Chapter 2
The Stochastic Schrödinger Equation

2.1 Introduction

In this chapter, we introduce the theory of measurements in continuous time (diffusive case) starting from the particular but important case of complete observation. This allows to present the Hilbert space formulation of the theory, where the state of the observed quantum system is described by a vector in the Hilbert space $\mathcal{H}$ of the system. Even if this is a special case of the more general theory presented in Chaps. 3, 4 and 5, it deserves a separate treatment for different reasons: it is instructive, it uses only the Hilbert space formulation of quantum mechanics, it is of interest on its own because the stochastic Scrödinger equation presented in this chapter has also been used in different contexts [1–6], some mathematical results of the following chapter will relay anyhow on the theory presented here, and Hilbert space SDEs are the key starting point for efficient numerical simulations of the dynamics of open quantum systems [1, 7].

First, we introduce the class of SDEs in Hilbert spaces which we are interested in and we present their mathematical properties. After that, we discuss their physical interpretation and start to develop the theory of continuous measurements.

Given the initial (pure) state $\psi_0 \in \mathcal{H}$ of the measured quantum system, the aim is to get two stochastic processes together with the probability distribution of their trajectories:

- the output $W(t)$ of the continuous measurement;
- the system state $\widehat{\psi}(t)$, whose evolution includes the continuous measurement and which is continuously conditioned on the observed output;
- the physical probability distribution of the processes $W(t)$ and $\widehat{\psi}(t)$.

The system state $\widehat{\psi}(t)$ is called a posteriori state, as it depends on the trajectory observed for $W(s)$ in the time interval $0 \leq s \leq t$. The knowledge of the physical probability distribution of $W(t)$ and $\widehat{\psi}(t)$ allows to consider and to compute mean values at a given time, just as correlations and multi-time moments.

There are two possible ways to develop the theory: to start from the nonlinear evolution equation of the a posteriori states $\widehat{\psi}(t)$ or from the linear evolution equation of the so-called non-normalised a posteriori states $\psi(t)$. We prefer to begin with

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this second approach which is the direct generalisation of the traditional description of an instantaneous measurement.

When a quantum system undergoes a "von Neumann measurement" of an observable represented by a self-adjoint operator X with discrete eigenvalues x_k and eigen-projections E_k, one usually fixes the space $\Omega = \{x_1, x_2, \ldots\}$ of the possible outcomes and, for every $x_k \in \Omega$, uses the corresponding projection E_k to introduce the linear state transformation (von Neumann reduction postulate):

$$\psi_0 \mapsto \psi_1(x_k) := E_k \psi_0.$$

Then, $\psi_1(x_k)$ gives both the physical probability distribution for the outcome X and the a posteriori state $\widehat{\psi}_1$: if ψ_0 is the initial system state, then

- $\|\psi_1(x_k)\|^2$ is the probability of observing $X = x_k$;
- $\widehat{\psi}_1(x_k) = \psi_1(x_k) / \|\psi_1(x_k)\|$ is the a posteriori state when $X = x_k$.

In order to generalise consistently such a representation of a measurement to the continuous time case, we use the powerful mathematical tools of stochastic calculus and thus we prefer to begin with their presentation.

Section 2.2 is devoted to the theory of homogeneous linear SDEs. To read this section, one needs the notions of filtration, stochastic process, martingale, stochastic integral with respect to a Wiener process and strong solution of an SDE; moreover, familiarity with the Itô formula is essential. All these topics of stochastic calculus are recalled in Sects. A.2, A.3 and A.4. In Sect. 2.3, the subclass of linear SDEs of our concern is presented and studied. Here the notions of exponential martingale, change of probability measure and Girsanov transformation are needed; they are recalled in Sect. A.5.

The SDE approach to the quantum theory of open systems and of continuous measurements is given in the rest of the chapter, starting from Sect. 2.4. In this chapter, only the Hilbert space formulation of quantum mechanics is needed, as it is presented in Sect. B.2. The key notion is "positive operator-valued measure", a mathematical object which represents a general quantum mechanical observable.

As already said in Sect. 1.3, we work in a finite dimensional Hilbert space, which is enough to give the main ideas of the stochastic approach to open systems and continuous measurements and to develop the simplest applications. For results and examples in infinite dimensional Hilbert spaces, see [7–27].

2.2 Linear Stochastic Differential Equations

Assumption 2.1. The Hilbert space of the quantum system is $\mathcal{H} = \mathbb{C}^n$.

The SDEs we consider are driven by white noise, the derivative of the Wiener process. So, let us introduce such a stochastic process and fix the framework needed for SDEs.

Assumption 2.2. We fix a stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$ satisfying usual conditions (Sect. A.2.2) and a continuous d-dimensional Wiener process $W = \{W(t), t \geq 0\}$, with increments independent of the past (Definition A.21). We assume

$$\mathcal{F} = \mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t \,. \tag{2.1}$$

The symbol $\mathbb{E}_\mathbb{Q}$ indicates the expectation with respect to the probability $\mathbb{Q}$.

2.2.1 An Homogeneous Linear SDE in Hilbert Space

Let us start by considering a generic homogeneous linear SDE with "multiplicative noise" for an $\mathcal{H}$-valued process $\psi = \{\psi(t),\ t \geq 0\}$:

$$\begin{cases} \mathrm{d}\psi(t) = K(t)\psi(t)\,\mathrm{d}t + \displaystyle\sum_{j=1}^{d} R_j(t)\psi(t)\,\mathrm{d}W_j(t)\,, \\ \psi(0) = \psi_0\,, \qquad\qquad\qquad\qquad\qquad \psi_0 \in \mathcal{H}. \end{cases} \tag{2.2}$$

Assumption 2.3. The initial condition ψ_0 is non random. The coefficients $R_j(t)$, $K(t)$ are (non-random) linear operators on $\mathcal{H}$. The functions $t \mapsto K(t)$ and $t \mapsto R_j(t)$ are measurable and such that $\forall T \in (0, +\infty)$

$$\sup_{t \in [0,T]} \|K(t)\| < +\infty, \qquad \sup_{t \in [0,T]} \Big\| \sum_j R_j(t)^* R_j(t) \Big\| < +\infty. \tag{2.3}$$

Theorem 2.4. *Under Assumption 2.3, the linear SDE* (2.2) *admits strong solutions in* $[0, +\infty)$*. Pathwise uniqueness and uniqueness in law hold. Moreover, for any* $p \geq 2$ *and* $T > 0$*, there exists a constant* $C(p, T)$ *such that*

$$\mathbb{E}_\mathbb{Q}\left[\sup_{t \in [0,T]} \|\psi(t)\|^p \right] \leq C(p, T)\big(1 + \|\psi_0\|^p\big)\,. \tag{2.4}$$

Proof. Let us make the identifications $b(x, t) = K(t)x$, $\sigma_j(x, t) = R_j(t)x$. We have the estimates

$$\|b(x, t)\| = \|K(t)x\| \leq \|K(t)\|\, \|x\|\,,$$

$$\sum_j \big\|\sigma_j(x, t)\big\|^2 = \sum_j \big\|R_j(t)x\big\|^2 = \big\langle x \big| \textstyle\sum_j R_j(t)^* R_j(t)x \big\rangle \leq \Big\| \textstyle\sum_j R_j(t)^* R_j(t) \Big\|\ \|x\|^2\,.$$

Obviously, we also have $\sum_j \big\|\sigma_j(x, t) - \sigma_j(y, t)\big\|^2 = \sum_j \big\|\sigma_j(x - y, t)\big\|^2$ and $\|b(x, t) - b(y, t)\| = \|b(x - y, t)\|$. Then, Hypotheses A.25, A.32 and A.34 hold with $L(T) = 2\max\Big\{\sup_{t\in[0,T]}\|K(t)\|^2,\ \sup_{t\in[0,T]}\Big\|\sum_j R_j(t)^* R_j(t)\Big\|\Big\}$, $M(T) = \sqrt{2L(T)}$, and Theorems A.36 and A.38 give the statements. □

Let us recall that the existence of strong solutions means that (2.2) admits a solution for every choice of the probability space, of the filtration and of the Wiener process (see Definition A.27). For the notions of uniqueness see Definitions A.28 and A.29.

In our construction, the stochastic basis and the Wiener process are fixed by Assumption 2.2. Then, by ψ we denote the continuous, adapted process (Itô process – see Sect. A.3.4) satisfying

$$\psi(t) = \psi_0 + \int_0^t K(s)\psi(s)\,\mathrm{d}s + \sum_{j=1}^d \int_0^t R_j(s)\psi(s)\,\mathrm{d}W_j(s)\,; \tag{2.5}$$

such a process is unique up to indistinguishableness (Sect. A.4.1).

Remark 2.5. In the following, the natural filtration of the increments of the Wiener process and its augmented version will be important: for $0 \le s \le t$, we define

$$\mathcal{D}_t^s := \sigma\{W(r) - W(s),\ r\in[s,t]\}, \qquad \overline{\mathcal{D}}_t^s := \mathcal{D}_t^s \vee \mathcal{N}; \tag{2.6}$$

$\mathcal{N}$ is the class of the $\mathbb{Q}$-null sets in $\mathcal{F}$.

Because of the properties of a Wiener process, the filtration $\{\overline{\mathcal{D}}_t^s,\ t\in[s,+\infty)\}$ satisfies the usual conditions: $\overline{\mathcal{D}}_t^s$ is independent of $\mathcal{F}_s$ and $\overline{\mathcal{D}}_t^s \subset \overline{\mathcal{D}}_t^0 \subset \mathcal{F}_t \subset \mathcal{F}$, for $0 \le s \le t$.

Because of the existence of strong solutions and of the fact that the initial condition is non-random, the continuous $(\mathcal{F}_t)$-adapted process ψ is also $\left(\overline{\mathcal{D}}_t^0\right)$-adapted.

2.2.2 The Stochastic Evolution Operator

Equation (2.2) being a linear equation, we can introduce a stochastic process of operators $A_t^0(\omega)$ giving the application $\psi_0 \mapsto \psi(t,\omega)$. Indeed, let us consider the operator-valued processes A_t^s, with $t \ge s \ge 0$, defined by the SDE

$$\begin{cases} \mathrm{d}A_t^s = K(t)A_t^s\,\mathrm{d}t + \displaystyle\sum_{j=1}^d R_j(t)A_t^s\,\mathrm{d}W_j(t)\,, \\ A_s^s = \mathbb{1}\,. \end{cases} \tag{2.7}$$

This is a linear SDE for an $n\times n$-dimensional complex process; so, exactly as for (2.2), in $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$ there is a pathwise unique, continuous, adapted solution.

Proposition 2.6. *Under Assumption 2.3, the linear SDE* (2.7) *admits strong solutions in* $[s, +\infty)$, $\forall s \geq 0$. *Pathwise uniqueness and uniqueness in law hold. Moreover, for any* $p \geq 2$ *and* $T > s$, *there exists a constant* $C(p, T)$ *such that*

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[s,T]} \left\|A_t^s\right\|_2{}^p\right] \leq C(p,T)\left(1+n^{p/2}\right). \tag{2.8}$$

Proof. Let us make the identifications $b(a,t) = K(t)a$, $\sigma_j(a,t) = R_j(t)a$, $a \in M_n$. Now a, b, σ_j are vectors whose components are labelled by a couple of indices; then, the relevant norm is the Hilbert–Schmidt one (B.3). We have the estimates

$$\begin{aligned}\|b(a,t)\|^2 = \|K(t)a\|_2{}^2 &= \operatorname{Tr}\left\{a^*K(t)^*K(t)a\right\} = \operatorname{Tr}\left\{K(t)^*K(t)aa^*\right\}\\ &\leq \left\|K(t)^*K(t)\right\| \left\|aa^*\right\|_1 = \|K(t)\|^2 \operatorname{Tr}\left\{aa^*\right\} = \|K(t)\|^2 \|a\|_2{}^2,\end{aligned}$$

$$\begin{aligned}\sum_j \left\|\sigma_j(a,t)\right\|^2 &= \sum_j \left\|R_j(t)a\right\|_2{}^2 = \sum_j \operatorname{Tr}\left\{a^*R_j(t)^*R_j(t)a\right\}\\ &= \operatorname{Tr}\left\{\textstyle\sum_j R_j(t)^*R_j(t)aa^*\right\} \leq \left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\| \|aa^*\|_1\\ &= \left\|\sum_j R_j(t)^*R_j(t)\right\| \|a\|_2{}^2.\end{aligned}$$

Then, the proof goes on as in Theorem 2.4, exactly with the same constants. Note that $\|\mathbb{1}\|_2{}^2 = \operatorname{Tr}\{\mathbb{1}\} = n$. □

Because of the properties stated in the following proposition, A_t^s is called *stochastic evolution operator*. In mathematical terms, A_t^0 is the *fundamental matrix* of the linear equation (2.2), while in the physical literature the term *propagator* is more used.

Proposition 2.7. *For* $0 \leq s \leq t$, A_t^s *is* $\mathbb{Q}$*-independent of* $\mathcal{F}_s$ *and* $\overline{\mathcal{D}}_t^s$*-measurable. Moreover, for every given* $0 \leq r \leq s$, *almost surely (a.s.) we have*

$$A_t^s A_s^r = A_t^r, \qquad \forall t \geq s, \tag{2.9}$$

$$\psi(t) = A_t^0 \psi_0, \qquad \forall t \geq 0. \tag{2.10}$$

More explicitly, the continuous processes $t \mapsto A_t^s A_s^r$ and $t \mapsto A_t^r$ are indistinguishable; the same holds for the processes $t \mapsto \psi(t)$ and $t \mapsto A_t^0\psi_0$.

Proof. Because of the existence of strong solutions and pathwise uniqueness, the random variable A_t^s is $\overline{\mathcal{D}}_t^s$-measurable; then, the statement about the independence follows from the independent increment property of the Wiener process.

