Chapter 2: Fixed Point Theory

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1 Introduction

Fixed points and fixed point computations occur in just about every field of computer science. Their widespread use is due to the fact that the semantics of recursion can be described by fixed points of functions or functionals, or more generally, functors or morphisms. Of course, the treatment of fixed points in mathematics goes well back before their first use in computer science: They frequently occur in analysis, algebra, geometry, and logic. One

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of the first occurrences of fixed points in the theory of automata and formal languages were probably the equational characterizations of regular and context-free languages as least solutions to right-linear and polynomial fixed point equations. Kleene's theorem for regular languages follows from the fixed point characterization just by a few simple equational properties of the fixed point operation. Many results in the theory of automata and languages can be derived from basic properties of fixed points.

The aim of this paper is to provide an introduction to that part of the theory of fixed points that has applications to weighted automata and weighted languages. We start with a treatment of fixed points in the ordered setting and review some basic theorems guaranteeing the existence of least (or greatest) fixed points. Then we establish several (equational) properties of the least fixed point operation including the Bekić identity asserting that systems of fixed point equations can be solved by the technique of successive elimination. Then we use the Bekić identity and some other basic laws to introduce the axiomatic frameworks of Conway and iteration theories. We provide several axiomatizations of these notions and review some completeness results showing that iteration theories capture the equational properties of the fixed point operation in a large class of models. In the last two sections, we treat fixed points of linear functions and affine functions over semirings and semimodules. The main results show that for such functions, the fixed point operation can be characterized by a star operation possibly in conjunction with an omega operation. We show that the equational properties of the fixed point operation are reflected by corresponding properties of the star and omega operations.

Some Notation

The composition of functions $f : A \to B$ and $g : B \to C$ is written $g \circ f$, $x \mapsto g(f(x))$. The identity function $A \to A$ will be denoted $\mathbf{id}_A$. When $f : A \to B$ and $g : A \to C$, the *target pairing* (or just *pairing*) of f and g is the function $\langle f, g\rangle : A \to B \times C$, $x \mapsto (f(x), g(x))$, $x \in A$. In the same way, one defines the *(target) tupling* $f = \langle f_1, \dots, f_n\rangle : A \to B_1 \times \cdots \times B_n$ of $n \geq 0$ functions $f_i : A \to B_i$. When $n = 0$, the Cartesian product $B_1 \times \cdots \times B_n$ is a singleton set and f is the unique function from A to this set. The ith *projection function* $A_1 \times \cdots \times A_n \to A_i$ will be denoted $\mathbf{pr}_{A_i}^{A_1 \times \cdots \times A_n}$, or $\mathbf{pr}_i^{A_1 \times \cdots \times A_n}$, or just $\mathbf{pr}_i$. A *base function* is any tupling of projections. When $f : A \to A'$ and $g : B \to B'$, $f \times g$ is the function $A \times B \to A' \times B'$ mapping each pair $(x, y) \in A \times B$ to $(f(x), g(y))$. Clearly, $f \times g = \langle f \circ \mathbf{pr}_A^{A \times B}, g \circ \mathbf{pr}_B^{A \times B}\rangle$.

2 Least Fixed Points

When A is a set, an *endofunction* over A is a function $A \to A$. We say that $a \in A$ is a *fixed point* of f if $f(a) = a$. We also say that a is a *solution of* or *solves the fixed point equation* $x = f(x)$. When A is partially ordered by a relation $\leq$, we also define *prefixed points* of f as those elements $a \in A$ with $f(a) \leq a$. Dually, we call $a \in A$ a *post-fixed point* of f if $a \leq f(a)$. Thus, a fixed point is both a prefixed point and a post-fixed point. A *least fixed point* of f is least among the fixed points of f, and a *least prefixed point* is least among all prefixed points of f. *Greatest fixed points* and *greatest post-fixed points* are defined dually. It is clear that the extremal (i.e., least or greatest) fixed points, prefixed points and post-fixed points are unique whenever they exist.

Least prefixed points give rise to the following *fixed point induction* principle. When P is a poset and $f : P \to P$ has a least prefixed point x, then we have $x \leq y$ whenever $f(y) \leq y$. As an application of the principle, we establish a simple fact.

Proposition 2.1. *Let P be a partially ordered set and let $f : P \to P$ be monotone. If f has a least prefixed point, then it is the least fixed point of f. Dually, if f has a greatest post-fixed point, then it is the greatest fixed point of f.*

Proof. We only prove the first claim since the second follows by reversing the order. Suppose that p is the least prefixed point of f. Then $f(p) \leq p$, and since f is monotone, $f(f(p)) \leq f(p)$. This shows that $f(p)$ is a prefixed point. Thus, by fixed point induction, $p \leq f(p)$. Since p is both a prefixed point and a post-fixed point, it is a fixed point. $\square$

Next, we provide conditions guaranteeing the existence of fixed points. Recall that a *directed set* in a partially ordered set P is a nonempty subset D of P such that any two elements of D have an upper bound in D. A *chain* in P is a linearly ordered subset of P. Note that every nonempty chain is a directed set.

Definition 2.2. *A* complete partial order*, or cpo is a partially ordered set $P = (P, \leq)$ which has a least element denoted $\perp_P$ or just $\perp$ such that each directed set $D \subseteq P$ has a supremum $\bigvee D$.*

It is known that a partially ordered set P is a cpo iff it has suprema of all chains, or suprema of well-ordered chains, cf. [43]. See also [16].

Definition 2.3. *Suppose that P, Q are partially ordered sets and $f : P \to Q$. We say that f is* continuous *if it preserves all existing suprema of directed sets: For all directed sets $D \subseteq P$, if $\bigvee D$ exists, then so does $\bigvee f(D)$, and*

$$f\Big(\bigvee D\Big) = \bigvee f(D).$$

Every continuous function $P \to Q$ is monotone, since for all $x, y \in P$ with $x \leq y$, $f(y) = f(\bigvee\{x, y\}) = \bigvee\{f(x), f(y)\}$, i.e., $f(x) \leq f(y)$. From [43], it is also known that a function $f : P \to Q$ is continuous iff it preserves suprema of nonempty chains, or suprema of nonempty well-ordered chains.

Remark 2.4. Suppose that P, Q are partially ordered sets and $f : P \to Q$ and $g : Q \to P$ are monotone functions. We say that (f, g) is a *Galois connection* if for all $x \in P$ and $y \in Q$, $f(x) \leq y$ iff $x \leq g(y)$. It is known that when (f, g) is a Galois connection then f preserves all existing suprema, and g preserves all existing infima. In particular, f is continuous, and when P has a least element $\bot_P$ then Q also has a least element $\bot_Q$, and $f(\bot_P) = \bot_Q$.

Theorem 2.5. *Suppose that P is a cpo and $f : P \to P$ is monotone. Then f possesses a least prefixed point (which is the least fixed point of f).*

Proof. Define $x_\alpha = f(x_\beta)$, if α is the successor of the ordinal β, and $x_\alpha = \bigvee\{x_\beta : \beta < \alpha\}$ if α is a limit ordinal. In particular, x_0 is the least element $\bot$. It is a routine matter to verify that $x_\alpha \leq x_\beta$ whenever $\alpha \leq \beta$. Thus, there is a (least) ordinal α with $x_\alpha = x_{\alpha+1}$. This element x_α is the least prefixed point of f. □

A partial converse of Theorem 2.5 is proved in [43]: If P is a partially ordered set such that any monotone endofunction $P \to P$ has a least fixed point, then P is a cpo.

The above rather straightforward argument makes use of the axiom of choice. An alternative proof which avoids using this axiom is presented in [19]. A special case of the theorem is the *Knaster–Tarski fixed point theorem* [50, 19] asserting that a monotone endofunction of a complete lattice L has a least (and by duality, also a greatest) fixed point.

When the endofunction f in Theorem 2.5 is continuous, the least fixed point can be constructed in ω steps.

Corollary 2.6. *Suppose that P is a cpo and $f : P \to P$ is continuous. Then the least prefixed point of f is $\bigvee\{f^n(\bot) : n \geq 0\}$ (which is the least fixed point of f).*

Proof. Using the above notation, we have by continuity that $f(x_\omega) = x_\omega$, where $x_\omega = \bigvee\{f^n(\bot) : n \geq 0\}$. □

Note that the same result holds if we only assume that P is a *countably complete* or *ω-complete* partially ordered set, i.e., when it has a least element and suprema of ω-chains, or equivalently, suprema of countable directed sets, and if f is *ω-continuous*, i.e., it preserves suprema of ω-chains, or suprema of countable directed sets.

Dually, if P is a partially ordered set which has infima of all chains, and thus a greatest element $\top$, and if $f : P \to P$ is monotone, then f has a greatest post-fixed point which is the greatest fixed point of f. This greatest fixed point can be constructed as the first x_γ with $x_\gamma = x_{\gamma+1}$, where $x_\alpha = f(x_\beta)$ if α is the successor of the ordinal β, and $x_\alpha = \bigwedge\{x_\beta : \beta < \alpha\}$ if α is a limit ordinal, the infimum of the set $\{x_\beta : \beta < \alpha\}$. Thus, $x_0 = \top$. If, in addition, f preserves infima of nonempty chains, then the greatest post-fixed point is $\bigwedge\{f^n(\top) : n \geq 0\}$.

Besides single fixed point equations $x = f(x)$, we will consider finite *systems of fixed point equations*:

$$\begin{aligned} x_1 &= f_1(x_1, \ldots, x_n), \\ &\vdots \\ x_n &= f_n(x_1, \ldots, x_n). \end{aligned}$$

Each component equation $x_i = f_i(x_1, \ldots, x_n)$ of such a system may be considered as a fixed point equation in the unknown x_i and the *parameters* $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. This leads to *parametric fixed point equations* of the sort $x = f(x, y)$, where f is a function $P \times Q \to P$.

Note that when P and Q are cpo's and A is a set, then $P \times Q$ and Q^A, equipped with the pointwise order, are cpo's. Moreover, the set of all continuous functions $P \to Q$ is also a cpo denoted $(P \to Q)$. For any partially ordered sets, P_1, P_2, Q and function $f : P_1 \times P_2 \to Q$, f is monotone or continuous iff it is *separately* monotone or continuous in either argument, i.e., when the functions ${}_{p_1}f : P_2 \to Q$ and $f_{p_2} : P_1 \to Q$, ${}_{p_1}f(y) = f(p_1, y)$, $f_{p_2}(x) = f(x, p_2)$, $p_1 \in P_1$, $p_2 \in P_2$ have the appropriate property. Moreover, a function $f = \langle f_1, f_2 \rangle : Q \to P_1 \times P_2$ is monotone or continuous iff both functions $f_i = \mathbf{pr}_i \circ f$, $i = 1, 2$ are monotone or continuous, where $\mathbf{pr}_1$ and $\mathbf{pr}_2$ denote the first and second projection functions $P_1 \times P_2 \to P_1$ and $P_1 \times P_2 \to P_2$.

Definition 2.7. *Suppose that P, Q are partially ordered sets such that $f : P \times Q \to P$ is monotone and for each $y \in P$ the endofunction $f_y : P \to P$, $f_y(x) = f(x, y)$ has a least prefixed point. Then we define $f^\dagger : Q \to P$ as the function mapping each $y \in Q$ to the least prefixed point of f_y.*

In a similar fashion, one could define a greatest (post)fixed point operation. Since the properties of this operation follow from the properties of the least (pre)fixed point operation by simple duality, below we will consider only the least fixed point operation. Nested least and greatest fixed points are considered in the μ-calculus, cf. Arnold and Niwinski [1]. It is known that over complete lattices, the alternation hierarchy obtained by nesting least and greatest fixed points is infinite.

Notice the pointwise nature of the above definition. For each $y \in Q$, $f^\dagger(y)$ is $(f_y)^\dagger$, the least prefixed point of the function $f_y : P \to P$ (which may be identified with a function $P \times R \to P$, where R has a single element).

The above definition of the *dagger operation* is usually applied in the case when P, Q are cpo's. In that case, the existence of the least prefixed point is guaranteed by Theorem 2.5.

