Chapter 2
Cyclic covers of the projective line

Recall that we will study variations of Hodge structures of families of cyclic coverings of the projective line. Moreover some families of such covers are suitable for the construction of families of Calabi-Yau manifolds with dense sets of complex multiplication fibers. In order to understand variations of Hodge structures of such families of cyclic coverings we need to understand the Hodge structure of a cyclic covering $C \to \mathbb{P}^1$.

A cyclic cover $\pi : C \to \mathbb{P}^1$ is given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}, \tag{2.1}$$

where each d_k is an integer satisfying $1 \leq d_k \leq m - 1$. The numbers d_k are not uniquely determined by the isomorphism class of a cover. However, these numbers determine the isomorphism class of a cover and we will use them for the computation of the variation of Hodge structures in the following chapters.

In Section 2.1 we give a general description of cyclic covers of $\mathbb{P}^1$ and explain which tuples $(d_1, \ldots, d_n)$ yield equivalent covers. We will see that the Galois group action of the cyclic covering yields an eigenspace decomposition of $\pi_*(\mathbb{C})$ over the complement of the branch points. In Section 2.2 we use the branch indices d_k for the description of the monodromy representations of these eigenspaces. We have also an eigenspace decomposition of $H^1(C, \mathbb{C})$ by the Galois group action, which can also be described by using the branch indices d_k, as we will do in Section 2.3. In the next chapter this eigenspace decomposition will be extended to an eigenspace decomposition of the VHS of our families of cyclic coverings of $\mathbb{P}^1$. In Section 2.4 we cover certain curves C given by (2.1) by a Fermat curve, which implies that each of these certain curves C has CM.

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2.1 Description of a cyclic cover of the projective line

Let us first repeat some known facts about Galois covers of $\mathbb{P}^1$.

Definition 2.1.1. Let T_1, T_2, and S be topological spaces resp., complex manifolds resp., algebraic varieties. The coverings $f_1 : T_1 \to S$ and $f_2 : T_2 \to S$, which are morphisms in the respective category, are called equivalent, if there is an isomorphism $g : T_1 \to T_2$ in the respective category such that $f_1 = f_2 \circ g$.

Proposition 2.1.2. *Let G be a finite group, and $S := \{a_1, \ldots, a_n\} \subset \mathbb{A}^1 \subset \mathbb{P}^1$. There is a correspondence between the following objects:*

1. *The isomorphism classes of Galois extensions of $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(x)$ with Galois group G and branch points contained in S.*
2. *The equivalence classes of (non-ramified) Galois coverings $f : R \to \mathbb{P}^1 \setminus S$ of topological spaces with deck transformation group isomorphic to G.*
3. *The normal subgroups in the fundamental group $\pi_1(\mathbb{P}^1 \setminus S)$ with quotient isomorphic to G.*

Proof. (see [62], Theorem 5.14) □

Remark 2.1.3. We will need to understand the correspondence of the preceding Proposition. The correspondence between (1) and (2) is given by the facts that a Galois covering $f : R \to \mathbb{P}^1 \setminus S$ (of topological spaces) yields a covering $f : \bar{R} \to \mathbb{P}^1$ of compact Riemann surfaces, and any morphism of compact Riemann surfaces corresponds to an embedding of their function fields.

The correspondence between (2) and (3) is given by the path lifting properties of coverings of Hausdorff spaces. Take $b \in R$. Let $p = f(b)$, and $\gamma \in \pi_1(\mathbb{P}^1 \setminus S, p)$, and $f^*(\gamma(0)) = b$. Then $f^*(\gamma(1)) = g \cdot b$ for some $g \in G \cong \mathrm{Deck}(R/(\mathbb{P}^1 \setminus P))$. This induces a homomorphism $\Phi_b : \pi_1(\mathbb{P}^1 \setminus S, p) \to G$ and a kernel of this homomorphism, which is a normal subgroup G.