Let us fix $s \geq r \geq 0$ and set

$$B_t := \begin{cases} A_t^r, & \text{if } r \leq t < s, \\ A_t^s A_s^r, & \text{if } t \geq s. \end{cases}$$

Then, by (2.7) we have for $t \geq s$

$$\begin{aligned} B_t &= A_t^s A_s^r = A_s^r + \int_s^t K(u) A_u^s A_s^r \, \mathrm{d}u + \sum_j \int_s^t R_j(u) A_u^s A_s^r \, \mathrm{d}W_j(u) \\ &= \mathbb{1} + \int_r^s K(u) A_u^r \, \mathrm{d}u + \sum_j \int_r^s R_j(u) A_u^r \, \mathrm{d}W_j(u) + \int_s^t K(u) B_u \, \mathrm{d}u \\ &\quad + \sum_j \int_s^t R_j(u) B_u \, \mathrm{d}W_j(u) = \mathbb{1} + \int_r^t K(u) B_u \, \mathrm{d}u + \sum_j \int_r^t R_j(u) B_u \, \mathrm{d}W_j(u); \end{aligned}$$

by the definition of B, the same equation holds also for $t < s$. Therefore, B_t and A_t^r satisfy the same equation and, by uniqueness, they are indistinguishable. This proves (2.9). Similarly $\psi(t)$ and $A_t^0 \psi_0$ satisfy the same equation and, so, (2.10) holds. □

Also the adjoint A_t^{s*} of the stochastic evolution operator is a continuous, adapted process and for $t \geq s$ it satisfies

$$\begin{cases} \mathrm{d}A_t^{s*} = A_t^{s*} K(t)^* \, \mathrm{d}t + \displaystyle\sum_{j=1}^d A_t^{s*} R_j(t)^* \, \mathrm{d}W_j(t), \\ A_s^{s*} = \mathbb{1}. \end{cases} \tag{2.11}$$

2.2.2.1 The Stochastic Liouville Formula

It is important to prove other properties of the stochastic evolution operator and in particular that the matrix A_t^s is a.s. invertible.

Proposition 2.8. *For every given initial time $s \geq 0$, the Wronskian determinant $D_t^s := \det A_t^s$ is given by the stochastic Liouville formula*

$$D_t^s = \exp\left(\int_s^t \mathrm{Tr}\left\{ K(r) - \frac{1}{2} \sum_j R_j(r)^2 \right\} \mathrm{d}r + \sum_j \int_s^t \mathrm{Tr}\left\{ R_j(r) \right\} \, \mathrm{d}W_j(r) \right). \tag{2.12}$$

This equality holds a.s. for every $t \geq s$ and, so, $\mathbb{Q}(D_t^s > 0, \ \forall t \geq s) = 1$. Then, the operator A_t^s is a.s. invertible and the process $(A_t^s)^{-1}$ satisfies the SDE

$$\mathrm{d}(A_t^s)^{-1} = (A_t^s)^{-1} \left[\textstyle\sum_j R_j(t)^2 - K(t) \right] \mathrm{d}t - \sum_{j=1}^d (A_t^s)^{-1} R_j(t) \mathrm{d}W_j(t). \tag{2.13}$$

Finally, for every $0 \le s \le t$, *the following representation holds a.s.:*

$$A_t^s = A_t^0 (A_s^0)^{-1}. \tag{2.14}$$

Proof. Let $s \ge 0$ be a given initial time. By differentiating the explicit expression of the determinant, which is a polynomial in the matrix elements of A_t^s, and by using the Itô formula for products, in the proof of Theorem 2.2 in [28] the following formula for the stochastic differential of D_t^s is obtained:

$$\mathrm{d}D_t^s = \left[\mathrm{Tr}\left\{K(t) - \frac{1}{2}\sum_j R_j(t)^2\right\} + \frac{1}{2}\sum_j \big(\mathrm{Tr}\left\{R_j(t)\right\}\big)^2\right] D_t^s \,\mathrm{d}t$$
$$+ \sum_j \mathrm{Tr}\left\{R_j(t)\right\} D_t^s \,\mathrm{d}W_j(t)\,. \tag{2.15}$$

But this is a one-dimensional linear SDE with initial condition $D_s^s = \mathbb{1}$. Again the solution is pathwise unique and it is an exercise in stochastic calculus to verify that (2.12) solves this linear SDE. Thus, $\mathbb{Q}(D_t^s > 0,\ \forall t \ge s) = 1$ and A_t^s is a.s. invertible for every $t \ge s$.

To prove (2.13), let us consider the equation

$$\mathrm{d}Z_t^s = Z_t^s\Big[\textstyle\sum_j R_j(t)^2 - K(t)\Big]\mathrm{d}t - \displaystyle\sum_{j=1}^d Z_t^s R_j(t)\mathrm{d}W_j(t)\,, \qquad Z_s^s = \mathbb{1}\,. \tag{2.16}$$

Once more the solution is unique. By Itô formula for products one gets $\mathrm{d}\left(Z_t^s A_t^s\right) = 0$. Together with $Z_s^s A_s^s = \mathbb{1}$ and continuity in t, this gives $Z_t^s A_t^s = \mathbb{1}$ for every $t \ge s$. By multiplying on the right by $\left(A_t^s\right)^{-1}$, which exists, we get $Z_t^s = \left(A_t^s\right)^{-1}$ for every $t \ge s$ and (2.13) is proved.

By using (2.9), we have $A_t^0(A_s^0)^{-1} = A_t^s A_s^0 (A_s^0)^{-1} = A_t^s$ and (2.14) is proved.

□

2.2.3 *The Square Norm of the Solution*

Let us now study the behaviour of the norm of $\psi(t)$, which will be a key object in the whole construction.

Proposition 2.9. *We have*

$$\|\psi(t)\|^2 = \|\psi_0\|^2 + \int_0^t \langle \psi(s) \big| \Big(K(s) + K(s)^* + \textstyle\sum_j R_j(s)^* R_j(s)\Big)\, \psi(s)\rangle \mathrm{d}s$$
$$+ \sum_{j=1}^d \int_0^t \langle \psi(s) | (R_j(s) + R_j(s)^*)\psi(s)\rangle \mathrm{d}W_j(s)\,. \tag{2.17}$$

Moreover, $\forall T \geq 0$,

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \sum_j \left|\langle\psi(t)|(R_j(t)+R_j(t)^*)\psi(t)\rangle\right|^2 \mathrm{d}t\right] < +\infty, \tag{2.18}$$

and the stochastic integral in (2.17) *is a square-integrable continuous martingale.*

Proof. By Itô formula, we get

$$\begin{aligned}
\mathrm{d}\,\|\psi(t)\|^2 &= \langle\psi(t)|\mathrm{d}\psi(t)\rangle + \langle\mathrm{d}\psi(t)|\psi(t)\rangle + \langle\mathrm{d}\psi(t)|\mathrm{d}\psi(t)\rangle \\
&= \langle\psi(t)|K(t)\psi(t)\rangle\mathrm{d}t + \sum_j\langle\psi(t)|R_j(t)\psi(t)\rangle\mathrm{d}W_j(t) + \langle K(t)\psi(t)|\psi(t)\rangle\mathrm{d}t \\
&\quad + \sum_j\langle R_j(t)\psi(t)|\psi(t)\rangle\mathrm{d}W_j(t) + \sum_j \left\|R_j(t)\psi(t)\right\|^2 \mathrm{d}t \\
&= \left\langle\psi(t)\right|\left(K(t)+K(t)^* + \textstyle\sum_j R_j(t)^*R_j(t)\right)\psi(t)\rangle\mathrm{d}t \\
&\quad + \sum_j\langle\psi(t)|(R_j(t)+R_j(t)^*)\psi(t)\rangle\mathrm{d}W_j(t)\,,
\end{aligned}$$

which gives (2.17).

For every $x \in \mathcal{H}$, let P_x be the one-dimensional orthogonal projection on the Hilbert ray containing x and recall that $R_j(t)^*P_xR_j(t) \geq 0$ and $R_j(t)^*(\mathbb{1}-P_x)R_j(t) \geq 0$. Then, we have

$$\begin{aligned}
\sum_j\langle x|(R_j(t)+R_j(t)^*)x\rangle^2 &\leq 4\sum_j\left|\langle x|R_j(t)x\rangle\right|^2 = 4\,\|x\|^2\sum_j\langle x|R_j(t)^*P_xR_j(t)x\rangle \\
&\leq 4\,\|x\|^2\sum_j\langle x|R_j(t)^*R_j(t)x\rangle \leq 4\,\|x\|^4\left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\|;
\end{aligned}$$

so, the following estimate holds: $\forall x \in \mathcal{H}$,

$$\sum_j\langle x|(R_j(t)+R_j(t)^*)x\rangle^2 \leq 4\,\|x\|^4\left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\|. \tag{2.19}$$

By using this inequality and the L^p estimate (2.4) given in Theorem 2.4, we get

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\int_0^T\sum_j\langle\psi(t)|(R_j(t)+R_j(t)^*)\psi(t)\rangle^2\mathrm{d}t\right] \\
&\leq 4\sup_{t\in[0,T]}\left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\|\,T\,\mathbb{E}_{\mathbb{Q}}\left[\sup_{0\leq t\leq T}\|\psi(t)\|^4\right] \\
&\leq 4\sup_{t\in[0,T]}\left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\|\,TC(4,T)\left(1+\|\psi_0\|^4\right) < +\infty,
\end{aligned}$$

and (2.18) is proved. Then, the integrand process $\langle\psi(t)|(R_j(t) + R_j(t)^*)\psi(t)\rangle$ belongs to the space $\mathcal{M}^2$ for every j (Sect. A.3.1), and the stochastic integral in (2.17) is a square-integrable continuous martingale (Sect. A.3.3). □

2.3 The Linear Stochastic Schrödinger Equation

For the physical interpretation anticipated in Section 2.1 and discussed in Sect. 2.4, we are not interested in (2.2) in general, but only when $\|\psi(t)\|^2$ is a martingale of mean one and can be interpreted as a probability density with respect to $\mathbb{Q}$.

2.3.1 A Key Restriction

In order to reduce $\|\psi(t)\|^2$ to a martingale, we need the vanishing of the integrand in the time integral in (2.17) for every initial condition, i.e.

$$K(t) + K(t)^* + \sum_j R_j(t)^* R_j(t) = 0,$$

which is equivalent to the following assumption.

Assumption 2.10. The operator $K(t)$ has the structure

$$K(t) = -\mathrm{i}H(t) - \frac{1}{2}\sum_{j=1}^{d} R_j(t)^* R_j(t)\,, \tag{2.20}$$

where $H(t)$ is a self-adjoint operator on $\mathcal{H}$, called *effective Hamiltonian* of the system.

By Assumptions 2.3 and 2.10, the function $t \mapsto H(t)$ is measurable and

$$\forall T \in (0, +\infty), \qquad \sup_{t\in[0,T]} \|H(t)\| < +\infty\,. \tag{2.21}$$

Proposition 2.8 gives $\|\psi(t)\| > 0$ and we can define the continuous processes

$$\widehat{\psi}(t) := \|\psi(t)\|^{-1}\ \psi(t)\,, \tag{2.22}$$

$$m_j(t) := \big\langle\widehat{\psi}(t)\big|\big(R_j(t) + R_j(t)^*\big)\widehat{\psi}(t)\big\rangle = 2\,\mathrm{Re}\,\big\langle\widehat{\psi}(t)\big|R_j(t)\widehat{\psi}(t)\big\rangle. \tag{2.23}$$

Theorem 2.11. *Under Assumptions 2.2 and 2.10, the square norm $\|\psi(t)\|^2$ of the solution of the SDE* (2.2) *is a positive, continuous martingale and*

$$\|\psi(t)\|^2 = \|\psi_0\|^2 \exp\left\{\sum_j \left[\int_0^t m_j(s)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t m_j(s)^2 \mathrm{d}s\right]\right\}. \tag{2.24}$$

Moreover, $\forall p \geq 1$,

$$\sup_{0\leq t\leq T} \mathbb{E}_{\mathbb{Q}}\left[\|\psi(t)\|^{2p}\right] \leq \mathbb{E}_{\mathbb{Q}}\left[\sup_{0\leq t\leq T} \|\psi(t)\|^{2p}\right] < +\infty. \tag{2.25}$$

Proof. Being an Itô process, ψ is continuous and this holds for its square norm.

By Assumption 2.10 and the definitions (2.22), (2.23), equation (2.17) reduces to

$$\|\psi(t)\|^2 = \|\psi_0\|^2 + \sum_j \int_0^t m_j(s)\, \|\psi(s)\|^2 \,\mathrm{d}W_j(s). \tag{2.26}$$

By Proposition 2.9, the positive continuous process $\|\psi(t)\|^2$ is a square-integrable martingale. By taking m as given, (2.26) is a Doléans equation whose solution is unique and given by (2.24) (cf. Proposition A.41 and (Eqs. (A.23), (A.24), (A.25), (A.26)).

By inequality (2.19), we have

$$\sum_j m_j(t)^2 \leq 4\left\|\textstyle\sum_j R_j(t)^* R_j(t)\right\|,$$

$$\int_0^T \sum_j m_j(t)^2 \mathrm{d}t \leq 4 \sup_{t\in[0,T]} \left\|\textstyle\sum_j R_j(t)^* R_j(t)\right\| T. \tag{2.27}$$

Then, the last statement follows from Proposition A.42. □

In the following, we shall call *linear stochastic Schrödinger equation* the original SDE (2.2) for an $\mathcal{H}$-valued process ψ under all Assumptions 2.1, 2.2, 2.3 and 2.10, i.e.

$$\begin{cases} \mathrm{d}\psi(t) = \left(-\mathrm{i}H(t) - \dfrac{1}{2}\displaystyle\sum_{j=1}^d R_j(t)^* R_j(t)\right)\psi(t)\,\mathrm{d}t + \displaystyle\sum_{j=1}^d R_j(t)\psi(t)\,\mathrm{d}W_j(t)\,, \\ \psi(0) = \psi_0\,, \qquad\qquad \psi_0 \in \mathcal{H}. \end{cases} \tag{2.28}$$

Of course, the solution is the continuous, adapted stochastic process $\psi(t) = A_t^0\psi_0$, where the stochastic evolution operator A_t^s and its adjoint A_t^{s*} still satisfy the SDEs (2.7) and (2.11) with $K(t) = -\mathrm{i}H(t) - \dfrac{1}{2}\displaystyle\sum_{j=1}^d R_j(t)^* R_j(t)$ and $H(t) = H(t)^*$.

2.3.2 A Change of Probability

Assumption 2.12. The initial condition is normalised: $\|\psi_0\| = 1$.

$\|\psi(t)\|^2$ being a positive martingale with $\mathbb{E}_{\mathbb{Q}}\left[\|\psi(t)\|^2\right] = 1$ by the discussion in Sect. A.5.3 and Remark A.46, we have the following.

Remark 2.13. For any $T > 0$ the equation

$$\widehat{\mathbb{P}}^T_{\psi_0}(F) := \int_F \|\psi(T,\omega)\|^2\,\mathbb{Q}(\mathrm{d}\omega) \equiv \mathbb{E}_{\mathbb{Q}}\left[1_F\|\psi(T)\|^2\right], \qquad F \in \mathcal{F}_T, \tag{2.29}$$

defines a new probability law $\widehat{\mathbb{P}}^T_{\psi_0}$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbb{Q}_T$, the restriction of $\mathbb{Q}$ to $\mathcal{F}_T$. Let us denote by $\widehat{\mathbb{E}}^T_{\psi_0}$ the expectation with respect to $\widehat{\mathbb{P}}^T_{\psi_0}$.