Proposition 2.8. *Suppose that $f : P \times Q \to P$ is monotone. Then $f^\dagger$ is also monotone. Moreover, when P and Q are cpo's and f is continuous, so is $f^\dagger$.*

Proof. Assume that $y \leq z$ in Q. If x is a prefixed point of f_z, then $f_y(x) = f(x,y) \leq f(x,z) = f_z(x) \leq x$, so that x is also a pre-fixed point of f_y. Thus, the least prefixed point of f_y is below the least prefixed point of f_z, i.e., $f^\dagger(y) \leq f^\dagger(z)$.

Assume now that P and Q are cpo's and f is continuous. Let D denote a directed subset of Q. We have

$$f^\dagger(y) = \bigvee\{f_y^n(\perp_P) : n \geq 0\} = \bigvee\{f^n(\perp_P, y) : n \geq 0\}, \quad \text{for all } y \in Q,$$

where we define $f^0(x,y) = x$ and $f^{n+1}(x,y) = f(f^n(x,y),y)$, for all $n \geq 0$. Now,

$$\begin{aligned}
\bigvee\Big\{f^n\Big(\perp_P, \bigvee D\Big) : n \geq 0\Big\} &= \bigvee\Big\{\bigvee\{f^n(\perp_P, y) : y \in D\} : n \geq 0\Big\} \\
&= \bigvee\{f^n(\perp_P, y) : n \geq 0,\ y \in D\} \\
&= \bigvee\Big\{\bigvee\{f^n(\perp_P, y) : n \geq 0\} : y \in D\Big\} \\
&= \bigvee\{f^\dagger(y) : y \in D\}.
\end{aligned}$$
□

It is also known that for cpo's P, Q, the function $((P \times Q) \to P) \to (Q \to P)$ which maps each continuous $f : P \times Q \to P$ to the continuous function $f^\dagger : Q \to P$ is itself continuous; see, e.g., [19].

The dagger operation satisfies several nontrivial equational properties. We list a few below. Let P, Q, R denote cpo's and $f, g, \ldots$ monotone or continuous functions whose sources and targets are specified below.

Fixed point identity

$$f^\dagger = f \circ \langle f^\dagger, \mathbf{id}_Q \rangle \tag{1}$$

where $f : P \times Q \to P$.

Parameter identity

$$\big(f \circ (\mathbf{id}_P \times g)\big)^\dagger = f^\dagger \circ g \tag{2}$$

where $f : P \times Q \to P$ and $g : R \to Q$.

Composition identity

$$\big(f \circ \langle g, \mathbf{pr}_R^{P\times R}\rangle\big)^\dagger = f \circ \big\langle \big(g \circ \langle f, \mathbf{pr}_R^{Q\times R}\rangle\big)^\dagger, \mathbf{id}_R \big\rangle \tag{3}$$

where $f : Q \times R \to P$, $g : P \times R \to Q$ and $\mathbf{pr}_R^{P\times R} : P \times R \to P$ and $\mathbf{pr}_R^{Q\times R} : Q \times R \to Q$ are projection functions.

Double dagger identity or Diagonal identity

$$\big(f^\dagger\big)^\dagger = \big(f \circ (\langle \mathbf{id}_P, \mathbf{id}_P\rangle \times \mathbf{id}_Q)\big)^\dagger, \tag{4}$$

where $f : P \times P \times Q \to P$.

Pairing identity or Bekić identity

$$\langle f, g\rangle^\dagger = \big\langle f^\dagger \circ \big\langle h^\dagger, \mathbf{id}_R\big\rangle, h^\dagger\big\rangle \tag{5}$$

where $f : P \times Q \times R \to P$, $g : P \times Q \times R \to Q$ and

$$h = g \circ \big\langle f^\dagger, \mathbf{id}_{Q\times R}\big\rangle. \tag{6}$$

Permutation identity

$$\big(\pi \circ f \circ \big(\pi^{-1} \times \mathbf{id}_Q\big)\big)^\dagger = \pi \circ f^\dagger, \tag{7}$$

where

$$f : P_1 \times \cdots \times P_n \times Q \to P_1 \times \cdots \times P_n \quad \text{and}$$
$$\pi = \big\langle \mathbf{pr}_{i_1}^{P_1\times\cdots\times P_n}, \ldots, \mathbf{pr}_{i_n}^{P_1\times\cdots\times P_n}\big\rangle$$

for some permutation $(i_1, \ldots, i_n)$ of the first n positive integers, and where π^{-1} is the inverse of π, i.e., $\pi^{-1} = \langle \mathbf{pr}_{j_1}^{P_1\times\cdots\times P_n}, \ldots, \mathbf{pr}_{j_n}^{P_1\times\cdots\times P_n}\rangle$ where $(j_1, \ldots, j_n)$ is the inverse of $(i_1, \ldots, i_n)$.

For these identities, we refer to [4, 20, 44, 45, 47, 51] and [9]. Each of the above identities can be explained using an ordinary functional language. For example, the fixed point identity (1) says that $f^\dagger(y)$ is a solution of the fixed point equation $x = f(x, y)$ in the unknown x and parameter y. It is customary to write this least solution as $\mu x.f(x, y)$. Using this μ-notation, the fixed point identity reads $f(\mu x.f(x, y), y) = \mu x.f(x, y)$. The parameter identity (2) is implicit in the μ-notation. It is due to the pointwise nature of the definition of dagger, and it says that solving $x = f(x, y)$ and then substituting $g(z)$ for y gives the same result as first substituting $g(z)$ for y and then solving $x = f(x, g(z))$. In the composition identity (3), one considers the equations $x = f(g(x, z), z)$ and $y = g(f(y, z), z)$, with least solutions $\mu x.f(g(x, z), z)$ and $\mu y.g(f(y, z), z)$. The composition identity asserts that these are related: $\mu x.f(g(x, z), z) = f(\mu y.g(f(y, z), z), z)$. The double dagger identity (4) asserts that the least solution of $x = f(x, x, z)$ is the same as the least solution of $y = f^\dagger(y, z)$, where $f^\dagger(y, z)$ is in turn the least solution of $x = f(x, y, z)$. In the μ-notation, $\mu x.\mu y.f(x, y, z) = \mu x.f(x, x, z)$. The Bekić identity (5) asserts that systems

$$x = f(x, y, z), \tag{8}$$
$$y = g(x, y, z) \tag{9}$$

can be solved by *Gaussian elimination* (or *successive elimination*). To find the least solution of the above system, where f, g are appropriate functions, one can proceed as follows. First, solve the first equation to obtain $x = f^\dagger(y, z)$, then substitute this solution for x in the second equation to obtain $y = g(f^\dagger(y, z), y, z) = h(y, z)$. The identity asserts that the second component of the least solution of the above system is the least solution of $y =$

$h(y,z)$, i.e., $h^\dagger(z)$. Moreover, it asserts that the first component is $f^\dagger(h^\dagger(z),z)$, which is obtained by back substituting $h^\dagger(z)$ for y in $f^\dagger(y,z)$, the solution of just the first equation. In the μ-notation, $\mu(x,y).(f(x,y,z),g(x,y,z)) = (\mu x.f(\mu y.h(y,z),z),\mu y.h(y,z))$, where $h(y,z) = g(\mu x.f(x,y,z),y,z)$.

We still want to illustrate the Bekić identity over semirings. So, suppose that S is a continuous semiring, cf. [21]. It will be shown later that there is a star operation $^* : S \to S$ such that least solutions of fixed point equations $x = ax+b$ can be expressed as a^*b. Suppose now that $f,g : S^2 \to S$, $f(x,y) = ax+by+e$ and $g(x,y) = cx+dy+f$ and consider the system of fixed point equations

$$\begin{aligned} x &= ax+by+e,\\ y &= cx+dy+f. \end{aligned}$$

Then $f^\dagger(y)$, the least solution of just the first equation is $a^*(by+e) = a^*by+a^*e$. Thus, $h(y) = g(f^\dagger(y),y)$ is $(d+ca^*b)y+ca^*e+f$, and $h^\dagger = (d+ca^*b)^*(ca^*e+f)$ is the second component of the least solution of the above system. The first component is $f^\dagger(h^\dagger) = a^*b(d+ca^*b)^*(ca^*e+f)+a^*e$. Using the matrix notation, the least solution of

$$\begin{pmatrix} x\\ y\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix} + \begin{pmatrix} e\\ f\end{pmatrix}$$

is

$$\begin{pmatrix} x\\ y\end{pmatrix} = \begin{pmatrix} a^*b(d+ca^*b)^*ca^*+a^* & a^*b(d+ca^*b)^*\\ (d+ca^*b)^*ca^* & (d+ca^*b)^*\end{pmatrix}\begin{pmatrix} e\\ f\end{pmatrix}.$$

We leave it to the reader to express the permutation identity (7) in the μ-notation.

A special case of the fixed point identity is

$$\left(f\circ \mathbf{pr}_Q^{P\times Q}\right)^\dagger = f, \tag{10}$$

where $f : Q \to P$, and a special case of the parameter identity is

$$\left(f\circ \mathbf{pr}_{P\times Q}^{P\times Q\times R}\right)^\dagger = f^\dagger\circ \mathbf{pr}_Q^{Q\times R}, \tag{11}$$

where $f : P\times Q \to P$. A special case of the permutation identity (7) is

Transposition identity

$$\left(\pi_{Q,P}^{P,Q}\circ\langle f,g\rangle\circ\left(\pi_{P,Q}^{Q,P}\times \mathbf{id}_R\right)\right)^\dagger = \pi_{Q,P}^{P,Q}\circ\langle f,g\rangle^\dagger, \tag{12}$$

where $f : P\times Q\times R \to P$ and $g : P\times Q\times R \to Q$, and where $\pi_{Q,P}^{P,Q} = \langle \mathbf{pr}_Q^{P\times Q}, \mathbf{pr}_P^{P\times Q}\rangle$ and $\pi_{P,Q}^{Q,P} = \langle \mathbf{pr}_P^{Q\times P}, \mathbf{pr}_Q^{Q\times P}\rangle$.

In the μ-notation, (10) can be written as $\mu x.f(y) = f(y)$, while the transposition identity (12) asserts that $\mu(x,y).(f(x,y,z), g(x,y,z))$ is the transposition of $\mu(y,x).(g(x,y,z), f(x,y,z))$. Equation (11), being a special case of the parameter identity, is implicit in the μ-notation.

Theorem 2.9. *All of the above identities hold for the least prefixed point operation.*

Proof. It is clear that the fixed point (1), parameter (2), and permutation (7) identities hold. We now establish the pairing identity (5). Suppose that $f : P \times Q \times R \to P$, $g : P \times Q \times R \to Q$ such that $f^\dagger$ and $h^\dagger$ exist. This means that for all $y \in Q$ and $z \in R$, $f^\dagger(y,z)$ is the least prefixed point solution of the single equation (8), and for all $z \in R$, $h^\dagger(z)$ is the least prefixed point solution of the equation

$$y = h(y,z).$$

We want to show that for all z, $(f^\dagger(h^\dagger(z),z), h^\dagger(z))$ is the least prefixed point solution of the system consisting of (8) and (9). But

$$f\big(f^\dagger\big(h^\dagger(z),z\big),h^\dagger(z),z\big) = f^\dagger\big(h^\dagger(z),z\big)$$

and

$$g\big(f^\dagger\big(h^\dagger(z),z\big),h^\dagger(z),z\big) = h\big(h^\dagger(z),z\big) = h^\dagger(z),$$

showing that $(f^\dagger(h^\dagger(z),z), h^\dagger(z))$ is a solution. Suppose that (x_0,y_0) is any prefixed point solution, so that $f(x_0,y_0,z) \le x_0$ and $g(x_0,y_0,z) \le y_0$. Then $f^\dagger(y_0,z) \le x_0$, and thus

$$h(y_0,z) = g\big(f^\dagger(y_0,z),z\big) \le g(x_0,y_0,z) \le y_0.$$

Thus, by fixed point induction, $h^\dagger(z) \le y_0$ and $f^\dagger(h^\dagger(z),z) \le f^\dagger(y_0,z) \le x_0$.