Remark 2.1.4. Let $f : R \to \mathbb{P}^1$ be a Galois covering with branch points $a_1, \ldots, a_n$. One can choose $\gamma_1, \ldots, \gamma_n \in \pi_1(\mathbb{P}^1 \setminus P)$ such that each γ_k is given by a loop running counterclockwise "around" exactly one a_k. Hence one has that

$$\gamma_n = \gamma_1^{-1} \ldots \gamma_{n-1}^{-1}$$

and we conclude that

$$\Phi_b(\gamma_n) = \Phi_b(\gamma_1)^{-1} \ldots \Phi_b(\gamma_{n-1})^{-1}.$$

From now on we consider only irreducible cyclic covers of $\mathbb{P}^1$. An irreducible cyclic cover can be given by a prime ideal

$$(y^m - (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}) \subset \mathbb{C}[x, y].$$

First this ideal defines only an affine curve in $\mathbb{A}^2$, which has singularities, if there are some $d_k > 1$. But there exists a unique smooth projective curve C birationally equivalent to this affine curve. By the natural projection onto the x-axis, one obtains a cyclic cover of the smooth curve C onto $\mathbb{P}^1$.

Remark 2.1.5. Let us consider the cover given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n},$$

and fix a $k_0 \in \{1, \ldots n\}$. By an automorphism of $\mathbb{P}^1$, one can put a_{k_0} onto 0. Let $\mu_{k_0} = \frac{d_{k_0}}{m} \in \mathbb{Q}$, and D a small disc centered in 0, which does not contain any other a_k with $k \neq k_0$. Take any point $p \in \partial D$ and remove the segment $[0, p]$. The topological space $D \setminus [0, p]$ is simply connected. Hence one can define root functions $z \to z^{\mu_{k_0}}$ on this space, which are given by:

$$z^{\mu_{k_0}} = |z|^{\mu_{k_0}} \exp(\frac{2\pi i t d_{k_0}}{m} + 2\pi i \frac{\ell}{m}) \text{ (with } \ell = 0, 1, \ldots, m-1 \text{ and } z = |z| \exp(2\pi i t))$$

Since the cover is given by $y^m = x^{d_{k_0}}$ resp., $y = x^{\mu_{k_0}}$ over a small disc around 0, we may lift a closed path around 0 to some path with starting point $(z, z^{\mu_{k_0}})$ and ending point $(z, e^{2\pi i \mu_{k_0}} z^{\mu_{k_0}})$.

Definition 2.1.6. Let $e^{2\pi i \mu_{k_0}}$ and d_{k_0} be given by Remark 2.1.5. Then $e^{2\pi i \mu_{k_0}}$ is the local monodromy datum of d_{k_0}.

Lemma 2.1.7. *Assume that $d_1, \ldots, d_n < m$. Let the (non-singular projective) curve C be given by*

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}.$$

Then the Galois group G is $\mathbb{Z}/(m)$, and the covering $C \to \mathbb{P}^1$ is given by the kernel of the homomorphism Φ given by $\gamma_k \to d_k \in \mathbb{Z}/(m)$. The point ∞ is a branch point and

$$\Phi(\gamma_\infty) = -\sum_{k=1}^{n} d_k \mod m,$$

if and only if m does not divide $\sum_{k=1}^{n} d_k$.

Proof. The last statement of the lemma follows by the preceding rest of the lemma and the Remark 2.1.4.

The Galois group and $\mathbb{Z}/(m)$ are obviously isomorphic. Let us remove the ramification points of C. Then we obtain a Riemann surface R. Now take a small loop γ_k around p_k, which starts and ends in $p \in \mathbb{P}^1$. Moreover take a point $b \in R$ with $f(b) = p$. The definition of R and Remark 2.1.5 imply that the lifting $f^*(\gamma_k)$ of the path γ_k starting in b ends in the point $d_k \cdot b$. Hence the statement follows from Proposition 2.1.2 and Remark 2.1.3. $\square$

Let $d \in \mathbb{Z}$ and $1 < m \in \mathbb{N}$. The residue class of d in $\mathbb{Z}/(m)$ is denoted by $[d]_m$.

Remark 2.1.8. Let $G = \mathbb{Z}/(m)$, and $[d]_m \in \mathbb{Z}/(m)^*$. We consider the kernels of the monodromy representations of the covers locally given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}$$

and

$$y^m = (x - a_1)^{[dd_1]_m} \cdot \ldots \cdot (x - a_n)^{[dd_n]_m}.$$

By the preceding lemma, these kernels coincide. Hence we conclude that both covers are equivalent.