Moreover, $\left\{\widehat{\mathbb{P}}^T_{\psi_0},\ T > 0\right\}$ is a consistent family of probabilities, in the sense that

$$0 < S < T,\ F \in \mathcal{F}_S \ \Rightarrow\ \widehat{\mathbb{P}}^T_{\psi_0}(F) = \widehat{\mathbb{P}}^S_{\psi_0}(F)\,. \tag{2.30}$$

Then, Girsanov theorem (Theorem A.45 and Proposition A.47) gives the following fundamental result. The class of integrand processes $\mathcal{L}^2$ is defined in Sect. A.3.1.

Theorem 2.14. *Under the law $\widehat{\mathbb{P}}^T_{\psi_0}$ defined by* (2.29)*, the continuous processes*

$$\widehat{W}_j(t) := W_j(t) - \int_0^t m_j(s)\mathrm{d}s, \qquad j = 1,\dots,d, \quad t \in [0,T], \tag{2.31}$$

are independent, standard Wiener processes with respect to the filtration $(\mathcal{F}_t)$.

Given d stochastically integrable processes $G_j(t)$, i.e. $G_j \in \mathcal{L}^2$, the Itô integrals $\sum_j \int_0^t G_j(s)\mathrm{d}\widehat{W}_j(s)$ and $\sum_j \int_0^t G_j(s)\mathrm{d}W_j(s)$ are defined for every $t \in [0,T]$, each one under its corresponding probability law, and we have $\mathbb{Q}$-a.s. and $\widehat{\mathbb{P}}^T_{\psi_0}$-a.s.

$$\sum_{j=1}^d \int_0^t G_j(s)\mathrm{d}\widehat{W}_j(s) = \sum_{j=1}^d \int_0^t G_j(s)\mathrm{d}W_j(s) - \sum_{j=1}^d \int_0^t G_j(s)m_j(s)\mathrm{d}s, \qquad \forall t \in [0,T]\,. \tag{2.32}$$

Proposition 2.15. *The processes $\widehat{\psi}$, m, $\widehat{W}$ are $(\overline{\mathcal{D}}^0_t)$-adapted.*

Proof. The statement follows immediately from the definitions (2.22), (2.23), (2.31) and Proposition 2.7. □

2.4 The Physical Interpretation

Let us begin with a list of the mathematical objects involved by the linear stochastic Schrödinger equation (2.28) and their heuristic interpretation in the theory of continuous measurements, in analogy with the traditional representation of an instantaneous discrete measurement (Sect. 2.1).

- ψ_0 is the initial state of the quantum system;
- $(\Omega, \mathcal{F})$ is the measurable space of the possible outcomes of the experiment;
- $\mathcal{F}_t$ is the collection of events verifiable already at time t;
- the d stochastic processes $W_j(t)$ are the output of the continuous measurement and their derivatives $\dot{W}_j(t)$ can be interpreted as instantaneous imprecise measurements of the quantum observables $R_j(t) + R_j(t)^*$ performed at time t;
- $\overline{\mathcal{D}}_t^0$ is the collection of events verifiable already at time t which effectively regard the continuous measurement;
- the stochastic linear state transformation $\psi_0 \mapsto \psi(t) = A_t^0 \psi_0$ gives both the probability of the events, which could occur up to time t, and the state of the quantum system conditioned on the observation in the time interval $[0, t]$:
  - $\widehat{\mathbb{P}}_{\psi_0}^T$ is the physical probability law of the events which could occur in $[0, T]$;
  - $\widehat{\psi}(t, \omega)$ is the state of the system at time t, conditioned on having observed the trajectory $s \mapsto W(s, \omega)$ up to time t.

When the canonical realisation of the Wiener process is used, i.e. when the only output of the experiment is the diffusive process W, the outcome ω itself can be identified with the trajectory of the output; indeed in this case we have $W(s, \omega) = \omega(s)$ (see Remark A.23). Then $\psi(t, \omega)$, $\|\psi(t, \omega)\|$ and $\widehat{\psi}(t, \omega)$ depend only on $\omega(s)$ for $0 \leq s \leq t$. In particular, $\|\psi(t, \omega)\|^2$ is the density of probability (with respect to the Wiener measure) of observing $W(s) = \omega(s)$ in the time interval $0 \leq s \leq t$.

When $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$ is bigger than the canonical realisation of the Wiener process, still $\psi(t, \omega)$, $\|\psi(t, \omega)\|$ and $\widehat{\psi}(t, \omega)$ depend only on $W(s)$ for $0 \leq s \leq t$ because the stochastic processes $\psi(t)$, $\|\psi(t)\|$ and $\widehat{\psi}(t)$ are adapted to $(\overline{\mathcal{D}}_t^0)$ and thus $\psi(t, \omega) = \psi(t, \omega')$ if $W(s, \omega) = W(s, \omega')$ for $0 \leq s \leq t$ (maybe except for a set of null probability).

Therefore, even if from a mathematical point of view it can be convenient to work with a Wiener process W with increments independent of the past in an arbitrary filtration $(\mathcal{F}_t)$, from a physical point of view the relevant filtration is always $(\overline{\mathcal{D}}_t^0)$: it contains all the events regarding the output W of the measurement and, moreover, only these events really condition the system state $\widehat{\psi}$.

What we have to do now is to show that this interpretation is consistent with the general formulation of quantum mechanics. However, let us first add two further remarks on the physical interpretation.

The use itself of linear SDEs to assign the evolution of $\psi(t)$ implies a Markovian hypothesis about the observed quantum system and the measurement process: for every $0 \leq s \leq t$, in spite of all the information available at time s (the initial state ψ_0 and all the events in $\mathcal{F}_s$), the conditioned state $\widehat{\psi}(s)$ at time s is sufficient to evaluate the conditional state $\widehat{\psi}(t)$ at time t (together with the output W in $[s, t]$, of course).

There are two typical but physically different interpretations of the linear stochastic Schrödinger equation (2.28). Sometimes it is obtained by starting from a free closed evolution of the quantum system and introducing the continuous measurement as a perturbation, by adding a stochastic term in the evolution equation for every continuously monitored quantum observable $R_j(t) + R_j(t)^*$. In this case, one can think of possibly switching off the measurement ($R_j(t) \equiv 0$), and the linear stochastic Schrödinger equation (2.28) reduces to an ordinary Schrödinger equation $\mathrm{d}\psi(t) = -\mathrm{i}H(t)\psi(t)\,\mathrm{d}t$. Other times, the linear stochastic Schrödinger equation (2.28) is obtained by starting from an open evolution of the quantum system $\mathcal{H}$ and introducing continuous measurements which acquire information on the system without introducing extra perturbations (e.g. the continuous monitoring of an atom by the detection of its fluorescence light). In this case, the "mean" evolution of the quantum system is not modified by the continuous measurement, but it is "unravelled" in many different trajectories according to the observed output W.

2.4.1 The POM of the Output and the Physical Probabilities

First, we introduce properly the positive operator-valued measure (see Definition B.1) associated with the continuous measurement in the time interval $[0, T]$. Taking the stochastic evolution operator A_T^0 associated with the linear stochastic Schrödinger equation (2.28), we can define

$$\widehat{E}_T(F) := \int_F A_T^0(\omega)^* A_T^0(\omega)\mathbb{Q}(\mathrm{d}\omega) \equiv \mathbb{E}_{\mathbb{Q}}\left[1_F A_T^{0*} A_T^0\right], \qquad F \in \mathcal{F}_T\,. \tag{2.33}$$

Then, $\widehat{E}_T$ is a positive operator-valued measure (POM) on the value space $(\Omega, \mathcal{F}_T)$. Indeed, it is positive and σ-additive by construction and, moreover,

$$\langle\psi_0|\widehat{E}_T(\Omega)\psi_0\rangle = \mathbb{E}_{\mathbb{Q}}\left[\langle\psi_0|A_T^{0*}A_T^0\psi_0\rangle\right] = \mathbb{E}_{\mathbb{Q}}\left[\|\psi(T)\|^2\right] = \|\psi_0\|^2, \qquad \forall\psi_0 \in \mathcal{H},$$

which implies $\widehat{E}_T(\Omega) = \mathbb{1}$ by the normalisation of ψ_0.

The POM $\widehat{E}_T$ assigns to each event in $\mathcal{F}_T$, according to the axioms of Sect. B.2.1, just the probability $\widehat{\mathbb{P}}_{\psi_0}^T$ that we called physical probability. Indeed, by (2.10) and (2.33) we get

$$\langle\psi_0|\widehat{E}_T(F)\psi_0\rangle = \mathbb{E}_{\mathbb{Q}}\left[\langle\psi_0|A_T^{0*}A_T^0\psi_0\rangle 1_F\right] = \int_F \|\psi(T,\omega)\|^2\,\mathbb{Q}(\mathrm{d}\omega) \tag{2.34}$$

and, by (2.29),

$$\langle\psi_0|\widehat{E}_T(F)\psi_0\rangle = \widehat{\mathbb{P}}_{\psi_0}^T(F), \qquad \forall F \in \mathcal{F}_T, \tag{2.35}$$

which is the standard formula for probabilities in the Hilbert space formulation of quantum mechanics.

Moreover, $\left\{\widehat{E}_T,\ T > 0\right\}$ is a consistent family of POMs, in the sense that

$$0 < S < T,\ F \in \mathcal{F}_S \Rightarrow \widehat{E}_T(F) = \widehat{E}_S(F)\,. \tag{2.36}$$

Indeed, for every $0 < S < T$, $F \in \mathcal{F}_S$, A_T^S being independent of $\mathcal{F}_S$, one gets

$$\begin{aligned}
\widehat{E}_T(F) &= \mathbb{E}_\mathbb{Q}\left[1_F A_T^{0*} A_T^0\right] = \mathbb{E}_\mathbb{Q}\left[1_F A_S^{0*} A_T^{S*} A_T^S A_S^0\right] \\
&= \mathbb{E}_\mathbb{Q}\left[1_F A_S^{0*}\, \mathbb{E}_\mathbb{Q}\left[A_T^{S*} A_T^S | \mathcal{F}_S\right] A_S^0\right] = \mathbb{E}_\mathbb{Q}\left[1_F A_S^{0*}\, \mathbb{E}_\mathbb{Q}\left[A_T^{S*} A_T^S\right] A_S^0\right] \\
&= \mathbb{E}_\mathbb{Q}\left[1_F A_S^{0*} A_S^0\right] = \widehat{E}_S(F).
\end{aligned}$$

Another way to look at (2.33) is to say that $A_t^{0*} A_t^0$ is the density (or Radon–Nikodym derivative) of the POM $\widehat{E}_t$ with respect to the probability measure

$$\mathbb{Q}_t := \mathbb{Q}\big|_{\mathcal{F}_t} . \tag{2.37}$$

By recalling that $A_t^{0*} A_t^0$ is $\mathcal{F}_t$-measurable, we can write

$$\frac{\widehat{E}_t(\mathrm{d}\omega)}{\mathbb{Q}_t(\mathrm{d}\omega)} = A_t^0(\omega)^* A_t^0(\omega) . \tag{2.38}$$

We already discussed the fact that the filtration $(\mathcal{F}_t)$ could be unnecessarily large: the natural value space, when the output of the continuous measurement is the process W in the time interval $[0, t]$, is $(\Omega, \overline{\mathcal{D}}_t^0)$. Moreover, we could perform the measurement only in the time interval $[s, t]$. As in the evolution equations only the increments of W appear (through the $\mathrm{d}W$ term), the natural candidate to be the output in the time interval $[s, t]$ is the process $W(r) - W(s)$, $r \in [s, t]$, which generates the set of events $\overline{\mathcal{D}}_t^s$. Thus, analogous to (2.33), we define a POM $\widehat{E}_t^s$ on the value space $(\Omega, \overline{\mathcal{D}}_t^s)$ by

$$\widehat{E}_t^s(F) := \int_F A_t^s(\omega)^* A_t^s(\omega)\, \mathbb{Q}(\mathrm{d}\omega) \equiv \mathbb{E}_\mathbb{Q}\left[1_F A_t^{s*} A_t^s\right], \qquad F \in \overline{\mathcal{D}}_t^s . \tag{2.39}$$

By this definition, we have that $\widehat{E}_t^0$ is the restriction of $\widehat{E}_t$ to $\overline{\mathcal{D}}_t^0$. Also the new POMs (2.39) are consistent with respect to t. In order to use $\widehat{E}_t^s$ for an arbitrary $s > 0$ one needs to know the system state at time s.

By noticing that the positive operator-valued random variable $A_t^{s*} A_t^s$ is $\overline{\mathcal{D}}_t^s$-measurable, we get that the analog of (2.38) is

$$\frac{\widehat{E}_t^s(\mathrm{d}\omega)}{\mathbb{Q}(\mathrm{d}\omega)\big|_{\overline{\mathcal{D}}_t^s}} = A_t^{s*}(\omega) A_t^s(\omega) . \tag{2.40}$$

Summing up, the POM representing the output of the continuous measurement in the time interval $[s, t]$ is $\widehat{E}_t^s$. Even if $s = 0$, the relevant POM is $\widehat{E}_t^0$, not $\widehat{E}_t$. We can also say that the physical probability, the probability of the events determined by the output in the time interval $[0, T]$, is $\widehat{\mathbb{P}}_{\psi_0}^T\big|_{\overline{\mathcal{D}}_t^0}$.

The probability $\widehat{\mathbb{P}}^T_{\psi_0}$ of events in the (augmented) natural filtration of W is obtained from a POM, as prescribed by quantum mechanics, whose value space is $\left(\Omega, \overline{\mathcal{D}}^0_T\right)$. We interpret W as the output of a continuous measurement performed on the quantum system $\mathcal{H}$ in the time interval $[0, T]$ and $\widehat{\mathbb{P}}^T_{\psi_0}$ as the corresponding physical probability. Moreover, from Girsanov formula (2.31) we have

$$W(t) = \widehat{W}(t) + \int_0^t m(s)\mathrm{d}s, \qquad t \in [0, T],$$

which says that the output process $W(t)$ decomposes to the sum of a Wiener process $\widehat{W}(t)$ and a process $\int_0^t m(s)\mathrm{d}s$ with trajectories of bounded variation. Let us remark that, even if it could be suggestive to interpret the two addenda as noise and signal, the two processes are typically not independent.

Remark 2.16. Here it is worthwhile to be more precise on the notion of output of the measurement. As already said, the choice of the two-time σ-algebras $\mathcal{D}^s_t$ or $\overline{\mathcal{D}}^s_t$, which are determined by the increments of W, reflects the fact that we consider as events which can be observed in the time interval $[s, t]$ only the events related to the increments of W with extreme times inside $[s, t]$, not the ones determined by $W(r)$ with $r \in [s, t]$. So, in this time interval, we observe the increments $W(r) - W(u)$, $s \leq u < r \leq t$, or functionals of these increments. “Morally” the output is the singular process $\dot{W}(r)$, $r \in [s, t]$. In the whole book we always understand this interpretation, even when we write that the output is W.