The double dagger and composition identities may be established directly using fixed point induction. Below, we show that these are already implied by (10), (11) and the pairing (5), and transposition (12) identities. First, note that by the pairing and transposition identities, we also have the following version of the pairing identity:

$$\langle f,g\rangle^\dagger = \big\langle k^\dagger, \big(g \circ \big(\pi^{Q,P}_{P,Q} \times \mathbf{id}_R\big)\big)^\dagger \circ \big\langle k^\dagger, \mathbf{id}_R\big\rangle\big\rangle \tag{13}$$

where

$$k = f \circ \big\langle \mathbf{pr}^{P\times R}_P, \big(g \circ \big(\pi^{Q,P}_{P,Q} \times \mathbf{id}_R\big)\big)^\dagger, \mathbf{pr}^{P\times R}_R\big\rangle. \tag{14}$$

Now, for the double dagger identity (4), assume that $f : P \times P \times Q \to P$. Let $g = \mathbf{pr}^{P\times P\times Q}_1$ and consider the function $\langle f,g\rangle : P \times P \times Q \to P \times P$. We can compute the second component of $\langle f,g\rangle^\dagger$ in two ways using the two versions of the pairing identity. The first version gives $f^{\dagger\dagger}$, while the second gives, using (10) and (11), $(f \circ (\langle \mathbf{id}_P, \mathbf{id}_P\rangle \times \mathbf{id}_Q))^\dagger$.

As for the composition identity (3), assume that $f : Q \times R \to P$, $g : P \times R \to P$. Then define $f' = f \circ \langle \mathbf{pr}_Q^{P\times Q\times R}, \mathbf{pr}_R^{P\times Q\times R}\rangle$ and $g' = g \circ \langle \mathbf{pr}_P^{P\times Q\times R}, \mathbf{pr}_R^{P\times Q\times R}\rangle$, and use the two versions of the pairing identity (and (10)) to compute the first component of $\langle f', g'\rangle^\dagger$ in two different ways. □

As already noted, the above identities are not all independent, (10), (11), (12) are instances of (1), (2), and (7), respectively. By the proof of Theorem 2.9, (10), (11) and the pairing (5), and transposition (12) identities imply (in conjunction with the usual laws of function composition and the Cartesian structure) the double dagger (4) and composition (3) identities. The fixed point identity is a particular instance of the composition identity (take $P = Q$ and $g = \mathbf{pr}_P^{P\times Q}$). In fact, the following systems are all equivalent, cf. [9]:

1. The system consisting of (10), (11), and the pairing (5), and transposition (12) (or permutation (7)) identities.
2. The system consisting of (10), (11) and the two versions of the pairing identity, (5) and (13).
3. The system consisting of the parameter (2), double dagger (4), and composition (3) identities.

Several other identities follow. For example, the following "symmetric version" of the Bekić identity follows. For all f, g as in the Bekić identity,

$$\langle f, g\rangle^\dagger = \langle k^\dagger, h^\dagger\rangle \tag{15}$$

where h and k are defined in (6) and (14). In the μ-notation, (15) can be written as

$$\begin{aligned}&\mu(x,y).\big(f(x,y,z), g(x,y,z)\big)\\ &\quad = \big(\mu x.f\big(x, \mu y.g(x,y,z), z\big), \mu y.g\big(\mu x.f(x,y,z), y, z\big)\big).\end{aligned}$$

3 Conway Theories

In most applications of fixed point theory, one considers a collection T of functions $f : A^n \to A^m$, for a *fixed set* A, sometimes equipped with additional structure, where n, m are nonnegative integers. For example, T may consist of the monotone, or continuous functions $P^n \to P^m$, where P is a cpo. When T contains the projection functions and is closed under composition and tupling, we call T a *Lawvere theory of functions*, or just a *theory of functions*. The collection of all functions $A^n \to A$ of a theory of functions is a *function clone*, cf. [16].

There is a more abstract notion due to Lawvere [42]. We may think of a theory T of functions over a set A as a category whose objects are not the sets A^n, but rather the nonnegative integers n. A morphism $n \to m$ in T is a function $A^n \to A^m$, subject to certain conditions. As such, T is a *category* with all finite products in the categorical sense (cf., e.g., [3]), with $n+m$ being the *product* of n and m, and 0 being the *terminal object*.

Definition 3.1. *A theory is a small category whose objects are the nonnegative integers such that each integer n is the n-fold product of object 1 with itself.*

Morphisms between theories are defined in the natural way. They preserve objects, composition, and the projections. It follows that morphisms also preserve tupling (and thus pairing) and the identity morphisms. Below, when T is a theory, we denote by $T(m,n)$ the set of morphisms $n \to m$ in T. (Note the reversal of the source and the target.)

Below, we will assume that each theory T comes with given *projection morphisms* $\mathbf{pr}_i^n : n \to 1$, $i = 1, \ldots, n$ making object n the n-fold product of 1 with itself. In a similar way, we write $\mathbf{pr}_n^{n,m}$ and $\mathbf{pr}_m^{n,m}$ for the projections $n+m \to n$ and $n+m \to m$, given by $\langle \mathbf{pr}_1^{n+m}, \ldots, \mathbf{pr}_n^{n+m} \rangle$ and $\langle \mathbf{pr}_{n+1}^{n+m}, \ldots, \mathbf{pr}_{n+m}^{n+m} \rangle$, respectively, and $\mathbf{id}_n = \langle \mathbf{pr}_1^n, \ldots, \mathbf{pr}_n^n \rangle$ for the identity morphism $n \to n$. Without loss of generality, we will assume that $\mathbf{id}_1 = \mathbf{pr}_1^1$. Since 0 is a terminal object, for each n, there is a unique morphism $n \to 0$. In any theory, a *base morphism* is a tupling of projection morphisms. For example, the identity morphisms and the morphisms with target 0 are base morphisms. Note that there is a base morphism $n \to m$ corresponding to each function $\rho : \{1, \ldots, m\} \to \{1, \ldots, n\}$, namely the morphism $\langle \mathbf{pr}_{\rho(1)}^n, \ldots, \mathbf{pr}_{\rho(m)}^n \rangle$. When ρ is bijective, injective, etc. we will also say that the corresponding base morphism has the appropriate property. When $f : p \to m$ and $g : q \to n$, $f \times g : p + q \to m + n$ is $\langle f \circ \mathbf{pr}_p^{p,q}, g \circ \mathbf{pr}_q^{p,q} \rangle$. When T is understood, we will just write $f : n \to m$ for $f \in T(m,n)$.

There is a representation theorem for theories (see, e.g., [9]) by which each theory is isomorphic to a theory of functions. But very often there are more natural ways of representing the morphisms of a theory (e.g., as matrices over a semiring).

When a theory T is equipped with a dagger operation $^\dagger : T(n, n+p) \to T(n,p)$, $n, p \geq 0$, we define when the fixed point identity and the other identities given above hold in T in the natural and expected way. For example, the fixed point identity (1) is given by

$$f^\dagger = f \circ \langle f^\dagger, \mathbf{id}_p \rangle, \tag{16}$$

where $f : n+p \to n$. As another example, the pairing identity (5) is understood in the form

$$\langle f, g \rangle^\dagger = \langle f^\dagger \circ \langle h^\dagger, \mathbf{id}_p \rangle, h^\dagger \rangle \tag{17}$$

where $f : n + m + p \to n$, $g : n + m + p \to m$ and

$$h = g \circ \langle f^\dagger, \mathbf{id}_{m+p} \rangle.$$

Definition 3.2. *A* Conway theory *is a theory T equipped with a dagger operation* $^\dagger : T(n, n+p) \to T(n, p)$ *which satisfies (10), (11), the pairing (5), and transposition (12) (or permutation (7)) identities.*

Morphisms of Conway theories, or theories equipped with a dagger operation, also preserve dagger. Two alternative axiomatizations of Conway theories are given below. By the discussion at the end of the preceding section, we have the following theorem.

Theorem 3.3. *Let T be a theory equipped with a dagger operation. The following are equivalent:*

1. *T is a Conway theory.*
2. *T satisfies (10), (11), and the two versions of the pairing identity, (5) and (13).*
3. *T satisfies the parameter (2), double dagger (4), and composition (3) identities.*

Corollary 3.4. *Any Conway theory satisfies all of the identities defined above.*

Yet another axiomatization can be derived from the following result.

Theorem 3.5. *Suppose that T is a theory equipped with a* scalar dagger operation $^\dagger : T(1, 1+p) \to T(1, p)$, $p \geq 0$ *satisfying the* scalar parameter *(18),* scalar composition *(19) and* scalar double dagger *(20) identities below.*

Scalar parameter identity

$$\big(f \circ (\mathbf{id}_1 \times g)\big)^\dagger = f^\dagger \circ g, \tag{18}$$

for all $f : 1+p \to 1$ and $g : q \to p$.
Scalar composition identity

$$\big(f \circ \langle g, \mathbf{pr}_p^{1,p}\rangle\big)^\dagger = f \circ \big\langle \big(g \circ \langle f, \mathbf{pr}_p^{1,p}\rangle\big)^\dagger, \mathbf{id}_p \big\rangle, \tag{19}$$

for all $f, g : 1+p \to 1$.
Scalar double dagger identity

$$f^{\dagger\dagger} = \big(f \circ (\langle \mathbf{id}_1, \mathbf{id}_1\rangle \times \mathbf{id}_p)\big)^\dagger, \tag{20}$$

for all $f : 2+p \to 1$.

Then there is a unique way to extend the dagger operation to all morphisms $n+p \to n$ for all $n, p \geq 0$ such that T becomes a Conway theory.

Proof. The unique extension is given by induction on n. When $n = 0$, $^\dagger : T(0,p) \to T(0,p)$ is the identity function on the singleton set $T(0,p)$. On morphisms in $T(1, 1+p)$, the dagger is already defined. Suppose that $n > 1$ and $f \in T(n, n+p)$. Then let $m = n-1$ and write f as $f = \langle f_1, f_2\rangle$ where $f_1 : m+1+p \to m$, $f_2 = m+1+p \to 1$. Then define $f^\dagger$ as $\langle f_1^\dagger \circ \langle h^\dagger, \mathbf{id}_p\rangle, h^\dagger\rangle$ where $h = g \circ \langle f_1^\dagger, \mathbf{pr}_1^{1+p}\rangle$. ∎

Corollary 3.6. *A theory T equipped with a dagger operation is a Conway theory iff T satisfies the scalar versions of the parameter, composition, and double dagger identities and the* scalar version of the pairing identity *(17), where f is arbitrary but g is scalar (i.e., $m = 1$).*

A detailed study of the identities true of all Conway theories is given in [5]. It is shown that there is an algorithm to decide whether an identity holds in all Conway theories, and that this decision problem is complete for PSPACE. The proof is based on a description of the structure of the free Conway theories using "aperiodic congruences" of flowchart schemes.

Remark 3.7. In any theory T equipped with a dagger operation, one may define a *feedback operation* $\uparrow : T(n+p, n+q) \to T(p,q)$, $n,p,q \geq 0$: Given $\langle f, g\rangle : n+q \to n+p$ with $f : n+q \to n$ and $g : n+q \to p$, we *define* $\uparrow\langle f, g\rangle = g \circ \langle f^\dagger, \mathbf{id}_q\rangle$. Then T, equipped with the feedback operation and the operation $\times$ as "tensor product" is a *traced monoidal category* [38]. The same notion was earlier defined under a different name in connection with flowcharts; see [49]. In fact, Conway theories correspond to traced monoidal categories whose tensor product is a (Cartesian) product. Another aspect of the connection is that traced monoidal categories are axiomatized by the identities that hold for flowchart schemes, and Conway theories by those that hold for flowchart schemes modulo aperiodic simulations (and the iteration theories defined in the next section are axiomatized by the identities that hold for flowchart schemes with respect to arbitrary simulations, or strong behavioral equivalence). Flowchart schemes were first axiomatized in [8]. For more information on the connection between Conway theories and traced monoidal categories, we refer to [36, 49]. See also Chap. 6, Sect. 8 in [9].