2.2 The local system corresponding to a cyclic cover

Now let us assume that our cover $\pi : C \to \mathbb{P}^1$ is given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n},$$

where m divides $d_1 + \ldots + d_n$ and ∞ is not a branch point. Moreover let

$$S := \{a_1, \ldots, a_n\}.$$

First let us consider the construction of a cyclic cover of an arbitrary algebraic manifold:

Remark 2.2.1. Let X be a complex algebraic manifold, $\mathcal{L}$ an invertible sheaf on X and

$$D = \sum b_k D_k$$

a normal crossing divisor on X, where $\mathcal{L}^m = \mathcal{O}(D)$ and $0 < b_k < m$ for each k. Then by $\mathcal{L}$ and D, one can construct a cyclic cover of degree m onto X (see [20], §3).

Definition 2.2.2. Let b_k and D_k be given by the previous remark. The number b_k is called the branch index of D_k with respect to this cyclic cover.

Example 2.2.3. In the case of

$$X = \mathbb{P}^1, \quad D = \sum_{k=1}^{n} d_k a_k, \quad \mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(\frac{1}{m}\sum_{k=1}^{n} d_k),$$

the cyclic cover of Remark 2.2.1 is given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}.$$

Next we describe the local system $\pi_*(\mathbb{C})|_{\mathbb{P}^1 \setminus S}$ and its monodromy.

Lemma 2.2.4. *Let V be a $\mathbb{C}$-vector space of dimension n, and X be an arcwise connected and locally simply connected topological space with $x \in X$. Then the monodromy representation provides a bijection between the set of isomorphism classes of local systems of stalk V on X and the set of representations*

$$\pi_1(X, x) \to \mathrm{GL}_n(\mathbb{C}),$$

modulo the action of $\mathrm{Aut}_{\mathbb{C}}(V)$ by conjugation.

Proof. (see [61], Remarque 15.12) □

Since $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^*$ is commutative, we can conclude:

Corollary 2.2.5. *The monodromy yields a bijection between the set of isomorphism classes of rank one local systems on $\mathbb{P}^1 \setminus S$ and the set of representations*

$$\pi_1(\mathbb{P}^1 \setminus S) \to \mathrm{GL}_1(\mathbb{C}).$$

The Galois group of our covering curve is isomorphic to $\mathbb{Z}/(m)$ and generated by a map ψ, which is given by $(x, y) \to (x, e^{2\pi i \frac{1}{m}} y)$ with respect to the above affine curve contained in $\mathbb{A}^2$, which is birationally equivalent to the covering curve. Hence a character χ of this group is determined by $\chi(\psi)$ with $\chi(\psi) \in \{e^{2\pi i \frac{j}{m}} | j = 0, 1, \ldots, m-1\}$. Thus the character group is isomorphic $\mathbb{Z}/(m)$ and we identify the character, which maps ψ to $e^{2\pi i \frac{j}{m}}$, with $j \in \mathbb{Z}/(m)$.[1]

Let D be an arbitrary disc contained in $\mathbb{P}^1 \setminus S$. The preimage of D is given by the disjoint union of discs D_r with $r = 0, 1, \ldots, m-1$ such that $\psi(D_r) = D_{[r+1]_m}$. The vector space $\pi_* \mathbb{C}_C|_{\mathbb{P}^1 \setminus S}(D)$ has the basis $\{v_j | j = 0, 1, \ldots, m-1\}$, where

$$v_j := (e^{\frac{2\pi j(m-1)}{m}}, \ldots, e^{\frac{2\pi j}{m}}, 1),$$

and the r-th. coordinate denotes the value of the corresponding section of $\pi^{-1}(D)$ on D_r. By the push-forward action, each v_j is an eigenvector with respect to the character given by j. Since D is arbitrary, one can glue the local eigenspaces, and obtain an eigenspace decomposition

$$\pi_* \mathbb{C}_C|_{\mathbb{P}^1 \setminus S} = \bigoplus_{j=0}^{m-1} \mathbb{L}_j$$