2.4.2 The A Posteriori States

Now we would like to justify the interpretation of $\widehat{\psi}(t)$ as the conditional state of the system at time t, i.e. as *a posteriori state* at time t (cf. Sect. B.4.3.2). We shall do this properly in Sect. 4.1.1, where, in a more general setup, we shall introduce explicitly the instruments. In the present paragraph, we only show that this interpretation is consistent with the present construction.

Let us consider an event F regarding the output in the time interval $[s, t]$, that is $F \in \overline{\mathcal{D}}^s_t$. If we evaluate its probability at the beginning of the experiment, when we only know that the initial state of the system is ψ_0, then we get $\widehat{\mathbb{P}}^T_{\psi_0}(F)$. On the other hand, if we reconsider the same event F at time s, when we have gathered all the information coming from the measurement in the time interval $[0, s]$, then its probability can be updated and it is given by $\widehat{\mathbb{P}}^T_{\psi_0}(F|\mathcal{F}_s) = \widehat{\mathbb{E}}^T_{\psi_0}[1_F|\mathcal{F}_s]$ (Sect. A.1.2.2). This is an $\mathcal{F}_s$-measurable random variable, as it depends on what is observed up to time s. The following proposition states that it can be computed using the POM $\widehat{E}^s_t$ defined by (2.39) and just $\widehat{\psi}(s)$ as the conditional state of the system at time s.

Proposition 2.17. *For all* $F \in \overline{\mathcal{D}}^s_t$, $0 \leq s < t \leq T$, *we have*

$$\widehat{\mathbb{P}}^T_{\psi_0}(F|\mathcal{F}_s) = \left\langle \widehat{\psi}(s) \middle| \widehat{E}^s_t(F)\widehat{\psi}(s) \right\rangle = \widehat{\mathbb{P}}^T_{\psi_0}(F|\overline{\mathcal{D}}^0_s). \tag{2.41}$$

Proof. For all $\mathcal{F}_s$-measurable bounded random variables Y, we get

$$\begin{aligned}\widehat{\mathbb{E}}^T_{\psi_0}\left[1_F Y\right] &= \mathbb{E}_{\mathbb{Q}}\left[\|\psi(t)\|^2\, 1_F Y\right] = \mathbb{E}_{\mathbb{Q}}\left[\langle\psi_0|A^{0*}_s A^{s*}_t A^s_t A^0_s \psi_0\rangle 1_F Y\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\langle\psi_0|A^{0*}_s\, \mathbb{E}_{\mathbb{Q}}\left[1_F A^{s*}_t A^s_t|\mathcal{F}_s\right] A^0_s\psi_0\rangle Y\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\langle\psi(s)|\, \mathbb{E}_{\mathbb{Q}}\left[1_F A^{s*}_t A^s_t\right] \psi(s)\rangle Y\right] \\ &= \widehat{\mathbb{E}}^T_{\psi_0}\left[\langle\widehat{\psi}(s)|\, \mathbb{E}_{\mathbb{Q}}\left[1_F A^{s*}_t A^s_t\right] \widehat{\psi}(s)\rangle Y\right];\end{aligned}$$

we have used the equality $\psi(t) = A^s_t A^0_s \psi_0$ and the independence of $1_F A^{s*}_t A^s_t$ from $\mathcal{F}_s$, which follows from Proposition 2.7. This computation proves that $\widehat{\mathbb{E}}^T_{\psi_0}\left[1_F|\mathcal{F}_s\right] = \left\langle\widehat{\psi}(s)\middle|\, \mathbb{E}_{\mathbb{Q}}\left[1_F A^{s*}_t A^s_t\right] \widehat{\psi}(s)\right\rangle$. By using the definition of $\widehat{E}^s_t$ we have $\widehat{\mathbb{E}}^T_{\psi_0}\left[1_F|\mathcal{F}_s\right] = \left\langle\widehat{\psi}(s)\middle|\widehat{E}^s_t(F)\widehat{\psi}(s)\right\rangle$. By the fact that $\overline{\mathcal{D}}^0_s \subset \mathcal{F}_s$ and that $\left\langle\widehat{\psi}(s)\middle|\widehat{E}^s_t(F)\widehat{\psi}(s)\right\rangle$ is $\overline{\mathcal{D}}^0_s$-measurable, we have

$$\begin{aligned}\widehat{\mathbb{E}}^T_{\psi_0}\left[1_F\middle|\overline{\mathcal{D}}^0_s\right] &= \widehat{\mathbb{E}}^T_{\psi_0}\left[\widehat{\mathbb{E}}^T_{\psi_0}\left[1_F|\mathcal{F}_s\right]\middle|\overline{\mathcal{D}}^0_s\right] \\ &= \widehat{\mathbb{E}}^T_{\psi_0}\left[\left\langle\widehat{\psi}(s)\middle|\widehat{E}^s_t(F)\widehat{\psi}(s)\right\rangle\middle|\overline{\mathcal{D}}^0_s\right] = \left\langle\widehat{\psi}(s)\middle|\widehat{E}^s_t(F)\widehat{\psi}(s)\right\rangle.\end{aligned}$$ □

Remark 2.18. As suggested in the presentation before the proposition, by comparing (2.41) with (2.35), we see that we can interpret the state $\widehat{\psi}(s)$ as the conditional state of the system at time s; we call $\widehat{\psi}(t)$ the *a posteriori state* at time t. Considering also (2.34), we call $\psi(t)$ the *non-normalised a posteriori state* at time t.

With this interpretation in mind, we consider again the output and, thanks to the representation

$$W_j(t) = \widehat{W}_j(t) + \int_0^t \left\langle\widehat{\psi}(s)\middle|\left(R_j(s) + R_j(s)^*\right)\widehat{\psi}(s)\right\rangle \mathrm{d}s, \qquad t \in [0,T], \tag{2.42}$$

we say that $\dot{W}_j(t)$ is an imprecise measurement of the quantum observable $R_j(t) + R_j(t)^*$. We shall consider again this interpretation in Sect. 4.3.

Remark 2.19 (A phase change). Let us consider now the normalised random vector $\widehat{\phi}(t,\omega) = \mathrm{e}^{\mathrm{i}\alpha(t,\omega)}\widehat{\psi}(t,\omega)$, where $\{\alpha(t),\ t \geq 0\}$ is an arbitrary $(\overline{\mathcal{D}}^0_t)$-adapted real process. By substituting $\widehat{\phi}(t)$ to $\widehat{\psi}(t)$ in (2.41), this formula continues to hold true. This means that $\widehat{\phi}(t)$ has the same right of $\widehat{\psi}(t)$ to the name of "a posteriori state". But this is nothing more than the stochastic version of the usual statement in quantum mechanics that a phase change of the state vector does not alter any physical quantity.

2.4.3 Infinite Time Horizon

Given the initial state ψ_0, we have a consistent set of probabilities $\widehat{\mathbb{P}}^T_{\psi_0}$, $T > 0$. As stated by Theorem 2.14, each $\widehat{\mathbb{P}}^T_{\psi_0}$ modifies the properties of the stochastic process

W, in the corresponding time interval $[0, T]$. A natural question is whether it is possible to have a unique probability for $T \to +\infty$. This would be useful, for instance, to study the long-time behaviour of W under the physical probability. By the discussion in Sect. A.5.5, we know that this is possible when the consistent measures are defined on standard Borel spaces. Therefore, if we consider the new probabilities restricted to $\mathcal{D}_t^0$, the natural (not augmented) filtration of W, we have a consistent set of probabilities on standard Borel spaces and we get that there exists a unique probability $\widehat{\mathbb{P}}_{\psi_0}^{\infty}$ on $\mathcal{D}_{\infty}^0 := \bigvee_{t>0} \mathcal{D}_t^0$ such that for every $T > 0$

$$\widehat{\mathbb{P}}_{\psi_0}^{\infty}(F) = \widehat{\mathbb{P}}_{\psi_0}^{T}(F), \qquad \forall F \in \mathcal{D}_T^0 . \tag{2.43}$$

Nevertheless, even if we choose $\mathcal{F} = \mathcal{D}_{\infty}^0$, each augmented σ-algebra $\overline{\mathcal{D}}_T^0$ is strictly greater than $\mathcal{D}_T^0$ and the limit probability $\widehat{\mathbb{P}}_{\psi_0}^{\infty}$ typically does not agree with $\widehat{\mathbb{P}}_{\psi_0}^{T}$ on the whole $\overline{\mathcal{D}}_T^0 \subset \mathcal{D}_{\infty}^0$. In order to work inside the filtration $(\mathcal{D}_t^0)$, it is enough to consider $(\mathcal{D}_t^0)$-adapted versions of the processes A_t^0, $\psi(t)$, $\widehat{\psi}(t)$, $m(t)$, $\widehat{W}(t)$. What we lose is that we are no more sure to have continuity in time for every ω.

Just to have an example of the differences, let us consider the POMs. By restricting $\widehat{E}_t^s$ to $\mathcal{D}_t^s$, we get from (2.40)

$$\widehat{E}_t^s(\mathrm{d}\omega)\big|_{\mathcal{D}_t^s} = \mathbb{E}_{\mathbb{Q}}\left[A_t^{s*}A_t^s|\mathcal{D}_t^s\right](\omega)\mathbb{Q}(\mathrm{d}\omega)\big|_{\mathcal{D}_t^s} , \tag{2.44}$$

but $\mathbb{E}_{\mathbb{Q}}\left[A_t^{s*}A_t^s|\mathcal{D}_t^s\right](\omega) = A_t^{s*}(\omega)A_t^s(\omega)$, $\mathbb{Q}$-a.s.

2.4.4 The Conservative Case

A very particular case is when the operators $R_j(t)$ are anti-selfadjoint [23]:

$$R_j(t) = -\mathrm{i}V_j(t) , \qquad V_j(t)^* = V_j(t) . \tag{2.45}$$

Equations (2.23), (2.26), (2.45) give $m_j(t) = 0$ and (for $\|\psi_0\| = 1$) $\|\psi(t)\| = 1$, $\forall t$. This implies $\widehat{\mathbb{P}}_{\psi_0}^{T} = \mathbb{Q}_T$, $\forall T > 0$, so that the randomness does not depend on the quantum system: the W_j are pure noises and there is no true measurement on the system.

The linear stochastic Shrödinger equation becomes

$$\mathrm{d}\psi(t) = -\mathrm{i}\left[H(t)\mathrm{d}t + \sum_j V_j(t)\mathrm{d}W_j(t)\right]\psi(t) - \frac{1}{2}\sum_j V_j(t)^2\psi(t)\mathrm{d}t , \tag{2.46}$$

and one can check that $(A_t^s)^*$ and $(A_t^s)^{-1}$ satisfy the same SDE: $(A_t^s)^*A_t^s \equiv \mathbb{1}$. Thus the system undergoes a stochastic unitary evolution: the quantum system has a unitary evolution in a random environment which determines the stochastic potential acting on the system. Even if W is observed, the measurement does not acquire any information on the quantum system itself, but it only detects which unitary evolution occurs among the possible ones.

This class of stochastic Shrödinger equations was introduced as a model of dissipative evolution, with W not observed. In this case, all the physical quantities are obtained with a mean with respect to W. For an example of this approach to quantum open systems, see [2].

2.5 The Stochastic Schrödinger Equation

A key point of the theory is to show that the a posteriori states satisfy an SDE, closed in $\widehat{\psi}(t)$ itself. The structure of such an equation is not of usual type, but it is possible, after some work, to arrive at a theorem giving existence and uniqueness of the solutions. Different approaches to the existence and uniqueness problem, in finite and infinite dimensional Hilbert spaces, are given in [24, 27, 29, 30].

2.5.1 The Stochastic Differential of the A Posteriori State

Let us compute the stochastic differential of the a posteriori state $\widehat{\psi}(t) = \|\psi(t)\|^{-1}\ \psi(t)$ under the physical probability $\widehat{\mathbb{P}}^T_{\psi_0}$ and in terms of the new Wiener process $\widehat{W}$. To put in full evidence the dependence of the differential on $\widehat{\psi}(t)$ itself, it is useful to introduce the quantities

$$n_j(t,x) := \langle x | R_j(t) x\rangle\,, \qquad t\in[0,+\infty), \quad x\in\mathcal{H}\,. \tag{2.47}$$

Note that

$$m_j(t) = 2\,\mathrm{Re}\, n_j\left(t, \widehat{\psi}(t)\right). \tag{2.48}$$

Proposition 2.20. *Under the probability $\widehat{\mathbb{P}}^T_{\psi_0}$, the stochastic differential of $\widehat{\psi}(t)$, $0\le t<T$, is*

$$\begin{aligned} \mathrm{d}\widehat{\psi}(t) &= \sum_j \left[R_j(t) - \mathrm{Re}\, n_j\left(t,\widehat{\psi}(t)\right)\right]\widehat{\psi}(t)\,\mathrm{d}\widehat{W}_j(t) \\ &\quad + \left[K(t) + \sum_j \left(\mathrm{Re}\, n_j\left(t,\widehat{\psi}(t)\right)\right) R_j(t) - \frac{1}{2}\sum_j \left(\mathrm{Re}\, n_j\left(t,\widehat{\psi}(t)\right)\right)^2\right]\widehat{\psi}(t)\,\mathrm{d}t\,. \end{aligned} \tag{2.49}$$

Proof. It is enough to apply Itô rules to $\widehat{\psi}(t) = \|\psi(t)\|^{-1}\ \psi(t)$ under the probability $\widehat{\mathbb{P}}^T_{\psi_0}$. By using (2.42) we can transform (2.2) into

$$d\psi(t) = \sum_j R_j(t)\psi(t)\,d\widehat{W}_j(t) + \Big(K(t) + \sum_j m_j(t)R_j(t)\Big)\psi(t)\,dt\,.$$

By using (2.24), (2.42) and the fact that $\|\psi(t)\| > 0$ with probability one, we get

$$\begin{aligned}\|\psi(t)\|^{-1} &= \exp\Big\{-\frac{1}{2}\sum_j\Big[\int_0^t m_j(s)dW_j(s) - \frac{1}{2}\int_0^t m_j(s)^2 ds\Big]\Big\}\\ &= \exp\Big\{-\frac{1}{2}\sum_j\Big[\int_0^t m_j(s)d\widehat{W}_j(s) + \frac{1}{2}\int_0^t m_j(s)^2 ds\Big]\Big\};\end{aligned}$$

by Itô formula, this gives

$$\begin{aligned}d\,\|\psi(t)\|^{-1} &= \|\psi(t)\|^{-1}\Big\{-\frac{1}{2}\sum_j\Big[m_j(t)d\widehat{W}_j(t) + \frac{1}{2}m_j(t)^2 dt\Big] + \frac{1}{8}\sum_j m_j(t)^2 dt\Big\}\\ &= -\frac{1}{2}\,\|\psi(t)\|^{-1}\sum_j\Big[m_j(t)d\widehat{W}_j(t) + \frac{1}{4}m_j(t)^2 dt\Big].\end{aligned}$$