4 Iteration Theories

The Conway identities do not capture all equational properties of the least (pre)fixed point operation. In order to achieve completeness, we now introduce the *commutative identity* in any theory T equipped with a dagger operation:

$$\mathbf{pr}_1 \circ \langle f \circ (\rho_1 \times \mathbf{id}_p), \ldots, f \circ (\rho_n \times \mathbf{id}_p)\rangle^\dagger = \big(f \circ (\rho \times \mathbf{id}_p)\big)^\dagger, \qquad (21)$$

where $f : n+p \to 1$, $n \geq 1$, each $\rho_i : n \to n$ is a base morphism (i.e., a tupling of projections), and ρ is the unique base morphism $1 \to n$, i.e., ρ is the *diagonal* $\langle \mathbf{id}_1, \ldots, \mathbf{id}_1\rangle$. Particular instances of the commutative identity are the *group identities*. Suppose that G is a finite group of order n with group operation denoted. Moreover, suppose for simplicity that the carrier of G is the set $\{1, \ldots, n\}$ with 1 being the unit element of G. For each i, define ρ_i as the tupling of the n projection morphisms $\mathbf{pr}^n_{i\cdot j}$, so that $\rho_i = \langle \mathbf{pr}^n_{i1}, \ldots, \mathbf{pr}^n_{in}\rangle$. Then the commutative identity above is called the group identity associated with G. (When the permutation identity holds, as will be the case below, it does not matter how the elements of the group G are enumerated.)

The commutative identity (21) can be explained in theories of continuous or monotone functions over cpo's as follows. Suppose that P is a cpo and $f : P^{n+p} \to P$ is continuous. Moreover, suppose that each $\rho_i : P^n \to P^n$ is a tupling of projections, i.e., a base function. Then consider the system of equations in n unknowns and p parameters:

$$\begin{aligned} x_1 &= f(x_{\rho_1(1)}, \dots, x_{\rho_1(n)}, y_1, \dots, y_p), \\ &\vdots \\ x_n &= f(x_{\rho_n(1)}, \dots, x_{\rho_n(n)}, y_1, \dots, y_p). \end{aligned}$$

The commutative identity asserts that the first component of the least solution of this parametric system is just the least solution of the single parametric equation

$$x = f(x, \dots, x, y_1, \dots, y_p).$$

When the permutation identity holds, the same is true for all other components.

Definition 4.1. *An iteration theory is a Conway theory satisfying the group identities.*

Morphisms of iteration theories are Conway theory morphisms. Iteration theories were defined in [6, 7] and independently in [24]. The axiomatization in [24] used the Conway theory identities and the "vector form" of the commutative identity; see below. The completeness of the group identities in conjunction with the Conway theory identities was proved in [27].

Theorem 4.2. *An identity involving the dagger operation holds in all theories of continuous functions on cpo's iff it holds in all theories of monotone functions on cpo's iff it holds in iteration theories.*

The proof is based on a concrete description of the free iteration theories as theories of *regular trees*, cf. [9], which are the unfoldings of finite flowchart schemes [8]. By this concrete description, it is known that there is a P-time algorithm to decide whether an identity holds in all iteration theories; see [18].

Theorem 4.2 can be generalized to a great extent. The following result was proved in [26].

Theorem 4.3. *The iteration theory identities are complete for the class of all theories T equipped with a partial order $\leq$ on each hom-set $T(n,m)$ and a dagger operation such that the operations of composition and tupling are monotone. Moreover, the fixed point identity (1), the parameter identity (2), and the fixed point induction axiom hold, so that*

$$f \circ \langle g, \mathbf{id}_p \rangle \leq g \quad \Longrightarrow \quad f^\dagger \leq g, \tag{22}$$

for all $f : n + p \to n$ and $g : p \to n$.

Thus, in such theories, called *Park theories* in [26], the fixed point equation $\xi = f \circ \langle \xi, \mathbf{id}_p \rangle$ has a least solution, namely $f^\dagger$. With the same argument as in the proof of Proposition 2.8, it follows that the dagger operation is also monotone. Equivalently, one may define Park theories as ordered theories as above satisfying the scalar parameter identity (18), the scalar versions of the fixed point and pairing identities, i.e., (16) with $n = 1$ and (17) with $m = 1$, and the fixed point induction axiom (22) for $n = 1$. See the proof of the Bekić identity. Moreover, the fixed point identity may be replaced by the inequality $f \circ \langle f^\dagger, \mathbf{id}_p \rangle \leq f^\dagger$, for all appropriate f. Instances of Park theories are the *continuous theories* and *rational theories*, cf. [9, 51]. In a continuous theory T, each $T(m, n)$ is a cpo and composition is continuous. The dagger operation is defined by least fixed points. In particular, the theory of continuous functions over a cpo is a continuous theory. A rational theory T is also ordered, but not all directed sets in $T(m, n)$ have suprema. But there are enough suprema to have least solutions of fixed point equations. It is known that each rational theory embeds in a continuous theory.

More generally, one often considers certain 2-categories, called 2-*theories*, such that for each $f : n + p \to n$ there is an *initial solution* of the fixed point equation $\xi = f \circ \langle \xi, \mathbf{id}_p \rangle$. The identities satisfied by such 2-theories are again those of iteration theories, cf. [34].

An essential feature of iteration theories is that the "vector form" of each identity true of iteration theories holds in all iteration theories. In a semantic setting, this means the following. Given a theory T and an integer k, we can form a new theory Tk whose morphisms $m \to n$ are the morphisms $mk \to nk$ of T. The composition operation in Tk is that inherited from T, and the ith projection morphism $n \to 1$ in Tk is $\langle \mathbf{pr}^{nk}_{(i-1)k+1}, \ldots, \mathbf{pr}^{nk}_{ik} \rangle$. If T is equipped with a dagger operation, then Tk is equipped with the dagger operation inherited from T, since if $f : n+p \to n$ in Tk, then f is a morphism $nk + pk \to nk$ in T and we may define $f^\dagger$ in Tk as the morphism $f^\dagger$ in T. For details, see [27].

Theorem 4.4. *When T is a Conway or iteration theory, so is Tk for each k.*

Proof. The claim is clear for Conway theories, since the vector form of each defining identity of Conway theories is also a defining identity. As for iteration theories, by the completeness of the iteration theory identities for the least fixed point operation on continuous functions on cpo's (Theorem 4.2), it suffices to prove that if T is the theory of continuous functions on a cpo P, equipped with the least fixed point operation, then each Tk is an iteration theory. But Tk is isomorphic to the theory of continuous functions over P^k, which is an iteration theory. $\square$

The commutative identity and the group identities seem to be extremely difficult to verify in practice. But in most cases, this is not so. The commutative identity, and thus the group identities are implied by certain quasi-identities, which are usually easy to establish.

Definition 4.5. *Let $\mathcal{C}$ be a set of morphisms in a theory T equipped with a dagger operation. We say that T has a* functorial dagger *with respect to $\mathcal{C}$ if*

$$f \circ (\rho \times \mathbf{id}_p) = \rho \circ g \quad \Longrightarrow \quad f^\dagger = \rho \circ g^\dagger, \tag{23}$$

for all $f : n+p \to n$, $g : m+p \to m$ in T and $\rho : m \to n$ is in $\mathcal{C}$.

When T has a functorial dagger with respect to the set of all base morphisms (all morphisms, respectively), we also say that T has a *weak* (*strong*, respectively) functorial dagger. It is known that every Conway theory has a functorial dagger with respect to the set of injective base morphisms. Moreover, if T has a strong functorial dagger, then it has a unique morphism $0 \to 1$. In [25], it is proved that if a Conway theory has a functorial dagger with respect to the set of base morphisms $1 \to n$, $n \geq 2$, then it has a weak functorial dagger.

Proposition 4.6. *If a Conway theory T has a weak functorial dagger, then T is an iteration theory.*

Proof. We show that under the assumptions, the commutative identity (21) holds. So, let $f : n+p \to 1$ and let $\rho_1, \ldots, \rho_n$ be base morphisms $n \to n$, and let ρ denote the unique base morphism $1 \to n$. Define $g = f \circ (\rho \times \mathbf{id}_p)$ and $h = \langle f \circ (\rho_1 \times \mathbf{id}_p), \ldots, f \circ (\rho_n \times \mathbf{id}_p)\rangle$. Then $h \circ (\rho \times \mathbf{id}_p) = \rho \circ g$, so that $h^\dagger = \rho \circ g^\dagger$, completing the proof. □

For other quasi-identities implying the commutative identity, we refer to [9, 11]. It is known that there exist iteration theories which do not have a weak functorial dagger.

Simpson and Plotkin [48] proved the following equational completeness result for iteration theories. Suppose that T is a nontrivial iteration theory equipped with a dagger operation, so that T has at least two morphisms $2 \to 1$, or equivalently, $\mathbf{pr}_1^2 \neq \mathbf{pr}_2^2$ in T. Then there are two cases. Either an identity holds in T iff it holds in all iteration theories, or it holds in all iteration theories with a unique morphism $0 \to 1$. It was argued in [9, 11] that all fixed point models satisfy at least the iteration theory identities. Thus, by the Plotkin–Simpson result, all nontrivial fixed point models either satisfy exactly the iteration theory identities, or the identities that hold in all iteration theories with a single "constant." Such iteration theories are, for example, the matrix theories over nontrivial iteration semirings defined below. Iteration theories of Boolean functions are described in [28].

5 Unique Fixed Points

Suppose that T is a theory. We say that a morphism $f = \langle f_1, \ldots, f_m\rangle : n \to m$ in T is *ideal* if none of the morphisms $f_i : n \to 1$ is a projection. Following Elgot [22], we call T an *ideal theory* if whenever f is ideal, then for all g in T with appropriate target, $f \circ g$ is ideal.

An important example of an ideal theory can be constructed over *complete metric spaces* $M = (M, d)$, where d denotes a distance function. It is clear that when (M, d) is complete, so is any finite power M^n of M equipped with the distance function d_n defined by $d_n((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max\{d(x_i, y_i) : i = 1, \ldots, n\}$. Now, a function $f : M \to M'$ between metric spaces $M = (M, d)$ and $M' = (M', d')$ is called a *proper contraction* if there is a constant $0 < c < 1$ such that $d'(f(x), f(y)) \leq cd(x, y)$ for all $x, y \in M$. The following simple but important fact is Banach's fixed point theorem [2].

Theorem 5.1. *When M is a complete metric space and $f : M \to M$ is a proper contraction, then f has a unique fixed point.*

Proof. If x, y are both fixed points, then $d(x, y) = d(f(x), f(y)) \leq cd(x, y)$ for some $0 < c < 1$. It follows that $d(x, y) = 0$, i.e., $x = y$.

To show that there is at least one fixed point, let $x_0 \in M$ and define $x_{n+1} = f(x_n)$ for all $n \geq 0$. Since f is a proper contraction, the sequence $(x_n)_n$ is a Cauchy sequence, and since M is complete, it has a limit x. Since f is a proper contraction, it follows that $f(x) = x$. $\square$

Let M be a complete metric space. Consider the collection T_M of all functions $M^n \to M^m$, $n, m \geq 0$ of the form $f = \langle f_1, \ldots, f_m \rangle$ such that each $f_i : M^m \to M$ is a proper contraction *or* a projection. It is clear that T_M is closed under composition and tupling, so that it is a theory of functions over M. Moreover, T_M is an ideal theory, since if M is nontrivial then a function $f = \langle f_1, \ldots, f_m \rangle$ is an ideal morphism iff each f_i is a proper contraction which implies that each component function of $f \circ g$ is also a proper contraction for any g in T_M with appropriate target.

Definition 5.2. *An iterative theory (cf. Elgot [22]) is an ideal theory T equipped with a dagger operation defined on ideal morphisms in $T(n, n + p)$, $n, p \geq 0$ such that for each ideal $f : n + p \to n$, the morphism $f^\dagger : p \to n$ is the unique solution of the fixed point equation $\xi = f \circ \langle \xi, \mathbf{id}_p \rangle$.*

Thus, the fixed point identity (16) and the *unique fixed point rule*

$$f \circ \langle g, \mathbf{id}_p \rangle = g \quad \Longrightarrow \quad g = f^\dagger$$

hold for all ideal $f : n + p \to n$ and all $g : p \to n$ in T.