[1] These two identifications with $\mathbb{Z}/(m)$ are obviously not canonical, but useful for the description of $\pi_* \mathbb{C}_C|_{\mathbb{P}^1 \setminus S}$ by using our explicit equation for $\pi : C \to \mathbb{P}^1$ as we will see a little bit later.

into rank 1 local systems, where $\mathbb{L}_j$ is the eigenspace with respect to the character given by $j \in \mathbb{Z}/(m)$. Hence the monodromy representation $\rho : \pi_1(\mathbb{P}^1 \setminus S) \to GL_m(\mathbb{C})$ has the corresponding decomposition

$$\rho = (\rho_0, \rho_1, \ldots, \rho_{m-1}) : \pi_1(\mathcal{X}) \to \prod_{i=0}^{m-1} \mathrm{GL}_1(\mathbb{C}),$$

where

$$\rho_j : \pi_1(\mathbb{P}^1 \setminus S) \to \mathrm{GL}_1(\mathbb{C})$$

is the monodromy representation of $\mathbb{L}_j$ for all $j = 0, 1, \ldots, m-1$.

Let us recall that our cyclic cover C is given by

$$y^m = (x - a_1)^{d_1} \ldots (x - a_n)^{d_n},$$

where ∞ is not a branch point. Now let $x \in \mathbb{P}^1 \setminus S$, and $x \in D$, where D is a sufficiently small open disc as above. Take a counterclockwise loop γ_k around a_k and cover the loop with a finite number of (sufficiently) small discs. The continuation of $\tilde{s}$ on the unification of these discs leads to a multisection. By Remark 2.1.5, the possible liftings $\gamma_k^{(r)}$ of the loop γ_k are paths with starting point $\gamma_k^{(r)}(0) = y_r$, where $y_r \in D_r$ and ending point $\gamma_k^{(r)}(1) = y_{[d_k+r]_m}$. This implies that the monodromy representation of $\mathbb{L}_j$ maps γ_k to $e^{\frac{2\pi j d_k}{m}}$. Hence we conclude:

Theorem 2.2.6. *Let the cyclic cover $\pi : C \to \mathbb{P}^1$, which is not branched over ∞, be given by*

$$y^m = (x - a_1)^{d_1} \ldots (x - a_n)^{d_n}. \tag{2.2}$$

Then the local system $\pi_* \mathbb{C}|_{\mathbb{P}^1 \setminus S}$ is given by the monodromy representation

$$\gamma_k \to \{(x_j)_{j=0,1\ldots,m-1} \to (e^{\frac{2\pi i j d_k}{m}} x_j)_{j=0,1\ldots,m-1}\}.$$

Remark 2.2.7. One can consider $\pi_*(\mathbb{Q}(e^{2\pi i \frac{1}{m}}))|_{\mathbb{P} \setminus S}$, too. Since a generator ψ of $\mathrm{Gal}(C; \mathbb{P}^1)$ satisfies $\psi^m = 1$, the minimal polynomial of its action on $\pi_*(\mathbb{Q}(e^{2\pi i \frac{1}{m}}))|_{\mathbb{P} \setminus S}$ decomposes into linear factors contained in $\mathbb{Q}(e^{2\pi i \frac{1}{m}})[x]$. Hence the eigenspace decomposition is defined over $\mathbb{Q}(e^{2\pi i \frac{1}{m}})$.

Each local system L of $\mathbb{C}$-vector spaces on any topological space X has a dual local system $L^\vee$ given by the sheafification of the presheaf

$$U \to \mathrm{Hom}_{\mathbb{C}}(L, \mathbb{C}).$$

Proposition 2.2.8. *One has*

$$\mathbb{L}_j^\vee = \bar{\mathbb{L}}_j.$$

Furthermore the monodromy representation $\mu_{\mathbb{L}_j^\vee}$ of $\mathbb{L}_j^\vee$ is given by $\mu_{\mathbb{L}_j^\vee}(\gamma_s) = \overline{\mu_{\mathbb{L}_j}(\gamma_s)}$ for all $s \in S$.