Finally, by using the Itô rules for the differential of a product, we obtain

$$\begin{aligned}d\widehat{\psi}(t) &= \sum_j R_j(t)\widehat{\psi}(t)\,d\widehat{W}_j(t) + \Big(K(t) + \sum_j m_j(t)R_j(t)\Big)\widehat{\psi}(t)\,dt\\ &\quad -\frac{1}{2}\sum_j m_j(t)\widehat{\psi}(t)\,d\widehat{W}_j(t) - \frac{1}{8}\sum_j m_j(t)^2\widehat{\psi}(t)\,dt - \frac{1}{2}\sum_j m_j(t)R_j(t)\widehat{\psi}(t)\,dt\\ &= \sum_j\Big[R_j(t) - \frac{m_j(t)}{2}\Big]\widehat{\psi}(t)\,d\widehat{W}_j(t)\\ &\quad + \Big[K(t) + \sum_j\frac{m_j(t)}{2}R_j(t) - \sum_j\frac{m_j(t)^2}{8}\Big]\widehat{\psi}(t)\,dt\,.\end{aligned}$$

By using the notation $n_j(t,x)$ introduced in Definition (2.47), we get (2.49). □

2.5.1.1 A Stochastic Phase Change

Let us stress that no physical consequence depends on the phase of $\widehat{\psi}(t)$: consider the presentation of quantum mechanics given in Appendix B, the POM (2.33), the probabilities (2.35), the processes m_j (2.23), the output W (2.42), etc. So, we are allowed to make any change of phase on $\widehat{\psi}(t)$, even a stochastic one. In this order of ideas, we introduce the new normalised vectors

$$\widehat{\phi}(t) := \exp\Big\{ -\mathrm{i}\sum_j \int_0^t \operatorname{Re} n_j\big(s, \widehat{\psi}(s)\big) \operatorname{Im} n_j\big(s, \widehat{\psi}(s)\big)\, \mathrm{d}s \\ -\mathrm{i}\sum_j \int_0^t \operatorname{Im} n_j\big(s, \widehat{\psi}(s)\big)\, \mathrm{d}\widehat{W}_j(s)\Big\}\widehat{\psi}(t)\,. \tag{2.50}$$

The vectors $\widehat{\phi}(t)$ have the same right to the name of "a posteriori states" as the vectors $\widehat{\psi}(t)$. By applying Itô formula to (2.50) and by using the differential (2.49) and the fact that $n_j\big(t, \widehat{\psi}(t)\big) \equiv n_j\big(t, \widehat{\phi}(t)\big)$, we get the stochastic differential of $\widehat{\phi}(t)$:

$$\mathrm{d}\widehat{\phi}(t) = \sum_j \big[R_j(t) - n_j\big(t, \widehat{\phi}(t)\big)\big]\widehat{\phi}(t)\, \mathrm{d}\widehat{W}_j(t) \\ + \Big[K(t) + \sum_j \overline{n_j\big(t, \widehat{\phi}(t)\big)}\, R_j(t) - \frac{1}{2}\sum_j \big|n_j\big(t, \widehat{\phi}(t)\big)\big|^2\Big]\widehat{\phi}(t)\, \mathrm{d}t. \tag{2.51}$$

Thus the choice (2.50) gives a simple expression for $\mathrm{d}\widehat{\phi}(t)$ which is commonly used in the literature, just as (2.49).

2.5.2 Four Stochastic Schrödinger Equations

Both equalities (2.49) and (2.51) are closed equations, in the stochastic processes $\widehat{\psi}$ and $\widehat{\phi}$ respectively, and both are known under the name of *stochastic Schrödinger equation* [9]. However, we got them for normalised vector processes and thus if we want to interpret them as SDEs for $\mathcal{H}$-vector processes, we need to extend them also to non-normalised vectors. There is not a unique way to do such an extension and we present for each of them two extensions, the most natural ones.

Equalities (2.49) and (2.51) involve the quantities $n_j(t, x)$ for normalised x. The first type of extension is to allow for a non-normalised x in the quadratic form (2.47) defining n_j; in this way polynomial coefficients are obtained. The second type of extension is to write $n_j(t, x)/\|x\|^2$ everywhere n_j appears in the differentials of normalised states and then to extend the resulting expressions in the natural way to non-normalised x; in this way we obtain coefficients with at most linear growth.

Thus, we obtain four nonlinear stochastic Schrödinger equations ($\ell = 1, 2, 3, 4$)

$$\begin{cases} \mathrm{d}X^\ell(t) = \sum_j L_j^\ell\big(t, X^\ell(t)\big)X^\ell(t)\, \mathrm{d}\widehat{W}_j(t) + K^\ell\big(t, X^\ell(t)\big)X^\ell(t)\, \mathrm{d}t\,, \\ X^\ell(0) = x_0\,, \quad x_0 \in \mathcal{H}\,, \end{cases} \tag{2.52}$$

where the quantities $L_j^\ell(t, x)$ and $K^\ell(t, x)$ are defined in the following; they are introduced in order to write the four SDEs always in a compact form.

The extension of the stochastic Schrödinger equation for $\widehat{\psi}$ with polynomial coefficients is obtained by taking

$$L_j^1(t,x) := R_j(t) - \operatorname{Re} n_j(t,x), \tag{2.53a}$$

$$\begin{aligned} K^1(t,x) &:= K(t) + \sum_j \big(\operatorname{Re} n_j(t,x)\big) R_j(t) - \frac{1}{2}\sum_j \big(\operatorname{Re} n_j(t,x)\big)^2 \\ &\equiv -\mathrm{i}\left[H(t) + \frac{\mathrm{i}}{2}\sum_j \big(\operatorname{Re} n_j(t,x)\big)\big(R_j(t) - R_j(t)^*\big)\right] \\ &\quad - \frac{1}{2}\sum_j L_j^1(t,x)^* L_j^1(t,x). \end{aligned} \tag{2.53b}$$

The extension of the stochastic Schrödinger equation for $\widehat{\psi}$ with linearly growing coefficients is given by the choice

$$L_j^2(t,x) := R_j(t) - \operatorname{Re}\frac{n_j(t,x)}{\|x\|^2}, \tag{2.54a}$$

$$\begin{aligned} K^2(t,x) &:= K(t) + \sum_j \left(\operatorname{Re}\frac{n_j(t,x)}{\|x\|^2}\right) R_j(t) - \frac{1}{2}\sum_j \left(\operatorname{Re}\frac{n_j(t,x)}{\|x\|^2}\right)^2 \\ &\equiv -\mathrm{i}\left[H(t) + \frac{\mathrm{i}}{2}\sum_j \left(\operatorname{Re}\frac{n_j(t,x)}{\|x\|^2}\right)\big(R_j(t) - R_j(t)^*\big)\right] \\ &\quad - \frac{1}{2}\sum_j L_j^2(t,x)^* L_j^2(t,x). \end{aligned} \tag{2.54b}$$

The extension of the stochastic Schrödinger equation for $\widehat{\phi}$ with polynomial coefficients is the one with

$$L_j^3(t,x) := R_j(t) - n_j(t,x), \tag{2.55a}$$

$$\begin{aligned} K^3(t,x) &:= K(t) + \sum_j \overline{n_j(t,x)}\, R_j(t) - \frac{1}{2}\sum_j \big|n_j(t,x)\big|^2 \\ &\equiv -\mathrm{i}\left[H(t) + \frac{\mathrm{i}}{2}\sum_j \left(\overline{n_j(t,x)}\, R_j(t) - n_j(t,x) R_j(t)^*\right)\right] \\ &\quad - \frac{1}{2}\sum_j L_j^3(t,x)^* L_j^3(t,x). \end{aligned} \tag{2.55b}$$

Finally, the extension of the stochastic Schrödinger equation for $\widehat{\phi}$ with linearly growing coefficients is obtained by taking

$$L_j^4(t,x) := R_j(t) - \frac{n_j(t,x)}{\|x\|^2}\,, \tag{2.56a}$$

$$\begin{aligned} K^4(t,x) &:= K(t) + \sum_j \frac{\overline{n_j(t,x)}}{\|x\|^2} R_j(t) - \sum_j \frac{\left|n_j(t,x)\right|^2}{2\,\|x\|^4} \\ &\equiv -\mathrm{i}\left[H(t) + \frac{\mathrm{i}}{2} \sum_j \left(\frac{\overline{n_j(t,x)}}{\|x\|^2} R_j(t) - \frac{n_j(t,x)}{\|x\|^2} R_j(t)^* \right) \right] \\ &\quad - \frac{1}{2} \sum_j L_j^4(t,x)^* L_j^4(t,x). \end{aligned} \tag{2.56b}$$

We are using the convention that

$$\frac{n_j(t,x)}{\|x\|^2} = 0 \qquad \text{for } x = 0. \tag{2.57}$$

When $\|x\| = 1$ we have $L^1(t,x) = L^2(t,x)$ and $K^1(t,x) = K^2(t,x)$ and the SDE (2.52) for $\ell = 1, 2$ reduces to (2.49) when $x_0 = \psi_0$, if one proves that the solution stays normalised for all t. However, the two equations are different when the initial condition has no norm one. Similarly, for $\|x\| = 1$ we have $L^3(t,x) = L^4(t,x)$ and $K^3(t,x) = K^4(t,x)$ and the SDE (2.52) for $\ell = 3, 4$ reduces to (2.51) when $x_0 = \phi_0$, if one proves that the solution stays normalised for all t.

2.5.2.1 The Conservative Case

This is the case $R_j(t)^* = -R_j(t)$ of Section 2.4.4, corresponding to dissipation, but no effective measurement. By setting $R_j(t) = -\mathrm{i}V_j(t)$, with $V_j(t)^* = V_j(t)$, we find

$$\operatorname{Re} n_j(t,x) = 0\,, \tag{2.58}$$

$$L_j^1(t,x) = L_j^2(t,x) = -\mathrm{i}V_j(t)\,, \tag{2.59a}$$

$$K^1(t,x) = K^2(t,x) = K(t) = -\mathrm{i}H(t) - \frac{1}{2}\sum_j V_j(t)^2\,, \tag{2.59b}$$

$$L_j^3(t,x) = -\mathrm{i}\left[V_j(t) - \langle x | V_j(t) x\rangle\right], \tag{2.60a}$$

$$L_j^4(t,x) = -\mathrm{i}\left[V_j(t) - \|x\|^{-2}\langle x | V_j(t) x\rangle\right], \tag{2.60b}$$

$$K^3(t,x) = -\mathrm{i}H(t) - \frac{1}{2}\sum_j L_j^3(t,x)^* L_j^3(t,x)\,, \tag{2.60c}$$

$$K^4(t,x) = -\mathrm{i}H(t) - \frac{1}{2}\sum_j L_j^4(t,x)^* L_j^4(t,x)\,. \tag{2.60d}$$

Then, for $\ell = 1, 2$ the stochastic Schrödinger equations (2.52) are linear and they coincide with the corresponding linear stochastic Schrödinger equation (2.46), while for $\ell = 3, 4$ they are nonlinear, but only due to a non-influent phase factor.

2.5.2.2 A Peculiar Case of Continuous Measurement

In the literature, when the case is considered of usual observables followed with continuity in time, a common choice is to take $R_j(t)^* = R_j(t)$ and to identify the continuously measured observables with $2R_j(t)$. In this case, the four stochastic Schrödinger equations (2.52) reduces to two, with a particularly simple form:

$$\operatorname{Im} n_j(t,x) = 0\,, \tag{2.61}$$

$$L_j^1(t,x) = L_j^3(t,x) = R_j(t) - n_j(t,x)\,, \tag{2.62a}$$

$$K^1(t,x) = K^3(t,x) = -\mathrm{i}H(t) - \frac{1}{2}\sum_j \big[R_j(t) - n_j(t,x)\big]^2\,, \tag{2.62b}$$

$$L_j^2(t,x) = L_j^4(t,x) = R_j(t) - \|x\|^{-2} n_j(t,x)\,, \tag{2.62c}$$

$$K^2(t,x) = K^4(t,x) = -\mathrm{i}H(t) - \frac{1}{2}\sum_j \big[R_j(t) - \|x\|^{-2} n_j(t,x)\big]^2\,. \tag{2.62d}$$

2.5.3 *Existence and Uniqueness of the Solution*

We have introduced four nonlinear SDEs (2.52) of the type of (A.14) with drift coefficients $b(x,t) = K^\ell(t,x)x$ and diffusion coefficients $\sigma_j(x,t) = L_j^\ell(t,x)x$, $\ell = 1,\ldots,4$, $j = 1,\ldots,d$, given by (2.53), (2.54), (2.55) and (2.56).

Remark 2.21. For every finite time horizon $T > 0$ the following statements hold.

- The drift and the diffusion coefficients of the four SDEs (2.52) satisfy Hypothesis A.25 (measurability condition).
- The expression $\langle x|b(x,t)\rangle + \frac{1}{2}\sum_j \left\|\sigma_j(x,t)\right\|^2$ goes into

$$\langle x|K^\ell(t,x)x\rangle + \frac{1}{2}\sum_j \left\|L_j^\ell(t,x)x\right\|^2$$
$$= \begin{cases} -\mathrm{i}\langle x|H(t)x\rangle + \mathrm{i}\sum_j \big(\operatorname{Re} n_j(t,x)\big)\big(\operatorname{Im} n_j(t,x)\big), & \ell = 1, \\ -\mathrm{i}\langle x|H(t)x\rangle + \dfrac{\mathrm{i}}{\|x\|^2}\sum_j \big(\operatorname{Re} n_j(t,x)\big)\big(\operatorname{Im} n_j(t,x)\big), & \ell = 2, \\ -\mathrm{i}\langle x|H(t)x\rangle, & \ell = 3, 4. \end{cases} \tag{2.63}$$

Therefore, the four sets of coefficients satisfy also the monotone condition (Hypothesis A.35) with $C(T) = 0$.

- By construction, the coefficients of the SDEs (2.52) with $\ell = 2, 4$ satisfy also the linear growth condition (Hypothesis A.34).

Lemma 2.22. *Let T be any finite time horizon. Then*

- *the coefficients of the SDEs* (2.52) *with* $\ell = 1, 3$ *satisfy the local Lipschitz condition (Hypothesis A.33);*
- *the coefficients of the SDEs* (2.52) *with* $\ell = 2, 4$ *satisfy the global Lipschitz condition (Hypothesis A.32).*

Proof. The coefficients of the SDEs (2.52) with $\ell = 1, 3$ are polynomials in the components of x; together with the boundedness Assumption 2.3, this gives by standard arguments that the local Lipschitz condition (A.17) holds.