Remark 5.3. Let T be an ideal theory. We say that $f : n + p \to n$ in T is a *power ideal* morphism if for some $k \geq 1$, f^k is ideal. When $f : n + p \to n$ is a power ideal morphism in an iterative theory T, then the fixed point equation $\xi = f \circ \langle \xi, \mathbf{id}_p \rangle$ has a unique solution, namely the solution of $\xi = f^k \circ \langle \xi, \mathbf{id}_p \rangle$, where f^k is ideal. See [22]. (Here, $f^0 = \mathbf{pr}_n^{n,p}$ and $f^{k+1} = f \circ \langle f^k, \mathbf{pr}_p^{n,p} \rangle$.)

The following result is from [13]; see also [9].

Theorem 5.4. *An ideal theory T is an iterative theory iff for each ideal morphism $f : 1 + p \to 1$ there is a unique solution of the equation $\xi = f \circ \langle \xi, \mathbf{id}_p \rangle$.*

Proof. The proof is based on a version of the pairing identity. One argues by induction. In the induction step, one shows that if the fixed point equation for ideal morphisms $n + q \to n$ and $m + q \to m$ have unique solutions, then the same holds for ideal morphisms $n + m + p \to n + m$. □

In an iterative theory, the dagger operation is only partially defined. In order to be able to solve all fixed point equations over an iterative theory T, there must be at least one morphism $0 \to 1$ in T.

Theorem 5.5. *Suppose that T is an iterative theory with at least one morphism $0 \to 1$. Then for each $\perp : 0 \to 1$, the dagger operation on T has a unique extension to all morphisms $n+p \to n$, $n, p \geq 0$ such that T becomes a Conway theory with $\mathbf{id}_1^\dagger = \perp$. Moreover, equipped with this unique extension, T is an iteration theory having a weak functorial dagger.*

This result was proved in [6, 7] and [24]. Iteration theories arising from Theorem 5.5 are called *pointed iterative theories*. One application of the theorem is the following.

Corollary 5.6. *Suppose that M is a complete metric space and consider the theory T_M defined above. Let x_0 be a point in M. Then there is a unique way to define a dagger operation on T_M such that T_M becomes a Conway theory with $\mathbf{id}_M^\dagger = x_0$. This unique Conway theory is an iteration theory with a weak functorial dagger.*

Without proof, we mention the following theorem.

Theorem 5.7. *An identity holds in all pointed iterative theories iff it holds in iteration theories.*

See [9, 24]. Thus, the equational properties of the least fixed point operation are the same as the equational properties of the unique fixed point operation.

6 Fixed Points of Linear Functions

Let S be a semiring. A function $S^n \to S$ is called *linear* if it is of the form

$$f(x_1, \ldots, x_n) = s_1 x_1 + \cdots + s_n x_n$$

for some $s_1, \ldots, s_n \in S$. A linear function $S^n \to S^m$ is a tupling of linear functions $S^n \to S$. Since any composition of linear functions is linear, it follows that linear functions over S determine a theory T_S.

The linear function f given above may be represented by the n-dimensional row matrix $(s_1, \ldots, s_n)$. More generally, any linear function $S^n \to S^m$ may be represented by an $m \times n$ matrix $M = (s_{ij})_{ij}$ over S: The linear function determined by M maps $x \in S^n$, an n-dimensional column vector to Mx, an m-dimensional column. It follows that T_S can be represented as the theory T with $T(m,n) = S^{m\times n}$, $m,n \geq 0$, the set of all $m \times n$ matrices over S, whose composition operation is matrix product. The projections are the row matrices with an entry equal to 1 and all other entries equal to 0. The identity morphism $\mathbf{id}_n$, $n \geq 0$ is the $n \times n$ unit matrix E_n. We denote this theory by $\mathbf{MAT}_S$ and call it the *matrix theory* over S.

Proposition 6.1. *T_S is isomorphic to $\mathbf{MAT}_S$.*

Note that in $\mathbf{MAT}_S$, a base morphism, also called a *base* or *functional matrix*, is a 0–1 matrix with a single occurrence of 1 in each row (at least when S is nontrivial). In particular, every *permutation matrix* is a base matrix. Note that the inverse of a permutation matrix π is its transpose, π^T. It is educational to see that for all $A \in \mathbf{MAT}_S(p,n)$ and $B \in \mathbf{MAT}_S(q,n)$,

$$\langle A, B\rangle = \begin{pmatrix} A \\ B \end{pmatrix},$$

and if $A \in \mathbf{MAT}_S(p,n)$ and $B \in \mathbf{MAT}_S(q,m)$ then

$$A \times B = \begin{pmatrix} A & 0_{pm} \\ 0_{qn} & B \end{pmatrix},$$

where 0_{pn} and 0_{qm} are zero matrices of appropriate dimension.

Below, we will show that any dagger operation on $\mathbf{MAT}_S$ satisfying the parameter identity determines and is determined by a star operation on $\mathbf{MAT}_S$, and, in fact, on the semiring S. Moreover, we show how to express the Conway identities and the commutative and group identities in terms of the star operation giving rise to Conway matrix theories, matrix iteration theories, and Conway and iteration semirings.

Proposition 6.2. *Suppose that $\mathbf{MAT}_S$ is equipped with a dagger operation such that the parameter identity holds. Then there is a unique* star operation *$A \mapsto A^*$ defined on the square matrices $A \in \mathbf{MAT}_S(n,n)$, $n \geq 0$ such that for all $(A,B) \in \mathbf{MAT}(n, n+p)$ with $A \in \mathbf{MAT}_S(n,n)$ and $B \in \mathbf{MAT}_S(n,p)$*

$$(A, B)^\dagger = A^* B. \tag{24}$$

If $\mathbf{MAT}_S$ is equipped with a star operation and if we define dagger by (24), then the parameter identity holds.

Proof. If the parameter identity (2) holds, then

$$(A \quad B)^\dagger = \left((A \quad E_n) \begin{pmatrix} E_n & 0 \\ 0 & B \end{pmatrix} \right)^\dagger = (A \quad E_n)^\dagger B.$$

Thus, we define $A^* = (A, E_n)^\dagger$. With this definition, (24) holds. Moreover, if $\mathbf{MAT}_S$ is equipped with a star operation and if we define dagger by (24), then the parameter identity holds. $\square$

Theorem 6.3. *Suppose that* $\mathbf{MAT}_S$ *is equipped with both a star and a dagger operation which are related by (24).*

1. *The fixed point identity (1) holds iff the* star fixed point identity *holds:*
$$A^* = AA^* + E_n \tag{25}$$
for all $A \in \mathbf{MAT}_S(n,n)$, $n \geq 0$.
2. *The double dagger identity (4) holds iff the* sum star identity *holds:*
$$(A+B)^* = (A^*B)^*A^* \tag{26}$$
for all $A, B \in \mathbf{MAT}_S(n,n)$, $n \geq 0$.
3. *The composition identity (3) holds iff the* product star identity *holds:*
$$(AB)^* = E_n + A(BA)^*B \tag{27}$$
for all $A \in \mathbf{MAT}_S(n,m)$, $B \in \mathbf{MAT}_S(m,n)$, $m,n \geq 0$.
4. *The identity (10) holds iff the* zero star identity *holds:*
$$0^*_{nn} = E_n, \tag{28}$$
where the entries of the $n \times n$ *matrix* 0_{nn} *are all* 0.
5. *The pairing identity (5) holds iff the* matrix star identity *holds:*
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{29}$$
where $A \in \mathbf{MAT}_S(n,n)$, $B \in \mathbf{MAT}_S(n,m)$, $C \in \mathbf{MAT}_S(m,n)$, *and* $D \in \mathbf{MAT}_S(m,m)$, *and where*
$$\alpha = A^*B\delta CA^* + A^*, \qquad \beta = A^*B\delta,$$
$$\gamma = \delta CA^*, \qquad \delta = (D + CA^*B)^*.$$
6. *The permutation identity (7) holds iff the* star permutation identity *holds:*
$$\left(\pi A \pi^T\right)^* = \pi A^* \pi^T, \tag{30}$$
where $A \in \mathbf{MAT}_S(n,n)$ *and where* $\pi \in \mathbf{MAT}_S(n,n)$ *is a permutation matrix with transpose* π^T.
7. *The transposition identity (12) holds iff the* star transposition identity *holds, i.e., the identity (30) when* $n = p + q$ *and* $\pi = \left(\begin{smallmatrix} 0 & E_p \\ E_q & 0 \end{smallmatrix}\right)$.
8. *The group identity associated with a finite group* G *of order* n *holds iff* star group identity *associated with* G *holds:*
$$e_1 M_G^* u_n = (a_1 + \cdots + a_n)^* \tag{31}$$
where M_G *is the* $n \times n$ *matrix whose* (i,j)*th entry is* $a_{i^{-1}j}$, *for all* $1 \leq i,j \leq n$, *and* $e_1 = \mathbf{pr}_1^n$ *is the* $1 \times n$ 0–1 *matrix whose first entry is* 1 *and whose other entries are* 0. *Finally,* u_n *is the* $n \times 1$ *matrix all of whose entries are* 1.

Definition 6.4. *A* Conway matrix theory (matrix iteration theory) *is a matrix theory* $\mathbf{MAT}_S$ *equipped with a star operation defined on square matrices such that when dagger is defined by (24) then it is a Conway theory (iteration theory, respectively). A morphism of Conway matrix theories or matrix iteration theories is a theory morphism which preserves star.*

It follows that morphisms also preserve the additive structure.

Note that $A^* = A^*A + E_n$ holds for all $n \times n$ matrices A in any Conway matrix theory. As an immediate corollary to Theorem 3.3, Theorem 3.5, Corollary 3.6, and Theorem 6.3 we obtain the following corollary.

Corollary 6.5. *A matrix theory T equipped with a star operation is a Conway theory iff one of the following three groups of identities holds in T.*

1. *The zero star (28), matrix star (29), and star transposition (or star permutation (30)) identities.*
2. *The product star (27) and sum star (26) identities.*
3. *The scalar versions of the product star, sum star, and matrix star identities, i.e., (27) with $m = n = 1$, (26) with $n = 1$, and (29) with $m = 1$.*

Thus, all identities (25)–(30) hold in Conway matrix theories. By adding to the axioms, the star group identities (31) associated with the finite groups, one obtains three sets of equational axioms for matrix iteration theories.

We note the following version of the matrix star identity:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} (A + BD^*C)^* & (A + BD^*C)^*BD^* \\ (D + CA^*B)^*CA^* & (D + CA^*B)^* \end{pmatrix}.$$

When $n = m = 1$, the product star and sum star identities only involve elements of the semiring S. This consideration gives rise to the following definitions; see also [21].

Definition 6.6. *A* *-semiring *is a semiring S equipped with a unary star operation $^* : S \to S$. A Conway semiring [9] is a *-semiring S which satisfies the (scalar) product star and sum star identities, i.e.,*

$$(ab)^* = a(ba)^*b + 1, \tag{32}$$

$$(a + b)^* = (a^*b)^*a^*, \quad a, b \in S. \tag{33}$$

An iteration semiring [9, 27] is a Conway semiring, which when the star of a square matrix is inductively defined by the scalar version of the matrix star identity (i.e., (29) with $m = 1$), satisfies each star group identity (31) associated with a finite group G. A morphism of *-semirings also preserves the star operation. A morphism of Conway or iteration semirings is a *-semiring morphism.

Corollary 6.7. *When* $\mathbf{MAT}_S$ *is a Conway matrix theory or a matrix iteration theory, then* S *is a Conway semiring or an iteration semiring. Suppose that* S *is a Conway semiring (iteration semiring, resp.). Then there is a unique way of extending the star operation on* S *to all square matrices over* S *such that* $\mathbf{MAT}_S$ *becomes a Conway matrix theory, or a matrix iteration theory.*

Proof. This follows from Theorems 3.5 and 6.3. □

Of course, the unique extension is given by the scalar version of the matrix star identity (29) with $m = 1$. Using the above results, one can show that the category of Conway semirings is equivalent to the category of Conway matrix theories, and that the category of iteration semirings is equivalent to the category of matrix iteration theories.

In any Conway matrix theory, the group identities follow from the functorial star conditions defined below.

