance of this became clear when Nirenberg-Treves [1] showed that the local solvability properties of arbitrary first order differential operators with analytic coefficients could be analysed by means of closely related examples. A few years later Nirenberg-Treves [2] extended their results to the higher order case and even to pseudo-differential operators. They proved that P is not solvable if with the notation in Theorem 26.4.7 $\operatorname{Im} qp$ changes sign from $-$ to $+$ at a zero of finite order on a bicharacteristic for $\operatorname{Re} qp$. For first order zeros this was known from Hörmander [11, 17]. The same necessary condition was found by Egorov [2]. A decisive point in this work is the theorem of Egorov [1] which allows a simplification of the principal symbol by conjugation with a Fourier integral operator. Nirenberg and Treves [2] also conjectured the necessity of condition (Ψ) for local solvability and proved its invariance. The idea of the full proof given here is due to Moyer [1]. It contains the invariance proof for condition (Ψ) as an essential component. The proof was previously presented in Hörmander [40].

Nirenberg and Treves [2] proved the sufficiency of condition (P) for local solvability in the analytic case. The analyticity assumption was removed by Beals-Fefferman [1] but the result remained local. Indeed, it did not even give local existence of C^∞ solutions for C^∞ right-hand sides. That such solutions exist was proved in Hörmander [37] where a semi-global existence theory was also added. The key to this is the proof of theorems on propagation of singularities. In the real constant coefficient case such results go back to Grušin [1] (see the notes to Chapter VIII) and were proved in general by Hörmander [25]. The detailed discussion of operators of real principal type in Section 26.1 is taken from Duistermaat-Hörmander [1] where it was given as an application of the theory of Fourier integral operators. The results for the involutive case in Section 26.2 were also proved there. Normal forms in the symplectic case were first given by Sato-Kawai-Kashiwara [1] in the analytic (hyperfunction) case. The C^∞ results in Section 26.3 are due to Duistermaat and Sjöstrand [1]. The geometrical arguments in Section 26.5 come from Hörmander [37]. Section 26.7 is an improvement of results there due to Dencker [1], and the key estimates in Section 26.8 are due to Nirenberg-Treves [2]. They are first used in Section 26.9 to prove the extension of the superharmonicity theorem of Duistermaat-Hörmander [1] given in Hörmander [37], and they are also essential in Section 26.10. The main result there is due to Dencker [1]. The methods of Beals-Fefferman [1] are also very essential in the proof. The standard conclusions in Section 26.11 are taken from Hörmander [37].

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