Proof. (see [19], Proposition 2) □

Hence by the respective monodromy representations, we obtain for all $j = 1, \ldots, m-1$:

Corollary 2.2.9.

$$\mathbb{L}_j^\vee = \mathbb{L}_{m-j}$$

Let $r|m$. We consider the $\mathbb{C}$-algebra endomorphism Φ_r of $\mathbb{C}[x,y]$ given by $x \to x$ and $y \to y^r$. The (non-singular) curve C is birationally equivalent to the affine variety given by $\mathrm{Spec}(\mathbb{C}[x,y]/I)$, where

$$I = (y^m - (x-a_1)^{d_1} \ldots (x-a_n)^{d_n}).$$

By Φ_r, we obtain the prime ideal

$$\Phi_r^{-1}(I) = (y^{\frac{m}{r}} - (x-a_1)^{d_1} \ldots (x-a_n)^{d_n}).$$

Let C_r be the irreducible projective non-singular curve birationally equivalent to the affine variety given by $\mathrm{Spec}(\mathbb{C}[x,y]/\Phi_r^{-1}(I))$.

Remark 2.2.10. By the equation above, we have a cover $\pi_r : C_r \to \mathbb{P}^1$ of degree $\frac{m}{r}$. The homomorphism Φ_r induces a cover $\phi_r : C \to C_r$ of degree r such that

$$\pi = \pi_r \circ \phi_r.$$

Proposition 2.2.11.

$$(\pi_r)_* \mathbb{C}_{C_r}|_{\mathbb{P}^1 \setminus S} = \bigoplus_{j=0}^{\frac{m}{r}-1} \mathbb{L}_{r \cdot j} \subset \pi_* \mathbb{C}_C|_{\mathbb{P}^1 \setminus S}.$$

Proof. Let $m_0 := \frac{m}{r}$. By Theorem 2.2.6, the monodromy representation of the local system $(\pi_r)_* \mathbb{C}_{C_r}|_{\mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}}$ is given by

$$\gamma_k \to \{(x_j)_{j=0,1\ldots,\frac{m}{r}-1} \to (e^{\frac{2\pi i j d_k}{m_0}} x_j)_{j=0,1\ldots,\frac{m}{r}-1} = (e^{\frac{2\pi i j r d_k}{m}} x_j)_{j=0,1\ldots,\frac{m}{r}-1}\}.$$

By the respective monodromy representations of the local systems $\mathbb{L}_j$, this yields the statement. □

2.3 The cohomology of a cover

In this section we discuss some known facts about the eigenspace decomposition of the Hodge structure of a curve C with respect to a cyclic cover $\pi : C \to \mathbb{P}^1$. The main reference for this section is given by §3 of the book [20] of H. Esnault and E. Viehweg. Section 2 of the essay [18] of P. Deligne and G. D. Mostow contains additional information about our case.

Let $\pi : C \to \mathbb{P}^1$ be given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}$$

such that ∞ is not a branch point,

$$S = \{a_1, \ldots, a_n\}, \ \ D = d_1 a_1 + \ldots + d_n a_n \ \text{ and } \ \mathcal{L}^{(j)} = \mathcal{O}_{\mathbb{P}^1}(j \frac{d_1 + \ldots + d_n}{m} - \sum_{k=1}^{n+3} [\frac{j}{m} \cdot d_k]).$$

Moreover let the generator ψ of the Galois group of π be given by $(x, y) \to (x, e^{2\pi i \frac{1}{m}} y)$ with respect to the explicit equation above, which yields π.

We fix some new notation: Let $q \in \mathbb{Q}$ and $[q]$ denote the largest integer, which is smaller than q. Then we define $[q]_1 := q - [q]$. Moreover we define

$$S_j := \{a \in S | [j\mu_a]_1 \neq 0\}.$$

Proposition 2.3.1. *The sheaves $\pi_*(\mathcal{O})$ and $\pi_*(\omega)$ have a decomposition into eigenspaces with respect to the Galois group representation, which are given by the sheaves ${\mathcal{L}^{(j)}}^{-1}$ and*

$$\omega_j := \omega_{\mathbb{P}^1}(\log D^{(j)}) \otimes {\mathcal{L}^{(j)}}^{-1} \quad \textit{with} \quad D^{(j)} := \sum_{a \in S_j} a$$

for $j = 0, 1, \ldots, m-1$ such that ψ acts via pull-back by the character $e^{2\pi i \frac{j}{m}}$ on ${\mathcal{L}^{(j)}}^{-1}$ resp., ω_j.