Let us now consider the case $\ell = 2, 4$. Given two vectors x, y in $\mathcal{H}$, let us set

$$\hat{x} := \frac{x}{\|x\|}, \qquad \hat{y} := \frac{y}{\|y\|}, \qquad P_x := |\hat{x}\rangle\langle\hat{x}|, \qquad P_y := |\hat{y}\rangle\langle\hat{y}|, \tag{2.64a}$$

$$\hat{x}_\perp := \frac{(\mathbb{1} - P_y)x}{\left\|(\mathbb{1} - P_y)x\right\|}, \qquad \hat{y}_\perp := \frac{(\mathbb{1} - P_x)y}{\|(\mathbb{1} - P_x)y\|}. \tag{2.64b}$$

With these notations we can write

$$L_j^4(t, x)x = (\mathbb{1} - P_x)R_j(t)x, \tag{2.65a}$$

$$L_j^2(t, x)x = L_j^4(t, x)x + \mathrm{i}\left[\operatorname{Im} n_j(t, \hat{x})\right]x, \tag{2.65b}$$

$$K^4(t, x)x = K(t)x + g(t, x) - \frac{1}{2}P_x g(t, x), \tag{2.65c}$$

$$g(t, x) := \sum_j R_j(t)P_x R_j(t)^* x, \tag{2.65d}$$

$$K^2(t, x) = K^4(t, x) + \mathrm{i}\sum_j \left[\operatorname{Im} n_j(t, \hat{x})\right] R_j(t)x + \frac{1}{2}\sum_j \left[\operatorname{Im} n_j(t, \hat{x})\right]^2 x. \tag{2.65e}$$

By using

$$\|x - y\|^2 = \left\|(\mathbb{1} - P_y)x\right\|^2 + \left\|P_y x - y\right\|^2 = \|(\mathbb{1} - P_x)y\|^2 + \|P_x y - x\|^2,$$
$$\|y\| = \|y - x + x\| \le \|y - x\| + \|x\|, \qquad \|x\| \le \|x - y\| + \|y\|,$$

we get

$$\left\|(\mathbb{1} - P_y)x\right\| \le \|y - x\|, \qquad \left\|(\mathbb{1} - P_y)\hat{x}\right\| \le 1, \qquad |\langle\hat{x}|\hat{y}\rangle| \le 1,$$
$$\|y\|\left\|(\mathbb{1} - P_y)\hat{x}\right\| \le \|y - x\|\left\|(\mathbb{1} - P_y)\hat{x}\right\| + \left\|(\mathbb{1} - P_y)x\right\| \le 2\|y - x\|,$$
$$\|(\mathbb{1} - P_x)y\| \le \|y - x\|, \qquad \|(\mathbb{1} - P_x)\hat{y}\| \le 1, \qquad \|x\|\,\|(\mathbb{1} - P_x)\hat{y}\| \le 2\|y - x\|,$$

$$\begin{aligned}\left||\langle\hat{y}|\hat{x}\rangle|^2\langle\hat{y}|x\rangle-\langle\hat{y}|y\rangle\right| &= \left|\langle\hat{y}|P_xP_yx\rangle-\langle\hat{y}|y\rangle\right|\\ &= \left|\langle\hat{y}|P_xP_yx\rangle-\langle\hat{y}|P_xy\rangle-\langle\hat{y}|(\mathbb{1}-P_x)y\rangle\right|\\ &\le \left|\langle\hat{y}|P_xP_y(x-y)\rangle\right|+|\langle\hat{y}|(\mathbb{1}-P_x)y\rangle|\\ &\le \|x-y\|+\|(\mathbb{1}-P_x)y\|\le 2\|x-y\|\,,\end{aligned}$$

$$\|R_i(t)\|^2=\left\|R_i(t)^*\right\|\|R_i(t)\|=\left\|R_i(t)^*R_i(t)\right\|\le\left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\|.$$

Let us check the global Lipschitz condition for the various coefficients. Consider first L^4:

$$\begin{aligned}\sum_j\left\|L_j^4(t,x)x-L_j^4(t,y)y\right\|^2 &= \sum_j\left\|(\mathbb{1}-P_x)R_j(t)x-(\mathbb{1}-P_y)R_j(t)y\right\|^2\\ &= \sum_j\left\|(\mathbb{1}-P_x)\left[R_j(t)x-(\mathbb{1}-P_y)R_j(t)y\right]\right\|^2\\ &\quad+\sum_j\left\|P_x(\mathbb{1}-P_y)R_j(t)y\right\|^2\\ &= \sum_j\left\|(\mathbb{1}-P_x)\left[(\mathbb{1}-P_y)R_j(t)(x-y)+P_yR_j(t)x\right]\right\|^2\\ &\quad+\sum_j\left|\langle(\mathbb{1}-P_y)\hat{x}|R_j(t)y\rangle\right|^2;\end{aligned}$$

we have

$$\begin{aligned}\sum_j\left|\langle(\mathbb{1}-P_y)\hat{x}|R_j(t)y\rangle\right|^2 &\le \sum_j\left\|(\mathbb{1}-P_y)\hat{x}\right\|^2\left\|R_j(t)y\right\|^2\\ &= \sum_j\left\|(\mathbb{1}-P_y)\hat{x}\right\|^2\langle y|R_j(t)^*R_j(t)y\rangle\\ \le\left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\|\left\|(\mathbb{1}-P_y)\hat{x}\right\|^2\|y\|^2 &\le 4\left\|\textstyle\sum_j R_j(t)^*R_j(t)\right\|\|y-x\|^2\end{aligned}$$

and

$$\begin{aligned}&\sum_j\left\|(\mathbb{1}-P_x)\left[(\mathbb{1}-P_y)R_j(t)(x-y)+P_yR_j(t)x\right]\right\|^2\\ &\quad\le\sum_j\left(\left\|(\mathbb{1}-P_x)(\mathbb{1}-P_y)R_j(t)(x-y)\right\|+\left\|(\mathbb{1}-P_x)P_yR_j(t)x\right\|\right)^2\\ &\quad\le\sum_j\left(\left\|R_j(t)(x-y)\right\|+\|(\mathbb{1}-P_x)\hat{y}\|\left|\langle\hat{y}|R_j(t)x\rangle\right|\right)^2\end{aligned}$$

$$
\begin{aligned}
&\leq 2 \sum_j \left(\left\| R_j(t)(x-y) \right\|^2 + \|(\mathbb{1} - P_x)\hat{y}\|^2 \left\| R_j(t)x \right\|^2 \right) \\
&\leq 2 \left\| \textstyle\sum_j R_j(t)^* R_j(t) \right\| \left(\|x-y\|^2 + \|(\mathbb{1} - P_x)\hat{y}\|^2 \|x\|^2 \right) \\
&\leq 10 \left\| \textstyle\sum_j R_j(t)^* R_j(t) \right\| \|x-y\|^2 ,
\end{aligned}
$$

and so

$$
\sum_j \left\| L_j^4(t,x)x - L_j^4(t,y)y \right\|^2 \leq 14 \sup_{t\in[0,T]} \left\| \textstyle\sum_j R_j(t)^* R_j(t) \right\| \|x-y\|^2 .
$$

Therefore, $L_\bullet^4(t,x)x$ is globally Lipschitz.

We now consider K^4: we have

$$
\begin{aligned}
\|g(t,x) - g(t,y)\| &= \left\| \textstyle\sum_j R_j(t)\big(P_x R_j(t)^* x - P_y R_j(t)^* y\big) \right\| \\
&\leq \left\| \textstyle\sum_j R_j(t)(\mathbb{1} - P_y) P_x R_j(t)^* x \right\| \\
&\quad + \left\| \textstyle\sum_j R_j(t) P_y P_x (\mathbb{1} - P_y) R_j(t)^* x \right\| \\
&\quad + \left\| \textstyle\sum_j R_j(t) P_y P_x P_y R_j(t)^* (\mathbb{1} - P_y) x \right\| \\
&\quad + \left\| \textstyle\sum_j R_j(t)\big(P_y P_x P_y R_j(t)^* P_y x - P_y R_j(t)^* y\big) \right\| \\
&= \Big\{ \left\| \textstyle\sum_j R_j(t) |\hat{x}_\perp\rangle\langle \hat{x}| R_j(t)^* \hat{x}\rangle \right\| \\
&\quad + \left\| \textstyle\sum_j R_j(t) |\hat{y}\rangle\langle \hat{x}_\perp| R_j(t)^* \hat{x}\rangle \right\| |\langle \hat{y}|\hat{x}\rangle| \\
&\quad + \left\| \textstyle\sum_j R_j(t) |\hat{y}\rangle\langle \hat{y}| R_j(t)^* \hat{x}_\perp\rangle \right\| |\langle \hat{y}|\hat{x}\rangle|^2 \Big\} \|(\mathbb{1} - P_y)x\| \\
&\quad + \left\| \textstyle\sum_j R_j(t) |\hat{y}\rangle\langle \hat{y}| R_j(t)^* \hat{y}\rangle \right\| \big| |\langle \hat{y}|\hat{x}\rangle|^2 \langle \hat{y}|x\rangle - \langle \hat{y}|y\rangle \big| \\
&\leq \Big\{ \left\| \textstyle\sum_j R_j(t) |\hat{x}_\perp\rangle\langle \hat{x}| R_j(t)^* \hat{x}\rangle \right\| + \left\| \textstyle\sum_j R_j(t) |\hat{y}\rangle\langle \hat{x}_\perp| R_j(t)^* \hat{x}\rangle \right\| \\
&\quad + \left\| \textstyle\sum_j R_j(t) |\hat{y}\rangle\langle \hat{y}| R_j(t)^* \hat{x}_\perp\rangle \right\| \\
&\quad + 2 \left\| \textstyle\sum_j R_j(t) |\hat{y}\rangle\langle \hat{y}| R_j(t)^* \hat{y}\rangle \right\| \Big\} \|x-y\| ,
\end{aligned}
$$

$$
\begin{aligned}
\|g(t,x) - g(t,y)\| &\leq 5 \sum_i \|R_i(t)\|^2 \|x-y\| \\
&\leq 5d \sup_{t\in[0,T]} \left\| \textstyle\sum_j R_j(t)^* R_j(t) \right\| \|x-y\| .
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\| P_x g(t,x) - P_y g(t,y) \right\| \\
&\quad = \left\| P_y P_x g(t,x) + (\mathbb{1} - P_y) P_x g(t,x) - P_y P_x g(t,y) - P_y (\mathbb{1} - P_x) g(t,y) \right\| \\
&\quad \le \left\| (\mathbb{1} - P_y) P_x g(t,x) \right\| + \left\| P_y (\mathbb{1} - P_x) g(t,y) \right\| + \left\| P_y P_x \big(g(t,x) - g(t,y) \big) \right\| \\
&\quad = \left\| (\mathbb{1} - P_y) P_x g(t,x) \right\| + \left| \langle \widehat{y} | (\mathbb{1} - P_x) g(t,y) \rangle \right| + \left\| P_y P_x \big(g(t,x) - g(t,y) \big) \right\| \\
&\quad \le \left\| (\mathbb{1} - P_y) x \right\| \left| \langle \hat{x} | g(t,\hat{x}) \rangle \right| + \| (\mathbb{1} - P_x) y \| \, \| g(t,\hat{y}) \| + \| g(t,x) - g(t,y) \| \\
&\quad \le 7d \sup_{t \in [0,T]} \left\| \textstyle\sum_j R_j(t)^* R_j(t) \right\| \, \| x - y \| \, .
\end{aligned}
$$

Therefore, $K^4(t,x)x$ is globally Lipschitz.

We now consider L^2 and K^2, which are related to the previous coefficients by (2.65). The differences with respect to the terms with $\ell = 4$ have similar structures; it is enough to check one of such differences:

$$
\begin{aligned}
\left\| n_j\big(t,\hat{x}\big) x - n_j\big(t,\hat{y}\big) y \right\| &\le \left| n_j\big(t,\hat{x}\big) \right| \, \| x - y \| + \left| n_j\big(t,\hat{x}\big) - n_j\big(t,\hat{y}\big) \right| \, \| y \| \\
&\le 2 \left\| R_j(t) \right\| \, \| x - y \| + \left| n_j\big(t,\hat{x}\big) \, \| y \| - \langle \hat{y} | R_j(t) x \rangle \right| \\
&\le 2 \left\| R_j(t) \right\| \, \| x - y \| + \Big\| \, \| y \| \, \hat{x} - \| x \| \, \hat{y} \Big\| \left\| R_j(t) \right\| \le 3 \left\| R_j(t) \right\| \, \| x - y \| \, .
\end{aligned}
$$

Therefore, $L^2(t,x)x$ and $K^2(t,x)x$ are globally Lipschitz. □

Theorem 2.23. *Every one of the four SDEs* (2.52) *admits a strong solution in the time interval* $[0,+\infty)$. *Pathwise uniqueness and uniqueness in law hold. Moreover, the norm of the solutions of the equations with* $\ell = 2, 4$ *is conserved,*

$$
\left\| X^2(t) \right\|^2 = \left\| X^2(0) \right\|^2 , \qquad \left\| X^4(t) \right\|^2 = \left\| X^4(0) \right\|^2 , \tag{2.66}
$$

while for $\ell = 1, 3$ *we have*

$$
\begin{aligned}
1 - \left\| X^\ell(t) \right\|^2 &= \left(1 - \left\| X^\ell(0) \right\|^2 \right) \\
&\quad \times \exp\left\{ -2 \sum_j \int_0^t \operatorname{Re} n_j \left(s, X^\ell(s) \right) \left[\mathrm{d}\widehat{W}_j(s) + \operatorname{Re} n_j \left(s, X^\ell(s) \right) \mathrm{d}s \right] \right\}.
\end{aligned} \tag{2.67}
$$

Proof. Uniqueness and existence of solutions is by Remark 2.21, Lemma 2.22 and Theorem A.36.