Definition 6.8. *Suppose that* $\mathbf{MAT}_S$ *is equipped with a star operation. Let* $\mathcal{C}$ *be a set of matrices in* $\mathbf{MAT}_S$*. We say that* $\mathbf{MAT}_S$ *satisfies the* functorial star *implication for* $\mathcal{C}$*, or that* $\mathbf{MAT}_S$ *has a functorial star with respect to* $\mathcal{C}$*, if for all* $A \in \mathbf{MAT}_S(n,n)$ *and* $B \in \mathbf{MAT}_S(m,m)$ *and all* $C \in \mathbf{MAT}_S(n,m)$ *in* $\mathcal{C}$*,*

$$AC = CB \quad \Longrightarrow \quad A^*C = CB^*.$$

When $\mathbf{MAT}_S$ *has a functorial star with respect to the set of all matrices in* $\mathbf{MAT}_S$*, then* $\mathbf{MAT}_S$ *is said to have a strong functorial star. And when* $\mathbf{MAT}_S$ *has a functorial star with respect to the set of all base matrices,* $\mathbf{MAT}_S$ *is said to have a weak functorial star.*

Proposition 6.9. *A Conway matrix theory* $\mathbf{MAT}_S$ *has a functorial star with respect to* $\mathcal{C}$ *iff it has functorial dagger with respect to* $\mathcal{C}$ *when dagger is defined by (24).*

Thus, $\mathbf{MAT}_S$ has a strong or weak functorial star iff it has a strong or weak functorial dagger. Moreover, $\mathbf{MAT}_S$ has a weak functorial star iff it has a weak functorial star with respect to all $n \times 1$ base matrices, $n \geq 2$.

Corollary 6.10. *If* $\mathbf{MAT}_S$ *is a Conway matrix theory with a weak functorial star, then* $\mathbf{MAT}_S$ *is a matrix iteration theory.*

We mention one more property of Conway and iteration semirings. The *dual* of a *-semiring S is equipped with the same sum and star operation and constants as S, but multiplication, denoted $\circ$, is defined by $a \circ b = ba$, the product of a and b in S in the reverse order.

Proposition 6.11. *The dual of a Conway or iteration semiring is also a Conway or iteration semiring.*

See [27]. In the rest of this section, we will exhibit three subclasses of iteration semirings.

6.1 Inductive *-Semirings

Recall from [21] that an *ordered monoid* is a commutative monoid $(M, +, 0)$ such that the sum operation is monotone. An *ordered semiring* is a semiring which is an ordered monoid such that the product operation is also monotone. Moreover, an ordered semiring S is *positively ordered* if $0 \leq s$ for all $s \in S$. A morphism of ordered semirings is a monotone semiring morphism. This section is based on [31].

Definition 6.12. *An* inductive *-semiring *is an ordered semiring S which is a *-semiring such that for any $a, b \in S$, a^*b is the least prefixed point of the function $S \to S$, $x \mapsto ax + b$. A morphism of inductive *-semirings is a morphism of ordered semirings which is a *-semiring morphism.*

Proposition 6.13. *Any inductive *-semiring S is positively ordered.*

Proof. The least solution of the equation $x = x$ is $1^* \cdot 0 = 0$. Since any element of S is a solution, it follows that 0 is the least element of S. □

Proposition 6.14. *When S is an inductive *-semiring, the star operation is monotone.*

Proof. This follows from Proposition 2.8. □

The dual of an inductive *-semiring is not necessarily an inductive *-semiring.

Definition 6.15. *A* symmetric inductive *-semiring *is an inductive *-semiring whose dual is also an inductive *-semiring.*

Proposition 6.16. *An inductive *-semiring S is symmetric iff for all $a, b, x \in S$, if $xa + b \leq x$, then $ba^* \leq x$.*

If S is an ordered semiring, $\mathbf{MAT}_S$ is equipped with the pointwise partial order. It is clear that the theory operations are monotone as is the sum operation on matrices.

Theorem 6.17. *Let S be an ordered semiring which is a *-semiring. Then S is an inductive *-semiring iff $\mathbf{MAT}_S$ is a Park theory when the dagger is defined by (24).*

Proof. This follows from Theorem 6.3 and the Bekić identity (5). □

Corollary 6.18. *Thus, when S is an inductive *-semiring, then for each $A \in \mathbf{MAT}_S(n, n)$ and $B \in \mathbf{MAT}_S(n, p)$, $(A, B)^\dagger = A^*B$ is the least prefixed point solution of the equation $X = AX + B$.*

Corollary 6.19. *Any inductive *-semiring S is an iteration semiring, so that $\mathbf{MAT}_S$ is a matrix iteration theory.*

Proof. By Theorem 4.3. □

Corollary 6.20. *If S is an inductive *-semiring, so is $S^{n\times n}$, for each $n \geq 0$.*

Recall from [21] that an ordered semiring S is *continuous* if S is a cpo with least element 0 and the sum and product operations are continuous.

Proposition 6.21. *Every continuous semiring is a symmetric inductive *-semiring where $a^* = \bigvee\{\sum_{i=1}^n a^i : n \geq 0\}$.*

Proof. This follows from Corollary 2.6. When S is a continuous semiring, then for each $a, b \in S$, the function $f(x) = ax + b$ is continuous with least prefixed point $\bigvee f^n(0, b)$. But for each n, $f^n(0, b) = \sum_{i=1}^n a^i b = (\sum_{i=1}^n a^i)b$, so that by continuity, $\bigvee f^n(0, b) = (\bigvee\{\sum_{i=1}^n a^i : n \geq 0\})b$. Since the dual of a continuous semiring is also continuous, it follows now that any continuous semiring is a symmetric inductive *-semiring. □

Kozen [40] defines a *Kleene algebra* as an idempotent symmetric inductive *-semiring. In [39], it is shown that there is an idempotent inductive *-semiring which is not a Kleene algebra.

Remark 6.22. Kozen showed in [40] that for each alphabet A, the semiring of regular languages over A, equipped with the partial order of set inclusion is the free Kleene algebra on A. Krob [41] proved that the same semiring is the free iteration semiring on A satisfying the identity $1^* = 1$, and thus also the free idempotent inductive *-semiring on A. See also [14, 15] and [10]. For recent extensions of these results, see [12, 29]. It is shown in [12] that for each alphabet A, the *-semiring of rational power series [33] over the semiring $\mathbb{N}^\infty$ is the free iteration semiring over A satisfying three additional simple identities. Moreover, an identity holds in these semirings iff it holds in all continuous (or complete, see below) semirings. And the same semirings, equipped with the sum order, are the free symmetric inductive *-semirings. The paper [12] also contains a characterization of the semirings of rational power series over the semiring $\mathbb{N}$ as the free "*partial iteration semirings*."

6.2 Complete Semirings

For the definition of *complete semirings* and their morphisms, we refer to [21] where original references can be found. When S is a complete semiring, then we may equip each hom-set $\mathbf{MAT}_S(n, m) = S^{n\times m}$ with the pointwise sum operation, so that the straightforward generalizations of the defining axioms of complete semirings hold. In particular,

$$\sum_{j\in J}\sum_{i\in I_j} A_i = \sum_{i\in\bigcup_{j\in J} I_j} A_i,$$
$$B\Big(\sum_{i\in I} A_i\Big) = \sum_{i\in I} BA_i,$$
$$\Big(\sum_{i\in I} A_i\Big)C = \sum_{i\in I} A_iC,$$

where I is any set which is the disjoint union of sets $I_j,\ j\in J$, and where $A_i,\ i\in I$ is a family of matrices in $\mathbf{MAT}_S(m,n)$, and $B\in\mathbf{MAT}_S(p,m)$ and $C\in\mathbf{MAT}_S(n,q)$.

Now, any complete semiring S can be turned into a *-semiring by defining $s^*=\sum s^k$. By the above remark, if S is complete, then each semiring $S^{n\times n}$ is also complete and is thus a *-semiring with $A^*=\sum A^k$, for each $A\in S^{n\times n}$. We can use the star operation on S and the scalar version of the matrix star identity to define another star operation on $S^{n\times n}$. However, the two star operations coincide as noticed in [17, 37]. We have the following result, cf. [9].

Theorem 6.23. *When S is a complete semiring, then S is an iteration semiring with a strong functorial star. Thus, S is an iteration semiring and $\mathbf{MAT}_S$ is a matrix iteration theory.*

In a *rationally additive semiring* S, only certain sums are required to exist including the geometric sums $s^*=\sum_{n\geq 0}s^n$, for all $s\in S$. It is shown in [30] that they are also iteration semirings with a strong functorial star.

6.3 Iterative Semirings

An *ideal* of a semiring S is a set $I\subseteq S$ which is closed under the sum operation and contains 0. Moreover, $SI=IS=I$. Let I be an ideal of S and S_0 a subsemiring of S. Below, we will say that S is the *direct sum* of S_0 and I if each $s\in S$ can be written in a unique way in the form s_0+a, where $s_0\in S_0$ and $a\in I$. The following result is from [9].

Theorem 6.24. *Suppose that S is the direct sum of S_0 and I, where S_0 is a subsemiring of S and I is an ideal. Moreover, suppose that each fixed point equation $x=ax+b$ with $a\in I$ has a unique solution in S. If S_0 is a Conway semiring, then there is a unique way to extend the star operation on S_0 to the whole semiring S such that S becomes a Conway semiring. Moreover, when S_0 is an iteration semiring, then so is S.*

Proof. First, we define a^* for all $a\in I$ as the unique solution of the equation $x=ax+1$. When $a\in I\cap S_0$, the star fixed point identity guarantees that this unique solution is just a^* taken in the Conway semiring S_0. Moreover, it follows that the unique solution of $x=ax+b$ is a^*b, for all b. Then the star operation on S is defined as follows. Given $s\in S$, write s in the unique way as a sum $s_0 + a$ with $s_0 \in S_0$ and $a \in I$. Then by the sum star identity we are forced to define $s^* = (s_0^* a)^* s_0^*$, where s_0^* is taken in S_0 and $s_0^* a$ is in I, since I is an ideal. For more details, the reader is referred to [9]. □

Thus, under the above assumptions, if S_0 is a Conway semiring, then $\mathbf{MAT}_S$ is a Conway matrix theory and if S_0 is an iteration semiring then $\mathbf{MAT}_S$ is a matrix iteration theory. In either case, we have the following proposition.

Proposition 6.25. *Under the assumptions of Theorem 6.24, for any matrices $A \in \mathbf{MAT}_S(n, n)$ and $B \in \mathbf{MAT}_S(n, p)$ such that each entry of A is in I, $A^* B$ is the unique solution of the fixed point equation $X = AX + B$.*

An application of Theorem 6.24 is that if S is a Conway semiring or an iteration semiring, then so is the power series semiring $S\langle\langle A^* \rangle\rangle$ (for the definition of power series semirings, see [21]) for any set A. This follows since $S\langle\langle A^* \rangle\rangle$ is the direct sum of S and the ideal I of proper power series, and when s is a proper power series and r is any power series in $S\langle\langle A^* \rangle\rangle$, the function $x \mapsto sx + r$ is a proper contraction with respect to the complete metric on $S\langle\langle A^* \rangle\rangle$ defined by $d(s, s') = 2^{-n}$, where n is the length of the shortest word w with $(s, w) \neq (s', w)$, for all distinct series s, s'.

Definition 6.26. *We call a semiring S an iterative semiring if S is the direct sum of an iteration semiring S_0 generated by 1 and an ideal I such that each equation $x = ax + b$ with $a \in I$ has a unique solution.*

Corollary 6.27. *Each iterative semiring is an iteration semiring.*

7 Fixed Points of Affine Functions

In this section, we will consider pairs (S, V) consisting of a semiring S and a (left) S-semimodule V. An *affine function* is a function $f : V^n \to V$ of the form

$$f(x_1, \ldots, x_n) = s_1 x_1 + \cdots + s_n x_n + v,$$

where each s_i is in S and v is in V. An affine function $V^n \to V^m$ is a target tupling of affine functions $V^n \to V$. The collection of all affine functions is a theory of functions over V denoted T_V.

Each affine function $V^n \to V^m$ may be represented by a pair (A, v) consisting of a matrix $A \in S^{m \times n}$ and a column vector $v \in V^m$. This representation gives rise to the following definition.