Proof. The eigenspace decomposition of $\pi_*(\mathcal{O})$ follows by [20], Corollary 3.11. Moreover [20], Lemma 3.16, d) yields the decomposition of $\pi_*(\omega)$ into the claimed sheaves. Since ${\mathcal{L}^{(j)}}^{-1}$ is an eigenspace with respect to the Galois group representation, ω_j is an eigenspace of the same eigenvalue. □

Remark 2.3.2. One has obviously $h^0(\omega_0) = 0$. By [20], 2.3, c), one concludes that

$$\omega_{\mathbb{P}^1}(\log D^{(j)}) = \omega_{\mathbb{P}^1}(D^{(j)})$$

for $j = 1, \ldots, m-1$. Hence for $j = 1, \ldots, m-1$ we obtain

$$h^0(\omega_j) = h^0(\mathcal{O}_{\mathbb{P}^1}(-2 + deg(D^{(j)}) - j\frac{d_1 + \ldots + d_{n+3}}{m} + \sum_{k=1}^{n+3}[\frac{j}{m} \cdot d_k]))$$
$$= -1 + |S_j| + \sum_{a \in S_j}(-j\mu_a + [j\mu_a]) = -1 + \sum_{a \in S_j}(1 - [j\mu_a]_1).$$

But here we want to determine our eigenspaces on $\pi_*(\omega_C)$ with respect to the push-forward action. Thus we put $\omega^{(j)} := \omega_{[m-j]_m}$, and we obtain

$$h_j^{1,0}(C) := h^0(\omega^{(j)}) = h^0(\omega_{[m-j]_m}) = -1 + \sum_{a \in S_j}(1 - [(m-j)\mu_a]_1) = -1 + \sum_{a \in S_j}[j\mu_a]_1.$$

Moreover let $H_j^{0,1}(C)$ denote the vector space of antiholomorphic 1-forms on C with respect to the corresponding character of the Galois group action. Since the push-forward action of the Galois group respects the alternating form of the polarization of the Hodge structure on $H^1(C,\mathbb{Z})$, one concludes that $H^{0,1}_{[m-j]_m}(C)$ is the dual of $H_j^{1,0}(C)$. Thus:

Proposition 2.3.3. *We have the eigenspace decomposition*

$$H^1(C,\mathbb{C}) = \bigoplus_{j=1}^{m-1} H_j^1(C,\mathbb{C}) \quad \textit{with} \quad H_j^{1,0}(C) \oplus H_j^{0,1}(C) = H_j^1(C,\mathbb{C}).$$

Moreover by $h_j^{0,1}(C) = h^{1,0}_{[m-j]_m}(C)$ and the preceding calculations, one concludes:

Proposition 2.3.4. *We have*

$$h_j^{1,0}(C) = \sum_{s \in S_j}[j\mu_s]_1 - 1, \quad \textit{and} \quad h_j^{0,1}(C) = \sum_{s \in S_j}(1 - [j\mu_s]_1) - 1.$$

The preceding two propositions imply:

Corollary 2.3.5.

$$h_j^1(C,\mathbb{C}) = |S_j| - 2$$

2.4 Cyclic covers with complex multiplication

Let us now search for examples of covers of $\mathbb{P}^1$ with complex multiplication. The family given by

$$\mathbb{P}^2 \supset V(y^m - x_1(x_1 - x_0)(x_1 - a_1x_0)\ldots(x_1 - a_{m-3}x_0))$$
$$\to (a_1, \ldots, a_{m-3}) \in (\mathbb{A}^1 \setminus \{0,1\})^{m-3} \setminus \{a_i = a_j | i \neq j\}$$

has obviously a fiber isomorphic to the Fermat curve $\mathbb{F}_m$, which is given by $V(y^m+x^m+1)$ and has complex multiplication (see [22] and [32]). For another family with a fiber with complex multiplication, we must work a little bit.