By computations similar to those in (2.17), one gets

$$
\begin{gathered}
\mathrm{d} \left\| X^\ell(t) \right\|^2 = 2 \left(1 - \left\| X^\ell(t) \right\|^2 \right) \sum_j \operatorname{Re} n_j \left(t, X^\ell(t) \right) \mathrm{d}\widehat{W}_j(t) \, , \ \text{for } \ell = 1, 3, \\
\mathrm{d} \left\| X^\ell(t) \right\|^2 = 0 \, , \ \text{for } \ell = 2, 4 \, .
\end{gathered}
$$

Then, the statements about the norm follow from Proposition A.41 applied to the stochastic processes $Z(t) = 1 - \left\| X^\ell(t) \right\|^2$. □

Thus, in the case of polynomial coefficients ($\ell = 1, 3$), the solutions X^ℓ of the stochastic Schrödinger equation move inside the unit ball if $\left\| X^\ell(0) \right\| < 1$, on the unit sphere if $\left\| X^\ell(0) \right\| = 1$ and outside the unit ball if $\left\| X^\ell(0) \right\| > 1$. In the case of linearly growing coefficients ($\ell = 2, 4$), the solutions X^ℓ of the stochastic Schrödinger equation move on the corresponding spheres of radius $\left\| X^\ell(0) \right\|$.

If we take the four equations with the same normalised initial condition, by uniqueness, we have that the solutions of the equations of number 1 and 2 coincide and the same holds for the solutions of numbers 3 and 4. Moreover, the solutions of 1 or 2 and of 3 or 4 are connected by (2.50).

Of course, when the stochastic Schrödinger equation (2.52) is considered in the probability space $\left(\Omega, \mathcal{F}_T, \widehat{\mathbb{P}}^T_\psi\right)$ for $\ell = 1, 2$ and normalised initial condition, its solution $\widehat{\psi}$ is the normalisation (2.22) of the solution ψ of the linear stochastic Schrödinger equation (2.28) in $(\Omega, \mathcal{F}_T, \mathbb{Q})$.

2.5.4 *The Stochastic Schrödinger Equation as a Starting Point*

By the results of the previous subsection, we have that both the SDEs for a posteriori states (2.49) and (2.51) with initial condition $\widehat{\psi}(0) = \widehat{\phi}(0) = \psi_0$, $\|\psi_0\| = 1$, have a unique (pathwise and in law) strong solution with $\left\|\widehat{\psi}(t)\right\| = \left\|\widehat{\phi}(t)\right\| = 1$. The solutions of the two equations are connected by the relation (2.50).

This point is very important because it gives the possibility of starting the whole theory from the nonlinear stochastic Schrödinger equation; we sketch this construction just below. For the theory of continuous measurements, this is only an alternative possibility, but conceptually this is needed when the nonlinear SDE is postulated for some reason, as for a modification of quantum mechanics [3, 6, 15, 31], or it is used for stochastic simulations of quantum dynamical semigroups as explained in Sect. 3.2.3.2. The problem of strong solutions, in the more general context of infinite dimensional Hilbert spaces and equations involving unbounded operators as coefficients, was already studied in [32].

Every one of the four stochastic Schrödinger equations (2.52) can be taken as starting point; let us choose the SDE with $\ell = 2$. Let us fix a stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,+\infty)}, \mathbb{P}\right)$ in usual hypotheses and let $\widehat{B}$ be a continuous Wiener process in this basis with increments independent of the past. Let $\widehat{\psi}$ be a solution of (2.52), $\ell = 2$, with the Wiener process $\widehat{B}$ and initial condition $\psi_0 \in \mathcal{H}$, $\|\psi_0\| = 1$. By Theorem 2.23, the solution is unique and its norm is conserved: $\left\|\widehat{\psi}(t)\right\| = 1$, $\forall t \in [0, +\infty)$. Due to the normalisation for every time, the stochastic differential of $\widehat{\psi}(t)$ reduces to (2.49), i.e.

$$\mathrm{d}\widehat{\psi}(t) = \sum_j \left[R_j(t) - \frac{1}{2} m_j(t) \right] \widehat{\psi}(t)\, \mathrm{d}\widehat{B}_j(t) + \left[K(t) + \frac{1}{2} \sum_j m_j(t) R_j(t) - \frac{1}{8} \sum_j m_j(t)^2 \right] \widehat{\psi}(t)\, \mathrm{d}t, \quad (2.68a)$$

$$m_j(t) = 2\,\mathrm{Re}\langle \widehat{\psi}(t) | R_j(t) \widehat{\psi}(t) \rangle. \quad (2.68b)$$

In the case of a continuous measurement, besides the stochastic evolution of the state $\widehat{\psi}(t)$, we have to introduce also the stochastic output and its relation with $\widehat{\psi}(t)$. The output is the stochastic process with components

$$B_j(t) = \widehat{B}_j(t) + \int_0^t m_j(s) \mathrm{d}s. \quad (2.69)$$

The physical probability is $\mathbb{P}$. Notice that, having chosen the nonlinear stochastic Schrödinger equation as a starting point, the system state $\widehat{\psi}(t)$ at time t depends on $\widehat{B}(s)$, $0 \le s \le t$, which is not the observed output. Anyway we are still allowed to interpret $\widehat{\psi}(t)$ as the system state at time t conditioned by the observation of the output $B(s)$ for $0 \le s \le t$ because the knowledge of $B(s)$, $0 \le s \le t$, is equivalent to the knowledge of $\widehat{B}(s)$, $0 \le s \le t$. Heuristically one can think that the knowledge of the trajectory of $B(s)$ in $[0, t]$ determines the corresponding trajectory of $\widehat{B}(s)$ and thus the value of $\widehat{\psi}(t)$. The correct mathematical statement is that the two processes generate the same augmented filtration:

$$\sigma\Big\{ B(s),\, s \in [0, t] \Big\} \vee \mathcal{N} = \sigma\Big\{ \widehat{B}(s),\, s \in [0, t] \Big\} \vee \mathcal{N}. \quad (2.70)$$

Indeed, the inclusion $\subset$ is obvious because of (2.69) and because the process $\widehat{\psi}$ is adapted to the augmented natural filtration of $\widehat{B}$ thanks to Theorem 2.23. The opposite inclusion $\supset$ follows from the possibility of recovering the linear stochastic Schrödinger equation and by its theorem of existence and uniqueness of strong solutions. Let us show this fact.

Given the initial state ψ_0 of the system, consider the positive continuous process

$$q(t) = \exp\left\{ -\frac{1}{2} \sum_j \left[\int_0^t m_j(s) \mathrm{d}\widehat{B}_j(s) + \frac{1}{2} \int_0^t m_j(s)^2 \mathrm{d}s \right] \right\}. \quad (2.71)$$

Its square $q(t)^2$ is a positive $\mathbb{P}$-martingale and

$$\mathbb{Q}^t_{\psi_0}(\mathrm{d}\omega) = q(t, \omega)^2\, \mathbb{P}(\mathrm{d}\omega) \quad (2.72)$$

defines a new probability on $(\Omega, \mathcal{F}_t)$; the probabilities $\mathbb{Q}^t_{\psi_0}$, $t \geq 0$, are consistent. By Girsanov theorem, under the law $\mathbb{Q}^T_{\psi_0}$ the process $B(t)$, $t \in [0, T]$, with components (2.69) is a multidimensional standard Wiener process.

Let us define

$$\psi(t) = q(t)^{-1}\,\widehat{\psi}(t)\,; \tag{2.73}$$

by Itô calculus we get, under $\mathbb{Q}^T_{\psi_0}$, the linear stochastic Schrödinger equation (2.28):

$$\mathrm{d}\psi(t) = \sum_j R_j(t)\psi(t)\mathrm{d}B_j(t) + K(t)\psi(t)\mathrm{d}t. \tag{2.74}$$

Thus, Theorem 2.4 guarantees that $\psi(t)$ is adapted to the augmented filtration of $B(t)$ and then the same is true for $q(t) = \|\psi(t)\|^{-1}$, $\widehat{\psi}(t) = q(t)\psi(t)$ and $\widehat{B}(t)$. This completes the proof of (2.70).

Finally, the uniqueness in law of the solutions of all the equations involved guarantees that, for every finite interval of time $[0, T]$, the law of B under $\mathbb{P}$ and the law of W under $\widehat{\mathbb{P}}^T_{\psi_0}$ coincide. So, the two approaches, the one starting from the linear stochastic Schrödinger equation and the one starting from the nonlinear one, are completely equivalent.

2.6 The Linear Approach Versus the Nonlinear One

As the theory can be formulated by starting either from the linear stochastic Schrödinger equation, or from the nonlinear one, let us give here just some hints of comparison between the two approaches.

- Advantages of the linear approach:
  - Direct generalisation of the traditional description of an instantaneous measurement.
  - Clear analytical relation between the a posteriori state and the observed output: if the canonical realisation of the Wiener process is used, then the a posteriori states $\psi(t)$ and $\widehat{\psi}(t)$ are explicitly functions of the trajectory of the output $W(s)$ for $0 \leq s \leq t$.
- Characteristic features of the linear approach:
  - The output process W is a fixed function from the sample space Ω to $C^d_0(0, \infty)$, the space of all $\mathbb{R}^d$-valued continuous functions of a positive variable. Its physical properties depend on the physical probability $\widehat{\mathbb{P}}^T_{\psi_0}$, which changes on $(\Omega, \mathcal{F})$ according to the choice of the initial system state ψ_0.

- Disadvantages of the linear approach:
  - The linear stochastic Schrödinger equation is not suitable for numerical simulations as the norm of the non-normalised a posteriori state $\psi(t)$ can become very small.
- Advantages of the nonlinear approach:
  - The stochastic Schrödinger equation directly gives the a posteriori state $\widehat{\psi}(t)$.
  - The stochastic Schrödinger equation is suitable for numerical simulations [7, 32–35].
- Characteristic features of the nonlinear approach:
  - The probability $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F})$ is fixed. The output B is a function from the sample space Ω to $C_0^d(0, \infty)$, the space of all $\mathbb{R}^d$-valued continuous functions of a positive variable, which changes according to the choice of the initial system state ψ_0 (thus modifying its physical properties).
- Disadvantages of the nonlinear approach:
  - Non-transparent relation between the a posteriori state $\widehat{\psi}(t)$ and the output $B(t)$.

2.7 Tricks to Simplify the Equations

In special cases, some peculiar time dependencies can be eliminated and/or more compact forms of the stochastic Schrödinger equation can be obtained. Let us see how.

2.7.1 Time-Dependent Coefficients and Unitary Transformations

A particularly interesting case is when the time dependence of the coefficients in the linear stochastic Schrödinger equation (2.28) can be eliminated by using a unitary transformation. Let us assume that there exists a self-adjoint operator H_0 such that

$$\mathrm{e}^{\mathrm{i}H_0 t} R_j(t) \mathrm{e}^{-\mathrm{i}H_0 t} = R_j(0) , \qquad \mathrm{e}^{\mathrm{i}H_0 t} H(t) \mathrm{e}^{-\mathrm{i}H_0 t} = H(0) . \tag{2.75}$$

In the physical literature, this transformation is known as the use of a (suitable) *interaction picture*. We define the "interaction Hamiltonian" $H_I := H(0) - H_0$ and

$$R_j^0 := R_j(0) , \qquad K^0 := K(0) - \mathrm{i}H_0 \equiv -\mathrm{i}H_I - \frac{1}{2} \sum_j R_j^{0*} R_j^0 . \tag{2.76}$$

By setting

$$\Phi(t) := \mathrm{e}^{\mathrm{i}H_0 t}\, \psi(t)\,, \tag{2.77}$$

we get $\mathrm{d}\Phi(t) = \mathrm{i}H_0\Phi(t)\mathrm{d}t + \mathrm{e}^{\mathrm{i}H_0 t}K(t)\psi(t)\mathrm{d}t + \mathrm{e}^{\mathrm{i}H_0 t}\sum_j R_j(t)\psi(t)\mathrm{d}W_j(t)$. By inserting before $\psi(t)$ the identity $\mathbb{1} = \mathrm{e}^{-\mathrm{i}H_0 t}\mathrm{e}^{\mathrm{i}H_0 t}$, we obtain the linear SDE with time-independent coefficients

$$\mathrm{d}\Phi(t) = K^0\Phi(t)\mathrm{d}t + \sum_j R_j^0\Phi(t)\mathrm{d}W_j(t)\,. \tag{2.78}$$

We can now redo the whole construction of probabilities and a posteriori states by starting from this equation instead of from (2.2). We have $\|\psi(t)\|^2 = \|\Phi(t)\|^2$, $m_j(t) = 2\,\mathrm{Re}\langle\widehat{\psi}(t)|R_j(t)\widehat{\psi}(t)\rangle = 2\,\mathrm{Re}\langle\widehat{\Phi}(t)|R_j^0\widehat{\Phi}(t)\rangle$ and nothing changes for what concerns the physical probabilities. We have only to recall that the a posteriori states are given by $\widehat{\psi}(t) = \mathrm{e}^{-\mathrm{i}H_0 t}\widehat{\Phi}(t)$.

In the example of Section 8.1, we use just this trick in order to simplify the time dependence of the coefficients.

2.7.2 Complex Noise

When one of the coefficients $R_j(t)$ in the linear stochastic Schrödinger equation (2.28) differs from another one only by a multiplicative factor i (imaginary unit), the equations assume a simpler form by introducing complex Wiener processes [5, 33, 36–38]. Let us illustrate this fact in the case $d = 2$.

Assume that we have

$$R_1(t) = \frac{1}{\sqrt{2}}\, R(t), \qquad R_2(t) = \frac{\mathrm{i}}{\sqrt{2}}\, R(t). \tag{2.79}$$

Then, we define the complex Wiener process

$$W(t) = \frac{1}{\sqrt{2}}\, W_1(t) + \frac{\mathrm{i}}{\sqrt{2}}\, W_2(t), \tag{2.80}$$

for which the Itô rules turn out to be $\mathrm{d}W(t)^2 = 0$, $\mathrm{d}\overline{W(t)}\mathrm{d}W(t) = \mathrm{d}t$. With these notations the linear SDE (2.28) becomes

$$\mathrm{d}\psi(t) = R(t)\psi(t)\mathrm{d}W(t) + K(t)\psi(t)\mathrm{d}t\,, \qquad K(t) = -\mathrm{i}H(t) - \frac{1}{2}\, R(t)^* R(t). \tag{2.81}$$

Also the nonlinear stochastic Schrödinger equation assumes a simpler form in this case, especially if we consider the a posteriori states $\widehat{\phi}(t)$ with a changed phase:

$$
\begin{aligned}
\mathrm{d}\widehat{\phi}(t) &= \big[R(t) - \langle \widehat{\phi}(t)|R(t)\widehat{\phi}(t)\rangle\big]\,\widehat{\phi}(t)\,\mathrm{d}\widehat{W}(t) \\
&\quad + \left[K(t) + \langle \widehat{\phi}(t)|R(t)^*\widehat{\phi}(t)\rangle R(t) - \frac{1}{2}\left|\langle \widehat{\phi}(t)|R(t)\widehat{\phi}(t)\rangle\right|^2\right]\widehat{\phi}(t)\,\mathrm{d}t, \qquad (2.82)
\end{aligned}
$$

$$
\widehat{W}(t) = \frac{1}{\sqrt{2}}\,\widehat{W}_1(t) + \frac{\mathrm{i}}{\sqrt{2}}\,\widehat{W}_2(t) = W(t) - \int_0^t \langle \widehat{\phi}(s)|R(s)^*\widehat{\phi}(s)\rangle \mathrm{d}s. \qquad (2.83)
$$

2.8 Summary: The Stochastic Schrödinger Equation

2.8.1 The Linear Stochastic Schrödinger Equation

2.8.1.1 Hilbert Space and System Operators

Assumptions 2.1, 2.3, 2.10.