Definition 7.1. *Let (S, V) be a semiring–semimodule pair. The* matricial theory *$\mathbf{Matr}_{S,V}$ [23] over (S, V) has as morphisms $n \to m$ all pairs (A, v) where $A \in \mathbf{MAT}_S(m, n)$ and $v \in V^m$. Composition is defined by*

$$(A, v) \circ (B, w) = (AB, v + Aw),$$

where AB is the usual matrix product and Aw is the action of A on w, i.e., $(Aw)_i = \sum_{j=1}^{n} A_{ij} w_j$. The projection morphism $\mathbf{pr}_i^n$ is the pair $(e_i, 0)$, where e_i is the ith n-dimensional unit row vector considered as a row matrix. Morphisms between matricial theories are theory morphisms which preserve the additive structure.

It can be seen that a morphism $\mathbf{Matr}_{S,V} \to \mathbf{Matr}_{S',V'}$ is completely determined by a semiring morphism $h_S : S \to S'$ and a semimodule morphism $h_V : V \to V'$ such that $(sv)h_V = (sh_S)(vh_V)$ for all $s \in S$ and $v \in V$. Thus, the category of matricial theories is equivalent to the category of semiring–semimodule pairs.

Proposition 7.2. *The theory T_V is a quotient of $\mathbf{Matr}_{S,V}$. A surjective theory morphism $\mathbf{Matr}_{S,V} \to T_V$ maps $(A, v) \in \mathbf{Matr}_{S,V}(m, n)$ to the function $\langle f_1, \dots, f_m \rangle : V^n \to V^m$ with $f_i(u_1, \dots, u_n) = A_{i1} u_1 + \cdots + A_{in} u_n + v_i$, for all i.*

The above morphism is usually not injective. To get a faithful representation, one can use $(A, v) \in \mathbf{Matr}_{S,V}(m, n)$ to induce a function $(S \times V)^n \to (S \times V)^m$. Indeed, we can map $(A, v) \in \mathbf{Matr}_{S,V}(m, n)$ to the function $g = \langle g_1, \dots, g_m \rangle : (S \times V)^n \to (S \times V)^m$, $g_i((x_1, u_1), \dots, (x_n, u_n)) = (A_{i1} x_1 + \cdots + A_{in} x_n, A_{i1} u_1 + \cdots + A_{in} u_n + v_i)$. This mapping $(A, v) \mapsto g$ is always injective.

Below, we will show that when $\mathbf{Matr}_{S,V}$ is equipped with a dagger operation such that it is a Conway or an iteration theory, then the dagger operation determines and is determined by a star and an omega operation satisfying certain natural axioms. For all omitted details we refer to [9]. Each matricial theory $\mathbf{Matr}_{S,V}$ has $\mathbf{MAT}_S$ as its *underlying matrix theory*.

Suppose that $\mathbf{Matr}_{S,V}$ is equipped with a dagger operation. Hence the dagger operation applied to $(A, v) \in \mathbf{Matr}_{S,V}(n, n + p)$ produces $f^\dagger = (C, z)$ where $C \in \mathbf{MAT}_S$ and $z \in V^n$. Two operations are implicitly defined by the dagger operation. For each $A \in \mathbf{MAT}_S$, consider $((A, E_n), 0^n)$, where all entries of $0^n \in V^n$ are 0. Then we define A^* and A^ω by

$$\big((A, E_n), 0^n\big)^\dagger = \big(A^*, A^\omega\big). \tag{34}$$

Thus, $A \mapsto A^*$ is a map $\mathbf{MAT}_S(n, n) \to \mathbf{MAT}_S(n, n)$, and $A \mapsto A^\omega$ is a map from $\mathbf{MAT}_S(n, n)$ to V^n, for each $n \geq 0$.

Theorem 7.3. *Suppose that $\mathbf{Matr}_{S,V}$ is equipped with a dagger operation and that the star and omega operations are defined as above. Then the parameter identity holds in T if and only if the dagger operation is determined by the star and omega operations:*

$$\big((A, B), v\big)^\dagger = \big(A^* B, A^* v + A^\omega\big), \tag{35}$$

for all $((A, B), v) \in \mathbf{Matr}_{S,V}(n, n+p)$ *with* $A \in \mathbf{MAT}_S(n, n)$, $B \in \mathbf{MAT}_S(n, p)$ *and* $v \in V^n$.

Suppose that the dagger, star, and omega operations are related by (35).

- *The star fixed point identity (25) and the* omega fixed point identity

$$AA^\omega = A^\omega, \quad A \in \mathbf{MAT}_S(n, n), \tag{36}$$

hold if and only if the fixed point identity holds.

- *The product star identity (27) and the* product omega identity

$$(AB)^\omega \;=\; A(BA)^\omega, \tag{37}$$

$A \in \mathbf{MAT}_S(n, m)$, $B \in \mathbf{MAT}_S(m, n)$, *hold if and only if the composition identity holds.*

- *The sum star identity and the* sum omega identity

$$(A + B)^\omega = (A^*B)^\omega + (A^*B)^*A^\omega, \tag{38}$$

$A, B \in \mathbf{MAT}_S(n, n)$, *hold if and only if the double dagger identity holds.*

- *The zero star identity and the* zero omega identity

$$0^\omega_{nn} = 0^n \tag{39}$$

hold if and only if (10) holds.

- *The matrix star identity (29) and the* matrix omega identity *(40) hold if and only if the pairing identity holds.*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\omega = \begin{pmatrix} A^*B(D + CA^*B)^\omega + A^*B(D + CA^*B)^*CA^\omega + A^\omega \\ (D + CA^*B)^\omega + (D + CA^*B)^*CA^\omega \end{pmatrix} \tag{40}$$

for all $A \in \mathbf{MAT}_S(n, n)$, $B \in \mathbf{MAT}_S(n, m)$, $C \in \mathbf{MAT}_S(m, n)$, $D \in \mathbf{MAT}_S(m, m)$.

- *The star permutation identity (30) and the* omega permutation identity *(41) hold if and only if the permutation identity holds.*

$$\left(\pi A \pi^T\right)^\omega = \pi A^\omega, \tag{41}$$

where $\pi \in \mathbf{MAT}_S(n, n)$ *is a permutation matrix and* $A \in \mathbf{MAT}_S(n, n)$.

- *The star transposition identity and the* omega transposition identity *hold iff the transposition identity holds, where the omega transposition identity is (41) with* π *restricted to matrices of the form* $\left(\begin{smallmatrix} 0 & E_p \\ E_q & 0 \end{smallmatrix}\right)$.
- *The star group identity (31) and the* omega group identity *(42) associated with a finite group* G *hold if and only if the group identity associated with* G *holds.*

$$e_1 M_G^\omega = (a_1 + \cdots + a_n)^\omega \tag{42}$$

where $a_1, \ldots, a_n \in S$ *and* M_G *is defined above.*

Conversely, if $\mathbf{Matr}_{S,V}$ *is a matricial theory equipped with star and omega operations defined for all square matrices in* $\mathbf{MAT}_S$*, and if the dagger operation is defined by (35), then* $\mathbf{Matr}_{S,V}$ *satisfies the parameter identity and all of the above equivalences hold.*

If the star and omega sum and product identities hold, then the omega pairing identity can be expressed in either of the following two forms:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\omega = \begin{pmatrix} (A+BD^*C)^\omega + (A+BD^*C)^*BD^\omega \\ (D+CA^*B)^\omega + (D+CA^*B)^*CA^\omega \end{pmatrix},$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\omega = \begin{pmatrix} (A^*BD^*C)^*A^\omega + (A^*BD^*C)^*A^*BD^\omega + (A^*BD^*C)^\omega \\ (D^*CA^*B)^*D^*CA^\omega + (D^*CA^*B)^*D^\omega + (D^*CA^*B)^\omega \end{pmatrix}.$$

Definition 7.4. *A* matricial iteration theory *is a matricial theory which is also an iteration theory. A* Conway matricial theory *is a matricial theory which is a Conway theory. Morphisms of matricial iteration theories and Conway matricial theories are matricial theory morphism which preserve dagger and thus star and omega.*

The following results follow from Theorem 7.3, and the axiomatization results in Sect. 3 (Theorems 3.3 and 3.5).

Corollary 7.5. *Let* $\mathbf{Matr}_{S,V}$ *be a matricial theory. Suppose that either* $\mathbf{Matr}_{S,V}$ *is equipped with a star and an omega operation and dagger is defined by (35), or that* $\mathbf{Matr}_{S,V}$ *is equipped with a dagger operation satisfying the parameter identity in which the star and omega operations are defined by (34). Then* T *is a Conway matricial theory if and only if* T *satisfies either of the following groups of equational axioms:*

1. *The zero star (28) and zero omega (39) identities, the matrix star (29) and matrix omega (40) identities, and the star and omega transposition identities.*
2. *The sum and product star and omega identities, (26), (27), (37), (38).*
3. *The scalar versions of the sum and product star and omega identities (i.e., the identities (26), (38), (27), and (37) with* $n = m = 1$*), and the scalar version of the matrix star and matrix omega identities, i.e., (29) and (40) with* $m = 1$*.*

Moreover, $\mathbf{Matr}_{S,V}$ *is a matricial iteration theory iff it is a Conway matricial theory satisfying the star and omega group identities (31), (42) associated with finite groups. In either case, the dagger, star, and omega operations are related by (34) and (35).*

Note that the star and omega fixed point identities hold in any Conway matricial theory. Also, if $\mathbf{Matr}_{S,V}$ is a Conway matricial theory, then $\mathbf{MAT}_S$ is a Conway matrix theory, and if $\mathbf{Matr}_{S,V}$ is a matricial iteration theory then $\mathbf{MAT}_S$ is a matrix iteration theory.

If $\mathbf{MAT}_S$ is equipped with a star operation, we may equip $\mathbf{Matr}_{S,V}$ with an omega operation such that $A^\omega = 0^n$, for all $A \in \mathbf{MAT}_S(n,n)$. When $\mathbf{MAT}_S$ is a matrix iteration theory, $\mathbf{Matr}_{S,V}$ is a matricial iteration theory. Similarly, if $\mathbf{MAT}_S$ is a Conway matrix theory, $\mathbf{Matr}_{S,V}$ is a Conway matricial theory. In particular, any matrix iteration theory $\mathbf{MAT}_S$ may be viewed as the matricial iteration theory $\mathbf{Matr}_{S,V}$ where $V = \{0\}$ is the trivial S-semimodule.

In a matricial theory $\mathbf{Matr}_{S,V}$, any morphism $0 \to 1$ may be identified with an element of V. Similarly, each morphism $1 \to 1$ in the underlying matrix theory $\mathbf{MAT}_S$ may be considered to be an element of the semiring S. Thus, when the matrix star and omega identities hold, the star and omega operations are determined by operations $^* : S \to S$ and $^\omega : S \to V$.

Definition 7.6. *A* Conway semiring–semimodule pair *consists of a Conway semiring S, an S semimodule V and an operation $^\omega : S \to V$ which satisfies the sum and product omega identities*

$$(a+b)^\omega = (a^*b)^*a^\omega + (a^*b)^\omega, \tag{43}$$
$$(ab)^\omega = a(ba)^\omega \tag{44}$$

for all a,b in S. An iteration semiring–semimodule pair *is a Conway semiring–semimodule pair such that S is an iteration semiring, which when star and omega on matrices are defined by the matrix star and matrix omega identities (29) and (40) with $m = 1$, satisfies the omega group identity associated with any finite group. Morphisms of Conway and iteration semiring–semimodule pairs are morphisms of semiring–semimodule pairs which preserve star and omega.*

Proposition 7.7.

- *When $\mathbf{Matr}_{S,V}$ is a matricial iteration theory, (S,V) is an iteration semiring–semimodule pair, and when $\mathbf{Matr}_{S,V}$ is a Conway matricial theory, (S,V) is a Conway semiring–semimodule pair.*
- *Let (S,V) be an iteration (or Conway) semiring–semimodule pair. There is a unique way to extend the star and omega operations on S to all square matrices in $\mathbf{MAT}_S$ so that $\mathbf{Matr}_{S,V}$ becomes a matricial iteration theory (or Conway matricial theory, respectively).*

In fact, the category of Conway matricial theories is equivalent to the category of Conway semiring–semimodule pairs, and the category of matricial iteration theories is equivalent to the category of iteration semiring–semimodule pairs.