Lemma 2.4.1. *If (V, h_1) and (W, h_2) are two $\mathbb{Q}$-Hodge structures of weight k, then*

$$\mathrm{Hg}(V \oplus W, h_1 \oplus h_2) \subset \mathrm{Hg}(V, h_1) \times \mathrm{Hg}(W, h_2) \subset \mathrm{GL}(V) \times \mathrm{GL}(W) \subset \mathrm{GL}(V \oplus W),$$

and the projections

$$\mathrm{Hg}(V \oplus W) \to \mathrm{Hg}(V), \quad \textit{and} \quad \mathrm{Hg}(V \oplus W) \to \mathrm{Hg}(W)$$

are surjective.

Proof. (see [58], Lemma 8.1) □

Lemma 2.4.2. *Let $V \subset W$ be a rational sub-Hodge structure of a polarized Hodge structure W. Then we have a direct sum decomposition*

$$W = V \oplus V',$$

where V' is also a rational sub-Hodge structure of W.

Proof. (see [61], Lemme 7.26) □

Lemma 2.4.3. *A curve C, which is covered by the Fermat curve $\mathbb{F}_m$ given by $V(x^m + y^m + z^m) \subset \mathbb{P}^2$ for some $1 \leq m \in \mathbb{N}$, has complex multiplication.*

Proof. A covering $\mathbb{F}_m \to C$ yields an injective vector space homomorphism

$$H^1(C, \mathbb{Q}) \to H^1(\mathbb{F}_m, \mathbb{Q}),$$

which extends to an embedding of Hodge structures (see [61], 7.3.2 for more details). This embedding induces a direct sum decomposition into two rational sub-Hodge structures of $H^1(\mathbb{F}_m, \mathbb{Q})$ (see Lemma 2.4.2). Hence by Lemma 2.4.1 and the fact that $\mathbb{F}_m$ has complex multiplication, one obtains the statement. □

Theorem 2.4.4. *Let $0 < d_1, d < m$, and ξ_k denote a primitive k-th. root of unity for all $k \in \mathbb{N}$. Then the curve C, which is given by*

$$y^m = x^{d_1} \prod_{i=1}^{n-2} (x - \xi_{n-2}^i)^d,$$

is covered by the Fermat curve $\mathbb{F}_{(n-2)m}$ given by $V(y^{(n-2)m} + x^{(n-2)m} + 1)$ and has complex multiplication.

Proof. Let C be the curve, which is given by

$$y^m = x^{d_1} \prod_{i=1}^{n-2} (x - \xi_{n-2}^i)^d,$$

and $\phi : \mathbb{A}^2 \to \mathbb{A}^2$ be the morphism, which is given by $y \to yx^{d_1}$ and $x \to x^m$. By a little abuse of notation, we denote by $C \cap \mathbb{A}^2$ the singular affine curve given by the equation above, which is birationally equivalent to C. The corresponding homomorphism $\phi^* : \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ sends the ideal, which defines $C \cap \mathbb{A}^2$, to the ideal generated by

$$y^m x^{m \cdot d_1} - x^{m \cdot d_1} \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^d.$$

This is contained in the ideal generated by

$$y^m - \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^d. \tag{2.3}$$

Let $m_0 := \frac{m}{gcd(m,d)}$, and $d_0 := \frac{d}{gcd(m,d)}$. It is obvious that

$$y^m - \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^d = \prod_{j=0}^{gcd(m,d)-1} (y^{m_0} - \xi_{gcd(m,d)}^j \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^{d_0}).$$

Now we take the curve C_1, which is given by

$$y^{m_0} = \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^{d_0}.$$

By the definitions of m_0 and d_0, and Remark 2.1.8, the curve C_1 is given by

$$y^{m_0} = \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i),$$

too. Hence this curve irreducible, and ϕ induces a cover $C_1 \to C$ resp., ϕ^* induces a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[C \cap \mathbb{A}^2] \to \mathbb{C}[C_1 \cap \mathbb{A}^2]$. By $x \to x$ and $y \to y^{n-2\frac{m}{m_0}}$, we get a cover of the Fermat curve $\mathbb{F}_{(n-2)m}$ given by $V(y^{(n-2)m} + x^{(n-2)m} + 1)$ onto C_1. Now we use the composition of these covers $\mathbb{F}_{(n-2)m} \to C_1$ and $C_1 \to C$, and Lemma 2.4.3. This yields the statement. □

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