- The Hilbert space of the quantum system under consideration is $\mathcal{H} = \mathbb{C}^n$.
- The effective Hamiltonian $H(t)$ and the system operators $R_j(t)$, $j = 1, \dots, d$, (dissipative terms) are non-random linear operators on $\mathcal{H}$; $H(t)$ is self-adjoint: $H(t)^* = H(t)$.
- The functions $t \mapsto H(t)$ and $t \mapsto R_j(t)$ are measurable and, for every $T \in (0, +\infty)$,

$$
\sup_{t\in[0,T]} \|H(t)\| < +\infty, \qquad \sup_{t\in[0,T]} \left\| \sum_j R_j(t)^* R_j(t) \right\| < +\infty.
$$

- We use the shorthand notation: $K(t) := -\mathrm{i}H(t) - \dfrac{1}{2}\displaystyle\sum_{j=1}^{d} R_j(t)^* R_j(t)$.

2.8.1.2 Reference Probability Space and Filtrations

Assumption 2.2, Remark 2.5.

- $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$ is a stochastic basis satisfying the usual conditions, which means that $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space, $(\mathcal{F}_t)$ is a filtration of sub-σ-algebras of $\mathcal{F}$, $\mathcal{F}_t = \displaystyle\bigcap_{s:s>t} \mathcal{F}_s$, $\mathbb{Q}(A) = 0 \Rightarrow A \in \mathcal{F}_t$, $\forall t \geq 0$.
- $\mathcal{F} = \mathcal{F}_\infty := \displaystyle\bigvee_{t\geq 0} \mathcal{F}_t$, $\qquad \mathcal{N} := \{B \in \mathcal{F} : \mathbb{Q}(B) = 0\}$.
- The symbol $\mathbb{E}_{\mathbb{Q}}$ indicates the expectation with respect to $\mathbb{Q}$.
- W is a continuous d-dimensional Wiener process defined in $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q}\big)$. In particular, the process W has increments independent of the past with respect to the filtration $(\mathcal{F}_t)$.
- The natural filtration of the increments of W: $\mathcal{D}_t^s := \sigma\{W(u) - W(s),\ u \in [s, t]\}$.

- The augmented natural filtration of the increments of W: $\overline{\mathcal{D}}_t^s := \mathcal{D}_t^s \vee \mathcal{N}$.
- The filtration $\{\overline{\mathcal{D}}_t^s,\ t \in [s, +\infty)\}$ satisfies the usual conditions: $\overline{\mathcal{D}}_t^s$ is independent of $\mathcal{F}_s$ and $\overline{\mathcal{D}}_t^s \subset \overline{\mathcal{D}}_t^0 \subset \mathcal{F}_t \subset \mathcal{F}$, for $0 \le s \le t$.

2.8.1.3 The Linear Stochastic Schrödinger Equation

Assumptions 2.2, 2.12, equations (2.2), (2.7), (2.11), Propositions 2.6, 2.7, 2.8, Theorem 2.11.

- The linear stochastic Schrödinger equation (2.28):

$$\mathrm{d}\psi(t) = K(t)\psi(t)\,\mathrm{d}t + \sum_{j=1}^{d} R_j(t)\psi(t)\,\mathrm{d}W_j(t)\,.$$

- Initial condition: a non-random $\psi_0 \in \mathcal{H}, \quad \|\psi_0\| = 1$.
- The solution is an $\mathcal{H}$-valued process ψ, which is continuous and $\left(\overline{\mathcal{D}}_t^0\right)$-adapted.
- $\|\psi(t)\|^2$ is a mean one, continuous martingale.
- The stochastic evolution operator, or propagator, A_t^s is a continuous process in $t \ge s$, which is $\left(\overline{\mathcal{D}}_t^s\right)$-adapted and independent of $\mathcal{F}_s$. It satisfies

$$\mathrm{d}A_t^s = K(t)A_t^s\,\mathrm{d}t + \sum_{j=1}^{d} R_j(t)A_t^s\,\mathrm{d}W_j(t)\,, \qquad A_s^s = \mathbb{1}.$$

- The adjoint operator $(A_t^s)^*$ satisfies

$$\mathrm{d}(A_t^s)^* = (A_t^s)^* K(t)^*\mathrm{d}t + \sum_{j=1}^{d} (A_t^s)^* R_j(t)^*\mathrm{d}W_j(t)\,, \qquad (A_s^s)^* = \mathbb{1}.$$

- $\psi(t) = A_t^0\psi_0, \qquad A_t^r = A_t^s A_s^r$ for $0 \le r \le s \le t$.
- $\det A_t^s > 0, \qquad A_t^s = A_t^r\left(A_s^r\right)^{-1}$ for $0 \le r \le s \le t$.
- The inverse operator $(A_t^s)^{-1}$ satisfies

$$\mathrm{d}(A_t^s)^{-1} = (A_t^s)^{-1}\left[\sum_j R_j(t)^2 - K(t)\right]\mathrm{d}t - \sum_{j=1}^{d} (A_t^s)^{-1} R_j(t)\mathrm{d}W_j(t)\,,$$

with $(A_s^s)^* = \mathbb{1}$.

2.8.1.4 The Physical Probability and the A Posteriori States

Equations (2.22), (2.23), (2.33), (2.39), (2.43), Theorems 2.11, 2.14, Proposition 2.17, Remarks 2.13, 2.16, 2.18.

- $\psi(t)$ is the non-normalised a posteriori state at time t.
- $\|\psi(t)\| > 0, \quad \widehat{\psi}(t) := \|\psi(t)\|^{-1}\ \psi(t)$.
- $\widehat{\psi}(t)$ is the a posteriori state at time t.
- $m_j(t) := \big\langle \widehat{\psi}(t) \big| \big(R_j(t) + R_j(t)^*\big) \widehat{\psi}(t) \big\rangle = 2\,\mathrm{Re}\big\langle \widehat{\psi}(t) \big| R_j(t) \widehat{\psi}(t) \big\rangle$.
- $\|\psi(t)\|^2 = \exp\Bigg\{ \sum_j \bigg[\int_0^t m_j(s)\mathrm{d}W_j(s) - \frac{1}{2}\int_0^t m_j(s)^2 \mathrm{d}s \bigg] \Bigg\}$.
- The expression $\widehat{\mathbb{P}}^T_{\psi_0}(\mathrm{d}\omega) = \|\psi(T,\omega)\|^2\, \mathbb{Q}(\mathrm{d}\omega)\Big|_{\mathcal{F}_T}$ defines the "physical" probability on $(\Omega, \mathcal{F}_T)$. The expectation with respect to $\widehat{\mathbb{P}}^T_{\psi_0}(\mathrm{d}\omega)$ is denoted by $\widehat{\mathbb{E}}^T_{\psi_0}$. The physical probability for the events regarding the output W up to time T is $\widehat{\mathbb{P}}^T_{\psi_0}\big|_{\overline{\mathcal{D}}^0_T}$.
- The family of probabilities $\Big\{ \widehat{\mathbb{P}}^T_{\psi_0},\ T > 0 \Big\}$ is consistent, which means that for any choice of $T > t \geq 0$ we have $\widehat{\mathbb{P}}^T_{\psi_0}(F) = \widehat{\mathbb{P}}^t_{\psi_0}(F)$, $\forall F \in \mathcal{F}_t$.
- There exists a unique probability $\widehat{\mathbb{P}}^\infty_{\psi_0}$ on $\mathcal{D}^0_\infty := \bigvee_{t>0} \mathcal{D}^0_t$ such that

$$\widehat{\mathbb{P}}^\infty_{\psi_0}(F) = \widehat{\mathbb{P}}^T_{\psi_0}(F), \qquad \forall T > 0, \quad \forall F \in \mathcal{D}^0_T\,.$$

- Under the physical law $\widehat{\mathbb{P}}^T_{\psi_0}$, the process with components

$$\widehat{W}_j(t) := W_j(t) - \int_0^t m_j(s)\mathrm{d}s, \qquad t \in [0, T],$$

  is a continuous Wiener processes with increments independent of the past. It is $(\overline{\mathcal{D}}^0_t)$-adapted.
- The stochastic integrals with respect to W and $\widehat{W}$ are linked by (2.32).
- POMs and probabilities:

$$\widehat{E}^s_t(F) := \int_F A^s_t(\omega)^* A^s_t(\omega)\mathbb{Q}(\mathrm{d}\omega), \qquad \forall F \in \overline{\mathcal{D}}^s_t\,,$$
$$\widehat{\mathbb{P}}^T_{\psi_0}(F) = \langle \psi_0 | \widehat{E}^0_T(F)\psi_0 \rangle, \qquad \forall F \in \overline{\mathcal{D}}^0_T\,.$$

- Consistency of the POMs:

$$0 \leq r < s < t, \quad F \in \overline{\mathcal{D}}^r_s \quad \Rightarrow \quad \widehat{E}^r_t(F) = \widehat{E}^r_s(F).$$

- For all $F \in \overline{\mathcal{D}}^s_t$, $0 \leq s < t \leq T$, we have

$$\widehat{\mathbb{P}}^T_{\psi_0}(F|\mathcal{F}_s) = \big\langle \widehat{\psi}(s) \big| \widehat{E}^s_t(F) \widehat{\psi}(s) \big\rangle = \widehat{\mathbb{P}}^T_{\psi_0}(F|\overline{\mathcal{D}}^0_s).$$

2.8.2 The Nonlinear Stochastic Schrödinger Equation

- Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,+\infty)}, \mathbb{P}\big)$ be a stochastic basis in the usual hypotheses and $\widehat{B}$ be a continuous Wiener process in this basis with increments independent of the past.
- Both the nonlinear SDEs

$$\begin{aligned}\mathrm{d}\widehat{\psi}(t) = {} & \sum_j \big[R_j(t) - \mathrm{Re}\langle\widehat{\psi}(t)|R_j(t)\widehat{\psi}(t)\rangle\big]\,\widehat{\psi}(t)\,\mathrm{d}\widehat{B}_j(t) + K(t)\widehat{\psi}(t)\,\mathrm{d}t \\ & + \sum_j \Big[\big(\mathrm{Re}\langle\widehat{\psi}(t)|R_j(t)\widehat{\psi}(t)\rangle\big)\,R_j(t) - \frac{1}{2}\big(\mathrm{Re}\langle\widehat{\psi}(t)|R_j(t)\widehat{\psi}(t)\rangle\big)^2\Big]\,\widehat{\psi}(t)\,\mathrm{d}t\end{aligned}$$

  and

$$\begin{aligned}\mathrm{d}\widehat{\phi}(t) = {} & \sum_j \big[R_j(t) - \langle\widehat{\phi}(t)|R_j(t)\widehat{\phi}(t)\rangle\big]\,\widehat{\phi}(t)\,\mathrm{d}\widehat{B}_j(t) + K(t)\widehat{\phi}(t)\,\mathrm{d}t \\ & + \sum_j \Big[\overline{\langle\widehat{\phi}(t)|R_j(t)\widehat{\phi}(t)\rangle}\,R_j(t) - \frac{1}{2}\big|\langle\widehat{\phi}(t)|R_j(t)\widehat{\phi}(t)\rangle\big|^2\Big]\widehat{\phi}(t)\,\mathrm{d}t,\end{aligned}$$

  with initial condition $\widehat{\psi}(0) = \widehat{\phi}(0) = \psi_0$, $\|\psi_0\| = 1$, have a unique (pathwise and in law) strong solution with $\big\|\widehat{\psi}(t)\big\| = \big\|\widehat{\phi}(t)\big\| = 1$. The solutions of the two equations are connected by the relation

$$\begin{aligned}\widehat{\phi}(t) = \exp\Big\{ & -\mathrm{i}\sum_j \int_0^t \mathrm{Re}\langle\widehat{\psi}(s)|R_j(s)\widehat{\psi}(s)\rangle\,\mathrm{Im}\langle\widehat{\psi}(s)|R_j(s)\widehat{\psi}(s)\rangle\,\mathrm{d}s \\ & -\mathrm{i}\sum_j \int_0^t \mathrm{Im}\langle\widehat{\psi}(s)|R_j(s)\widehat{\psi}(s)\rangle\,\mathrm{d}\widehat{B}_j(s)\Big\}\widehat{\psi}(t).\end{aligned}$$

- The output of the measurement is the process $B(t)$, $t \geq 0$, under the law $\mathbb{P}$, with components

$$B_j(t) = \widehat{B}_j(t) + \int_0^t m_j(s)\mathrm{d}s\,,$$

  where

$$m_j(t) = 2\,\mathrm{Re}\langle\widehat{\phi}(t)|R_j(t)\widehat{\phi}(t)\rangle = 2\,\mathrm{Re}\langle\widehat{\psi}(t)|R_j(t)\widehat{\psi}(t)\rangle.$$

  More precisely, the output is the collection of the increments of B in the interval of observation; heuristically, the output is the time derivative of B.

- The square of

$$q(t) = \exp\left\{-\frac{1}{2}\sum_j\left[\int_0^t m_j(s)\mathrm{d}\widehat{B}_j(s) + \frac{1}{2}\int_0^t m_j(s)^2\mathrm{d}s\right]\right\}$$

  is a positive $\mathbb{P}$-martingale, and

$$\mathbb{Q}^t_{\psi_0}(\mathrm{d}\omega) = q(t,\omega)^2\,\mathbb{P}(\mathrm{d}\omega)$$

  defines a new probability on $(\Omega, \mathcal{F}_t)$; the probabilities $\mathbb{Q}^t_{\psi_0}$, $t \geq 0$, are consistent.
- Under the law $\mathbb{Q}^T_{\psi_0}$ the process $B(t)$, $t \in [0, T]$, is a multidimensional standard Wiener process.
- The random vector

$$\psi(t) = q(t)^{-1}\,\widehat{\psi}(t),$$

  under the law $\mathbb{Q}^T_{\psi_0}$, satisfies the linear SDE

$$\mathrm{d}\psi(t) = \sum_j R_j(t)\psi(t)\mathrm{d}B_j(t) + K(t)\psi(t)\mathrm{d}t.$$

- In particular cases, the SDEs involved in the theory can be simplified by some tricks, for instance by using unitary transformations (in the case in Sect. 2.7.1) or complex Wiener processes (in the case in Sect. 2.7.2).

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