In any Conway matricial theory, the group identities follow from the functorial star and omega conditions.

Definition 7.8. *Suppose that $\mathbf{Matr}_{S,V}$ is equipped with a star and omega operation. Let $\mathcal{C}$ be a set of matrices in $\mathbf{MAT}_S$. We say that $\mathbf{Matr}_{S,V}$ satisfies the functorial star implication for $\mathcal{C}$, or has a functorial star with respect to $\mathcal{C}$ if*

$\mathbf{MAT}_S$ *does. We say that* $\mathbf{Matr}_{S,V}$ *satisfies the functorial omega implication for* $\mathcal{C}$*, or that* $\mathbf{Matr}_{S,V}$ *has a functorial omega with respect to* $\mathcal{C}$*, if for all* $A \in \mathbf{MAT}_S(n,n)$ *and* $B \in \mathbf{MAT}_S(m,m)$ *and all* $C \in \mathbf{MAT}_S(n,m)$ *in* $\mathcal{C}$*,*

$$AC = CB \quad \Longrightarrow \quad A^\omega = CB^\omega .$$

When $\mathbf{Matr}_{S,V}$ *has a functorial star and omega with respect to the set of all matrices (all base matrices, respectively) in* $\mathbf{MAT}_S$*, then* $\mathbf{Matr}_{S,V}$ *is said to have a strong functorial star and omega (*weak functorial star and omega*, respectively).*

Proposition 7.9. *Suppose that* $\mathbf{Matr}_{S,V}$ *is a Conway matricial theory.*

- *For any set* $\mathcal{C} \subseteq \mathbf{MAT}_S$*,* $\mathbf{Matr}_{S,V}$ *has a functorial dagger with respect to* $\mathcal{C}$ *if and only if* $\mathbf{Matr}_{S,V}$ *has a functorial star and omega with respect to* $\mathcal{C}$*.*
- $\mathbf{Matr}_{S,V}$ *has a functorial star and omega with respect to all injective base matrices.*
- *If* $\mathbf{Matr}_{S,V}$ *has a functorial star and omega with respect to all* $n \times 1$ *base matrices,* $n \geq 2$*, then the star and omega group identities hold in* $\mathbf{Matr}_{S,V}$*.*
- $\mathbf{Matr}_{S,V}$ *has a weak functorial star and omega if and only if* $\mathbf{Matr}_{S,V}$ *has a functorial star and omega with respect to all* $n \times 1$ *base matrices,* $n \geq 2$*.*

Corollary 7.10. *Any Conway matricial theory with a weak functorial star and omega is a matricial iteration theory.*

We end this section by exhibiting two classes of iteration semiring–semimodule pairs.

7.1 Complete Semiring–Semimodule Pairs

This section is based on [32]. Recall the definition of a complete monoid and that of a complete semiring. We call a semiring–semimodule pair (S, V) a *complete semiring–semimodule pair* if S is a complete semiring, V is a complete monoid, and the action is *completely distributive*, so that $(\sum_{i\in I} s_i)(\sum_{j\in J} v_j) = \sum_{(i,j)\in I\times J} s_i v_j$. Moreover, we require that an *infinite product operation* $S \times S \times \cdots \to S$,

$$(s_1, s_2, \ldots) \mapsto \prod_{j\geq 1} s_j$$

is given mapping infinite sequences over S to V subject to the following conditions:

$$\prod_{j\geq 1} s_j = \prod_{j\geq 1} (s_{n_{j-1}+1} \cdots s_{n_j}), \tag{45}$$

$$s_1 \cdot \prod_{i\geq 1} s_{i+1} = \prod_{i\geq 1} s_i, \tag{46}$$

$$\prod_{j\geq 1} \sum_{i_j \in I_j} s_{i_j} = \sum_{(i_1,i_2,\ldots)\in I_1\times I_2\times\cdots} \prod_{j\geq 1} s_{i_j}, \tag{47}$$

where in the first equation $0 = n_0 \leq n_1 \leq n_2 \leq \cdots$ and $I_1, I_2, \ldots$ are sets. (Complete semimodules of complete semirings without an infinitary product operation on the semiring are studied in Chap. 23 of [35]. When (S, V) is a complete semiring–semimodule pair, then equipped only with the binary and infinitary multiplication operations, (S, V) is an *ω-semigroup* [46].)

Suppose that (S, V) is complete. Then we define

$$s^* = \sum_{i \geq 0} s^i \quad \text{and} \quad s^\omega = \prod_{i \geq 1} s,$$

for all $s \in S$.

Theorem 7.11. *Every complete semiring–semimodule pair (S, V) is an iteration semiring–semimodule pair.*

Proof. We already know that S is an iteration semiring. We establish the sum omega and product omega identities and leave the proof of the group identities to the reader. So, suppose that $a, b \in S$. We also consider the set $\{a, b\}$ as an alphabet Σ. When w is a finite or infinite word over this alphabet, we let $\overline{w}$ denote the corresponding product over S which is either an element of S (finite product) or an element in V (infinite product). Our proof of the sum omega identity uses the fact that $\{a, b\}^\omega = (\{a\}^*\{b\})^\omega \cup (\{a\}^*\{b\})^*\{a\}^\omega = K \cup L$ holds over the alphabet Σ.[3]

$$\begin{aligned}
(a+b)^\omega &= \prod_{j \geq 1} (a+b) \\
&= \sum_{w \in \{a,b\}^\omega} \overline{w} \\
&= \sum_{u \in K} \overline{u} + \sum_{v \in L} \overline{v} \\
&= \prod_{j \geq 1} \sum_{u \in \{a\}^*\{b\}} \overline{u} + \Big(\sum_{v \in (\{a\}^*\{b\})^*} \overline{v} \Big) \prod_{j \geq 1} a \\
&= (a^*b)^\omega + (a^*b)^* a^\omega .
\end{aligned}$$

As for the product omega identity, let $c_j = a$ if $j \geq 1$ is odd, and let $c_j = b$ if $j \geq 1$ is even. Then

$$(ab)^\omega = \prod_{j \geq 1} (ab) = \prod_{j \geq 1} c_j = a \prod_{j \geq 2} c_j = a(ba)^\omega. \quad \square$$

Thus, $\mathbf{Matr}_{S,V}$ is a matricial iteration theory, so that when the dagger is defined by (24), then all iteration theory identities hold over any complete semiring–semimodule pair. Without proof, we mention the following proposition.

[3] Here, for any language X of nonempty finite words, we denote by X^ω the set $\{x_1 x_2 \ldots : x_i \in X\}$ of ω-words.

Proposition 7.12. *When (S,V) is a complete semiring–semimodule pair then for each n, $(S^{n\times n},V^n)$ is also a complete semimodule pair with infinitary product such that for each $A_1,A_2,\ldots \in S^{n\times n}$, and for each i, the ith entry of $A_1\cdot A_2\cdots$ is the sum of all elements of the form $(A_1)_{i,j_1}\cdot(A_2)_{j_1,j_2}\cdots$. Moreover, for each $A\in \mathbf{MAT}_S(n,n)$, A^ω in $\mathbf{Matr}_{S,V}$ is the same as A^ω in the complete semiring–semimodule pair $(S^{n\times n},V^n)$.*

7.2 Bi-inductive Semiring–Semimodule Pairs

This section is based on [32]. We call a semiring–semimodule pair (S,V) ordered if S is an ordered semiring and V is an ordered monoid, ordered by $\leq$, such that $sv\leq s'v'$ whenever $s\leq s'$ in S and $v\leq v'$ in V.

Definition 7.13. *Suppose that (S,V) is an ordered semiring–semimodule pair equipped with a star operation $^*: S\to S$ and an omega operation $^\omega : S\to V$ such that*

$$aa^*+1\leq a^* \tag{48}$$

$$ax+y\leq x \quad\implies\quad a^*y\leq x, \tag{49}$$

for all $a\in S$ and $x,y\in S$ or $x,y\in V$, and

$$aa^\omega \geq a^\omega \tag{50}$$

$$ax+y\geq x \quad\implies\quad a^\omega + a^*y\geq x, \tag{51}$$

for all $a\in S$ and $x,y\in V$. Then we call (S,V) a bi-inductive semiring–semimodule pair. *A morphism of bi-inductive semiring–semimodule pairs is a morphism of semiring–semimodule pairs which preserves the order and the star and omega operations.*

The terminology is due to the fact that bi-inductive semiring–semimodule pairs satisfy both an induction axiom (49) and a coinduction axiom (51). Affine functions $x\mapsto ax+v$ over V have both a least prefixed point and a greatest post-fixed point, namely a^*v and $a^\omega + a^*v$, where a^* is the least prefixed point solution of $x=ax+1$ over S and a^ω is the greatest post-fixed point solution of $x=ax$ over V. Note that if (S,V) is a bi-inductive semiring–semimodule pair then S is an inductive *-semiring.

Proposition 7.14. *If (S,V) is bi-inductive, then 0 is the least and 1^ω is the greatest element of V.*

Proof. The fact that 0 is least follows from Proposition 6.13. The fact that 1^ω is the greatest element of V follows by noting that any element of V solves the equation $x=1x$. □

Theorem 7.15. *Every bi-inductive semiring–semimodule pair (S,V) is an iteration semiring–semimodule pair. Moreover, the star and omega operations are monotone.*

Proof. We already know that S is an iteration semiring and that * is monotone. The fact that $^\omega$ is monotone follows from (the dual of) Proposition 2.8.

We prove that the product omega identity holds. Indeed, if $a, b \in S$, then $aba(ba)^\omega = a(ba)^\omega$, thus $(ab)^\omega \geq a(ba)^\omega$. Thus, $(ab)^\omega \geq a(ba)^\omega \geq ab(ab)^\omega = (ab)^\omega$, proving $(ab)^\omega = a(ba)^\omega$.

Next, we prove that the sum omega identity holds. Given $a, b \in S$,

$$\begin{aligned}
&(a+b)[(a^*b)^*a^\omega + (a^*b)^\omega]\\
&\quad = a(a^*b)^*a^\omega + a(a^*b)^\omega + b(a^*b)^*a^\omega + b(a^*b)^\omega\\
&\quad = a[a^*(ba^*)^*b + 1]a^\omega + aa^*(ba^*)^\omega + (ba^*)^*ba^\omega + (ba^*)^\omega\\
&\quad = aa^*(ba^*)^*ba^\omega + aa^\omega + aa^*(ba^*)^\omega + (ba^*)^*ba^\omega + (ba^*)^\omega\\
&\quad = (aa^* + 1)(ba^*)^*ba^\omega + (aa^* + 1)(ba^*)^\omega + a^\omega\\
&\quad = [a^*(ba^*)^*b + 1]a^\omega + a^*(ba^*)^\omega\\
&\quad = (a^*b)^*a^\omega + (a^*b)^\omega.
\end{aligned}$$

It follows by (51) that $(a+b)^\omega \geq (a^*b)^*a^\omega + (a^*b)^\omega$. As for the reverse inequality, note that for all $x \in V$, if $(a+b)x = ax + bx \geq x$, then $a^\omega + a^*bx \geq x$, so that $(a^*b)^\omega + (a^*b)^*a^\omega \geq x$. Taking $x = (a+b)^\omega$, we have $(a^*b)^\omega + (a^*b)^*a^\omega \geq (a+b)^\omega$.

We omit the verification of the group identities. □

Thus, when (S, V) is a bi-inductive semiring–semimodule pair, then $\mathbf{Matr}_{S,V}$ is an iteration semiring–semimodule pair. Thus, $\mathbf{Matr}_{S,V}$ is a matricial iteration theory, so that when dagger is defined by (24), then all iteration theory identities hold.

Theorem 7.16. *Suppose that (S, V) is a bi-inductive semiring–semimodule pair. Then for any $(A, v) : n \to n$ in $\mathbf{Matr}_{S,V}$, A^*v is the least prefixed point solution and $A^\omega + A^*v$ is the greatest post-fixed point solution of $x = Ax + v$. Thus, each $(S^{n\times n}, V^n)$ is also a bi-inductive semiring–semimodule pair.*

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